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Stability analysis of asymptotic profiles for sign-changing solutions to fast diffusion equations

Goro Akagi* and Ryuji Kajikiya†

Abstract

Every solution $u = u(x,t)$ of the Cauchy-Dirichlet problem for the fast diffusion equation, $\partial_t(|u|^{m-2}u) = \Delta u$ in $\Omega \times (0,\infty)$ with a smooth bounded domain Ω of $\mathbb{R}^N$ and $2 < m < 2^* := 2N/(N-2)_+$, vanishes in finite time at a power rate. This paper is concerned with asymptotic profiles of sign-changing solutions and a stability analysis of the profiles. Our method of proof relies on a detailed analysis of a dynamical system on some surface in the usual energy space as well as energy method and variational method.

Keywords: fast diffusion equation; asymptotic profile; stability analysis.

MSC: 35K67, 35B40, 35B35

1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We are concerned with the Cauchy-Dirichlet problem for fast diffusion equations of

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the form

$$\partial_t \left(|u|^{m-2}u\right) = \Delta u \qquad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

$$u = 0 \qquad \text{on } \partial\Omega \times (0,\infty), \tag{1.2}$$

$$u(\cdot,0) = u_0 \qquad \text{in } \Omega, \tag{1.3}$$

where $\partial_t = \partial/\partial t$ and $m > 2$. By putting $w = |u|^{m-2}u$, equation (1.1) can be rewritten in a usual form of fast diffusion equation,

$$\partial_t w = \Delta \left(|w|^{r-2}w\right) \qquad \text{in } \Omega \times (0,\infty) \tag{1.4}$$

with the exponent $r = m/(m-1) \in (1,2)$. Fast diffusion equations arise in the studies of plasma physics, kinetic theory of gases, solid state physics (see [4], [6] and also [36]), and moreover, there are a vast amount of contributions (see [25, 26, 28], [16], [15], [39], [44], [13], [35], [19], [20], [18], [17], [29] and so on). Sabinina [37] studied the one-dimensional case and found that classical solutions vanish at a finite time t_*, which is called *extinction time*, depending on initial data, and moreover, the existence of finite extinction time was extended to more general cases (see [3], [14], [24]). Berryman and Holland [5] established the optimal extinction rate of solutions $u = u(x,t)$ vanishing at a finite time t_* under $m \leq 2^* := 2N/(N-2)_+$, more precisely, it holds that

$$c_1(t_* - t)^{1/(m-2)} \leq \|u(t)\|_{H^1_0(\Omega)} \leq c_2(t_* - t)^{1/(m-2)} \qquad \text{for all } t \in [0, t_*]$$

with $c_1, c_2 > 0$, provided that $u_0 \not\equiv 0$. Moreover, they revealed asymptotic profiles $\phi(x) := \lim_{t \nearrow t_*}(t_* - t)^{-1/(m-2)}u(x,t)$ of smooth positive solutions and proved the stability of ϕ only for nonnegative solutions. Kwong [27] extended the results of [5] to nonnegative weak solutions. Furthermore, Savaré and Vespri proved the convergence in $L^m(\Omega)$ of $(t_* - t)^{-1/(m-2)}u(x,t)$ as $t \nearrow t_*$ for sign-changing solutions in [38], where a generalized equation is treated. We also refer the reader to [41] as a survey of this field. However, there seems no contribution to the stability analysis of asymptotic profiles for sign-changing solutions. This paper addresses such an extinction property as well as a stability analysis of asymptotic profiles for sign-changing energy solutions to the fast diffusion equation (1.1)–(1.3).

In Section 2, we summarize preliminary facts on H^{-1}-solutions $u = u(x,t)$ for (1.1)–(1.3) as well as the extinction time $t_* = t_*(u_0)$ of u. In Section 3, we revisit asymptotic profiles of sign-changing H^{-1}-solutions for (1.1)–(1.3) which vanish at a finite time $t_* = t_*(u_0)$. To this end, we use the following change of variables,

$$v(x,s) = (t_* - t)^{-1/(m-2)}u(x,t) \quad \text{with} \quad s = \log(t_*/(t_* - t)), \tag{1.5}$$

and investigate the convergence of $v(x,s)$ as $s \to \infty$ along a subsequence. Then (1.1)–(1.3) will be rewritten as

$$\partial_s \left(|v|^{m-2}v\right) - \Delta v = \lambda_m |v|^{m-2} v \qquad \text{in } \Omega \times (0,\infty), \tag{1.6}$$

$$v = 0 \qquad \text{on } \partial\Omega \times (0,\infty), \tag{1.7}$$

$$v(\cdot,0) = v_0 \qquad \text{in } \Omega \tag{1.8}$$

with $\lambda_m := (m-1)/(m-2) > 0$ and the initial data

$$v_0 = t_*(u_0)^{-1/(m-2)} u_0. \tag{1.9}$$

Here it is noteworthy that the functional $J : H^1_0(\Omega) \to \mathbb{R}$ given below acts as a Lyapunov functional for this system:

$$J(w) := \frac{1}{2} \int_\Omega |\nabla w(x)|^2 dx - \frac{\lambda_m}{m} \int_\Omega |w(x)|^m dx \quad \text{ for } \ w \in H^1_0(\Omega).$$

By using a standard energy method, one can prove that $v(s)$ converges in $H^1_0(\Omega)$ as $s \to \infty$ along a subsequence to a stationary solution ϕ of (1.6)–(1.8), which solves

$$-\Delta \phi = \lambda_m |\phi|^{m-2} \phi \qquad \text{in } \Omega, \tag{1.10}$$

$$\phi = 0 \qquad \text{on } \partial\Omega \tag{1.11}$$

and plays as a critical point of J, and therefore ϕ is an asymptotic profile of $u = u(x,t)$ as $t \nearrow t_*$.

In Section 4, we investigate the dynamical system generated by (1.6)–(1.8) on the peculiar phase space

$$\mathcal{X} := \{t_*(u_0)^{-1/(m-2)} u_0;\ u_0 \in H^1_0(\Omega) \setminus \{0\}\},$$

which is equivalently rewritten by $\mathcal{X} = \{v_0 \in H^1_0(\Omega);\ t_*(v_0) = 1\}$ (see Proposition 4.2) to obtain several important facts to be used in our stability analysis of asymptotic profiles. Moreover, we also provide some representation of extinction time for solutions of separable form as an independent interest. Furthermore, a by-product of our analysis also completely classifies $H^1_0(\Omega)$ into the stable and unstable sets and their separatrix in terms of large-time behaviors of solutions for the Cauchy-Dirichlet problem (1.6)–(1.8). Then the set $\mathcal{X}$ acts as the separatrix and contacts the Nehari manifold associated with the functional J only at the set of non-trivial solutions of (1.10), (1.11) (see also Proposition 4.11 and Remark 4.12).

In Section 5, we perform a stability analysis of asymptotic profiles for (1.1)–(1.3). We first give definitions for the (asymptotic) stability and instability of profiles, which coincide with those in Lyapunov's sense of equilibria of the dynamical system generated by (1.6)–(1.8) in the phase space $\mathcal{X}$. Our analysis performed in Section 4 will be crucial here to investigate the restriction by $\mathcal{X}$ on behaviors of solutions to (1.6)–(1.8). Due to these materials, we prove that every least energy solution ϕ of (1.10), (1.11) is a stable (resp., asymptotically stable) profile, if ϕ is isolated in $H^1_0(\Omega)$ from all the other least energy (resp., sign-definite) solutions. In particular, if (1.10), (1.11) has a unique positive solution, it is an asymptotically stable profile (see §5.3 for known results on the uniqueness of positive solutions of (1.10), (1.11)). Moreover, we also prove that every sign-changing solution ϕ of (1.10), (1.11) is not an asymptotically stable profile. In addition, if ϕ is isolated in $H^1_0(\Omega)$ from all the other nontrivial solutions of (1.10), (1.11) with lower energies than $J(\phi)$, it is an unstable profile. Furthermore, we prove that sign-changing least energy solutions of (1.10), (1.11) are always unstable profiles. In the final subsection, we study the one dimensional case and give a representation of all stationary solutions of (1.6)–(1.8). By using it, we prove that any solution $v(s)$ of (1.6)–(1.8) with $N = 1$ converges to a stationary solution as $s \to \infty$, and furthermore, a positive or negative stationary solution is asymptotically stable and all the other solutions are unstable.

Here we emphasize that (isolated) positive and negative solutions of (1.10), (1.11) are asymptotically stable equilibria of the dynamical system generated by (1.6)–(1.8) on the phase space $\mathcal{X}$. In contrast, nontrivial solutions (particularly, positive and negative solutions) of (1.10), (1.11) are saddle points of J in $H^1_0(\Omega)$, and hence they could be unstable equilibria of the dynamical system in the whole $H^1_0(\Omega)$. One can say that the geometry of $\mathcal{X}$ peculiarly affects the stability and instability of equilibria.

As for porous medium equations, i.e., the case of $1 < m < 2$, solutions decay at the optimal rate, $t^{1/(m-2)}$, as $t \to \infty$, and moreover, asymptotic profiles of solutions have already been studied also for sign-changing data (see [2], [40] and also [42] as a survey of this field). In this case, by applying the change of variables, $v(x,s) = (t+1)^{1/(m-2)}u(x,t)$ and $s = \log(t+1)$, one also treats (1.6)–(1.8); then, v_0 coincides with u_0.

Notation. We write $(s)_+ := \max\{s, 0\} \geq 0$ for $s \in \mathbb{R}$. Let $H^1_0(\Omega)$ be the closure of $C^\infty_0(\Omega)$ in the usual Sobolev space $H^1(\Omega) = W^{1,2}(\Omega)$. Let us denote by $\|\cdot\|_m$ the usual norm of L^m-spaces, and moreover, $\|u\|_{1,2} := \|\nabla u\|_2$ and $\|\cdot\|_{-1,2}$ stand for the norms of $H^1_0(\Omega)$ and of its dual space $H^{-1}(\Omega)$, respectively. The value of a functional $u \in H^{-1}(\Omega)$ at $v \in H^1_0(\Omega)$ is denoted by $\langle u, v\rangle_{H^1_0}$. We put $F := (-\Delta)^{-1}$, where Δ is the Laplace operator from $H^1_0(\Omega)$ into $H^{-1}(\Omega)$, to be a duality mapping, i.e., $\langle w, F(w)\rangle_{H^1_0} = \|w\|^2_{-1,2} = \|F(w)\|^2_{1,2}$ for $w \in H^{-1}(\Omega)$.

