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DOUBLY NONLINEAR PARABOLIC EQUATIONS INVOLVING VARIABLE EXPONENTS

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Abstract. This paper is concerned with doubly nonlinear parabolic equations involving variable exponents. The existence of solutions is proved by developing an abstract theory on doubly nonlinear evolution equations governed by gradient operators. In contrast to constant exponent cases, two nonlinear terms have inhomogeneous growth and some difficulty may occur in establishing energy estimates. Our method of proof relies on an efficient use of Legendre-Fenchel transforms of convex functionals and an energy method.

1. **Introduction.** Differential equation with nonstandard growth is one of the fastest growing topics in the recent developments of nonlinear analysis. The reader is referred to [21] for an overview of differential equations with nonstandard growth. This field is supported by the longtime study of Lebesgue and Sobolev spaces with variable exponents. There are a vast amount of contribution to elliptic equations with variable exponents. On the other hand, parabolic problems have not been studied so well, and they are attracting more attention from mathematical interests as well as from engineering applications.

Let Ω be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$ and let $p(\cdot)$ and $m(\cdot)$ be variable exponents defined in Ω with values in $(1,\infty)$. The so-called $p(\cdot)$-*Laplacian* $\Delta_{p(\cdot)}$ is a typical example of nonlinearity with nonstandard growth, and it is defined by

$$\Delta_{p(\cdot)}\phi = \nabla\cdot\left(|\nabla\phi(x)|^{p(x)-2}\nabla\phi(x)\right) \quad \text{for } x\in\Omega.$$

This paper is concerned with the following doubly nonlinear parabolic problem (P):

$$\partial_t\left(|u|^{m(\cdot)-2}u\right) - \Delta_{p(\cdot)}u = 0 \quad \text{in } \Omega\times(0,T), \tag{1}$$

$$|u|^{m(\cdot)-2}u = v_0 \quad \text{on } \Omega\times\{0\}, \tag{2}$$

where $\partial_t = \partial/\partial t$, $T>0$, $v_0 = v_0(x)$ is a given initial data and

$$|u|^{m(\cdot)-2}u = |u(x,t)|^{m(x)-2}u(x,t) \quad \text{for } (x,t)\in\Omega\times(0,T),$$

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with either the Dirichlet condition

$$u = 0 \quad \text{on} \;\; \partial\Omega \times (0,T), \tag{3}$$

or the Neumann condition

$$|\nabla u|^{p(\cdot)-2}\partial_n u = 0 \quad \text{on} \;\; \partial\Omega \times (0,T), \tag{4}$$

where $\partial_n u$ denotes the outward normal derivative of u on $\partial\Omega$. We denote by (P) the initial-boundary value problem (1), (2) with either the Dirichlet condition (3) or the Neumann condition (4).

Parabolic equations involving the $p(\cdot)$-Laplacian appear in the field of image restoration (see [14]) and in some model of electrorheological fluids (see [32]). Then these equations have been mathematically studied in [1], [7], [19], [11], [36], [3] (see also the references of [3]). Some porous medium type equation with variable exponents is also studied in [8], where the well-posedness is proved and asymptotic behaviors of solutions are investigated.

The study of doubly nonlinear parabolic equations dates back to 1970s (see [30], [25], [20] and also [5], [34]). Equation (1) can be regarded as a generalized form of two sorts of nonlinear diffusion equations: porous medium equation ($m(\cdot) \equiv m$, $p(\cdot) \equiv 2$) and p-Laplace parabolic equation ($m(\cdot) \equiv 2$, $p(\cdot) \equiv p$), and moreover, it also appears in some model of non-Newtonian fluid dynamics. This field has also encouraged the developments of the theory of nonlinear evolution equations (see, e.g., [10], [15], [23], [27], [31], [35], [2], [4]). However, to the best of the author's knowledge, there is no contribution to doubly nonlinear problems with nonstandard growth such as (1) except for [6] and [12]. In [6], a doubly nonlinear parabolic equation of the form:

$$\partial_t v - \nabla \cdot \left(a(\cdot,\cdot,v)|v|^{\alpha(\cdot,\cdot)}|\nabla v|^{p(\cdot,\cdot)-2}\nabla v \right) = f \quad \text{in} \;\; \Omega \times (0,T) \tag{5}$$

is studied for given functions $a = a(x,t,v)$, $f = f(x,t)$ and (x,t)-dependent exponents $p = p(x,t)$, $\alpha = \alpha(x,t)$, and then, bounded weak solutions of the Cauchy-Dirichlet problem are constructed for $L^\infty(\Omega)$-data by using Galerkin's method. However, Equation (1) does not seem to be directly covered by their frame due to the x-dependence of the variable exponent m, although (1) can be rewritten as (5) by setting $v = |u|^{m-2}u$, $f \equiv 0$, $\alpha(x,t) = (p(x)-1)(m'-2)$ and $a(x,t,v) = (m'-1)^{p(x)-1}$, provided that $m(\cdot) \equiv m$ is a constant exponent in (1). In [12], the existence of solutions is proved for a doubly nonlinear equation involving variable exponents $m(\cdot)$, $q(\cdot)$ such as

$$\partial_t \left(u + |u|^{m(\cdot)-2}u \right) - \Delta_p u + |u|^{q(\cdot)-2}u = f \quad \text{in} \;\; \Omega \times (0,T)$$

with a standard p-Laplacian.

Let us mention a couple of difficulties arising from variable exponents of doubly nonlinear problems. It is often useful in energy methods to test doubly nonlinear equations by operands of the time-differential operator ∂_t (e.g., $|u|^{m(\cdot)-2}u$ for (1)). In constant exponent cases of (1), i.e., $m(x) \equiv m$, with (3) or (4), one can formally calculate

$$\begin{aligned} \int_\Omega \left(-\Delta_{p(\cdot)}u\right)|u|^{m-2}u \, \mathrm{d}x &= \int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\left(|u|^{m-2}u\right)\mathrm{d}x \\ &= (m-1)\int_\Omega |\nabla u|^{p(x)}|u|^{m-2}\mathrm{d}x \geq 0, \end{aligned}$$

and then, this observation plays a crucial role to establish $L^2(\Omega)$-estimates for the nonlinear term $|u|^{m-2}u$ (see, e.g., [30], [10], [2], [4]). On the other hand, in variable exponent cases, an additional term appears in the same process,

$$\int_\Omega \left(-\Delta_{p(\cdot)}u\right) |u|^{m(\cdot)-2}u \, \mathrm{d}x = \int_\Omega (m(x)-1)|\nabla u|^{p(x)}|u|^{m(x)-2}\mathrm{d}x + \int_\Omega |\nabla u|^{p(x)-2} \left(\nabla u \cdot \nabla m\right) |u|^{m(x)-2}u \log |u| \mathrm{d}x,$$

and the last term is more difficult to be controlled. In the language of maximal monotone operator theory, for constant exponent cases, two nonlinear operators $A : u \mapsto -\Delta_{p(\cdot)}u$ and $B : u \mapsto |u|^{m(\cdot)-2}u$ comply with some angle condition, which is related to the maximality of the sum of two operators (see, e.g., [13]). On the other hand, for variable exponent cases, such a condition might be violated, and hence, even for proving the existence of solutions, previous approaches developed for constant exponent cases might not work well.

