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Symmetry and stability of asymptotic profiles for fast diffusion equations in annuli

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Abstract

This paper is concerned with stability analysis of asymptotic profiles for (possibly sign-changing) solutions vanishing in finite time of the Cauchy–Dirichlet problems for fast diffusion equations in annuli. It is proved that the unique positive radial profile is not asymptotically stable, and moreover, it is unstable for the two-dimensional annulus. Furthermore, the method of stability analysis presented here will be also applied to exhibit symmetry breaking of least energy solutions.

MSC: 35K67; 35J61; 35B40; 35B35; 35B06

Keywords: Fast diffusion equation; Semilinear elliptic equation; Asymptotic profile; Stability analysis; Symmetry breaking

1. Introduction

Let us consider the Cauchy–Dirichlet problem for the *Fast Diffusion Equation* (*FDE* for short),

$$\partial_t\left(|u|^{m-2}u\right) = \Delta u \quad \text{in } \Omega \times (0,\infty), \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{2}$$

$$u(\cdot,0) = u_0 \quad \text{in } \Omega, \tag{3}$$

where $2 < m < 2^* := 2N/(N-2)_+$ and Ω is an annulus $A_N(a,b)$,

$$\Omega = A_N(a,b) := \left\{x \in \mathbb{R}^N \colon a < |x| < b\right\}, \tag{4}$$

with $N \geqslant 2$ and $0 < a < b$. By putting $w = |u|^{m-2}u$, Eq. (1) is transformed to a usual form,

$$\partial_t w = \Delta\left(|w|^{m'-2}w\right) \quad \text{in } \Omega \times (0,\infty),$$

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where m' denotes the Hölder conjugate of m, i.e., $m' = m/(m-1) \in (1,2)$. FDEs arise in the studies of plasma physics, kinetic theory of gases, solid state physics (see [2,4] and also [20]) and so on. There are many contributions to the fast diffusion equation (see [3,5–9,13,16,21]).

From here to the end of Section 3, Ω is assumed to be any bounded smooth domain in $\mathbb{R}^N$. According to the celebrated work of Berryman and Holland [3], every solution $u = u(x,t)$ of (1)–(3) vanishes at a finite time $t_* = t_*(u_0)$ with the rate of $(t_* - t)_+^{1/(m-2)}$, and moreover, such a vanishing solution $u = u(x,t)$ has an *asymptotic profile* $\phi = \phi(x)$ as the limit

$$\phi(x) := \lim_{t \nearrow t_*} (t_* - t)^{-1/(m-2)} u(x,t) \tag{5}$$

(see also [16,21]). Each asymptotic profile ϕ is characterized as a nontrivial solution of the Dirichlet problem for the *Emden–Fowler equation* (or *Lane–Emden equation*) with the constant $\mathfrak{c}_m := (m-1)/(m-2) > 0$,

$$-\Delta\phi = \mathfrak{c}_m |\phi|^{m-2}\phi \quad \text{in } \Omega, \tag{6}$$

$$\phi = 0 \quad \text{on } \partial\Omega, \tag{7}$$

which is the Euler–Lagrange equation of the *energy* functional,

$$J(u) := \frac{1}{2}\int_\Omega \left|\nabla u(x)\right|^2 dx - \frac{\mathfrak{c}_m}{m}\int_\Omega \left|u(x)\right|^m dx \quad \text{for } u \in H_0^1(\Omega).$$

Moreover, the set of all asymptotic profiles of solutions for (1)–(3) coincides with the set $\mathcal{S}$ of all nontrivial solutions for (6), (7) (see [1] for more details). We also refer the reader to [13] and [7] for more detailed analysis of the convergences (5) to asymptotic profiles.

In [1], the authors introduced the notions of (asymptotic) stability and instability of asymptotic profiles for FDEs (see Definition 3.4 in Section 3) and also provided the following criteria (see Theorems 3.5 and 3.6 in Section 3 for more details):

- Each least energy solution ϕ of (6), (7) is (resp., asymptotically) stable in the sense of asymptotic profiles for FDEs, if ϕ is isolated from the other least energy (resp., sign-definite) solutions. In particular, if ϕ is a unique positive solution, then it is asymptotically stable.
- All sign-changing solutions are not asymptotically stable profiles. Moreover, they are unstable, if they are isolated from the other profiles having lower energies.

These criteria enable us to determine the (asymptotic) stability and the instability of profiles in many cases. For instance, when Ω is a ball in $\mathbb{R}^N$, due to Gidas, Ni and Nirenberg [14], the Dirichlet problem (6), (7) admits the unique positive solution ϕ, which is radially symmetric. Then from the stability criteria stated above, ϕ turns out to be asymptotically stable. As for the one-dimensional case, all the nontrivial solutions of (6), (7) are completely classified as follows: unique positive and negative solutions are asymptotically stable and all sign-changing solutions are unstable.

However, the stability criteria described above do not necessarily cover all situations. Indeed, when Ω is an annulus, the Dirichlet problem (6), (7) admits a unique positive radial solution; however, it does not take the least energy among nontrivial solutions (the least energy is attained by another positive solution without radial symmetry), provided that the annulus is sufficiently thin (see [11]). Such a positive radial solution is beyond the scope of the foregoing stability criteria.

The main purpose of this paper is to investigate the stability of the positive radial profile for FDEs in the annulus. As a by-product, we shall also exhibit symmetry breaking of least energy solutions for the Emden–Fowler equation (6), (7) in annuli by applying an argument developed here for the stability analysis of asymptotic profiles for FDEs. Symmetry breaking of least energy solutions for (6), (7) in annuli has already been observed by Coffman [11], Li [17] and Byeon [10], provided that the thickness of the annulus is *sufficiently thin*. Our method of proof can provide a quantitative sufficient condition on the thickness of annuli for symmetry breaking.

The content of this paper is the following. Section 2 provides preliminary facts on the variational analysis of the Emden–Fowler equation. Basic formulations of our stability analysis will be reviewed in Section 3. The main results are given in Section 4, where we shall prove that the unique radial profile is not asymptotically stable, and moreover it is unstable when $N = 2$.

Notation. The positive part of each number $r \in \mathbb{R}$ is denoted by $(r)_+$, i.e., $(r)_+ := \max\{r, 0\}$. We denote the $L^p(\Omega)$-norm by $\|\cdot\|_p$ and the $H_0^1(\Omega)$-norm by $\|\cdot\|_{1,2}$, which is defined by

$$\|u\|_{1,2} := \|\nabla u\|_2 = \left(\int_\Omega |\nabla u|^2\, dx\right)^{1/2}.$$

When Ω is an annulus $A_N(a,b) := \{x \in \mathbb{R}^N\colon\ a < |x| < b\}$, we denote by $L^2_r(\Omega)$ the set of all radial functions, $u(x) = u(|x|)$, in $L^2(\Omega)$ equipped with the norm,

$$\|u\|_{L^2_r(\Omega)} := \left(\int_a^b u(r)^2 r^{N-1}\, dr\right)^{1/2}.$$

Furthermore, we denote by $H^1_{0,r}(\Omega)$ the set of all radial functions in $H_0^1(\Omega)$.