We set constants: $\lambda_m := (m-1)/(m-2) > 0$,

$$2^* := \frac{2N}{N-2} \quad \text{if} \quad N \geq 3 \qquad \text{and} \qquad 2^* = \infty \quad \text{if} \quad N = 1, 2,$$

where 2^* is the so-called Sobolev critical exponent associated with the continuous embedding $H^1(\Omega) \hookrightarrow L^m(\Omega)$ for $m \in [1, 2^*]$. Furthermore, define the Rayleigh quotient,

$$R(w) := \frac{\|w\|_{1,2}}{\|w\|_m} \quad \text{for} \quad w \in H_0^1(\Omega) \setminus \{0\},$$

associated with the Sobolev-Poincaré inequality

$$\|w\|_m \leq C_m \|w\|_{1,2} \quad \text{for } w \in H_0^1(\Omega), \tag{1.12}$$

provided that $m \in [1, 2^*]$, with the best possible constant C_m which is the supremum of $R(w)^{-1}$ over $w \in H_0^1(\Omega) \setminus \{0\}$.

2 Preliminary facts on H^{-1}-solutions

In this paper we are concerned with H^{-1}-solutions for (1.1)–(1.3) defined by

Definition 2.1 (H^{-1}-solutions). *A function $u : \Omega \times (0, \infty) \to \mathbb{R}$ is said to be an H^{-1}-solution (simply, solution) for* (1.1)–(1.3) *if the following conditions are all satisfied*:

- $u \in L^2(0, T; H_0^1(\Omega)) \cap L^m(0, T; L^m(\Omega))$ *and* $|u|^{m-2}u \in W^{1,2}(0, T; H^{-1}(\Omega))$ *for all* $T > 0$.
- *It follows that*

$$\langle \left(|u|^{m-2}u\right)'(t), \phi \rangle_{H_0^1} + \int_\Omega \nabla u(x,t) \cdot \nabla \phi(x) dx = 0 \tag{2.1}$$

$$\text{for a.a. } t \in (0, \infty) \text{ and for all } \phi \in H_0^1(\Omega),$$

  where $\langle \cdot, \cdot \rangle_{H_0^1}$ *denotes the duality pairing between* $H_0^1(\Omega)$ *and its dual space and* $' = d/dt$.
- $|u|^{m-2}u(\cdot, t) \to |u_0|^{m-2}u_0$ *strongly in* $H^{-1}(\Omega)$ *as* $t \to +0$.

The existence and uniqueness of H^{-1}-solutions have already been proved for any $|u_0|^{m-2}u_0 \in H^{-1}(\Omega)$ by using the theory of evolution equations due to Brézis [7]. As pointed out in [3], every solution $u = u(x,t)$ vanishes at a finite time t_* and remains to be zero for $t \geq t_*$. Throughout this paper we call t_* the *extinction time* of u. From the uniqueness of a solution, one can write $t_* = t_*(u_0)$.

Furthermore, we can derive energy inequalities in a standard way.

Proposition 2.2 (Energy inequalities). *Let u be an H^{-1}-solution for* (1.1)–(1.3) *with $u_0 \in H^1_0(\Omega) \cap L^m(\Omega)$. Then u belongs to $C([0,\infty); H^1_0(\Omega) \cap L^m(\Omega))$ such that $u(0) = u_0$ and the following energy inequalities hold for a.a. $t \in (0,\infty)$:*

$$\frac{1}{m'} \frac{d}{dt} \|u(t)\|_m^m + \|u(t)\|_{1,2}^2 = 0, \tag{2.2}$$

$$\mu_m \left\| \frac{d}{dt} |u|^{(m-2)/2} u(t) \right\|_2^2 + \frac{1}{2} \frac{d}{dt} \|u(t)\|_{1,2}^2 \leq 0, \tag{2.3}$$

with the constant $\mu_m := 4/(mm') > 0$ and $m' := m/(m-1)$.

Identity (2.2) immediately follows from the multiplication of (1.1) and $u(t)$. To derive (2.3), we first establish a corresponding energy inequality by testing suitable approximations of (1.1) on $\partial_t u$, and then derive the convergence of such approximations. Here, we also exploited the following elementary inequality:

$$\mu_m \left| |a|^{(m-2)/2} a - |b|^{(m-2)/2} b \right|^2 \leq (a-b) \left(|a|^{m-2} a - |b|^{m-2} b \right) \quad \text{for all } a, b \in \mathbb{R},$$

provided that $m \in (1,\infty)$.

As in [5] and [25] (see also [16], [38]), one can prove that

$$R(u(t)) \text{ is nonincreasing for } t \geq 0, \tag{2.4}$$

as well as the following proposition.

Proposition 2.3 (Estimates from above and below). *For $m > 2$, let u be an H^{-1}-solution of* (1.1)–(1.3) *with an initial data $u_0 \in H^1_0(\Omega) \cap L^m(\Omega) \setminus \{0\}$. Let t_* be the extinction time of u. Then it follows that*

$$\|u(t)\|_m \leq \left(\frac{R(u_0)^2}{\lambda_m} \right)^{1/(m-2)} (t_* - t)^{1/(m-2)} \quad \textit{for all } t \in [0, t_*]. \tag{2.5}$$

In addition, if $m \leq 2^*$, then it holds that

$$\|u(t)\|_m \geq \left(\frac{C_m^{-2}}{\lambda_m} \right)^{1/(m-2)} (t_* - t)^{1/(m-2)} \quad \textit{for all } t \in [0, t_*]. \tag{2.6}$$

Moreover, we have

$$\begin{aligned} \left(\frac{C_m^{-m}}{\lambda_m} \right)^{1/(m-2)} (t_* - t)^{1/(m-2)} &\leq \|u(t)\|_{1,2} \\ &\leq \left(\frac{R(u_0)^m}{\lambda_m} \right)^{1/(m-2)} (t_* - t)^{1/(m-2)}. \end{aligned} \tag{2.7}$$

This fact has already been proved (see, e.g., [38], where pointwise estimates are also established for nonnegative solutions and their gradients); however, for the reader's convenience, we briefly give a proof.

Proof. Since $R(u(t))$ is nonincreasing, we use (2.2) to get

$$\begin{aligned} 0 &= \frac{1}{m'}\frac{d}{dt}\|u(t)\|_m^m + R(u(t))^2\|u(t)\|_m^2 \\ &\leq \frac{1}{m'}y'(t) + R(u_0)^2 y(t)^{2/m}, \end{aligned}$$

where we have put $y(t) := \|u(t)\|_m^m$. Noting that $y(t_*) = 0$, we get (2.5).

By (2.2) and (1.12), we have

$$y'(t) + m' C_m^{-2} y(t)^{2/m} \leq 0.$$

This shows (2.6). By the nonincrease of $R(u(t))$ and (1.12), we have

$$\|u(t)\|_{1,2} \leq R(u_0)\|u(t)\|_m, \quad \|u(t)\|_m \leq C_m \|u(t)\|_{1,2}.$$

Then (2.5) and (2.6) imply (2.7). □

By putting $t = 0$ into (2.5) and (2.6), we can immediately obtain the following corollary, which is also exhibited in [28].

COROLLARY 2.4 (Extinction rate and extinction time). *Assume that* $2 < m \leq 2^*$. *Let u be an H^{-1}-solution of* (1.1)–(1.3) *with $u_0 \in H_0^1(\Omega) \setminus \{0\}$. The optimal extinction rate of u is $(t_* - t)^{1/(m-2)}$ as $t \nearrow t_*$. Moreover, the extinction time $t_* = t_*(u_0)$ is estimated in terms of u_0 as follows*:

$$\lambda_m \frac{\|u_0\|_m^m}{\|u_0\|_{1,2}^2} \leq t_*(u_0) \leq \lambda_m C_m^2 \|u_0\|_m^{m-2}. \tag{2.8}$$

We next show some scaling property of extinction times by using the fact that equation (1.1) is invariant under the following transformation:

$$u \mapsto u_\lambda(x,t) := \lambda^{-1/(m-2)} u(x, \lambda t) \quad \text{for } \lambda > 0.$$

PROPOSITION 2.5 (Scaling property of t_*). *For any $\mu > 0$ and $|u_0|^{m-2}u_0 \in H^{-1}(\Omega)$, it follows that*

$$t_*(\mu u_0) = \mu^{m-2} t_*(u_0).$$

Let us discuss the continuous dependence of the extinction time t_* on initial data u_0. To the best of our knowledge, the following result could be new, and it will play a crucial role with the scaling property in our stability analysis of asymptotic profiles.

Proposition 2.6 (Continuity of t_*). *Assume that* $2 < m \leq 2^*$. *Let* $u_0, u_{0,n} \in H_0^1(\Omega)$ *be such that* $(u_{0,n})$ *is bounded in* $H_0^1(\Omega)$ *and* $u_{0,n} \to u_0$ *strongly in* $L^m(\Omega)$. *Then* $t_*(u_{0,n})$ *converges to* $t_*(u_0)$. *In particular, if* $m < 2^*$, *then* $t_*(\cdot)$ *is weakly sequentially continuous in* $H_0^1(\Omega)$.

Proof. Set $\tau_n := t_*(u_{0,n})$. By (2.8), we have

$$\lambda_m \frac{\|u_{0,n}\|_m^m}{\|u_{0,n}\|_{1,2}^2} \leq \tau_n \leq \lambda_m C_m^2 \|u_{0,n}\|_m^{m-2}.$$

If $u_0 = 0$, then the inequality above implies that τ_n converges to 0. Let $u_0 \neq 0$. Since $\|u_{0,n}\|_{1,2}$ is bounded by some constant $C > 0$ and $u_{0,n} \to u_0$ strongly in $L^m(\Omega)$, up to a subsequence, τ_n converges to a limit $\tau > 0$ such that

$$\tau_n \to \tau \quad \text{and} \quad 0 < \lambda_m C^{-2} \|u_0\|_m^m \leq \tau \leq \lambda_m C_m^2 \|u_0\|_m^{m-2}. \tag{2.9}$$

We are next concerned with the convergence of solutions u_n of (1.1)–(1.3) with initial data $u_{0,n}$. Let u be the solution of (1.1)–(1.3) with u_0. Then subtract (1.1) from that with $u = u_n$ and multiply this by $F(|u_n|^{m-2}u_n(t) - |u|^{m-2}u(t))$ (see §1 for the definition of F) to get

$$\frac{1}{2}\frac{d}{dt} \left\| |u_n|^{m-2}u_n(t) - |u|^{m-2}u(t) \right\|_{-1,2}^2 \leq 0 \quad \text{for a.a. } t > 0,$$

which leads us to

$$\begin{aligned} &\sup_{t \geq 0} \left\| |u_n|^{m-2}u_n(t) - |u|^{m-2}u(t) \right\|_{-1,2}^2 \\ &\leq \; C_m^2 \left\| |u_{0,n}|^{m-2}u_{0,n} - |u_0|^{m-2}u_0 \right\|_{m'}^2 \to 0. \end{aligned} \tag{2.10}$$

The convergence of the last term follows from the fact that $u_{0,n} \to u_0$ strongly in $L^m(\Omega)$. Furthermore, one can easily observe by (2.3) that

$$\sup_{t \geq 0} \|u_n(t)\|_{1,2} \leq \|u_{0,n}\|_{1,2} \leq C,$$

and moreover, by Tartar's inequality,

$$\begin{aligned} &\omega \|u_n(t) - u(t)\|_m^m \\ &\leq \; \left\langle |u_n|^{m-2}u_n(t) - |u|^{m-2}u(t), u_n(t) - u(t) \right\rangle_{H_0^1} \to 0 \end{aligned}$$

with a constant $\omega > 0$. Thus $u_n \to u$ strongly in $C([0,\infty); L^m(\Omega))$.