In [27], some abstract framework free from the energy technique mentioned above is also established for doubly nonlinear parabolic equations, and it imposes uniform power growth conditions on nonlinear operators A, B instead of angle conditions (cf. a similar attempt was originally made by [15]). However, for variable exponent cases, two operators A, B are not homogeneous and might have different lower and upper growth orders (cf. in constant exponent cases, they are homogeneous, e.g., $\Delta_p(cu) = c^{p-1}\Delta_p u$). Hence, doubly nonlinear problems such as (P) involving variable exponents do not immediately fall within the framework of [27].

In this paper, we prove the existence of solutions for (P) by developing an abstract theory on evolution equations governed by gradient operators of convex functionals. To cope with the preceding difficulties, we shall efficiently employ Legendre-Fenchel transforms of convex functionals and apply energy technique developed by the author in [2, 4]. Our abstract theory does not rely on neither angle conditions between two nonlinear operators nor uniform power growth conditions. Instead, we introduce a *joint coercivity condition of convex functionals.* Moreover, our framework is built on two reflexive Banach spaces in a common ambient space; however, we do not assume any embedding between them. We finally remark that, in contrast with [6], our result is concerned with more energetic solutions; indeed, initial data belong to a natural energy class, but might not belong to $L^\infty(\Omega)$. On the other hand, we impose the so-called Sobolev subcritical condition on variable exponents in compensation. Moreover, our abstract frame can handle both the Cauchy-Dirichlet and -Neumann problems in a unified fashion.

In the next section, we briefly review Lebesgue and Sobolev spaces with variable exponents as well as selected topics of convex analysis for latter use. In Section 3, we reduce (P) into the Cauchy problem for an abstract doubly nonlinear evolution equation. Section 4 is devoted to establishing an existence result for the Cauchy problem. In Section 5, the preceding abstract theory will be applied to (P), both the Dirichlet and Neumann cases, under a subcritical condition of variable exponents.

2. **Preliminaries.**

2.1. **Lebesgue and Sobolev spaces with variable exponents.** This subsection is devoted to some preliminary results on Lebesgue and Sobolev spaces with variable exponents (see [16] for a survey). Let Ω be a domain in $\mathbb{R}^d$. We denote by $\mathcal{P}(\Omega)$

the set of all measurable functions $p : \Omega \to [1, \infty]$. For $p \in \mathcal{P}(\Omega)$, we write

$$p^+ := \operatorname*{ess\,sup}_{x\in\Omega}\, p(x), \quad p^- := \operatorname*{ess\,inf}_{x\in\Omega}\, p(x).$$

Throughout this subsection, we assume that $p \in \mathcal{P}(\Omega)$. Define the Lebesgue space with a variable exponent $p(\cdot)$ as follows:

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R}\colon \text{ measurable in } \Omega \text{ and } \int_\Omega |u(x)|^{p(x)} \mathrm{d}x < \infty \right\}$$

with a Luxemburg-type norm

$$\|u\|_{p(\cdot)} := \inf \left\{ \lambda > 0\colon \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} \mathrm{d}x \le 1 \right\}.$$

Then $L^{p(\cdot)}(\Omega)$ is a special sort of Musielak-Orlicz spaces (see [29]) and sometimes called *Nakano space*. For $p^+ < \infty$, the dual space of $L^{p(\cdot)}(\Omega)$ is identified with $L^{p'(\cdot)}(\Omega)$ with the dual variable exponent $p' \in \mathcal{P}$ given by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \quad \text{ for a.e. } x \in \Omega,$$

where we write $1/\infty = 0$.

Hölder's inequalities also hold for variable exponent Lebesgue spaces (see Lemma 3.2.20 of [16]).

Proposition 1 (Hölder's inequality)**.** *For $s, p, q \in \mathcal{P}(\Omega)$, it holds that*

$$\|fg\|_{s(\cdot)} \le 2\|f\|_{p(\cdot)}\|g\|_{q(\cdot)} \quad \textit{for all } \ f \in L^{p(\cdot)}(\Omega),\ g \in L^{q(\cdot)}(\Omega),$$

provided that

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)} \quad \textit{for a.e. } x \in \Omega.$$

In particular, if Ω is bounded and $p(x) \le q(x)$ for a.e. $x \in \Omega$, then $L^{q(\cdot)}(\Omega)$ is continuously embedded in $L^{p(\cdot)}(\Omega)$.

The following proposition plays an important role to establish energy estimates (see, e.g., Theorem 1.3 of [17] for a proof).

Proposition 2. *It holds that*

$$\sigma^-(\|w\|_{p(\cdot)}) \le \int_\Omega |w(x)|^{p(x)} \mathrm{d}x \le \sigma^+(\|w\|_{p(\cdot)}) \quad \textit{for all } \ w \in L^{p(\cdot)}(\Omega)$$

with the strictly increasing functions

$$\sigma^-(s) := \min\{s^{p^-}, s^{p^+}\}, \quad \sigma^+(s) := \max\{s^{p^-}, s^{p^+}\} \quad \textit{for } s \ge 0.$$

We next define variable exponent Sobolev spaces $W^{1,p(\cdot)}(\Omega)$ as follows:

$$W^{1,p(\cdot)}(\Omega) := \left\{ u \in L^{p(\cdot)}(\Omega)\colon \frac{\partial u}{\partial x_i} \in L^{p(\cdot)}(\Omega) \quad \text{for all } \ i = 1, 2, \dots, d \right\}$$

with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \left(\|u\|_{p(\cdot)}^2 + \|\nabla u\|_{p(\cdot)}^2 \right)^{1/2},$$

where $\|\nabla u\|_{p(\cdot)}$ denotes the $L^{p(\cdot)}(\Omega)$-norm of $|\nabla u|$. Furthermore, let $W^{1,p(\cdot)}_0(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Here we note that the space $W^{1,p(\cdot)}_0(\Omega)$ is usually defined in a slightly different way for the variable exponent case. However, both definitions are equivalent under (6) given below (see [16] and also [37] for an unusual phenomena of discontinuous exponents).