2. Variational analysis of the Emden–Fowler equation

In this section, we summarize notation and definitions related to the variational analysis of the Dirichlet problem (6), (7) for later use. Let us first define the Lagrangian functional $J : H_0^1(\Omega) \to \mathbb{R}$ by

$$J(u) := \frac{1}{2}\|\nabla u\|_2^2 - \frac{\mathfrak{c}_m}{m}\|u\|_m^m \quad \text{for } u \in H_0^1(\Omega),$$

whose critical points solve (6), (7) in the distribution sense. We denote by $\mathcal{S}$ the set of all nontrivial solutions of (6), (7), i.e., nontrivial critical points of J. Moreover, a solution u is called a *least energy solution* if it minimizes J among all the nontrivial solutions of (6), (7). As will be explained in Proposition 2.1 below, every least energy solution is sign-definite. Throughout this paper, a least energy solution is always supposed to be positive without any loss of generality.

The *Nehari manifold* $\mathcal{N}$ associated with J is defined by

$$\begin{aligned}\mathcal{N} &:= \left\{u \in H_0^1(\Omega) \setminus \{0\}\colon\ \left\langle J'(u), u\right\rangle_{H_0^1(\Omega)} = 0\right\}\\ &= \left\{u \in H_0^1(\Omega) \setminus \{0\}\colon\ \|\nabla u\|_2^2 = \mathfrak{c}_m\|u\|_m^m\right\},\end{aligned}$$

and then, $\mathcal{N}$ includes $\mathcal{S}$ by definition. Furthermore, we find by the definitions of J and $\mathcal{N}$ that

$$\|\nabla u\|_2^2 = \mathfrak{c}_m\|u\|_m^m = \frac{2m}{m-2}J(u) \quad \text{for } u \in \mathcal{N}. \tag{8}$$

Now, let us define the Rayleigh quotient,

$$R(u) := \frac{\|\nabla u\|_2^2}{\|u\|_m^2} \quad \text{for } u \in H_0^1(\Omega) \setminus \{0\}. \tag{9}$$

Then $R(u)$ has a positive lower bound in $H_0^1(\Omega) \setminus \{0\}$ by the Sobolev imbedding. For any $u \in H_0^1(\Omega) \setminus \{0\}$, there exists a unique constant $c > 0$ such that $cu \in \mathcal{N}$. Moreover, $R(cu) = R(u)$ for all $c > 0$. Therefore it holds that

$$\inf\left\{R(u)\colon\ u \in H_0^1(\Omega) \setminus \{0\}\right\} = \inf\left\{R(u)\colon\ u \in \mathcal{N}\right\} > 0. \tag{10}$$

The following proposition is well known.

Proposition 2.1 *(Variational properties of least energy solutions). The following* (i)–(iii) *are equivalent*:

(i) *u is a least energy solution of* (6), (7),
(ii) *u is a minimizer of J over $\mathcal{N}$*,
(iii) *u is a minimizer of R over $\mathcal{N}$*.

Furthermore, every least energy solution is sign-definite.

3. Stability analysis of asymptotic profiles

In this section, we briefly review related materials on the stability analysis of asymptotic profiles of vanishing solutions for (1)–(3). Throughout this paper, we are concerned with solutions of (1)–(3) defined by

Definition 3.1 *(Solutions of FDEs).* A function $u : \Omega \times (0, \infty) \to \mathbb{R}$ is said to be a solution for (1)–(3) if the following conditions are all satisfied:

- $u \in C([0, T]; H_0^1(\Omega))$ and $|u|^{m-2}u \in C^1([0, T]; H^{-1}(\Omega))$ for all $T > 0$.
- It follows that
$$\left\langle \left(|u|^{m-2}u\right)'(t), \phi\right\rangle_{H_0^1} + \int_\Omega \nabla u(x,t) \cdot \nabla\phi(x)\,dx = 0 \quad \text{for all } t \in (0,\infty) \text{ and } \phi \in H_0^1(\Omega),$$
where $\langle\cdot,\cdot\rangle_{H_0^1}$ denotes the duality pairing between $H_0^1(\Omega)$ and its dual space $H^{-1}(\Omega)$ and $' = d/dt$.
- $u(\cdot, 0) = u_0$.

We denote by $t_*(u_0)$ the *extinction time* of the unique solution $u = u(x,t)$ of (1)–(3) for each initial data u_0 and simply write t_* if no confusion can arise. Then $t_*(\cdot)$ can be regarded as a functional defined on $H_0^1(\Omega)$, and moreover, $t_*(u_0)$ is positive and finite for any $u_0 \not\equiv 0$ and $t_*(0) = 0$.

The asymptotic profile of a vanishing solution $u = u(x,t)$ is defined in the following:

Definition 3.2 *(Asymptotic profiles).* Let $u_0 \in H_0^1(\Omega) \setminus \{0\}$ and let $u = u(x,t)$ be a solution for (1)–(3) vanishing at a finite time $t_* > 0$. A function $\phi \in H_0^1(\Omega)$ is called an *asymptotic profile* (or a profile, if no confusion arises) of u if there exists an increasing sequence $t_n \to t_*$ such that

$$\lim_{t_n \to t_*} \left\| (t_* - t_n)^{-1/(m-2)} u(t_n) - \phi \right\|_{1,2} = 0.$$

The following transformation is useful to investigate asymptotic profiles of vanishing solutions $u = u(x,t)$ for (1)–(3):

$$v(x,s) = (t_* - t)^{-1/(m-2)} u(x,t) \quad \text{and} \quad s = \log\bigl(t_*/(t_* - t)\bigr). \tag{11}$$

Then a limit of $v(\cdot, s_n)$ as $s_n \to \infty$ (equivalently, $t_n \nearrow t_*$) is an asymptotic profile of $u(x,t)$. Moreover, (1)–(3) is rewritten in terms of v as

$$\partial_s\left(|v|^{m-2}v\right) = \Delta v + \mathfrak{c}_m |v|^{m-2} v \quad \text{in } \Omega \times (0,\infty), \tag{12}$$
$$v = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{13}$$
$$v(\cdot, 0) = v_0 \quad \text{in } \Omega, \tag{14}$$

where $\mathfrak{c}_m$ and v_0 are given by

$$\mathfrak{c}_m := \frac{m-1}{m-2} > 0 \quad \text{and} \quad v_0 := t_*(u_0)^{-1/(m-2)} u_0. \tag{15}$$

Here multiplying (12) by $\partial_s v$ and integrating this over Ω, one can obtain

$$\frac{d}{ds} J\bigl(v(s)\bigr) \leqslant 0 \quad \text{for a.e. } s > 0, \tag{16}$$

which implies that the function $s \mapsto J(v(s))$ is non-increasing (see Section 1 or Section 2 for the definition of J), that is, J becomes a Lyapunov functional.