Recall (2.5) and (2.6) with $u = u_n$, $u_0 = u_{0,n}$ and $t_* = \tau_n$. Then since $(u_{0,n})$ is bounded in $H_0^1(\Omega)$ and $u_{0,n} \to u_0$ strongly in $L^m(\Omega)$, by passing to the limit as $n \to \infty$, one can derive that

$$c_1(\tau - t)^{1/(m-2)} \leq \|u(t)\|_m \leq c_2(\tau - t)^{1/(m-2)} \quad \text{for all } t \in [0, \tau]$$

with some constants $c_1, c_2 > 0$. Thus we conclude that $\tau = t_*(u_0)$.

In addition, if $m < 2^*$, then $H_0^1(\Omega)$ is compactly embedded in $L^m(\Omega)$, and therefore, if $u_{0,n} \to u_0$ weakly in $H_0^1(\Omega)$, then $t_*(u_{0,n}) \to t_*(u_0)$. $\square$

3 Asymptotic profiles

This section is devoted to investigating asymptotic profiles of H^{-1}-solutions $u = u(x,t)$ of (1.1)–(1.3), which vanish in finite time t_* at the rate $(t_* - t)^{1/(m-2)}$ (see Proposition 2.3). We are concerned with asymptotic profiles defined as follows.

Definition 3.1 (Asymptotic profiles). *Let $u_0 \in H_0^1(\Omega) \setminus \{0\}$ and let $u = u(x,t)$ be an H^{-1}-solution for* (1.1)–(1.3) *vanishing at a finite time $t_* > 0$. A function $\phi \in H_0^1(\Omega) \setminus \{0\}$ is called an asymptotic profile of u if there exists an increasing sequence $t_n \to t_*$ such that*

$$\lim_{t_n \nearrow t_*} \|(t_* - t_n)^{-1/(m-2)} u(t_n) - \phi\|_{1,2} = 0.$$

Asymptotic profiles for positive solutions of (1.1)–(1.3) are already studied in [5] and [27]. Moreover, those for sign-changing solutions are studied in [38], where the convergence of $(t_* - t_n)^{-1/(m-2)} u(\cdot, t_n)$ is proved in the strong topology of $L^m(\Omega)$ as $t_n \nearrow t_*$. The same conclusion can be also proved for sign-changing solutions in the strong topology of $H_0^1(\Omega)$ by simply combining the both arguments; however, we shall give a proof, because its argument may provide important hints for our stability analysis of asymptotic profiles.

In order to pursue our analysis, we exploit the change of variables (1.5) and derive the Cauchy-Dirichlet problem (1.6)–(1.8) from (1.1)–(1.3). The notion of H^{-1}-solution is defined also for (1.6)–(1.8) as in Definition 2.1. Let $2 < m \le 2^*$. Then as in (1.1)–(1.3), the following energy inequalities hold for a.a. $s \in (0, \infty)$:

$$\frac{1}{m'} \frac{d}{ds} \|v(s)\|_m^m + \|v(s)\|_{1,2}^2 = \lambda_m \|v(s)\|_m^m, \tag{3.1}$$

$$\mu_m \left\| \frac{d}{ds} |v|^{(m-2)/2} v(s) \right\|_2^2 + \frac{d}{ds} J(v(s)) \le 0, \tag{3.2}$$

where $J(\cdot)$ is a functional defined on $H_0^1(\Omega)$ by

$$J(w) := \frac{1}{2} \|w\|_{1,2}^2 - \frac{\lambda_m}{m} \|w\|_m^m \quad \text{for } w \in H_0^1(\Omega).$$

It is well-defined because $H_0^1(\Omega) \subset L^m(\Omega)$ by $m \le 2^*$. Moreover, multiplying (1.6) by $F(\partial_s(|v|^{m-2}v))$, we also derive

$$\begin{aligned} &\left\| \frac{d}{ds}|v|^{m-2}v(s) \right\|_{-1,2}^2 + \frac{1}{m'}\frac{d}{ds}\|v(s)\|_m^m \\ &= \lambda_m \left\langle |v|^{m-2}v(s), \frac{d}{ds}F\left(|v|^{m-2}v(s)\right) \right\rangle_{H_0^1} \\ &= \frac{\lambda_m}{2}\frac{d}{ds}\left\|F\left(|v|^{m-2}v(s)\right)\right\|_{1,2}^2. \end{aligned}$$

Thus it holds that

$$\left\| \frac{d}{ds}|v|^{m-2}v(s) \right\|_{-1,2}^2 + \frac{d}{ds}K(v(s)) = 0 \tag{3.3}$$

with the functional $K : L^m(\Omega) \to \mathbb{R}$ given by

$$K(w) := \frac{1}{m'}\|w\|_m^m - \frac{\lambda_m}{2}\left\|F\left(|w|^{m-2}w\right)\right\|_{1,2}^2 \quad \text{for } w \in L^m(\Omega).$$

Since $m \le 2^*$, the space $L^{m/(m-1)}(\Omega)$ is embedded in $H^{-1}(\Omega)$. Hence for $w \in L^m(\Omega)$, $|w|^{m-2}w$ belongs to $H^{-1}(\Omega)$ and $F(|w|^{m-2}w)$ is well-defined. Moreover, if w is in a bounded subset of $L^m(\Omega)$, then $F(|w|^{m-2}w)$ is bounded in $H_0^1(\Omega)$ and so is $K(w)$. By (2.4) we obtain

$$R(v(s)) = R(u(t)) \ \text{ is nonincreasing for } \ s \ge 0. \tag{3.4}$$

We also recall the Dirichlet problem (1.10), (1.11) as a stationary problem for (1.6)–(1.8). Equation (1.10) is called the Lane-Emden (or Emden-Fowler) equation. Here we remark that (1.10), (1.11) can be regarded as an Euler-Lagrange equation for the functional $J(\cdot)$. There are a number of contributions to this elliptic problem, and particularly, it is well known (see, e.g., [34]) that (1.10), (1.11) admits infinitely many nontrivial solutions and a positive solution, provided that $m < 2^*$. Moreover, any solution has a $C^2(\overline{\Omega})$ regularity by the standard regularity theorem of elliptic equation.

Now, we are ready to state a theorem on asymptotic profiles of solutions $u = u(x,t)$ for (1.1)–(1.3) as $t \to t_*$, equivalently, the convergence of solutions $v = v(x,s)$ for (1.6)–(1.8) as $s \to \infty$ (cf. [5], [27], [38]).

Theorem 3.2 (Asymptotic profiles). *Assume that $2 < m < 2^*$. Let u be an H^{-1}-solution of (1.1)–(1.3) with $u_0 \in H_0^1(\Omega) \setminus \{0\}$. Then for any increasing sequence $t_n \to t_*$, there exist a subsequence (n') of (n) and a solution $\phi \in H_0^1(\Omega) \setminus \{0\}$ of (1.10), (1.11) such that*

$$\lim_{t_{n'} \to t_*} \|(t_* - t_{n'})^{-1/(m-2)}u(t_{n'}) - \phi\|_{1,2} = 0. \tag{3.5}$$

Proof. Let $t_n \nearrow t_*$ and put $s_n := \log(t_*/(t_* - t_n)) \nearrow \infty$. By (2.5) and (2.6), it follows that

$$0 < c_1 \le \|v(s)\|_m \le c_2 \quad \text{for all } s \ge 0$$

with some constants $c_1, c_2 > 0$. Therefore once one proves (3.5), ϕ is nontrivial. Since $s \mapsto J(v(s))$ is nonincreasing by (3.2), we obtain

$$\sup_{s\in[0,\infty)} \|v(s)\|_{1,2} < \infty. \tag{3.6}$$

As stated before (3.4), $K(v(s))$ is bounded. Then (3.3) leads us to

$$\int_0^\infty \left\| \frac{d}{ds} |v|^{m-2} v(s) \right\|_{-1,2}^2 ds < \infty.$$

Let $\mathcal{T}$ be a measurable subset of $(0, \infty)$ such that $(0, \infty) \setminus \mathcal{T}$ has Lebesgue measure zero and (1.6), (3.1)–(3.3) hold for all $s \in \mathcal{T}$. For each $n \in \mathbb{N}$ we can take $\theta_n \in [s_n, s_n + 1] \cap \mathcal{T}$ such that

$$\frac{d}{ds} |v|^{m-2} v(\theta_n) \to 0 \quad \text{strongly in } H^{-1}(\Omega).$$

Hence by (1.6), it follows that

$$-\Delta v(\theta_n) - \lambda_m |v|^{m-2} v(\theta_n) \to 0 \quad \text{strongly in } H^{-1}(\Omega).$$

By (3.6), we can take a subsequence denoted by $\theta_n \to \infty$ and $\phi \in H_0^1(\Omega)$ such that

$$v(\theta_n) \to \phi \quad \text{weakly in } H_0^1(\Omega) \text{ and strongly in } L^m(\Omega),$$

since $H_0^1(\Omega)$ is compactly embedded in $L^m(\Omega)$ by assumption. Moreover,

$$|v|^{m-2} v(\theta_n) \to |\phi|^{m-2} \phi \quad \text{strongly in } L^{m'}(\Omega).$$

Therefore ϕ solves (1.10), (1.11).

To derive the strong convergence of $v(\theta_n)$ in $H_0^1(\Omega)$, we remark by (1.6) that

$$\|v(s)\|_{1,2}^2 = \lambda_m \|v(s)\|_m^m - \left\langle \frac{d}{ds} |v|^{m-2} v(s), v(s) \right\rangle_{H_0^1} \quad \text{for } s \in \mathcal{T}.$$

Put $s = \theta_n \in \mathcal{T}$ to get

$$\|v(\theta_n)\|_{1,2}^2 \to \lambda_m \|\phi\|_m^m = \|\phi\|_{1,2}^2.$$

Hence since $H_0^1(\Omega)$ is uniformly convex, $v(\theta_n) \to \phi$ strongly in $H_0^1(\Omega)$.