The following proposition is concerned with the uniform convexity of $L^{p(\cdot)}$- and $W^{1,p(\cdot)}$-spaces.

Proposition 3 ([16])**.** *If $p^+ < \infty$, then $L^{p(\cdot)}(\Omega)$ is a separable Banach space. In addition, if $p^- > 1$ and $p^+ < \infty$, then $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are uniformly convex. Hence they are reflexive.*

Let us exhibit Poincaré and Sobolev inequalities. To do so, we introduce the *log-Hölder condition*:

$$|p(x) - p(x')| \leq \frac{A}{\log(e + 1/|x - x'|)} \quad \text{for all } x, x' \in \Omega \tag{6}$$

with some constant $A > 0$ (see [16]). This condition follows from a Hölder continuity of p over $\overline{\Omega}$ with any Hölder exponent and it implies $p \in C(\overline{\Omega})$ and $p^+ < \infty$. We denote by $\mathcal{P}_{\log}(\Omega)$ the set of all $p \in \mathcal{P}(\Omega)$ satisfying the log-Hölder condition (6).

Proposition 4 ([16])**.** *Let Ω be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$ and let $p \in \mathcal{P}_{\log}(\Omega)$.*

(i) *There exists a constant $C \geq 0$ such that*

$$\|w\|_{p(\cdot)} \leq C\|\nabla w\|_{p(\cdot)} \quad \textit{for all} \ \ w \in W_0^{1,p(\cdot)}(\Omega).$$

In particular, the space $W_0^{1,p(\cdot)}(\Omega)$ has a norm $\|\cdot\|_{1,p(\cdot)}$ given by

$$\|w\|_{1,p(\cdot)} := \|\nabla w\|_{p(\cdot)} \quad \textit{for} \ \ w \in W_0^{1,p(\cdot)}(\Omega),$$

which is equivalent to $\|\cdot\|_{W^{1,p(\cdot)}(\Omega)}$.

(ii) *Let $q : \Omega \to [1, \infty)$ be a measurable and bounded function and suppose that*

$$q(x) \leq p^*(x) := dp(x)/(d - p(x))_+ \quad \textit{for a.e. } x \in \Omega,$$

where $(s)_+ := \max\{s, 0\}$ for $s \in \mathbb{R}$. Then $W^{1,p(\cdot)}(\Omega)$ is continuously embedded in $L^{q(\cdot)}(\Omega)$.

In addition, assume that

$$\operatorname*{ess\,inf}_{x\in\Omega} \ \big(p^*(x) - q(x)\big) > 0.$$

Then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact.

Remark 1. In [28], it is proved that the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact when $p^*(x)$ coincides with $q(x)$ on some thin part of Ω and the difference between two variable exponents are appropriately controlled on the other part (see also [24]).

As for the Kadec-Klee (or Radon-Riesz) property in terms of uniformly convex modulars of $L^{p(\cdot)}$-spaces, let us give the following proposition, which is a direct consequence of Lemma 2.4.17 and Theorem 3.4.9 of [16] and Proposition 2.

Proposition 5 (The Kadec-Klee property of uniformly convex modulars)**.** *Let $p \in \mathcal{P}(\Omega)$ be such that $1 < p^- \leq p^+ < \infty$. Let (u_n) be a sequence in $L^{p(\cdot)}(\Omega)$ and $u \in L^{p(\cdot)}(\Omega)$. If $u_n \to u$ weakly in $L^{p(\cdot)}(\Omega)$ and*

$$\int_\Omega |u_n(x)|^{p(x)} \mathrm{d}x \to \int_\Omega |u(x)|^{p(x)} \mathrm{d}x,$$

then $u_n \to u$ strongly in $L^{p(\cdot)}(\Omega)$.

2.2. **Convex analysis.** Let ϕ be a proper (i.e., $\phi \not\equiv \infty$), lower semicontinuous convex functional from a normed space E into $(-\infty, \infty]$. We denote by $D(\phi)$ the *effective domain* of ϕ, i.e., $D(\phi) := \{u \in E \colon \phi(u) < \infty\}$. Moreover, ϕ is said to be *coercive* in E, if it holds that

$$\lim_{|u|_E \to \infty} \frac{\phi(u)}{|u|_E} = \infty.$$

A functional $\phi : E \to \mathbb{R}$ is said to be *Gâteaux differentiable* at u (respectively, in E), if there exists $\xi \in E^*$ such that

$$\lim_{h \to 0} \frac{\phi(u + he) - \phi(u)}{h} = \langle \xi, e \rangle_E \quad \text{for all } e \in E$$

at u (respectively, for all $u \in E$). Then ξ is called the *Gâteaux derivative* of ϕ at u and denoted by $\mathrm{d}\phi(u)$. Here we note that ϕ is Gâteaux differentiable, if it is Fréchet differentiable. The *gradient operator* $\mathrm{d}\phi : E \to E^*$ of a Gâteaux differentiable functional ϕ maps u to $\mathrm{d}\phi(u)$.

The *subdifferential* operator $\partial\phi : E \to E^*$ of ϕ is defined by

$$\partial\phi(u) := \{\xi \in E^* \colon \phi(v) - \phi(u) \geq \langle \xi, v - u \rangle_E \ \text{ for all } \ v \in E\}$$

with the domain $D(\partial\phi) := \{u \in D(\phi) \colon \partial\phi(u) \neq \emptyset\}$. Subdifferential is a generalized notion of Fréchet (or Gâteaux) derivative, and they coincide with each other when ϕ is Fréchet (or Gâteaux) differentiable. It is well known that $\partial\phi$ is maximal monotone in $E \times E^*$. Throughout this paper, we denote by A the graph of a possibly multivalued operator $A : E \to E^*$. Hence $[u, \xi] \in A$ means that $u \in D(A)$ and $\xi \in A(u)$.

The *Legendre-Fenchel transform* (or *convex conjugate*) ϕ^* of a proper lower semicontinuous convex functional $\phi : E \to (-\infty, \infty]$ is given by

$$\phi^*(f) := \sup_{v \in E} \left\{ \langle f, v \rangle_E - \phi(v) \right\} \quad \text{for } f \in E^*.$$

Let us list up several useful properties of ϕ^* (see, e.g., [9]):

(i) ϕ^* is proper, lower semicontinuous and convex in E^*;
(ii) $\phi^*(f) = \langle f, u \rangle_E - \phi(u)$ for all $[u, f] \in \partial\phi$;
(iii) $u \in \partial\phi^*(f)$ if and only if $f \in \partial\phi(u)$.