The following proposition is concerned with the existence of asymptotic profiles and their characteristics (see [3, 16,21] and also [1] for its proof).

Proposition 3.3 *(Asymptotic profiles). Assume that $2 < m < 2^*$. Let u be a solution of (1)–(3) with $u_0 \in H_0^1(\Omega) \setminus \{0\}$. Then for any increasing sequence $t_n \to t_*$, there exist a subsequence (n') of (n) and a solution $\phi \in H_0^1(\Omega) \setminus \{0\}$ of (6), (7) such that*

$$\lim_{t_{n'} \to t_*} \left\| (t_* - t_{n'})^{-1/(m-2)} u(t_{n'}) - \phi \right\|_{1,2} = 0,$$

equivalently,

$$\lim_{s_{n'} \to \infty} \left\| v(s_{n'}) - \phi \right\|_{1,2} = 0$$

with a function $v(\cdot)$ and a sequence (s_n) given by the transformation (11).

In [1], the authors introduced the notions of *stability* and *instability of asymptotic profiles* for vanishing solutions of (1)–(3), and moreover, they actually performed stability analysis of profiles. An asymptotic profile ϕ is said to be *stable*, if the rescaled function $(t_* - t)^{-1/(m-2)} u(x,t)$ of any solution $u = u(x,t)$ of (1)–(3) with the extinction time t_* will stay within an arbitrarily small neighborhood of ϕ whenever the initial data $u(x,0) = u_0(x)$ lies in a sufficiently small neighborhood of ϕ. Moreover, ϕ is said to be *unstable*, if it is not stable. By virtue of the transformation (11), these notions of stability and instability for asymptotic profiles can be translated to those for stationary solutions of (12)–(14). Then we have to pay attention to the fact that, due to the relation (15), the initial data v_0 of $v = v(x,s)$ must lie on the set,

$$\begin{aligned} \mathcal{X} &:= \left\{ t_*(u_0)^{1/(m-2)} u_0 \colon u_0 \in H_0^1(\Omega) \setminus \{0\} \right\} \\ &= \left\{ v_0 \in H_0^1(\Omega) \setminus \{0\} \colon t_*(v_0) = 1 \right\} \end{aligned}$$

(see Proposition 6 of [1]), even though u_0 can be taken from the whole of the energy space $H_0^1(\Omega)$. We notice that if $v_0 \in \mathcal{X}$, then the corresponding solution v of (12)–(14) stays in $\mathcal{X}$ for all $t \geqslant 0$ (see Proposition 5 of [1]). Hence $\mathcal{X}$ is an invariant set, and therefore, (12)–(14) generates a dynamical system in $\mathcal{X}$. Moreover, all the nontrivial stationary solutions belong to $\mathcal{X}$. It was also proved in [1] that $\mathcal{X}$ is an unbounded surface surrounding the origin in $H_0^1(\Omega)$ and homeomorphic to the unit sphere of $H_0^1(\Omega)$ (see Proposition 10 of [1]).

Now, the (asymptotic) stability and instability of asymptotic profiles are defined as follows (see [1]):

Definition 3.4 *(Stability and instability of profiles).* Let $\phi \in H_0^1(\Omega) \setminus \{0\}$ be an asymptotic profile of a vanishing solution for (1)–(3), equivalently, ϕ is a nontrivial solution of (6), (7).

(i) ϕ is said to be *stable*, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any solution v of (12), (13) satisfies

$$\sup_{s \in [0,\infty)} \left\| v(s) - \phi \right\|_{1,2} < \varepsilon,$$

whenever $v(0) \in \mathcal{X}$ and $\|v(0) - \phi\|_{1,2} < \delta$.

(ii) ϕ is said to be *unstable*, if ϕ is not stable.

(iii) ϕ is said to be *asymptotically stable*, if ϕ is stable, and moreover, there exists $\delta_0 > 0$ such that any solution v of (12), (13) satisfies

$$\lim_{s \nearrow \infty} \left\| v(s) - \phi \right\|_{1,2} = 0,$$

whenever $v(0) \in \mathcal{X}$ and $\|v(0) - \phi\|_{1,2} < \delta_0$.

Here we stress that the positivity of solution is not supposed in this frame. Let us recall the stability criteria obtained in [1].

Theorem 3.5 *(Stability of profiles). Let ϕ be a least energy solution of (6), (7).*

(i) *If ϕ is isolated in $H_0^1(\Omega)$ from all the other least energy solutions of (6), (7), then it is a stable profile.*

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(ii) *If ϕ is isolated in $H_0^1(\Omega)$ from all the other sign-definite solutions of (6), (7), then it is an asymptotically stable profile. Especially, if (6), (7) has a unique positive solution, it is asymptotically stable.*

When Ω is a ball, a positive solution is unique and it becomes radially symmetric, due to Gidas, Ni and Nirenberg [14]; hence it is asymptotically stable by (ii) of Theorem 3.5. The following theorem is concerned with the instability of sign-changing profiles (see [1]).

Theorem 3.6 *(Instability of profiles). Let ψ be a sign-changing solution of (6), (7).*

(i) *ψ is not asymptotically stable.*
(ii) *If ψ is isolated in $H_0^1(\Omega)$ from all nontrivial solutions w of (6), (7) satisfying $J(w) < J(\psi)$, then ψ is an unstable profile.*

The following proposition (see [1] for more details) played an important role to prove the stability criteria above, and moreover, it will be employed in later sections as well.

Proposition 3.7 *(Properties of $t_*(\cdot)$ and $\mathcal{X}$). The following* (i)–(iii) *hold.*

(i) *The functional $t_*(\cdot)$ is continuous in $H_0^1(\Omega)$.*
(ii) *It holds that $t_*(\phi) = 1$ for any nontrivial solution ϕ of (6), (7).*
(iii) *Let v be a solution of (12)–(14). If $v_0 \in \mathcal{X}$, then $v(s) \in \mathcal{X}$ for all $s \geqslant 0$. Moreover, for any sequence $s_n \to \infty$, a subsequence of $v(s_n)$ converges to a nontrivial solution of (6), (7) strongly in $H_0^1(\Omega)$.*

4. Main results

In this section, we state main results. Hereafter, let Ω be an annulus $A_N(a,b)$ defined by (4). Dancer [12] considered a domain having a small hole, in particular, an annulus $A_N(a,b)$ which is close to a ball, that is, $(b-a)/a$ is large enough. Then he proved the next result.