In particular, we observe that $J(v(\theta_n)) \to J(\phi)$. Since $J(v(\cdot))$ is nonincreasing, $J(v(s_n))$ also converges to $J(\phi)$.

We finally prove the strong convergence of $v(s_n)$ in $H_0^1(\Omega)$. Since $s_n \le \theta_n \le s_n + 1$, we deduce that

$$
\begin{aligned}
&\left\| |v|^{m-2}v(s_n) - |v|^{m-2}v(\theta_n) \right\|_{-1,2} \\
\le\ & \left(\int_{s_n}^{\theta_n} \left\| \frac{d}{ds} |v|^{m-2} v(s) \right\|_{-1,2}^2 ds \right)^{1/2} \sqrt{\theta_n - s_n} \\
\le\ & \left(\int_{s_n}^{\infty} \left\| \frac{d}{ds} |v|^{m-2} v(s) \right\|_{-1,2}^2 ds \right)^{1/2} \to 0.
\end{aligned}
$$

Therefore by Tartar's inequality, we find that

$$
\omega \|v(s_n) - v(\theta_n)\|_m^m \le \left\langle |v|^{m-2}v(\theta_n) - |v|^{m-2}v(s_n), v(\theta_n) - v(s_n) \right\rangle_{H_0^1} \to 0,
$$

which implies that $v(s_n) \to \phi$ strongly in $L^m(\Omega)$, and weakly in $H_0^1(\Omega)$ by (3.6). Hence it holds that

$$
\begin{aligned}
\frac{1}{2}\|v(s_n)\|_{1,2}^2 &= J(v(s_n)) + \frac{\lambda_m}{m}\|v(s_n)\|_m^m \\
&\to J(\phi) + \frac{\lambda_m}{m}\|\phi\|_m^m = \frac{1}{2}\|\phi\|_{1,2}^2.
\end{aligned}
$$

Thus $v(s_n)$ converges to ϕ strongly in $H_0^1(\Omega)$. The proof is complete. □

Remark 3.3. (i) Let ϕ be a nontrivial solution for (1.10), (1.11). Then $u(x,t) := (1-t)_+^{1/(m-2)}\phi(x)$ solves (1.1)–(1.3) with $u_0 = \phi$ and vanishes at $t = 1$ (hence $t_*(\phi) = 1$). Moreover, $\phi(x)$ is the asymptotic profile of $u(x,t)$. On the other hand, by virtue of Theorem 3.2, every asymptotic profile solves (1.10), (1.11). Hence the set of all asymptotic profiles coincides with that of all nontrivial solutions for (1.10), (1.11).

(ii) This theorem does not say anything on the uniqueness of asymptotic profiles for each solution. However, in the one-dimensional case, since every nontrivial solution for (1.10), (1.11) is distinct from each other, every solution has a unique profile (see §5.4 for more details).

4 Dynamical systems generated by (1.6)–(1.8)

In this section, we study a dynamical system generated by (1.6)–(1.8) in the phase space $\mathcal{X}$ given by

$$
\mathcal{X} := \{t_*(w)^{-1/(m-2)}w;\ w \in H_0^1(\Omega) \setminus \{0\}\}.
$$

As we saw in §3, every asymptotic profile of solutions for (1.1)–(1.3) is characterized as a stationary point of the dynamical system. Moreover, in the next section, we shall introduce the notion of stability/instability for asymptotic profiles, which is equivalently rewritten by that for stationary points of the dynamical system on $\mathcal{X}$ (see Remark 5.2). Hence it is crucial to investigate the dynamical system, in particular, the geometry of the phase space $\mathcal{X}$.

On the other hand, we also provide some representation formula of t_* and classify asymptotic behaviors of solutions for (1.6)–(1.8) in terms of initial data as an independent interest.

Throughout this section, we assume that

$$2 < m \leq 2^*.$$

Let us start with the well-definedness of the dynamical system on $\mathcal{X}$.

Proposition 4.1 (Dynamical system on $\mathcal{X}$). *The solution operator $T(s) : v_0 \mapsto v(s)$ associated with* (1.6)–(1.8) *generates a dynamical system on the set $\mathcal{X}$.*

Proof. Let $v_0 \in \mathcal{X}$, i.e., $v_0 = t_*(u_0)^{-1/(m-2)} u_0$ with some $u_0 \in H^1_0(\Omega) \setminus \{0\}$. From the definition of extinction time, it is obvious that $t_*(u(t)) = t_*(u_0) - t$, where u is the unique H^{-1}-solution of (1.1)–(1.3) vanishing at $t_*(u_0)$. Hence we observe that

$$v(s) = (t_*(u_0) - t)^{-1/(m-2)} u(t) = t_*(u(t))^{-1/(m-2)} u(t).$$

Thus $v(s)$ lies on $\mathcal{X}$ for all $s \geq 0$. □

One can easily observe that

Proposition 4.2 (Representation of $\mathcal{X}$). *It follows that*

$$\mathcal{X} = \{w \in H^1_0(\Omega);\ t_*(w) = 1\}.$$

Proof. If $v \in \mathcal{X}$, then there exists $w \in H^1_0(\Omega) \setminus \{0\}$ such that $v = t_*(w)^{-1/(m-2)} w$. By Proposition 2.5, we have $t_*(v) = 1$. Conversely, if $v \in H^1_0(\Omega)$ and $t_*(v) = 1$, then we can write $v = t_*(v)^{-1/(m-2)} v \in \mathcal{X}$. □

Then by Proposition 2.6, one can immediately verify the following proposition.

Proposition 4.3 (Weak sequential closedness of $\mathcal{X}$). *Suppose that $m < 2^*$. The set $\mathcal{X}$ is sequentially closed in the weak topology of $H^1_0(\Omega)$, i.e., if $w_n \in \mathcal{X}$ and $w_n \to w$ weakly in $H^1_0(\Omega)$, then $w \in \mathcal{X}$.*

Throughout this paper, we employ the following notation,

$$
\begin{aligned}
j(w) &:= \lambda_m \|w\|_m^m - \|w\|_{1,2}^2 \quad \text{for } \ w \in H_0^1(\Omega), \\
\mathcal{S} &:= \{\phi \in H_0^1(\Omega);\ \phi \text{ is a nontrivial solution of (1.10), (1.11)}\}, \\
\mathcal{N} &:= \{w \in H_0^1(\Omega) \setminus \{0\};\ j(w) = 0\}, \\
d &:= \inf_{w \in \mathcal{N}} J(w), \\
\mathcal{V} &:= \left\{w \in H_0^1(\Omega);\ J(w) < d \ \text{ and } \ j(w) \le 0\right\}, \\
\mathcal{J}^+ &:= \left\{w \in H_0^1(\Omega);\ j(w) > 0\right\}.
\end{aligned}
\tag{4.1}
$$

The set $\mathcal{N}$ is called *Nehari manifold* and $\mathcal{N} = \{w \in H_0^1(\Omega)\setminus\{0\};\ \langle J'(w), w\rangle = 0\}$, where J' denotes the Fréchet derivative of J. For any $w \in \mathcal{N}$, by (1.12) we have

$$\|w\|_{1,2}^2 = \lambda_m \|w\|_m^m \le \lambda_m C_m^m \|w\|_{1,2}^m,$$

which shows

$$J(w) = \frac{m-2}{2m}\|w\|_{1,2}^2 \ge \frac{m-2}{2m}(\lambda_m C_m^m)^{-2/(m-2)} \quad \text{for any } w \in \mathcal{N}.$$

Thus

$$d = \frac{m-2}{2m}\left(\lambda_m C_m^m\right)^{-2/(m-2)}. \tag{4.2}$$

This value is attained by a certain solution of (1.10), (1.11) provided that $m < 2^*$ (see, e.g., [43]).

Here let us recall some preliminary facts on the Dirichlet problem (1.10), (1.11) to be used later. We call ϕ a *least energy solution* if it minimizes $J(\phi)$ over the set of all nontrivial solutions of (1.10), (1.11). Least energy solutions exist whenever $m < 2^*$.

Lemma 4.4. *Then the following* (i)–(iii) *are equivalent*:

(i) *ϕ is a least energy solution of* (1.10)*,* (1.11)*,*

(ii) *ϕ is a minimizer of J over $\mathcal{N}$,*

(iii) *ϕ is a minimizer of the Rayleigh quotient R over $\mathcal{N}$.*

Moreover, least energy solutions are sign-definite.

Proof. By the definition of $\mathcal{N}$, we have the relation,

$$\|w\|_{1,2}^2 = \frac{2m}{m-2}J(w) = \lambda_m^{-2/(m-2)} R(w)^{2m/(m-2)} \quad \text{for } w \in \mathcal{N}.$$

Since $\mathcal{S} \subset \mathcal{N}$, we have the equivalence among (i)–(iii).

If ϕ is a least energy solution, so is $|\phi|$. Hence $|\phi|$ is a critical point of J, i.e., it is a weak solution of (1.10), (1.11). Then it belongs to $C^2(\overline{\Omega})$ by the elliptic regularity theorem. By the strong maximum principle, $|\phi(x)| > 0$ in Ω. Thus ϕ is a positive or negative solution. $\square$

Remark 4.5. Coffman [10], Li [30] and Byeon [8] studied the case that Ω is the annulus domain Ω_R, $R < |x| < R+1$, and proved that a positive radial solution ϕ_{rad} of (1.10), (1.11) with $\Omega = \Omega_R$ is not a least energy solution if $R > 0$ is large enough. Indeed, they obtained many non-radial positive solutions $\psi_1, \ldots, \psi_k$, which are not rotationally equivalent, satisfying

$$J(\psi_1) < J(\psi_2) < \cdots < J(\psi_k) < J(\phi_{rad}),$$

provided that $R > 0$ is large enough, where ϕ_{rad} is a unique positive radial solution (the uniqueness of positive radial solutions in the annulus domain was proved by Ni [32]). Therefore least energy solutions are not radially symmetric.

Let ϕ be a least energy solution of (1.10), (1.11) and $O(N)$ the orthogonal group. Then the set $\{\phi(gx);\ g \in O(N)\}$ is not a single point but a continuum in $H^1_0(\Omega)$. Since $\phi(gx)$ also minimizes J over $\mathcal{N}$, least energy solutions are not unique and ϕ is not isolated from all the other least energy solutions. We shall also state known results on the uniqueness of the least energy solution in Remark 5.11.

We next investigate the dynamical system generated by (1.6)–(1.8) in a usual energy space, that is, $H^1_0(\Omega)$.