Moreover, we observe that $\phi^*(f) \geq -\phi(0)$ for all $f \in E^*$ when $0 \in D(\phi)$.

3. **Reduction of (P) to an abstract Cauchy problem.** Let us first state our basic assumptions (H):

$$p \in \mathcal{P}_{\log}(\Omega), \quad m \in \mathcal{P}(\Omega), \quad 1 < m^- \leq m^+ < \infty, \quad 1 < p^- \leq p^+ < \infty.$$

We now set up function spaces:

$$V := \begin{cases} W_0^{1,p(\cdot)}(\Omega) & \text{for the Dirichlet condition (3)}, \\ W^{1,p(\cdot)}(\Omega) & \text{for the Neumann condition (4)}, \end{cases} \qquad W := L^{m(\cdot)}(\Omega) \tag{7}$$

with dual spaces V^* and W^*, respectively.

Let us next introduce functionals,

$$\varphi(u) := \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} \mathrm{d}x \quad \text{for } u \in V \tag{8}$$

and

$$\psi(u) := \int_\Omega \frac{1}{m(x)} |u(x)|^{m(x)} \mathrm{d}x \quad \text{for } u \in W. \tag{9}$$

Then φ and ψ are Fréchet (hence Gâteaux) differentiable in V and W, respectively, and moreover, the Fréchet (Gâteaux) derivative $\mathrm{d}\varphi(u)$ coincides with $-\Delta_{p(\cdot)}u$ equipped with the homogeneous Dirichlet or Neumann condition in V^*, and $\mathrm{d}\psi(u) = |u|^{m(\cdot)-2}u$ in W^*. Therefore (P) is reduced into the following abstract Cauchy problem,

$$v'(t) + \mathrm{d}\varphi(u(t)) = 0, \quad v(t) = \mathrm{d}\psi(u(t)), \quad 0 < t < T, \tag{10}$$

$$v(0) = v_0. \tag{11}$$

4. **Abstract Cauchy problem.** In this section, we prove the existence of solutions for doubly nonlinear evolution equations such as (10), (11) (equivalently, (P)). Here we work in a more general frame. Throughout this section, let V and W be reflexive Banach spaces in a common ambient space such that

$$X := V \cap W \neq \emptyset.$$

We set the norm $|\cdot|_X := |\cdot|_V + |\cdot|_W$. Denote by V^*, W^* and X^* the dual spaces of V, W and X, respectively. Then it holds that $X \hookrightarrow V$, W and V^*, $W^* \hookrightarrow X^*$ continuously. Due to the presence of the common ambient space, V^* and W^* have a non-empty intersection.

Moreover, let $\varphi : V \to \mathbb{R}$ and $\psi : W \to \mathbb{R}$ are Gâteaux differentiable, continuous and convex. Then we treat the following abstract Cauchy problem (CP):

$$v'(t) + \mathrm{d}\varphi(u(t)) = 0 \ \text{ in } X^*, \quad v(t) = \mathrm{d}\psi(u(t)) \ \text{ in } W^*, \quad 0 < t < T, \tag{12}$$

$$v(0) = v_0 \ \text{ in } X^*, \tag{13}$$

where $\mathrm{d}\varphi : V \to V^*$ and $\mathrm{d}\psi : W \to W^*$ are Gâteaux derivatives of φ and ψ, respectively. We are concerned with strong solutions for (CP) in the following sense:

Definition 4.1 (Strong solutions)**.** A pair of functions $(u, v) : [0, T] \to X \times W^*$ is said to be a *strong solution* of (CP) on $[0, T]$ if the following (i)–(iii) hold true:

(i) v is an X^*-valued absolutely continuous function on $[0, T]$;
(ii) $v'(t) + \mathrm{d}\varphi(u(t)) = 0$ in X^* and $v(t) = \mathrm{d}\psi(u(t))$ in W^* for a.a. $t \in (0, T)$;
(iii) $v(0) = v_0$ in X^*, i.e., $v(t) \to v_0$ strongly in X^* as $t \to 0_+$.

In order to state our result, let us introduce the following assumptions:

(A1) (i) $\varphi + \psi$ is coercive in X.
(ii) Let (u_n) be a sequence in V and let $v_n = \mathrm{d}\psi(u_n)$ be such that $\varphi(u_n)$ and $\psi^*(v_n)$ are bounded for all $n \in \mathbb{N}$. Then (u_n) and (v_n) are bounded in X and W^*, respectively.

(A2) There exists a non-decreasing function ℓ_1 in $\mathbb{R}$ such that

$$|\mathrm{d}\varphi(u)|_{V^*} \le \ell_1(\varphi(u)) \quad \text{ for all } u \in V.$$

(A3) For every $\lambda \in \mathbb{R}$, the sublevel set $[\psi^* \le \lambda] = \{v \in W^* \colon \psi^*(v) \le \lambda\}$ is precompact in X^*.

Remark 2 (Assumptions)**.** In (A1), we impose *joint coercivity conditions* on φ and ψ. It is noteworthy that (A1) could hold even if neither φ nor ψ is not coercive (e.g., the Neumann case (4) of (P)). Condition (A2) can be regarded as a boundedness of the gradient operator $\mathrm{d}\varphi$ from V into V^*. Finally, (A3) provides some compactness to be used for proving the strong convergence of approximate solutions. It will require a subcritical condition between variable exponents in an application to (P) for initial data belonging to an energy class.

Remark 3 (Coercivity and smoothness of convex conjugates)**.** The Legendre-Fenchel transform ψ^* could not inherit its coercivity from ψ. Indeed, define a convex function $\psi : \mathbb{R} \to [0, \infty]$ by

$$\psi(x) = \begin{cases} \infty & \text{if } \ x < 0, \\ \dfrac{x^2}{2} & \text{if } \ x \geq 0. \end{cases}$$

Obviously, ψ is coercive in $\mathbb{R}$. On the other hand, the convex conjugate ψ^* of ψ reads,

$$\psi^*(\xi) = \max_{x \in \mathbb{R}} (x\xi - \psi(x)) = \max_{x \geq 0} \left(x\xi - \frac{x^2}{2} \right) = \begin{cases} 0 & \text{if } \ \xi < 0, \\ \dfrac{\xi^2}{2} & \text{if } \ \xi \geq 0. \end{cases}$$

Then ψ^* is not coercive in $\mathbb{R}$.

This example also exhibits that the smoothness of functionals could not be preserved under Legendre-Fenchel transform. Indeed, ψ $(= \psi^{**})$ is nonsmooth in $\mathbb{R}$ although ψ^* is of class $C^1(\mathbb{R})$.