Proposition 4.1. *Let $\Omega = A_N(a,b)$. If the ratio $(b-a)/a$ is large enough, then (6), (7) has a unique positive solution and it is radially symmetric.*

Roughly speaking, Proposition 4.1 says that if the hole in the annulus is small enough, or equivalently, the shell is thick enough, then a positive solution is unique. Combining Theorem 3.5 and Proposition 4.1, we obtain

Proposition 4.2. *Let $\Omega = A_N(a,b)$. If $(b-a)/a$ is large enough, then the unique positive radial asymptotic profile is asymptotically stable.*

Let us consider the opposite case where $(b-a)/a$ is small, that is, the hole in the annulus is large enough, or equivalently the shell is thin. A positive radial solution $\phi = \phi(r)$ with $r = |x|$ of (6), (7) in $A_N(a,b)$ becomes a solution of the following two point boundary value problem:

$$\phi'' + \frac{N-1}{r}\phi' + \mathfrak{c}_m \phi^{m-1} = 0 \quad \text{in } (a,b), \tag{17}$$

$$\phi(a) = \phi(b) = 0, \tag{18}$$

where $\phi' = d\phi/dr$. The next result is valid for any $0 < a < b$, which has been proved by Ni [19].

Proposition 4.3 *(Existence and uniqueness of positive radial solution). Let $\Omega = A_N(a,b)$. Then (6), (7) has a unique positive radial solution ϕ.*

We explain the difference between Propositions 4.1 and 4.3. Proposition 4.3 says that a solution is unique in the class of positive radial solutions. On the other hand, Proposition 4.1 asserts that a solution is unique in the class of positive (possibly non-radial) solutions if $(b-a)/a$ is large enough.

Our first main result is as follows.

Theorem 4.4. *Let $\Omega = A_N(a,b)$ and assume that*

$$C_{a,b} := \left(\frac{b}{a}\right)^{(N-3)_+}\left(\frac{b-a}{\pi a}\right)^2 < \frac{m-2}{N-1}. \tag{19}$$

Then the unique positive radial asymptotic profile ϕ is not asymptotically stable.

Since $(N-3)_+ = \max\{N-3, 0\}$, the constant $C_{a,b}$ stands for $C_{a,b} = (b/a)^{N-3}((b-a)/\pi a)^2$ if $N \geqslant 4$ and $C_{a,b} = ((b-a)/\pi a)^2$ if $N = 2, 3$. Theorem 4.4 means that if the ratio of the thickness $b-a$ of the shell to the inside diameter a is small enough, the positive radial profile ϕ is not asymptotically stable.

Remark 4.5 *(Sign-changing profiles).* It has already been proved in [1] that sign-changing profiles are not asymptotically stable (see Theorem 3.6). Hence all radial profiles are not asymptotically stable.

Our method to prove Theorem 4.4 is based on the comparison between the global least energy and the radial least energy. Therefore it gives us the next result as a by-product.

Corollary 4.6 *(Symmetry breaking of least energy solutions). Under assumption (19), a least energy solution of (6), (7) with $\Omega = A_N(a,b)$ is not radially symmetric.*

As stated in Introduction, symmetry breaking of least energy solutions for (6), (7) in annuli has already been proved by Coffman [11], Li [17] and Byeon [10], provided that the thickness of the annulus is sufficiently thin. However, our result can provide a quantitative sufficient condition on the thickness of annuli.

For $N = 2$, we obtain a stronger assertion than Theorem 4.4.

Theorem 4.7 *(Instability of positive radial profiles). Let $N = 2$ and $\Omega = A_2(a,b)$. If $(b-a)/a$ is small enough, then the unique positive radial asymptotic profile ϕ is unstable.*

5. Proof of Theorem 4.4 and Corollary 4.6

In this section, we shall prove Theorem 4.4 and Corollary 4.6. Our method of proof is based on the next lemma.

Lemma 5.1. *Let ϕ be a nontrivial solution of (6), (7) and let (ϕ_ε) be a sequence in $H_0^1(\Omega)$ such that*

$$\phi_\varepsilon \to \phi \quad \text{strongly in } H_0^1(\Omega) \text{ as } \varepsilon \to 0 \tag{20}$$

and

$$J(c\phi_\varepsilon) < J(\phi) \quad \text{for any } \varepsilon \in (0, \varepsilon_0) \text{ and } c > c_0 \tag{21}$$

with some constants $c_0 \in (0,1)$ and $\varepsilon_0 > 0$. Then ϕ is not asymptotically stable in the sense of asymptotic profiles for FDEs.

Proof. Set $v_{0,\varepsilon} := t_*(\phi_\varepsilon)^{-1/(m-2)}\phi_\varepsilon \in \mathcal{X}$ and $c_\varepsilon := t_*(\phi_\varepsilon)^{-1/(m-2)}$. By (20) and Proposition 3.7, we have $t_*(\phi_\varepsilon) \to t_*(\phi) = 1$ and $v_{0,\varepsilon} \to \phi$ strongly in $H_0^1(\Omega)$ as $\varepsilon \to 0$. Moreover, c_ε is sufficiently close to 1 for $\varepsilon > 0$ small enough. Choose $\varepsilon_1 \in (0, \varepsilon_0)$ such that $c_\varepsilon > c_0$ for all $\varepsilon \in (0, \varepsilon_1)$. Then it follows from (21) that

$$J(v_{0,\varepsilon}) = J(c_\varepsilon\phi_\varepsilon) < J(\phi) \quad \text{for all } \varepsilon \in (0, \varepsilon_1).$$

Let $v_\varepsilon(s)$ be the solution of (12)–(14) with the initial data $v_{0,\varepsilon}$. Since $s \mapsto J(v_\varepsilon(s))$ is non-increasing, $v_\varepsilon(s)$ cannot converge to ϕ whereas $v_{0,\varepsilon}$ is sufficiently close to ϕ. This shows that ϕ is not asymptotically stable. $\square$

To construct perturbed functions ϕ_ε in Lemma 5.1, we introduce the N-dimensional polar coordinate:

$$\begin{aligned} x_1 &= r\cos\theta_1, \\ x_2 &= r\sin\theta_1\cos\theta_2, \\ &\ \vdots \end{aligned}$$

$$x_i = r\sin\theta_1 \cdots \sin\theta_{i-1}\cos\theta_i \quad (\text{for } i = 3, \ldots, N-1),$$
$$\vdots$$
$$x_N = r\sin\theta_1 \cdots \sin\theta_{N-2}\sin\theta_{N-1}$$

for $r \in (a,b)$, $\theta_i \in [0,\pi]$ for $i = 1,2,\ldots,N-2$ and $\theta_{N-1} \in [0,2\pi)$. In what follows, we write

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_{N-1}), \qquad d\boldsymbol{\theta} = d\theta_1\, d\theta_2 \ldots d\theta_{N-1}.$$

We define Θ by the set of $\boldsymbol{\theta} \in \mathbb{R}^{N-1}$ satisfying $\theta_i \in [0,\pi]$ for $1 \leqslant i \leqslant N-2$ and $\theta_{N-1} \in [0,2\pi)$ if $N \geqslant 3$ and put $\Theta = [0,2\pi)$ if $N = 2$.