Lemma 4.6 (Large-time behavior of v). *Let v be an H^{-1}-solution of* (1.6)–(1.8) *with an initial data $v_0 \in H^1_0(\Omega)$.*

(i) *If $v_0 \in \mathcal{V}$, then v vanishes in finite time.*

(ii) *If $v_0 \in \mathcal{J}^+$, then v is unbounded in $L^m(\Omega)$ as $s \to \infty$.*

Proof. From the definition of d, one can say $\mathcal{N} \cap \mathcal{V} = \emptyset$, and hence,

$$\mathcal{V} = \left\{\phi \in H^1_0(\Omega);\ J(\phi) < d \ \text{ and } \ j(\phi) < 0\right\} \cup \{0\}.$$

If $v_0 = 0$, the assertion (i) is obvious. If $v_0 \in \mathcal{V} \setminus \{0\}$, then $v(s) \in \mathcal{V}$ for all $s \geq 0$. Indeed, since $J(v(s))$ is nonincreasing, $J(v(s)) < d$ for all $s \geq 0$, and therefore $v(s) \notin \mathcal{N}$. Hence $j(v(s)) < 0$.

We have

$$\begin{aligned} \frac{1}{2}\|v(s)\|^2_{1,2} &= J(v(s)) + \frac{1}{m}\left[j(v(s)) + \|v(s)\|^2_{1,2}\right] \\ &< J(v_0) + \frac{1}{m}\|v(s)\|^2_{1,2}, \end{aligned}$$

which implies

$$\|v(s)\|_{1,2} \le \left(\frac{2m}{m-2} J(v_0) \right)^{1/2}.$$

Hence we derive from (3.1) with (1.12) that

$$\frac{1}{m'} \frac{d}{ds} \|v(s)\|_m^m + \|v(s)\|_{1,2}^2 \le \lambda_m C_m^m \left(\frac{2m}{m-2} J(v_0) \right)^{(m-2)/2} \|v(s)\|_{1,2}^2.$$

Since $J(v_0) < d$, we find by (4.2) that

$$\delta := 1 - \lambda_m C_m^m \left(\frac{2m}{m-2} J(v_0) \right)^{(m-2)/2} > 0.$$

Hence it follows that

$$\frac{d}{ds} \|v(s)\|_m^m + m' \delta C_m^{-2} \|v(s)\|_m^2 \le 0 \quad \text{for a.a.} \quad s > 0,$$

which implies that $\|v(s)\|_m$ vanishes in finite time.

As to (ii), since $R(v(s)) \le R(v_0)$ by (3.4), we derive from (3.1) that

$$\frac{1}{m'} \frac{d}{ds} \|v(s)\|_m^m + R(v_0)^2 \|v(s)\|_m^2 \ge \lambda_m \|v(s)\|_m^m.$$

Now the following ODE of Bernoulli-type,

$$y' - m' \lambda_m y = -m' R(v_0)^2 y^{2/m} \quad \text{for} \quad s \ge 0 \quad \text{and} \quad y(0) = \|v_0\|_m^m,$$

is explicitly solved by

$$y(s) = \left\{ \left(\|v_0\|_m^{m-2} - \frac{R(v_0)^2}{\lambda_m} \right) e^s + \frac{R(v_0)^2}{\lambda_m} \right\}^{m/(m-2)}.$$

Here we note by the fact that $j(v_0) > 0$ that

$$\|v_0\|_m^{m-2} - \frac{R(v_0)^2}{\lambda_m} = \frac{j(v_0)}{\lambda_m \|v_0\|_m^2} > 0.$$

By the comparison principle, $y(s)^{1/m} \le \|v(s)\|_m$ for all $s \ge 0$. $\square$

From these observations, we can characterize the phase space $\mathcal{X}$ in the following proposition, which will play a crucial role in our stability analysis to be performed in §5.

Proposition 4.7 (Variational property of $\mathcal{X}$). *It holds that*

$$\mathcal{X} \subset [d \leq J] := \{w \in H_0^1(\Omega);\ d \leq J(w)\}.$$

Moreover, if $w \in \mathcal{X}$ and $J(w) = d$, then $J'(w) = 0$.

Proof. For each $v_0 \in \mathcal{X}$, i.e., $v_0 = t_*(u_0)^{-1/(m-2)} u_0$ with some $u_0 \in H_0^1(\Omega)$, the function $v(s) := (t_* - t)^{-1/(m-2)} u(t)$ uniquely solves (1.6)–(1.8), where u is a solution of (1.1)–(1.3) with $u(0) = u_0$, and moreover, $v(s)$ is uniformly away from zero and bounded for all $s \geq 0$ by (2.5) and (2.6). Hence by Lemma 4.6, we deduce that $v_0 \notin \mathcal{V} \cup \mathcal{J}^+$. Thus $\mathcal{X} \subset [d \leq J]$.

Suppose that $v_0 \in \mathcal{X}$ and $J(v_0) = d$ but $J'(v_0) \neq 0$. Let $v(s)$ be the unique solution of (1.6)–(1.8) with the initial data v_0. Since $J'(v_0) \neq 0$, $v(s)$ is not a stationary solution. Hence $(d/ds)J(v(s)) < 0$ by (3.2), and $J(v(s)) < J(v_0) = d$ for $s > 0$. Moreover, by Proposition 4.1, $v(s) \in \mathcal{X}$. These facts contradict $\mathcal{X} \subset [d \leq J]$. It completes our proof. □

Remark 4.8. (i) Only in the case of the subcritical, $m < 2^*$, one can obtain the conclusion of Proposition 4.7 in an easier way. Indeed, by Theorem 3.2, for any $v_0 \in \mathcal{X}$, we can take a sequence $s_n \to \infty$ such that $v(s_n)$ converges to $\phi \in \mathcal{S}$ strongly in $H_0^1(\Omega)$. Since ϕ solves (1.10), (1.11), it holds that $d \leq J(\phi)$. Since $J(v(\cdot))$ is nonincreasing, we can conclude that $d \leq J(v_0)$. Thus $v_0 \in [d \leq J]$.

(ii) Lemma 4.6 also provides further information that $j(w) \leq 0$ for all $w \in \mathcal{X}$. Hence by (3.1), $\|v(\cdot)\|_m$ is nonincreasing for the solution v of (1.6)–(1.9) with any $v_0 \in \mathcal{X}$. Moreover, since $J(v(\cdot))$ is nonincreasing, so is $\|v(\cdot)\|_{1,2}$.

The rest of this section is devoted to some results of independent interest. The following lemma provides a representation of t_*.

Lemma 4.9. *For $u_0 \in H_0^1(\Omega) \setminus \{0\}$, let u be an H^{-1}-solution of* (1.1)–(1.3), *and let $t_* = t_*(u_0)$ be the extinction time for u. Then it holds that*

$$\frac{1}{t_*} = \frac{1}{\lambda_m \|u_0\|_m^{m-2}} \int_0^\infty R(v(s))^2 e^{-s} ds, \tag{4.3}$$

where $v(s) := (t_* - t)^{-1/(m-2)} u(t)$ with $s = \log(t_*/(t_* - t)) \geq 0$ for $t \in [0, t_*)$.

Proof. By (2.2), we have

$$\frac{1}{m'} \frac{d}{dt} \|u(t)\|_m^m + R(u(t))^2 \|u(t)\|_m^2 = 0 \quad \text{for a.a. } t \in (0, t_*).$$

Then it follows that

$$\|u(t)\|_m = \left(\|u_0\|_m^{m-2} - \frac{1}{\lambda_m} \int_0^t R(u(\tau))^2 d\tau \right)_+^{1/(m-2)} \quad \text{for all } \ t \geq 0.$$

Putting $t = t_*$ and using the fact that $R(u(t)) = R(v(s))$, we can deduce

$$\lambda_m \|u_0\|_m^{m-2} = t_* \int_0^\infty R(v(s))^2 e^{-s} ds,$$

which implies (4.3). □

In the next proposition, solutions $u = u(x,t)$ of separable type (equivalently, solutions for initial data $u_0 \in \mathbb{R}(\mathcal{S})$, i.e., $u_0 = r\psi$ with some $r \in \mathbb{R}$ and $\psi \in \mathcal{S}$) are characterized by an explicit formula of the extinction time.

Proposition 4.10 (Extinction time for solutions of separable type). *Let $u_0 \in H_0^1(\Omega) \setminus \{0\}$ be fixed. Then $u_0 \in \mathbb{R}(\mathcal{S})$ if and only if*

$$t_*(u_0) = \lambda_m \frac{\|u_0\|_m^m}{\|u_0\|_{1,2}^2}. \tag{4.4}$$

Proof. The "only if part" follows from Proposition 2.5 and the fact that $t_*(\phi) = 1$ and $\lambda_m \|\phi\|_m^m = \|\phi\|_{1,2}^2$ for $\phi \in \mathcal{S}$. As for the "if part", by (4.3) and assumption, we obtain

$$\int_0^\infty R(v(s))^2 e^{-s} ds = \frac{\lambda_m \|u_0\|_m^{m-2}}{t_*(u_0)} = R(u_0)^2 = \int_0^\infty R(u_0)^2 e^{-s} ds.$$

Since $R(v(0)) = R(u_0)$, we observe by (3.4) that

$$R(u(t)) = R(v(s)) = R(u_0) \quad \text{for all } \ s \geq 0,$$

which yields

$$\frac{d}{dt} R(u(t))^2 = 0 \quad \text{for all } t \in [0, t_*).$$

We note by (2.2) and (2.3) that

$$\begin{aligned} 0 = \frac{d}{dt} R(u(t))^2 &= \frac{\|u(t)\|_m^m \frac{d}{dt} \|u(t)\|_{1,2}^2 - \frac{2}{m} \|u(t)\|_{1,2}^2 \frac{d}{dt} \|u(t)\|_m^m}{\|u(t)\|_m^{m+2}} \\ &\leq -2\mu_m \frac{\|u(t)\|_m^m \left\| \partial_t \left(|u|^{(m-2)/2} u \right)(t) \right\|_2^2 - \left(\frac{1}{2} \frac{d}{dt} \|u(t)\|_m^m \right)^2}{\|u(t)\|_m^{m+2}}. \end{aligned}$$

Since $(d/dt)\|u(t)\|_m^m \le 0$ by (2.2), the inequality above implies

$$\|u(t)\|_m^{m/2} \left\|\partial_t \left(|u|^{(m-2)/2}u\right)(t)\right\|_2 \le -\frac{1}{2}\frac{d}{dt}\|u(t)\|_m^m.$$

Here it also follows that

$$\|u(t)\|_m^m = \left\||u|^{(m-2)/2}u(t)\right\|_2^2,$$

and

$$\left(\partial_t \left(|u|^{(m-2)/2}u\right)(t), |u|^{(m-2)/2}u(t)\right)_{L^2} = \frac{1}{2}\frac{d}{dt}\|u(t)\|_m^m \le 0.$$

Therefore we obtain

$$\begin{aligned} &-\left(\partial_t \left(|u|^{(m-2)/2}u\right)(t), |u|^{(m-2)/2}u(t)\right)_{L^2} \\ &= \left\|\partial_t \left(|u|^{(m-2)/2}u\right)(t)\right\|_2 \left\||u|^{(m-2)/2}u(t)\right\|_2. \end{aligned}$$

Since $\|u(t)\|_m > 0$ for all $t \in [0, t_*)$, we can deduce that

$$\partial_t \left(|u|^{(m-2)/2}u\right)(x,t) = \lambda(t)|u|^{(m-2)/2}u(x,t) \quad \text{for a.a. } x \in \Omega \text{ and } t \in [0, t_*)$$

with $\lambda(t) := -\|\partial_t(|u|^{(m-2)/2}u)(t)\|_2/\|u(t)\|_m^{m/2}$. Thus

$$|u|^{(m-2)/2}u(x,t) = |u_0|^{(m-2)/2}u_0(x) \exp\left(\int_0^t \lambda(\tau)d\tau\right)$$

for a.a. $x \in \Omega$ and $t \in [0, t_*)$. Therefore $|u|^{(m-2)/2}u$ is a function of separable form, and therefore, so is u, i.e., $u_0 \in \mathbb{R}(\mathcal{S})$. $\square$

We finally show that $\mathcal{X}$ is a separatrix between the *stable set* and the *unstable set* for (1.6)–(1.8), that is, two regions of initial data for which $v(s)$ vanishes in finite time and grows up at infinity, respectively.