Now our result reads,

Theorem 4.2 (Existence of solutions for abstract Cauchy problems)**.** *Assume that* (A1)–(A3) *hold and let* $v_0 = \mathrm{d}\psi(u_0) \in D(\psi^*)$ *be given by an arbitrary* $u_0 \in X$*. Then the Cauchy problem* (CP)(= (12), (13)) *admits at least one strong solution* (u, v) *satisfying*

$$u \in L^\infty(0, T; X), \quad v \in W^{1,\infty}(0, T; X^*) \cap C_w([0, T]; W^*),$$
$$v' \in L^\infty(0, T; V^*), \quad \psi^*(v(\cdot)) \in W^{1,\infty}(0, T),$$

where $C_w([0, T]; W^*)$ *denotes the class of all weakly continuous functions on* $[0, T]$ *with values in* W^**.*

Remark 4 (Generalization of (A2))**.** In Theorem 4.2, (A2) can be replaced by a slightly weaker condition $(\mathrm{A2})'$:

$(\mathrm{A2})'$ There exists a non-decreasing function ℓ_2 in $\mathbb{R}$ such that

$$|\mathrm{d}\varphi(u)|_{V^*} \leq \ell_2(\varphi(u) + |u|_X + \psi^*(v)) \quad \text{ for all } u \in X \ \text{ with } \ v := \mathrm{d}\psi(u).$$

One may explicitly find where (A2) is used and confirm the possibility of the replacement in the following proof (see (26) below).

For simplicity, let us assume $\varphi \geq 0$, $\psi \geq 0$ and V and W are separable (see, e.g., [9]). However, they are not essential and can be removed by slightly modifying the following arguments.

The rest of this section is devoted to a proof of Theorem 4.2, which is based on a time-discretization. To this end, define functionals $J(\cdot; g) : X \to [0, \infty)$ for each $g \in X^*$ by

$$J(u; g) := \frac{1}{h}\psi(u) + \varphi(u) - \langle g, u \rangle_X \quad \text{ for } \ u \in X,$$

where $\langle \cdot, \cdot \rangle_X$ denotes a duality pairing between X and X^*. Here we note that the restrictions $\hat{\psi}, \hat{\varphi} : X \to [0, \infty)$ of ψ, φ, respectively, onto X are Gâteaux differentiable and $\mathrm{d}\hat{\psi}(u) = \mathrm{d}\psi(u)$, $\mathrm{d}\hat{\varphi}(u) = \mathrm{d}\varphi(u)$ for all $u \in X$. Then $J(\cdot; g)$ is Gâteaux differentiable, continuous and convex in X. Moreover, by (i) of (A1), $J(\cdot; g)$ is coercive in X, and hence, $J(\cdot; g)$ admits a minimizer for each $g \in X^*$.

Let $N \in \mathbb{N}$ and set $h := T/N > 0$. Recall that $u_0 \in X$ and $v_0 = \mathrm{d}\psi(u_0) \in D(\psi^*)$. For each $n = 0, 1, 2, \ldots, N-1$, let us iteratively take a unique minimizer $u_{n+1} \in X$ of the functional $J(\cdot; g_n)$ with

$$g_n := \frac{v_n}{h} \in W^* \hookrightarrow X^* \quad \text{and} \quad v_n := \mathrm{d}\psi(u_n).$$

Then it follows that

$$\frac{v_{n+1} - v_n}{h} + \mathrm{d}\varphi(u_{n+1}) = 0 \ \text{ in } X^*, \quad v_{n+1} = \mathrm{d}\psi(u_{n+1}) \tag{14}$$

for each $n = 0, 1, \ldots, N-1$.

Remark 5 (Gâteaux differentiability of functionals)**.** In many studies of doubly nonlinear evolution equations such as (12), (13), two functionals φ and ψ are assumed to be proper, lower semicontinuous and convex, and moreover, gradient operators are replaced by subdifferential operators. However, in this paper, we always assume the Gâteaux differentiability of φ and ψ. The Gâteaux differentiability is used at two points of our construction of solutions for discretized problems (14). One is for the sum rule $\mathrm{d}(\hat{\psi} + \hat{\varphi}) = \mathrm{d}\hat{\psi} + \mathrm{d}\hat{\varphi}$ of Gâteaux differential. This property also holds for subdifferentials under the maximality of the sum of two subdifferentials. The other is for the coincidence between $\mathrm{d}\varphi$ and $\mathrm{d}\hat{\varphi}$. Such a coincidence could be violated for subdifferentials (cf. it holds that $\partial\varphi(u) \subset \partial\hat{\varphi}(u)$ for $u \in X$).

Then we have the following estimates:

Lemma 4.3 (Estimates for solutions of discretized problems)**.** *There exists a constant $C \geq 0$ independent of n, N and h such that*

$$\max_n \varphi(u_n) \leq \varphi(u_0), \tag{15}$$

$$\max_n \psi^*(v_n) \leq \psi^*(v_0) + \varphi(0)T, \tag{16}$$

$$\max_n |u_n|_X \leq C, \quad \max_n |v_n|_{W^*} \leq C, \tag{17}$$

where ψ^* stands for the convex conjugate of ψ in W.

Proof. Multiply (14) by $(u_{n+1} - u_n)/h \in X$ to get

$$\left\langle \frac{v_{n+1} - v_n}{h}, \frac{u_{n+1} - u_n}{h} \right\rangle_W + \frac{\langle \mathrm{d}\varphi(u_{n+1}), u_{n+1} - u_n \rangle_V}{h} = 0.$$

By using the monotonicity of $\mathrm{d}\psi$ and the convexity of φ, we deduce

$$\frac{\varphi(u_{n+1}) - \varphi(u_n)}{h} \leq 0,$$

which implies (15).

Moreover, test (14) by $u_{n+1} \in X$. Then we observe

$$\left\langle \frac{v_{n+1} - v_n}{h}, u_{n+1} \right\rangle_W + \langle \mathrm{d}\varphi(u_{n+1}), u_{n+1} \rangle_V = 0.$$

Here note that $u_{n+1} \in \partial\psi^*(v_{n+1})$ (see (iii) of §2.2). By the convexity of ψ^* and φ, we obtain

$$\frac{\psi^*(v_{n+1}) - \psi^*(v_n)}{h} + \varphi(u_{n+1}) \leq \varphi(0),$$

which yields (16).