Let $\phi(r)$ be the unique positive solution of (17), (18). Then we define $\phi_\varepsilon : \Omega \to \mathbb{R}$ with $\varepsilon \in (0,1)$ by

$$\phi_\varepsilon(x) = \sigma_\varepsilon(\theta_1)\phi(r) \quad \text{for } x = x(r,\boldsymbol{\theta}) \in \Omega \tag{22}$$

with the function

$$\sigma_\varepsilon(\theta_1) = 1 + \varepsilon\cos\theta_1 \quad \text{for } \boldsymbol{\theta} \in \Theta,\ \varepsilon \in (0,1). \tag{23}$$

Remark 5.2 *(Well-definedness of ϕ_ε).* Each function $\phi_\varepsilon(x)$ is well defined in Ω. In the polar coordinate, when $\theta_1 = 0$, the point $(r, 0, \theta_2, \ldots, \theta_{N-1})$ for any $\theta_2, \ldots, \theta_{N-1}$ corresponds to the common point $(x_1 \ldots, x_N) = (r, 0, \ldots, 0)$. Therefore $\phi_\varepsilon(r, 0, \theta_2, \ldots, \theta_{N-1})$ should be independent of $\theta_2, \ldots, \theta_{N-1}$. This fact holds for $\theta_1 = \pi$ also. In the same reason, $\phi_\varepsilon(r, \theta_1, \ldots, \theta_{N-1})$ with $\theta_i = 0, \pi$ should be independent of $\theta_{i+1}, \ldots, \theta_{N-1}$. Moreover, ϕ_ε should be 2π periodic in θ_{N-1}. Our definition of ϕ_ε obeys these rules and $\phi_\varepsilon(x)$ is well defined.

Proposition 5.3 *(Decrease of energy by perturbations). Under the same assumptions as in Theorem 4.4, there exist $c_0 \in (0,1)$ and $\varepsilon_0 \in (0,1]$ such that $J(c\phi_\varepsilon) < J(\phi)$ for any $\varepsilon \in (0,\varepsilon_0)$ and $c > c_0$.*

Proof. Here and henceforth, if $n > m$, we mean $\sum_{i=n}^m a_i = 0$ and $\prod_{i=n}^m a_i = 1$ for any sequence (a_i). From the N-dimensional polar coordinate transformation, it follows that

$$\begin{aligned}|\nabla\phi_\varepsilon|^2 &= \left|\frac{\partial\phi_\varepsilon}{\partial r}\right|^2 + \sum_{j=1}^{N-1} \frac{1}{r^2\prod_{i=1}^{j-1}\sin^2\theta_i}\left|\frac{\partial\phi_\varepsilon}{\partial\theta_j}\right|^2 \\ &= \sigma_\varepsilon(\theta_1)^2\phi'(r)^2 + \varepsilon^2\sin^2\theta_1\frac{\phi(r)^2}{r^2}.\end{aligned}$$

Moreover, the Jacobian of this transformation is given by

$$\frac{\partial(x_1, x_2, \ldots, x_{N-1}, x_N)}{\partial(r, \theta_1, \ldots, \theta_{N-2}, \theta_{N-1})} = r^{N-1}\operatorname{Jac}(\boldsymbol{\theta}) \quad \text{with } \operatorname{Jac}(\boldsymbol{\theta}) = \prod_{i=1}^{N-2}\sin^{N-1-i}\theta_i.$$

For any $c > 0$, we derive

$$\begin{aligned}J(c\phi_\varepsilon) = {}& \frac{c^2}{2}\left(\int_\Theta \sigma_\varepsilon(\theta_1)^2\operatorname{Jac}(\boldsymbol{\theta})\,d\boldsymbol{\theta}\right)\int_a^b \phi'(r)^2 r^{N-1}\,dr + \frac{c^2}{2}\varepsilon^2\mu_N\int_a^b \phi(r)^2 r^{N-3}\,dr \\ &- \frac{c^m}{m}\mathfrak{c}_m\left(\int_\Theta \sigma_\varepsilon(\theta_1)^m\operatorname{Jac}(\boldsymbol{\theta})\,d\boldsymbol{\theta}\right)\int_a^b \phi(r)^m r^{N-1}\,dr\end{aligned} \tag{24}$$

with the constant

$$\mu_N := \int_\Theta \sin^2\theta_1\operatorname{Jac}(\boldsymbol{\theta})\,d\boldsymbol{\theta}.$$

To compute μ_N, we use the formula

$$\int_0^\pi \sin^j t\,dt = \frac{\sqrt{\pi}\,\Gamma((j+1)/2)}{\Gamma((j+2)/2)},$$

where $\Gamma(\cdot)$ denotes the Gamma function. Since in the definition of μ_N the intervals of integrations are $[0,\pi]$ for θ_i with $i \leqslant N-2$ and $[0,2\pi]$ for θ_{N-1}, it holds that

$$\mu_N = \frac{2\pi^{N/2}\Gamma((N+1)/2)}{\Gamma((N+2)/2)\Gamma((N-1)/2)} = \frac{2\pi^{N/2}}{\Gamma(N/2)}\cdot\frac{N-1}{N},$$

where we have used the relation $\Gamma(n+1) = n\Gamma(n)$.

For ϕ (and any radial solutions), by (8), we have

$$\int_a^b \phi'(r)^2 r^{N-1}\,dr = \mathfrak{c}_m \int_a^b \phi(r)^m r^{N-1}\,dr = \frac{2m}{m-2}\frac{J(\phi)}{\omega_N}. \tag{25}$$

Here ω_N denotes the surface area of the unit sphere of $\mathbb{R}^N$, i.e.,

$$\omega_N := \int_\Theta \mathrm{Jac}(\boldsymbol{\theta})\,d\boldsymbol{\theta} = \frac{2\pi^{N/2}}{\Gamma(N/2)}. \tag{26}$$