Proposition 4.11 (Stable set and unstable set). *Assume that $2 < m < 2^*$. The following* (i)–(iii) *are satisfied.*

(i) *The sets $\mathcal{X}$ and $\mathcal{N}$ are unbounded surfaces surrounding the origin and homeomorphic to the unit sphere of $H_0^1(\Omega)$. The space $H_0^1(\Omega)$ is separated by the surface $\mathcal{X}$ into the inside $\mathcal{X}^- := \{w \in H_0^1(\Omega);\ t_*(w) < 1\}$ and the outside $\mathcal{X}^+ := \{w \in H_0^1(\Omega);\ t_*(w) > 1\}$. Furthermore, $\mathcal{N} \subset \mathcal{X} \cup \mathcal{X}^+$ and $\mathcal{N} \cap \mathcal{X} = \mathcal{S}$.*

(ii) *The set $\mathcal{X}$ coincides with the set of initial data v_0 for which the solution $v(s)$ of* (1.6)–(1.8) *is bounded and uniformly away from the origin in terms of the L^m-norm for all $s \ge 0$. The same assertion also holds in terms of the H_0^1-norm.*

(iii) *Each solution* $v(s)$ *of* (1.6)–(1.8) *vanishes in finite time if and only if* $v_0 \in \mathcal{X}^-$, *and* $v(s)$ *grows up at infinity if and only if* $v_0 \in \mathcal{X}^+$.

Proof. In order to show (i), we set

$$x(w) := t_*(w)^{-1/(m-2)} w, \quad n(w) := \left(\frac{\|w\|_{1,2}^2}{\lambda_m \|w\|_m^m} \right)^{1/(m-2)} w$$

for $w \in H_0^1(\Omega) \setminus \{0\}$. Moreover, define the ray from the origin through a point $w \in H_0^1(\Omega) \setminus \{0\}$ by $\mathbb{R}^+(w) := \{\lambda w;\ \lambda \geq 0\}$. Then by Proposition 4.2 and the definition of $\mathcal{N}$, it follows that

$$\mathcal{X} \cap \mathbb{R}^+(w) = \{x(w)\}, \qquad \mathcal{N} \cap \mathbb{R}^+(w) = \{n(w)\}. \tag{4.5}$$

Hence $H_0^1(\Omega)$ is separated by $\mathcal{X}$ into two disjoint open sets $\mathcal{X}^-$, $\mathcal{X}^+$ given as in the statement of (i). Moreover, one can show that the mappings x and n are homeomorphic from the unit sphere S to $\mathcal{X}$ and $\mathcal{N}$, respectively (further details of proofs are left to the reader).

Combining the first inequality of (2.8) with (4.5), we deduce that $\mathcal{N} \subset \mathcal{X} \cup \mathcal{X}^+$. Let us show that $\mathcal{N} \cap \mathcal{X} = \mathcal{S}$. Since $t_*(\phi) = 1$ for $\phi \in \mathcal{S}$ by Remark 3.3, it holds that $\mathcal{S} \subset \mathcal{N} \cap \mathcal{X}$. It remains to show the converse inclusion. For any $w \in \mathcal{N} \cap \mathcal{X}$ (hence $t_*(w) = 1$ and $j(w) = 0$), thanks to Proposition 4.10, one can write $w = c\phi$ with some $c \in \mathbb{R}$ and $\phi \in \mathcal{S}$, which together with the fact that $t_*(w) = 1$ implies $c = \pm 1$, i.e., $w \in \mathcal{S}$. Hence $\mathcal{N} \cap \mathcal{X} = \mathcal{S}$. Since $\mathcal{S}$ is unbounded in $H_0^1(\Omega)$ (see [34]), so are $\mathcal{X}$ and $\mathcal{N}$. Thus (i) is proved.

As to (ii) and (iii), let $v_0 \in H_0^1(\Omega) \setminus \{0\}$ and let $u(x,t)$ and $v(x,s)$ be the solutions of (1.1)–(1.3) and (1.6)–(1.8), respectively, with the common initial data v_0. Then the relation (1.5) holds with $t_* = 1$. In case $v_0 \in \mathcal{X}^-$ (i.e., $t_*(v_0) < 1$), put $s_* = \log(1/(1 - t_*(v_0))) < \infty$. We have

$$\|v(s_*)\|_m = (1 - t_*(v_0))^{-1/(m-2)} \|u(t_*(v_0))\|_m = 0,$$

since $u(x,t)$ vanishes at $t = t_*(v_0)$. In case $v_0 \in \mathcal{X}^+$ (i.e., $t_*(v_0) > 1$), since $\|u(1)\|_m$ is positive, we deduce that

$$\|v(s)\|_m = (1-t)^{-1/(m-2)} \|u(t)\|_m \to \infty \ \text{ as } \ s \to \infty \text{ (equivalently, } t \nearrow 1).$$

In case $v_0 \in \mathcal{X}$ (i.e., $t_*(v_0) = 1$), Proposition 2.3 guarantees that $\|v(s)\|_m$ and $\|v(s)\|_{1,2}$ are bounded and uniformly away from zero for all $s \geq 0$. Combining all the facts above, we obtain (ii) and (iii). □

Remark 4.12. It is noteworthy that the separatrix $\mathcal{X}$ of the stable set and the unstable set of initial data for (1.6)–(1.8) is different from the so-called Nehari manifold $\mathcal{N}$ associated with the functional J, and $\mathcal{X}$ is inside $\mathcal{N}$. Moreover, their intersection coincides with $\mathcal{S}$, the set of nontrivial stationary solutions.

5 Stability analysis of profiles

In this section, we perform a stability analysis of asymptotic profiles of solutions for (1.1)–(1.3). Throughout this section, we assume that

$$2 < m < 2^*.$$

We start with explicitly giving a definition of the (asymptotic) stability and instability of profiles.

Definition 5.1 (Stability and instability of profiles). *Let $\phi \in H^1_0(\Omega) \setminus \{0\}$ be an asymptotic profile of a nontrivial solution for* (1.1)–(1.3)*, equivalently, ϕ is a nontrivial solution of* (1.10), (1.11).

(i) *ϕ is said to be stable, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any solution u of* (1.1)–(1.3) *and its extinction time t_* satisfy*

$$\sup_{t \in [0,t_*)} \|(t_* - t)^{-1/(m-2)} u(t) - \phi\|_{1,2} < \varepsilon,$$

whenever $\|u(0) - \phi\|_{1,2} < \delta$.

(ii) *ϕ is said to be unstable, if ϕ is not stable.*

(iii) *ϕ is said to be asymptotically stable, if ϕ is stable, and moreover, there exists $\delta_0 > 0$ such that any solution u of* (1.1)–(1.3) *and its extinction time t_* satisfy*

$$\lim_{t \nearrow t_*} \|(t_* - t)^{-1/(m-2)} u(t) - \phi\|_{1,2} = 0,$$

whenever $\|u(0) - \phi\|_{1,2} < \delta_0$.

Remark 5.2. Recall (1.5) and (1.9). Then we can translate the notion of stability/instability of asymptotic profiles for solutions of (1.1)–(1.3) into that of stationary points of the dynamical systems generated by (1.6)–(1.8) on $\mathcal{X}$. Indeed, since $t_*(\phi) = 1$ for all $\phi \in \mathcal{S}$ and $t_*(\cdot)$ is continuous in $H^1_0(\Omega)$ by Proposition 2.6, one can equivalently replace $\|u(0) - \phi\|_{1,2}$ in (i) and (iii) of Definition 5.1 by $\|v(0) - \phi\|_{1,2}$ with $v(0) = t_*(u(0))^{-1/(m-2)} u(0) \in \mathcal{X}$. Hence (i) and (iii) stated above can be rewritten by the following (i)$'$ and (iii)$'$, respectively:

(i)$'$ ϕ is said to be stable, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any solution v of (1.6)–(1.8) satisfies

$$\sup_{s \in [0,\infty)} \|v(s) - \phi\|_{1,2} < \varepsilon,$$

whenever $v(0) \in \mathcal{X}$ and $\|v(0) - \phi\|_{1,2} < \delta$.

(iii)′ ϕ is said to be asymptotically stable, if ϕ is stable, and moreover, there exists $\delta_0 > 0$ such that any solution v of (1.6)–(1.8) satisfies

$$\lim_{s \nearrow \infty} \|v(s) - \phi\|_{1,2} = 0,$$

whenever $v(0) \in \mathcal{X}$ and $\|v(0) - \phi\|_{1,2} < \delta_0$.

The next lemma is elementary but important for later arguments.

LEMMA 5.3. *Let ϕ_n and ϕ be solutions of* (1.10), (1.11) *such that ϕ_n converges to ϕ in $H_0^1(\Omega)$. If $\phi > 0$ in Ω, then $\phi_n(x) > 0$ in Ω for n large enough.*

In particular, sign-definite solutions of (1.10), (1.11) *are isolated in $H_0^1(\Omega)$ from all sign-changing solutions of* (1.10), (1.11).

Proof. Since ϕ_n converges to ϕ in $H_0^1(\Omega)$, it converges in $C^2(\overline{\Omega})$ also by the elliptic regularity theorem. From the Hopf maximum principle, it follows that $\partial\phi/\partial\nu < 0$ on $\partial\Omega$, where $\partial\phi/\partial\nu$ denotes the outward normal derivative. Then the $C^2(\overline{\Omega})$-convergence assures that $\phi_n > 0$ for n large enough. $\square$

In the rest of this section, we shall use the following notation:

$$\begin{aligned} [J \le a] &:= \{w \in H_0^1(\Omega);\ J(w) \le a\}, \\ B(\phi; r) &:= \{w \in H_0^1(\Omega);\ \|w - \phi\|_{1,2} < r\} \end{aligned}$$

for $a \in \mathbb{R}$, $r > 0$ and $\phi \in H_0^1(\Omega)$. Moreover, $[J < a]$ can be defined in a similar way.