Finally, using (ii) of (A1), one can derive (17) from (15) and (16). □

Now, let us introduce interpolants of $\{u_n\}$ and $\{v_n\}$. Let $\overline{v}_N : [0,T] \to W^*$ and $v_N : [0,T] \to W^*$ be a piecewise forward constant interpolant and a piecewise linear interpolant of $\{v_n\}$, respectively. More precisely, for each n, we set $t_n := nh$ (hence $t_0 = 0$ and $t_N = T$) and put

$$v_N(t) := \frac{(t-t_n)v_{n+1} + (t_{n+1}-t)v_n}{h}, \quad \overline{v}_N(t) := v_{n+1} \quad \text{for } \ t \in (t_n, t_{n+1}]$$

with $v_N(0) = \overline{v}_N(0) = v_0$. We also define a piecewise forward constant interpolant $\overline{u}_N : [0,T] \to X$ of $\{u_n\}$ in a similar way. Then (14) is rewritten as

$$v_N'(t) + \mathrm{d}\varphi(\overline{u}_N(t)) = 0 \ \text{ in } X^*, \quad 0 < t < T, \tag{18}$$

$$\overline{v}_N(t) = \mathrm{d}\psi(\overline{u}_N(t)) \ \text{ in } W^*, \quad 0 < t < T, \tag{19}$$

$$v_N(0) = v_0 \ \text{ in } W^*. \tag{20}$$

By virtue of Lemma 4.3, we have

$$\varphi(\overline{u}_N(t)) \le \varphi(u_0) \quad \text{for all } \ t \in [0,T], \tag{21}$$

$$\sup_{t\in[0,T]} \psi^*(\overline{v}_N(t)) \le \psi^*(v_0) + \varphi(0)T, \tag{22}$$

$$\sup_{t\in[0,T]} |\overline{u}_N(t)|_X \le C, \tag{23}$$

$$\sup_{t\in[0,T]} |v_N(t)|_{W^*} \le C, \quad \sup_{t\in[0,T]} |\overline{v}_N(t)|_{W^*} \le C \tag{24}$$

with some $C \ge 0$ independent of t, N and h. By the convexity of ψ^*, we see

$$\psi^*(v_N(t)) \le \frac{t-t_n}{h}\psi^*(v_{n+1}) + \frac{t_{n+1}-t}{h}\psi^*(v_n) \quad \text{for all } \ t \in (t_n, t_{n+1}],$$

which together with (16) implies

$$\sup_{t\in[0,T]} \psi^*(v_N(t)) \le \psi^*(v_0) + \varphi(0)T. \tag{25}$$

We further deduce by (A2) (or (A2)$'$) that

$$\sup_{t\in[0,T]} |\mathrm{d}\varphi(\overline{u}_N(t))|_{V^*} \le C, \tag{26}$$

which together with (18) implies

$$\max_n \left| \frac{v_{n+1} - v_n}{h} \right|_{X^*} = \sup_{t\in[0,T]} |v_N'(t)|_{X^*} \le C \tag{27}$$

with some $C \ge 0$ independent of t, N and h.

From these estimates, passing to the limit as $N \to \infty$ (equivalently, $h \to 0$), up to a subsequence, we have the following convergences: there exist $v_1 \in W^{1,\infty}(0,T;X^*) \cap L^\infty(0,T;W^*)$, $v_2 \in L^\infty(0,T;W^*)$, $u \in L^\infty(0,T;X)$ and $\xi \in L^\infty(0,T;V^*)$ such that

$$v_N \to v_1 \quad \text{weakly star in } W^{1,\infty}(0,T;X^*), \tag{28}$$

$$\text{weakly star in } L^\infty(0,T;W^*), \tag{29}$$

$$\overline{v}_N \to v_2 \quad \text{weakly star in } L^\infty(0,T;W^*), \tag{30}$$

$$\overline{u}_N \to u \quad \text{weakly star in } L^\infty(0,T;X), \tag{31}$$

$$\mathrm{d}\varphi(\overline{u}_N(\cdot)) \to \xi \quad \text{weakly star in } L^\infty(0,T;V^*). \tag{32}$$

Thus $v_1' + \xi = 0$ in X^* by (18) and $v_1' \in L^\infty(0,T;V^*)$.

Note by (27) that $v_N(\cdot)$ is equicontinuous in $C([0,T];X^*)$. From Assumption (A3) together with (25), we deduce by Ascoli's theorem (see, e.g., [33]) that

$$v_N \to v_1 \quad \text{strongly in } C([0,T];X^*), \tag{33}$$

which also yields $v_1(0) = v_0$, and moreover, by (24),

$$v_N(T) \to v_1(T) \quad \text{weakly in } W^*. \tag{34}$$

Here we claim that $v_1 = v_2 =: v$. Indeed, by a simple calculation, we infer that, for any $t \in (t_n, t_{n+1}]$,

$$\begin{aligned} v_N(t) - \overline{v}_N(t) &= \frac{(t-t_n)v_{n+1} + (t_{n+1}-t)v_n}{h} - v_{n+1} \\ &= \frac{(t-t_n-h)v_{n+1} + (t_{n+1}-t)v_n}{h} \\ &= -(t_{n+1}-t)\frac{v_{n+1}-v_n}{h}, \end{aligned}$$

which together with (27) implies

$$\begin{aligned} \sup_{t\in(t_n,t_{n+1}]} |v_N(t) - \overline{v}_N(t)|_{X^*} &= \sup_{t\in(t_n,t_{n+1}]} \left|\frac{t_{n+1}-t}{h}\right| |v_{n+1}-v_n|_{X^*} \\ &= |v_{n+1}-v_n|_{X^*} \le Ch \to 0 \quad \text{as } N \to \infty. \end{aligned}$$

Thus $v_1 = v_2 =: v \in W^{1,\infty}(0,T;X^*) \cap L^\infty(0,T;W^*)$ and

$$\overline{v}_N \to v \quad \text{strongly in } L^\infty(0,T;X^*). \tag{35}$$

Furthermore, by [26, Lemma 8.1, p.275], we note that

$$L^\infty(0,T;W^*) \cap C([0,T];X^*) \subset C_w([0,T];W^*).$$

Therefore v belongs to $C_w([0,T];W^*)$.