Then we also have the relation,

$$\mu_N = \frac{N-1}{N}\omega_N. \tag{27}$$

Observe that $(\pi/(b-a))^2$ is the first eigenvalue of the problem

$$-u'' = \lambda u \quad \text{in } (a,b), \qquad u(a) = u(b) = 0.$$

Therefore it holds that

$$\big(\pi/(b-a)\big)^2 \int_a^b u(r)^2\,dr \leqslant \int_a^b u'(r)^2\,dr,$$

for any $u \in H_0^1(a,b)$. Using this inequality, we have, for $N \geqslant 3$,

$$\begin{aligned}\int_a^b \phi(r)^2 r^{N-3}\,dr &\leqslant b^{N-3}\big((b-a)/\pi\big)^2 \int_a^b \phi'(r)^2\,dr \\ &\leqslant C_{a,b}\int_a^b \phi'(r)^2 r^{N-1}\,dr,\end{aligned} \tag{28}$$

with the constant $C_{a,b} > 0$ given by (19). The inequality above is still valid for $N = 2$. Using (25) and (28), we derive from (24) that

$$J(c\phi_\varepsilon) \leqslant \left\{\int\!\!\int_\Theta \left(\frac{c^2}{2}\sigma_\varepsilon(\theta_1)^2 - \frac{c^m}{m}\sigma_\varepsilon(\theta_1)^m\right)\mathrm{Jac}(\boldsymbol{\theta})\,d\boldsymbol{\theta} + \frac{c^2}{2}\varepsilon^2\mu_N C_{a,b}\right\} \times \frac{2m}{m-2}\frac{J(\phi)}{\omega_N}. \tag{29}$$

We shall show that the right hand side of (29) is less than $J(\phi)$ for sufficiently small $\varepsilon > 0$ and c close to 1. For any $s > -1$, there exists a constant $\delta \in (0,1)$ by the Taylor theorem such that

$$(1+s)^m = 1 + ms + \frac{m(m-1)}{2}(1+\delta s)^{m-2}s^2.$$

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Let $\alpha > 0$ be slightly less than 1 and it will be determined later on. Then one can choose $\varepsilon_0 = \varepsilon_0(\alpha) \in (0,1)$ so small that $(1-\varepsilon_0)^{m-2} \geqslant \alpha$. Hence it follows by $\delta \in (0,1)$ that

$$(1+\delta s)^{m-2} \geqslant (1-\varepsilon_0)^{m-2} \geqslant \alpha \quad \text{if } |s| < \varepsilon_0. \tag{30}$$

Putting $s = \varepsilon \cos\theta_1$ for an arbitrary $\varepsilon \in (0, \varepsilon_0)$, we have

$$(1+\varepsilon\cos\theta_1)^m \geqslant 1 + m\varepsilon\cos\theta_1 + \frac{m(m-1)}{2}\alpha\varepsilon^2\cos^2\theta_1. \tag{31}$$

Then we find that

$$\frac{c^2}{2}\sigma_\varepsilon(\theta_1)^2 - \frac{c^m}{m}\sigma_\varepsilon(\theta_1)^m \leqslant \frac{c^2}{2} - \frac{c^m}{m} + \left(c^2 - c^m\right)\varepsilon\cos\theta_1 + \left(\frac{c^2}{2} - \frac{m-1}{2}c^m\alpha\right)\varepsilon^2\cos^2\theta_1.$$

Substitute the inequality above into (29) and use the inequality $c^2/2 - c^m/m \leqslant (m-2)/2m$ for all $c \geqslant 0$. Moreover, observe that the integral of $\cos\theta_1 \operatorname{Jac}(\boldsymbol{\theta})$ over Θ vanishes, because $\int_0^\pi \cos\theta_1 \sin^{N-2}\theta_1\, d\theta_1 = 0$. Then we derive

$$J(c\phi_\varepsilon) \leqslant J(\phi) + \frac{mc^2\varepsilon^2}{(m-2)\omega_N}\zeta(c)J(\phi). \tag{32}$$

Here $\zeta(c)$ is given by

$$\zeta(c) := \left\{1 - (m-1)c^{m-2}\alpha\right\}\nu_N + \mu_N C_{a,b} \tag{33}$$

with

$$\nu_N := \int_\Theta \cos^2\theta_1 \operatorname{Jac}(\boldsymbol{\theta})\, d\boldsymbol{\theta} = \omega_N - \mu_N = \frac{\omega_N}{N}, \tag{34}$$

where we have used (27). From (27) again, we note

$$\zeta(c) = \frac{\omega_N}{N}\left\{(N-1)C_{a,b} + 1 - (m-1)c^{m-2}\alpha\right\}.$$

Now, we choose α, c_0, ε_0 in the following way. Assumption (19) is rewritten as $(N-1)C_{a,b} < m-2$. First, we determine $\alpha \in (0,1)$ slightly less than 1 such that

$$(N-1)C_{a,b} < (m-1)\alpha - 1.$$

Next, we take $\varepsilon_0 \in (0,1]$ satisfying (30). Finally, we choose $c_0 \in (0,1)$ slightly less than 1 such that

$$(N-1)C_{a,b} < (m-1)c_0^{m-2}\alpha - 1.$$

Then $\zeta(c) < 0$ for all $c > c_0$. Thus we conclude that $J(c\phi_\varepsilon) < J(\phi)$ for all $c > c_0$ and $\varepsilon \in (0, \varepsilon_0)$. This completes the proof. □

Proof of Theorem 4.4. Let ϕ_ε be as in (22). Then all the assumptions of Lemma 5.1 are satisfied. Hence the unique positive radial asymptotic profile ϕ is not asymptotically stable. □

Remark 5.4 *(Stability analysis under nonnegativity).* The positive radial asymptotic profile ϕ remains to be not asymptotically stable, even though we further impose the nonnegativity of initial data, $v(0) \geqslant 0$, in Definition 3.4 (i.e., the definition of $\mathcal{X}$ is replaced by $\{v_0 \in H_0^1(\Omega) \setminus \{0\}\colon\ v_0 \geqslant 0,\ t_*(v_0) = 1\}$). Indeed, the perturbed functions $\phi_\varepsilon(x) = \sigma_\varepsilon(\theta_1)\phi(r)$ are positive in Ω for all $\varepsilon \in (0,1)$, and hence, so are initial data $v_{0,\varepsilon}$ in the proof above. Moreover, the positivity will be also conserved by the fast diffusion flow.

Proof of Corollary 4.6. Let ϕ_ε be as in (22) and $v_\varepsilon(s)$ be as in the proof of Lemma 5.1. By (iii) of Proposition 3.7, we choose a diverging sequence $s_n \nearrow \infty$ such that $v_\varepsilon(s_n)$ converges to a certain limit ψ in $H_0^1(\Omega)$. Then $J(\psi) < J(\phi)$ and ψ is a positive solution of (6), (7). Since ϕ takes the least energy among all radial nontrivial solutions, the limit ψ (and hence, every least energy solution) is not radially symmetric. □

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6. Proof of Theorem 4.7

We have proved that the unique positive radial profile ϕ is not asymptotically stable. In this section, we further prove the instability of ϕ for the two-dimensional case, $N = 2$. To this end, we shall verify that ϕ is isolated from the other asymptotic profiles with a certain partial symmetry and employ the compactness and the symmetry-preservation of fast diffusion flow.