5.1 Stable profiles

In this subsection, we show that isolated least energy solutions of (1.10), (1.11) are (asymptotically) stable profiles of solutions for (1.1)–(1.3). Examples of isolated least energy solutions will be given in Remark 5.11. Recall that every least energy solution of (1.10), (1.11) is sign-definite by Lemma 4.4.

THEOREM 5.4 (Stable profiles). *Let ϕ be a least energy solution of* (1.10), (1.11).

(i) *If ϕ is isolated in $H_0^1(\Omega)$ from all the other least energy solutions of* (1.10), (1.11), *then it is a stable profile.*

(ii) *If ϕ is isolated in $H_0^1(\Omega)$ from all the other sign-definite solutions of* (1.10), (1.11), *then it is an asymptotically stable profile.*

Proof. Let ϕ satisfy the assumption of (i). We choose an $r > 0$ so small that there is no least energy solution in $B(\phi; r)$ except for ϕ. We claim that for any $\varepsilon \in (0, r)$,

$$c := \inf\{J(w);\ w \in \mathcal{X},\ \|w - \phi\|_{1,2} = \varepsilon\} > d. \tag{5.1}$$

By Proposition 4.7, we have already known that $c \geq d$. Hence suppose on the contrary that there is a sequence $w_n \in \mathcal{X}$ such that $\|w_n - \phi\|_{1,2} = \varepsilon$ and $J(w_n)$ converges to d. Then up to a subsequence, w_n converges to a limit w_∞ weakly in $H_0^1(\Omega)$ and strongly in $L^m(\Omega)$ by $m < 2^*$. The limit w_∞ belongs to $\mathcal{X}$ because of the sequential closedness of $\mathcal{X}$ in the weak topology of $H_0^1(\Omega)$ (see Proposition 4.3), and so $J(w_\infty) \geq d$ by Proposition 4.7. Therefore we obtain

$$\begin{aligned} \frac{1}{2}\|w_n\|_{1,2}^2 \quad &= \quad J(w_n) + \frac{\lambda_m}{m}\|w_n\|_m^m \\ &\to \quad d + \frac{\lambda_m}{m}\|w_\infty\|_m^m \leq J(w_\infty) + \frac{\lambda_m}{m}\|w_\infty\|_m^m = \frac{1}{2}\|w_\infty\|_{1,2}^2, \end{aligned}$$

which leads us to

$$\limsup_{n\to\infty} \|w_n\|_{1,2} \leq \|w_\infty\|_{1,2}.$$

This result together with the lower semicontinuity of $\|\cdot\|_{1,2}$ implies that $\|w_n\|_{1,2} \to \|w_\infty\|_{1,2}$. From the uniform convexity of $H_0^1(\Omega)$, we derive $w_n \to w_\infty$ strongly in $H_0^1(\Omega)$. Then $J(w_\infty) = d$, $w_\infty \in \mathcal{X}$ and $\|w_\infty - \phi\|_{1,2} = \varepsilon$. Hence Proposition 4.7 asserts that $J'(w_\infty) = 0$, i.e., w_∞ is a least energy solution. However, this contradicts the choice of r. Thus we have (5.1).

Let $\varepsilon > 0$ and choose $\delta \in (0, \varepsilon)$ so small that $J(w) < c$ for all $w \in B(\phi; \delta)$. Let $v_0 \in \mathcal{X} \cap B(\phi; \delta)$ (hence, $J(v_0) < c$). By Proposition 4.1, the solution $v(s)$ of (1.6)–(1.8) lies on $\mathcal{X}$ for all $s \geq 0$. We claim that $v(s)$ remains in $B(\phi; \varepsilon)$ for all $s \geq 0$. Indeed, if $v(s_0) \in \partial B(\phi; \varepsilon)$ at some $s_0 > 0$, then we have $c \leq J(v(s_0)) \leq J(v_0)$, which contradicts the fact that $J(v_0) < c$. Therefore $v(s) \in B(\phi; \varepsilon)$ for all $s \geq 0$. By Remark 5.2, this proves the stability of ϕ.

We next show (ii). Let ϕ be a least energy solution which is isolated in $H_0^1(\Omega)$ from all the other sign-definite solutions of (1.10), (1.11). By Lemma 5.3, ϕ is also isolated in $H_0^1(\Omega)$ from all the other solutions, and hence, one can choose an $r > 0$ so small that there is no solution of (1.10), (1.11) on $\overline{B(\phi; r)}$ except for ϕ. By (i), the solution $v(s)$ remains in $B(\phi; \varepsilon)$ forever, where we choose ε less than r. Then the ω-limit set of $v(s)$ consists of a single point ϕ only. Thus by Theorem 3.2, $v(s)$ converges to ϕ, and therefore ϕ is asymptotically stable. $\square$

5.2 Unstable profiles

We next discuss the instability of profiles for solutions of (1.1)–(1.3). It is well known that (1.10), (1.11) has an unbounded sequence of solutions in $H_0^1(\Omega)$, which is proved by Ambrosetti and Rabinowitz [1] (see [34] also), and moreover, the set of all sign-definite solutions has an a priori bound in $L^\infty(\Omega)$, hence in $H_0^1(\Omega)$ also by Gidas and Spruck [22]. Therefore (1.10), (1.11) has an unbounded sequence of sign-changing solutions.

Theorem 5.5 (Unstable profiles). *Let ψ be a sign-changing profile of a solution of* (1.1)–(1.3)*. Then ψ is not asymptotically stable. Moreover, if ψ is isolated in $H_0^1(\Omega)$ from all $w \in \mathcal{S}$ satisfying $J(w) < J(\psi)$, i.e., there exists $R > 0$ such that*

$$\overline{B(\psi; R)} \cap \mathcal{S} \cap [J < J(\psi)] = \emptyset, \tag{5.2}$$

then ψ is unstable.

Proof. Let ψ be any sign-changing profile, i.e., it is a sign-changing solution of (1.10), (1.11). Then ψ has a $C^2(\overline{\Omega})$-regularity, and each connected component of $\{x \in \Omega;\ \psi(x) \neq 0\}$ is called a *nodal domain*. By assumption, ψ has at least two nodal domains. Let D be one of them. Then D and $\Omega \setminus \overline{D}$ are nonempty open sets. Define the functional J_D by

$$J_D(w) := \int_D \left(\frac{1}{2}|\nabla w(x)|^2 - \frac{\lambda_m}{m}|w(x)|^m \right) dx.$$

Since ψ satisfies (1.10) in D and vanishes on ∂D, it holds that

$$\int_D |\nabla \psi(x)|^2 dx = \lambda_m \int_D |\psi(x)|^m dx.$$

Then we have the relation,

$$J_D(t\psi) = \left(\frac{t^2}{2} - \frac{t^m}{m} \right) \int_D |\nabla \psi(x)|^2 dx \quad \text{for } t \geq 0,$$

which attains its maximum only at $t = 1$ over $t \in [0, \infty)$. The same assertion holds for $J_{\Omega \setminus \overline{D}}(t\psi)$ also. We define

$$\psi_\mu(x) := \begin{cases} \mu\psi(x) & \text{if } x \in D, \\ \psi(x) & \text{if } x \in \Omega \setminus D \end{cases}$$

for $\mu \geq 0$. Clearly, ψ_μ belongs to $H_0^1(\Omega)$ because ψ vanishes on $\partial D \cup \partial\Omega$. Then we see easily that

$$J(c\psi_\mu) = J_D(c\mu\psi) + J_{\Omega \setminus \overline{D}}(c\psi) < J_D(\psi) + J_{\Omega \setminus \overline{D}}(\psi) = J(\psi) \tag{5.3}$$

if $\mu \geq 0$, $\mu \neq 1$ and $c \geq 0$.

We are now in a position to show that ψ is not asymptotically stable. It is obvious that ψ_μ converges to ψ in $H_0^1(\Omega)$ as $\mu \to 1$. Put $u_{0,\mu} := \psi_\mu$, $\tau_\mu := t_*(u_{0,\mu})$ and $v_{0,\mu} := \tau_\mu^{-1/(m-2)} u_{0,\mu} \in \mathcal{X}$. By Proposition 2.6, we have $\tau_\mu \to t_*(\psi) = 1$, and therefore

$$v_{0,\mu} \to \psi \quad \text{strongly in } H_0^1(\Omega) \text{ as } \mu \to 1.$$

Let $v_\mu(s)$ be the solution of (1.6)–(1.8) with the initial data $v_{0,\mu}$. Since $J(v_\mu(s))$ is nonincreasing, by (5.3) we obtain

$$J(v_\mu(s)) \leq J(v_{0,\mu}) < J(\psi) \quad \text{if } \mu \neq 1, \tag{5.4}$$

which implies that $v_\mu(s)$ never converges to ψ as $s \to \infty$. Consequently, by Remark 5.2, ψ is not asymptotically stable.

We next assume (5.2). Let $u_{0,\mu}$, $v_{0,\mu}$ and $v_\mu(s)$ be as above. Then $v_\mu(s) \notin B(\psi; R)$ for all s large enough. Indeed, if there exists a sequence (s_n) diverging to ∞ such that $v_\mu(s_n) \in B(\psi; R)$, then by Theorem 3.2, $v_\mu(s_n)$ converges to an element $\phi \in \overline{B(\psi; R)} \cap \mathcal{S}$ along a subsequence. Moreover, $J(\phi) \leq J(v_{0,\mu}) < J(\psi)$. However, this contradicts (5.2). Therefore $v_\mu(s) \notin B(\psi; R)$ for all s large enough. This proves that ψ is an unstable profile. $\square$

As stated at the beginning of this subsection, the set of all sign-changing solutions for (1.10), (1.11) is nonempty. Denote it by $\mathcal{SC}$.

Definition 5.6 (Sign-changing least energy solution). *We define the sign-changing least energy d_2 by*

$$d_2 := \inf\{J(\psi);\ \psi \in \mathcal{SC}\}. \tag{5.5}$$

We call ψ a sign-changing least energy solution if $J(\psi) = d_2$ and $\psi \in \mathcal{SC}$.

The existence of a sign-changing least energy solution has already been proved by Castro, Cossio, and Neuberger [9]. However, for the reader's convenience we give a proof, which is simpler than that of [9].