Now, it remains to prove $v(t) = \mathrm{d}\psi(u(t))$ and $\xi(t) = \mathrm{d}\varphi(u(t))$ for a.a. $t \in (0,T)$. We first prove $v(t) = \mathrm{d}\psi(u(t))$ by recalling that $\overline{v}_N(t) = \mathrm{d}\psi(\overline{u}_N(t))$ and observing from (31) and (35) that

$$\begin{aligned} \int_0^T \langle \overline{v}_N(t), \overline{u}_N(t)\rangle_W \mathrm{d}t &= \int_0^T \langle \overline{v}_N(t), \overline{u}_N(t)\rangle_X \mathrm{d}t \\ &\to \int_0^T \langle v(t), u(t)\rangle_X \mathrm{d}t = \int_0^T \langle v(t), u(t)\rangle_W \mathrm{d}t. \end{aligned}$$

Thus utilizing the maximal monotonicity of $\mathrm{d}\psi$ and applying Proposition 1.1 of [22], we conclude that $v(t) = \mathrm{d}\psi(u(t))$ for a.a. $t \in (0,T)$. As for the latter relation, using the convexity of ψ, we see

$$\begin{aligned} \int_0^T \langle \mathrm{d}\varphi(\overline{u}_N(t)), \overline{u}_N(t)\rangle_V \mathrm{d}t &\overset{(18)}{=} -\int_0^T \langle v_N'(t), \overline{u}_N(t)\rangle_X \mathrm{d}t \\ &= -\sum_{n=0}^{N-1} h \left\langle \frac{v_{n+1}-v_n}{h}, u_{n+1}\right\rangle_W \\ &\le -\sum_{n=0}^{N-1} \Big(\psi^*(v_{n+1}) - \psi^*(v_n)\Big) \\ &= -\psi^*(v_N(T)) + \psi^*(v_0). \end{aligned}$$

Therefore taking the *limsup* in both sides and using a chain rule based on subdifferentials (see Lemma 4.4 below), we deduce from (34) that

$$\begin{aligned}\limsup_{N\to\infty}\int_0^T \langle \mathrm{d}\varphi(\overline{u}_N(t)), \overline{u}_N(t)\rangle_V \mathrm{d}t &\leq -\psi^*(v(T)) + \psi^*(v_0)\\ &= \int_0^T \langle -v'(t), u(t)\rangle_X \mathrm{d}t\\ &= \int_0^T \langle \xi(t), u(t)\rangle_V \mathrm{d}t.\end{aligned}$$

Thus it follows from (31) and (32) that $\mathrm{d}\varphi(u(t)) = \xi(t)$ for a.a. $t \in (0,T)$. Consequently, (u,v) solves (10), (11).

Let us close this proof by the following lemma.

Lemma 4.4. *Let $u \in L^\infty(0,T;X)$ and $v \in W^{1,r}(0,T;X^*)$ be with $r \in (1,\infty]$ such that $v(t) = \mathrm{d}\psi(u(t))$ for a.a. $t \in (0,T)$. Then the function $t \mapsto \psi^*(v(t))$ belongs to $W^{1,r}(0,T)$, and moreover, it holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi^*(v(t)) = \langle v'(t), u(t)\rangle_X \quad \textit{for a.a. } t \in (0,T).$$

Proof. By the definition of subdifferentials, it follows that

$$\langle v(t+h) - v(t), u(t+h)\rangle_W \geq \psi^*(v(t+h)) - \psi^*(v(t)) \geq \langle v(t+h) - v(t), u(t)\rangle_W$$

for a.a. $t \in (0,T)$. Hence by assumptions, one can derive

$$\begin{aligned}&|\psi^*(v(\cdot+h)) - \psi^*(v(\cdot))|_{L^r(0,T-h)}\\ &\leq \sup_{t\in[0,T]} |u(t)|_X \, \|v(\cdot+h) - v(\cdot)\|_{L^r(0,T-h;X^*)} \leq Ch,\end{aligned}$$

which implies $\psi^*(v(\cdot)) \in W^{1,r}(0,T)$. Moreover, for $h > 0$, we have

$$\frac{\psi^*(v(t+h)) - \psi^*(v(t))}{h} \geq \left\langle \frac{v(t+h) - v(t)}{h}, u(t)\right\rangle_X \to \langle v'(t), u(t)\rangle_X,$$

which together with the differentiability of $\psi^*(v(\cdot))$ implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi^*(v(t)) \geq \langle v'(t), u(t)\rangle_X \quad \text{for a.a. } t \in (0,T).$$

The inverse inequality can be obtained by taking $h < 0$ and passing to the limit as $h \to 0_-$. $\square$

This lemma also yields $\psi^*(v(\cdot)) \in W^{1,\infty}(0,T)$. Thus we have proved Theorem 4.2.

5. **Solvability of (P).** In this section, we apply the preceding abstract theory to (P). Recall the reduction of (P) to an abstract Cauchy problem (10), (11) in Section 3 (particularly, (7)–(9)), and let us prove in advance that (A1) and (A2) are satisfied under the assumptions (H).

We next prove (A2). It holds that

$$|-\Delta_{p(\cdot)}u|_{V^*} \leq 2\left\| |\nabla u|^{p(\cdot)-1}\right\|_{p'(\cdot)} \leq 2\left(\|\nabla u\|_{p(\cdot)} + 1\right)^{p^+-1}.$$

Indeed, by Hölder's inequality (see Proposition 1), we note that, for any $v \in V$,

$$\begin{aligned}\langle -\Delta_{p(\cdot)}u, v\rangle_V &= \int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x)\cdot\nabla v(x)\mathrm{d}x\\ &\le 2\left\| |\nabla u|^{p(\cdot)-2}\nabla u\right\|_{p'(\cdot)} \|\nabla v\|_{p(\cdot)},\end{aligned}$$

where $p'(\cdot) \in \mathcal{P}_{\log}(\Omega)$ is given by $1/p(x) + 1/p'(x) = 1$ (see Definition 4.1.4 and Remark 4.1.5 of [16]). Here put $\lambda := (\|\nabla u\|_{p(\cdot)} + 1)^{p^+-1} \ge 1$. Then it follows that

$$\begin{aligned}\int_\Omega \left|\frac{|\nabla u|^{p(x)-1}}{\lambda}\right|^{p'(x)} \mathrm{d}x &= \int_\Omega \left|\frac{|\nabla u|}{\lambda^{1/(p(x)-1)}}\right|^{p(x)} \mathrm{d}x\\ &\le \int_\Omega \left|\frac{|\nabla u|}{\lambda^{1/(p^+-1)}}\right|^{p(x)} \mathrm{d}x = \int_\Omega \left|\frac{|\nabla u|}{\|\nabla u\|_{p(\cdot)}+1}\right|^{p(x)} \mathrm{d}x \le 1,\end{aligned}$$

where the last inequality follows from the definition of $\|\cdot\|_{p(\cdot)}$ in Subsection 2.1. Hence we obtain

$$\left\| |\nabla u|^{p(\cdot)-1}\right\|_{p'(\cdot)} \le (\|\nabla u\|_{p(\cdot)} + 1)^{p^+-1}.$$

Since σ^- is strictly increasing, by Proposition 2 it follows that

$$\|\nabla u\|_{p(\cdot)} \le (\sigma^-)^{-1}\left(\int_\Omega |\nabla u|^{p(x)}\mathrm{d}x\right) \le \left(p^+\varphi(u)+1\right)^{1/p^-} \qquad \text{for all } u \in V.$$

Thus (A2) holds with, e.g., $\ell_1(s) = 2(p^+s+2)^{p^+-1}$.