The polar coordinate is written as

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta.$$

Here another type of perturbation with partial symmetry is applied to the radial profile $\phi = \phi(r)$. Define $\phi_{k,\varepsilon} : \Omega \to \mathbb{R}$ for $\varepsilon > 0$ and $k \in \mathbb{N}$ by

$$\phi_{k,\varepsilon}(x) = \sigma_{k,\varepsilon}(\theta)\phi(r) \quad \text{for } x = x(r,\theta) \in \Omega \tag{35}$$

with

$$\sigma_{k,\varepsilon}(\theta) = 1 + \varepsilon\cos k\theta \quad \text{for } \theta \in [0, 2\pi).$$

Then the new functions $\phi_{k,\varepsilon}$ have *k-fold symmetry*, i.e., $\phi_{k,\varepsilon}$ is invariant under the rotation $\theta \mapsto \theta + 2\pi/k$, since $\sigma_{k,\varepsilon}(\theta + 2\pi/k) = \sigma_{k,\varepsilon}(\theta)$. As in the proof of Proposition 5.3, one can prove the following proposition by noting that the gradient and the Jacobian are computed as

$$|\nabla\phi_{k,\varepsilon}|^2 = \left|\frac{\partial\phi_{k,\varepsilon}}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial\phi_{k,\varepsilon}}{\partial\theta}\right|^2, \qquad \frac{\partial(x_1, x_2)}{\partial(r,\theta)} = r.$$

Proposition 6.1 *(Decrease of energy by k-fold symmetric perturbations). Let $N = 2$, $\Omega = A_2(a,b)$ and let k be a positive integer. Assume that*

$$k^2\left(\frac{b-a}{\pi a}\right)^2 < m - 2. \tag{36}$$

Then there exist $c_0 \in (0,1)$ and $\varepsilon_0 \in (0,1]$ such that $J(c\phi_{k,\varepsilon}) < J(\phi)$ for any $\varepsilon \in (0,\varepsilon_0)$ and $c > c_0$.

In what follows, we work in function spaces with finite rotational symmetries. Let G be a subgroup of the orthogonal group $O(2)$. We call Ω a *G invariant domain* if $g(\Omega) = \Omega$ for $g \in G$, where $g(\Omega)$ denotes the image of Ω under the orthogonal transformation $g \in G$. A function $u : \Omega \to \mathbb{R}$ is said to be *G invariant* if $u(gx) = u(x)$ for all $g \in G$ and $x \in \Omega$. Moreover, we define

$$H_0^1(\Omega, G) := \{u \in H_0^1(\Omega):\ u(gx) = u(x) \text{ for } g \in G\}.$$

The following proposition holds for general N.

Proposition 6.2 *(Symmetry-preservation of fast diffusion flow). Let G be a subgroup of $O(N)$. Let v be the unique solution of (12)–(14) with an initial data v_0. If v_0 is G invariant, then so is $v(s) = v(\cdot, s)$ for each $s > 0$.*

Proof. Since $-\Delta$ and the map $u \mapsto |u|^{m-2}u$ are G equivariant, i.e., each $g \in G$ and these operators are commutative, we find that $gv(x,s) := v(g^{-1}x, s)$ also solves (12), (13) with v_0 replaced by $gv_0(x) := v_0(g^{-1}x)$. Here we recall that v_0 is G invariant and the solution of (12)–(14) is unique. Hence gv coincides with v for any $g \in G$. Therefore $v(s) = v(\cdot, s)$ is G invariant for all $s > 0$. $\square$

For a positive integer k, we set

$$G_k := \{g(2\pi j/k):\ j = 0, 1, 2, \dots, k-1\}$$

with

$$g(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

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Then we remark that $\phi_{k,\varepsilon}$ given by (35) is G_k invariant. Put

$$\mathcal{S}(G_k) := \mathcal{S} \cap H_0^1(\Omega, G_k).$$

Here we recall that $\mathcal{S}$ denotes the set of nontrivial solutions of (6), (7). The set $\mathcal{S}(G_k)$ is not empty, because the positive radial solution ϕ belongs to it.

In what follows, K stands for the largest positive integer k satisfying (36). Then we have the following alternative:

(A) For some $k \in \{1, 2, \dots, K\}$, ϕ is isolated in $H_0^1(\Omega)$ from the other solutions in $\mathcal{S}(G_k)$.
(B) For all $k \in \{1, 2, \dots, K\}$, ϕ is an accumulation point of $\mathcal{S}(G_k)$ in $H_0^1(\Omega)$.

Lemma 6.3 *(Case (A)). If* (A) *holds, then ϕ is unstable.*

Proof. By (A), there exists a constant $\delta > 0$ such that

$$\overline{B(\phi, \delta)} \cap \mathcal{S}(G_k) = \{\phi\}, \tag{37}$$

where $B(\phi, \delta) := \{u \in H_0^1(\Omega)\colon \|u - \phi\|_{1,2} < \delta\}$. Let $\phi_{k,\varepsilon}$ be as in (35). Set $v_{0,\varepsilon} := t_*(\phi_{k,\varepsilon})^{-1/(m-2)}\phi_{k,\varepsilon}$ and $c_\varepsilon := t_*(\phi_{k,\varepsilon})^{-1/(m-2)}$. Then $v_{0,\varepsilon}$ belongs to $H_0^1(\Omega, G_k) \cap \mathcal{X}$. Since $\phi_{k,\varepsilon} \to \phi$ strongly in $H_0^1(\Omega)$ and $c_\varepsilon \to 1$ as $\varepsilon \to 0$, $v_{0,\varepsilon}$ converges to ϕ strongly in $H_0^1(\Omega)$. Let c_0 and ε_0 be constants given by Proposition 6.1 and choose $\varepsilon_1 \in (0, \varepsilon_0)$ such that $c_\varepsilon > c_0$ for all $\varepsilon \in (0, \varepsilon_1)$. Then by Proposition 6.1, one can assure that

$$J(v_{0,\varepsilon}) = J(c_\varepsilon \phi_{k,\varepsilon}) < J(\phi) \quad \text{for all } \varepsilon \in (0, \varepsilon_1).$$

Now, let $\varepsilon \in (0, \varepsilon_1)$ be fixed and let v_ε be the unique solution of (12)–(14) with the initial data $v_{0,\varepsilon}$. We claim that

$$\left\| v_\varepsilon(s) - \phi \right\|_{1,2} \geqslant \delta \quad \text{for all } s > S \text{ with some } S > 0. \tag{38}$$