Proposition 5.7. *The Dirichlet problem* (1.10), (1.11) *always admits a sign-changing least energy solution.*

Proof. It is known (we refer the reader to [34]) that J satisfies the Palais-Smale condition, i.e., any sequence (w_n) in $H_0^1(\Omega)$ has a subsequence strongly convergent in $H_0^1(\Omega)$ whenever $J(w_n)$ is bounded and $J'(w_n) \to 0$ as $n \to \infty$. Let ψ_n be a minimizing sequence for d_2, i.e., $\psi_n \in \mathcal{SC}$ and $J(\psi_n)$ converges to d_2. Then the Palais-Smale condition assures that, up to a subsequence, ψ_n converges to a critical point ψ of J. Then $J'(\psi) = 0$ and $J(\psi) = d_2$. We claim that ψ is sign-changing. Indeed, if ψ is positive, then by Lemma 5.3, $\psi_n > 0$ for n large enough. This contradicts $\psi_n \in \mathcal{SC}$. In case $\psi < 0$ also, a similar contradiction occurs. Therefore ψ is a sign-changing least energy solution. $\square$

Now, we have a corollary of Theorem 5.5.

Corollary 5.8 (Instability of sign-changing least energy solutions). *Every sign-changing least energy solution of* (1.10), (1.11) *is an unstable profile.*

Proof. Let ψ be a sign-changing least energy solution of (1.10), (1.11). Then we claim that ψ is isolated in $H_0^1(\Omega)$ from all $w \in \mathcal{S}$ satisfying $J(w) < J(\psi)$. Suppose on the contrary that there exists a sequence (w_n) in $\mathcal{S}$ such that $J(w_n) < J(\psi)$ and $w_n \to \psi$ strongly in $H_0^1(\Omega)$. Then w_n must be sign-definite because of the definition of sign-changing least energy solutions. However, since w_n converges also in $C^2(\overline{\Omega})$ by the bootstrap argument, ψ must be sign-definite, which yields a contradiction. Thus ψ is isolated. Therefore Theorem 5.5 leads us to our desired conclusion. $\square$

Remark 5.9. The sign-changing least energy d_2 is not necessarily the second least energy of nontrivial solutions for (1.10), (1.11). However, if (1.10), (1.11) has a unique positive solution, then d_2 is the second least energy, that is, there is no critical value of J in the interval (d, d_2).

5.3 Remarks on the uniqueness of positive profile

In this subsection, we give a couple of remarks on the uniqueness of positive solutions for (1.10), (1.11), which together with the preceding results imply the asymptotic stability of positive profiles.

A nontrivial solution ϕ of (1.10), (1.11) is said to be *nondegenerate* if the linearized operator does not admit zero as an eigenvalue, i.e., the linear problem,

$$\begin{cases} -\Delta z = (m-1)\lambda_m |\phi|^{m-2} z & \text{in } \Omega, \\ z = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{5.6}$$

admits only the trivial solution $z \equiv 0$.

Corollary 5.10. (i) *Every nondegenerate least energy solution of* (1.10), (1.11) *is an asymptotically stable profile. Moreover, all nondegenerate sign-changing solutions of* (1.10), (1.11) *are unstable profiles.*

(ii) *If* (1.10), (1.11) *has a unique positive solution, it is an asymptotically stable profile.*

Proof. It is well known that every nondegenerate solution is isolated in $H_0^1(\Omega)$ from all the other solutions, but for the reader's convenience, we give a proof. Assume that a sequence ϕ_n of solutions of (1.10), (1.11) converges to a nondegenerate solution ϕ in $H_0^1(\Omega)$. By the bootstrap argument with the elliptic regularity theorem, it converges in $C^2(\overline{\Omega})$ also. Put $z_n := (\phi_n - \phi)/\|\phi_n - \phi\|_\infty$ and $f(t) = |t|^{m-2}t$. Then z_n satisfies

$$-\Delta z_n = \lambda_m \frac{f(\phi_n) - f(\phi)}{\phi_n - \phi} z_n \quad \text{in } \Omega.$$

Since the right hand side is bounded in $L^\infty(\Omega)$, the elliptic regularity theorem guarantees that z_n is bounded in $W^{2,p}(\Omega)$ for all $p < \infty$. Thus a subsequence of z_n converges in $C^1(\overline{\Omega})$ to a limit z_∞, which solves (5.6). Moreover, $\|z_\infty\|_\infty = 1$ because $\|z_n\|_\infty = 1$. Thus (5.6) has a nontrivial solution and this fact contradicts the nondegeneracy of ϕ. Therefore every nondegenerate solution is isolated in $H_0^1(\Omega)$ from all the other solutions. Then (i) follows from Theorems 5.4 and 5.5.

Since least energy solutions are sign-definite, the uniqueness of positive solutions assures that of positive least energy solutions. Then (ii) follows from Theorem 5.4. □

Remark 5.11 (Uniqueness and nondegeneracy of positive solutions). The uniqueness and the nondegeneracy of positive (or least energy) solutions of (1.10), (1.11) rely on the shape of Ω and the growth order m of the nonlinear term. In this remark, we state known sufficient conditions of Ω and m for the uniqueness of positive solutions or the nondegeneracy of least energy solutions of (1.10), (1.11). For such Ω and m, Corollary 5.10 shows that least energy solutions are asymptotically stable profiles.

(i) (Gidas, Ni and Nirenberg, [21]). Let Ω be a ball and $2 < m < 2^*$. Then any positive solution must be radially symmetric, and moreover, a positive radial solution is unique (see [21, Lemma 2.3]).

(ii) (Dancer [11, Theorem 5]). Let $2 < m < 2^*$ and Ω be a bounded convex domain in $\mathbb{R}^2$, which is symmetric with respect to the coordinate axes. Then a positive solution is unique and nondegenerate (see Pacella [33] also).

(iii) (Lin [31]). Let $2 < m < 2^*$ and Ω be a bounded convex domain in $\mathbb{R}^2$. Then a least energy solution is unique and nondegenerate.

(iv) (Zou, [45]). Let $2 < m < 2 + \delta$ with a small $\delta > 0$ and Ω be close to a ball. Then a positive solution is unique.

(v) (Grossi, [23]). Let $N \geq 3$ and $2^* - \delta < m < 2^*$ with a small $\delta > 0$. Let $\Omega \subset \mathbb{R}^N$ be convex in x_i and symmetric with respect to the hyperplane $x_i = 0$ for each $1 \leq i \leq N$. Then a positive solution is unique and nondegenerate.

(vi) (Dancer, [12, Lemma 1]). Let $2 < m < 2 + \delta$ with a small $\delta > 0$ and Ω be any bounded smooth domain in $\mathbb{R}^N$. Then a positive solution is unique and nondegenerate.

5.4 Stability analysis in the one dimensional case

The final subsection addresses the one-dimensional case, i.e.,

$$N = 1, \quad \Omega = (0, 1).$$

Then we can give a representation of all nontrivial solutions of (1.10), (1.11), equivalently, all profiles of solutions for (1.1)–(1.3), and we also reveal the stability or the instability of each profile. Our analysis basically relies on a full picture of solutions to the one-dimensional stationary problem for (1.6)–(1.8), i.e.,

$$-\phi''(x) = \lambda_m |\phi|^{m-2} \phi(x) \quad \text{for } x \in (0, 1), \qquad \phi(0) = \phi(1) = 0. \tag{5.7}$$

Let us first show an explicit representation of solutions for (5.7). To do so, we define

$$g(\theta) := \frac{2}{m} \int_0^\theta (\sin \sigma)^{(2-m)/m} \, d\sigma, \qquad 0 \leq \theta \leq \frac{\pi}{2}$$

and put $\theta(x) := g^{-1}(x)$. Moreover, set

$$\Phi(x) := \begin{cases} (\sin \theta(x))^{2/m} & \text{if } \ 0 \leq x \leq T := g(\pi/2), \\ -(\sin \theta(-x))^{2/m} & \text{if } \ -T \leq x < 0. \end{cases}$$

One can further extend Φ to be an anti-periodic function in $\mathbb{R}$ with the period $2T$, that is, $\Phi(x + 2T) = -\Phi(x)$. Then Φ is of class C^2 and solves

$$-\Phi''(x) = \frac{m}{2} |\Phi|^{m-2} \Phi(x) \quad \text{for } x \in \mathbb{R}. \tag{5.8}$$

Since $m > 2$, the function $\Psi(x) := c_m \Phi(x)$ with $c_m := (2\lambda_m/m)^{-1/(m-2)}$ solves

$$-\Psi''(x) = \lambda_m |\Psi|^{m-2} \Psi(x) \quad \text{for } x \in \mathbb{R}. \tag{5.9}$$

Moreover, we notice that equation (5.9) is invariant under the scaling $\Psi_\mu(x) := \mu^{2/(m-2)}\Psi(\mu x)$ for $\mu > 0$. In order to satisfy the boundary condition, $\Psi_\mu(1) = 0$, we specify the scaling parameter $\mu = \mu_n := 2nT$ with $n \in \mathbb{N}$. By virtue of a standard argument for the uniqueness of solutions of the Cauchy problem for ODEs, one can ensure that every nontrivial solution of the Dirichlet problem (5.7) is explicitly written as $\phi = \Psi_{\mu_n}$ or $-\Psi_{\mu_n}$, which has exactly $n-1$ zeros in $(0,1)$, with some $n \in \mathbb{N}$. In the rest of the present paper, we write $\phi_n := \Psi_{\mu_n}$ for $n \in \mathbb{N}$.

Furthermore, we find that

$$J(\pm\phi_n) = \kappa_m n^{2m/(m-2)},$$

with some positive constant κ_m independent of n. This means that

$$J(\pm\phi_1) < J(\pm\phi_2) < \cdots < J(\pm\phi_n) < \cdots \nearrow \infty.$$

Thus each solution $\pm\phi_n$ is isolated in $H^1_0(0,1)$ from all the other solutions. For any solution $v(s)$ of (1.6)–(1.8), $J(v(s))$ is nonincreasing and converges as $s \to \infty$. Combining these facts, we obtain the uniqueness of the ω-limit point of $v(s)$ and get the stability or the instability of each profile as below.

Corollary 5.12 (Asymptotic profiles for $N = 1$). *For each solution $u = u(x,t)$ of* (1.1)–(1.3) *with $u_0 \in H^1_0(0,1) \setminus \{0\}$, there exists a unique solution $\phi \in H^1_0(0,1) \setminus \{0\}$ of* (5.7) *such that*

$$\lim_{t \nearrow t_*} \|(t_* - t)^{-1/(m-2)} u(t) - \phi\|_{1,2} = 0.$$

Corollary 5.13 (Stability/instability of profiles for $N = 1$). *The sign-definite profiles $\pm\phi_1$ are asymptotically stable and the other profiles $\pm\phi_i$ (for $i \neq 1$) are unstable.*

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