Finally, let us check (A1). By Proposition 2, for any $u \in W$ and $v \in W^* = L^{m'(\cdot)}(\Omega)$ with $m'(x) = m(x)/(m(x)-1)$, it follows that

$$\psi(u) = \int_\Omega \frac{1}{m(x)}|u(x)|^{m(x)}\mathrm{d}x \ge \frac{1}{m^+}\sigma^-\left(\|u\|_{m(\cdot)}\right) \tag{36}$$

and also

$$\psi^*(v) = \int_\Omega \frac{1}{m'(x)}|v(x)|^{m'(x)}dx \ge \frac{1}{(m'(\cdot))^+}\sigma^-\left(\|v\|_{m'(\cdot)}\right). \tag{37}$$

We further observe

$$\varphi(u) = \int_\Omega \frac{1}{p(x)}|\nabla u(x)|^{p(x)}\mathrm{d}x \ge \frac{1}{p^+}\sigma^-(\|\nabla u\|_{p(\cdot)}) \qquad \text{for all } u \in V. \tag{38}$$

Let (u_n) be a sequence in V and let $v_n := \mathrm{d}\psi(u_n)$ be such that $\varphi(u_n)$ and $\psi^*(v_n)$ are bounded. Then the boundedness of (v_n) in W^* follows from (37). Moreover, we observe

$$\begin{aligned}\psi^*(v_n) &= \int_\Omega \frac{1}{m'(x)}\left||u_n|^{m(x)-2}u_n(x)\right|^{m'(x)}\mathrm{d}x\\ &= \int_\Omega \frac{1}{m'(x)}|u_n(x)|^{m(x)}\mathrm{d}x \ge \frac{1}{(m'(\cdot))^+}\sigma^-(\|u_n\|_{L^{m(\cdot)}}).\end{aligned}$$

Hence (u_n) is bounded in $W = L^{m(\cdot)}(\Omega)$. Since $\varphi(u_n)$ is bounded, so is (∇u_n) in $(L^{p(\cdot)}(\Omega))^d$ by (38). It remains to show the boundedness of (u_n) in $L^{p(\cdot)}(\Omega)$, which is obvious in the Dirichlet case (3) by (i) of Proposition 4. As for the Neumann case (4), note that $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ compactly and $L^{p(\cdot)}(\Omega) \hookrightarrow L^{m(\cdot)\wedge p(\cdot)}(\Omega)$

continuously, where $m(x) \wedge p(x) := \min\{m(x), p(x)\}$. By Ehrling's lemma, for any $\varepsilon > 0$ there exists a constant $C_\varepsilon \geq 0$ such that

$$\begin{aligned} \|u\|_{p(\cdot)} &\leq \varepsilon \|u\|_{W^{1,p(\cdot)}(\Omega)} + C_\varepsilon \|u\|_{p(\cdot)\wedge m(\cdot)} \\ &\leq \varepsilon \|u\|_{W^{1,p(\cdot)}(\Omega)} + C_\varepsilon C \|u\|_{m(\cdot)} \quad \text{for all} \ \ u \in W^{1,p(\cdot)}(\Omega) \end{aligned}$$

with some constant $C \geq 0$ independent of ε and u. Here we remark that usual interpolation inequalities for Lebesgue norms are no longer valid for variable exponent spaces. So we employed Ehrling's lemma instead. Therefore choosing $\varepsilon > 0$ sufficiently small, we deduce that

$$\|u\|_{p(\cdot)} \leq C \left(\|\nabla u\|_{p(\cdot)} + \|u\|_{m(\cdot)} \right) \quad \text{for all} \ \ u \in W^{1,p(\cdot)}(\Omega)$$

with some constant $C \geq 0$. Thus (u_n) is bounded in $L^{p(\cdot)}(\Omega)$, and therefore, (ii) of (A1) holds. Condition (i) of (A1) can be also checked from (36) and (38) in a similar way.

Now, we have:

Theorem 5.1 (Solvability of (P)). *In addition to* (H), *assume that*

$$\operatorname*{ess\,inf}_{x \in \Omega} \ (p^*(x) - m(x)) > 0. \tag{39}$$

Set

$$V = \begin{cases} W_0^{1,p(\cdot)}(\Omega) & \textit{for the Dirichlet condition (3),} \\ W^{1,p(\cdot)}(\Omega) & \textit{for the Neumann condition (4).} \end{cases}$$

Let $T > 0$ and let $v_0 \in L^{m'(\cdot)}(\Omega)$ be such that $v_0 = |u_0|^{m(\cdot)-2} u_0$ with some $u_0 \in V$. Then there exists at least one solution $(u, v) : [0, T] \to V \times L^{m'(\cdot)}(\Omega)$ of (P), *i.e.,*

$$u \in L^\infty(0, T; V), \quad v \in W^{1,\infty}(0, T; V^*) \cap C([0, T]; L^{m'(\cdot)}(\Omega)),$$

and for every $\phi \in V$ it holds that

$$\langle v'(t), \phi \rangle_V + \int_\Omega |\nabla u(x, t)|^{p(x)-2} \nabla u(x, t) \cdot \nabla \phi(x) \mathrm{d}x = 0 \quad \textit{for a.a. } t \in (0, T)$$

with $v(x, t) = |u(x, t)|^{m(x)-2} u(x, t)$ for a.e. $(x, t) \in \Omega \times (0, T)$ and the initial condition $v(\cdot, 0) = v_0$ in Ω.

Proof. Conditions (A1), (A2) have already been checked. Condition (A3) follows immediately from (37) and the compact embedding $W^* \hookrightarrow V^*$ (equivalently, $V \hookrightarrow W$ by (39) and Proposition 4). Moreover, the initial data v_0 satisfies the assumption of Theorem 4.2, and hence, the Cauchy problem (10), (11) admits at least one strong solution. The continuity of $v(\cdot)$ in $L^{m'(\cdot)}(\Omega)$ follows from that of $\psi^*(v(\cdot))$, the fact that $v \in C_w([0, T]; L^{m'(\cdot)}(\Omega))$ and the uniform convexity of the modular of $L^{m'(\cdot)}(\Omega)$ (see Proposition 5). □

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