Indeed, if this claim is false, then there exists a sequence $s_n \to \infty$ such that $v_\varepsilon(s_n) \in B(\phi, \delta)$. By (iii) of Proposition 3.7, the solution $v_\varepsilon(s_n)$ converges to a nontrivial stationary solution ψ_ε along a subsequence. Hence $\psi_\varepsilon \in \overline{B(\phi, \delta)}$. Moreover, since $v_{0,\varepsilon}$ is G_k invariant, so is $v_\varepsilon(s)$ by Proposition 6.2. Thus we have $\psi_\varepsilon \in \mathcal{S}(G_k)$. Since $J(v_\varepsilon(\cdot))$ is nonincreasing, we have $J(\psi_\varepsilon) \leqslant J(v_{0,\varepsilon}) < J(\phi)$. Thus $\psi_\varepsilon \neq \phi$. Still, this contradicts (37). Consequently, (38) holds with the constant δ independent of ε, and therefore, ϕ is unstable. □

In view of Lemma 6.3, to prove the instability of ϕ, it is enough to remove the possibility of (B). To do so, we investigate the eigenvalue problem for the linearized operator $-\Delta - \mathfrak{c}_m(m-1)\phi^{m-2}$,

$$\left\{-\Delta - \mathfrak{c}_m(m-1)\phi^{m-2}\right\}u = \lambda u \quad \text{in } \Omega, \tag{39}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{40}$$

The two-dimensional Laplacian in polar coordinates is written as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Here we look for eigenfunctions in the separable form $u(x) = v(r)w(\theta)$. Then (39) is converted into

$$-\frac{w_{\theta\theta}}{w} = \frac{r^2}{v}\left\{v_{rr} + \frac{1}{r}v_r + \mathfrak{c}_m(m-1)\phi^{m-2}v + \lambda v\right\}. \tag{41}$$

Since w is 2π periodic, we have a nonnegative integer j such that $-w_{\theta\theta}/w = j^2$. If $j = 0$, then w is constant; if $j \geqslant 1$, then w is a linear combination of $\cos j\theta$ and $\sin j\theta$. Moreover, the radial part v solves

$$\left\{-\Delta_r - \mathfrak{c}_m(m-1)\phi^{m-2} + \frac{j^2}{r^2}\right\}v = \lambda v \quad \text{in } (a, b), \tag{42}$$

$$v(a) = v(b) = 0, \tag{43}$$

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where Δ_r is given by

$$\Delta_r := \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}.$$

For each fixed $j \geqslant 0$, by the Sturm–Liouville theory, the problem above has eigenvalues $\lambda_{1,j} < \lambda_{2,j} < \cdots$ and the corresponding (normalized) eigenfunctions $v_{1,j}, v_{2,j}, \ldots$, which forms a complete orthonormal system in $L^2_r(\Omega)$. Here $L^2_r(\Omega)$ denotes the set of all radial functions in $L^2(\Omega)$ (see Notation of Section 1). Therefore $\lambda_{i,j}$ ($i \geqslant 1$, $j \geqslant 0$) cover all the eigenvalues of (39), (40), and moreover, the system

$$\left\{\frac{1}{\sqrt{2\pi}} v_{i,0}(r)\right\}, \quad \left\{\frac{1}{\sqrt{\pi}} v_{i,j}(r)\cos j\theta\right\}, \quad \left\{\frac{1}{\sqrt{\pi}} v_{i,j}(r)\sin j\theta\right\}$$

(for $i, j = 1, 2, 3, \ldots$) is a complete orthonormal system in $L^2(\Omega)$.

Lemma 6.4 *(Case (B)). Assume that* (B) *holds. Then there exists a common multiple j of all natural numbers up to K (i.e., $\{k \in \mathbb{N}\colon k \leqslant K\}$) such that the eigenvalue problem (42), (43) with j has a zero eigenvalue $\lambda = 0$.*

To prove the lemma above, we check the signs of the first and second eigenvalues μ_1, μ_2 for the linearized problem in the radial form,

$$\left\{-\Delta_r - \mathfrak{c}_m(m-1)\phi^{m-2}\right\}u = \mu u \quad \text{in } (a,b), \tag{44}$$

$$u(a) = u(b) = 0. \tag{45}$$

Lemma 6.5 *(Signs of eigenvalues). It holds that $\mu_1 < 0 < \mu_2$.*

This lemma will be proved after proving Theorem 4.7. Let us give a proof of Lemma 6.4.

Proof of Lemma 6.4. Let $1 \leqslant k \leqslant K$ be fixed. By (B), there exists a sequence (ϕ_n) in $\mathcal{S}(G_k) \setminus \{\phi\}$ converging to ϕ strongly in $H^1_0(\Omega)$. Then (ϕ_n) converges also in $C^1(\overline{\Omega})$ by the elliptic regularity theorem. We put $u_n := (\phi_n - \phi)/\|\phi_n - \phi\|_\infty$ and $f(t) = \mathfrak{c}_m|t|^{m-2}t$. Then u_n solves

$$-\Delta u_n = \frac{f(\phi_n) - f(\phi)}{\phi_n - \phi} u_n \quad \text{in } \Omega, \qquad u_n = 0 \quad \text{on } \partial\Omega.$$

Since the right hand side is bounded in $L^\infty(\Omega)$, the elliptic regularity theorem guarantees that u_n is bounded in $W^{2,p}(\Omega)$ for all $1 \leqslant p < \infty$. By the compact imbedding $W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$ for large p, a subsequence of (u_n) converges in $C^1(\overline{\Omega})$ to a limit u_∞, which solves

$$\left\{-\Delta - \mathfrak{c}_m(m-1)\phi^{m-2}\right\}u_\infty = 0 \quad \text{in } \Omega, \qquad u_\infty = 0 \quad \text{on } \partial\Omega.$$

Moreover, $\|u_\infty\|_\infty = 1$ because $\|u_n\|_\infty = 1$. Since u_n is G_k invariant, u_∞ belongs to $H^1_0(\Omega; G_k)$ (this fact will be used later). Thus (39), (40) has a zero eigenvalue, i.e., $\lambda_{i,j} = 0$ with some i, j, and then, we denote by $v_{i,j}(r)$ the eigenfunction of (42), (43) corresponding to $\lambda_{i,j} = 0$. Then it holds that $j \geqslant 1$. Indeed, if $j = 0$, then (42) with $j = 0$ coincides with (44) and therefore $\lambda_{i,j} = 0$ becomes an eigenvalue of (44), (45). This contradicts Lemma 6.5. Thus $j \geqslant 1$.

For $V \in C[a,b]$, we denote by $\mu_k(-\Delta_r + V)$ the k-th eigenvalue of

$$\left\{-\Delta_r + V(r)\right\}u = \mu u \quad \text{in } (a,b), \qquad u(a) = u(b) = 0.$$

Since $j \geqslant 1$, we use Lemma 6.5 again to get

$$\mu_2\left(-\Delta_r - \mathfrak{c}_m(m-1)\phi^{m-2} + j^2/r^2\right) > \mu_2\left(-\Delta_r - \mathfrak{c}_m(m-1)\phi^{m-2}\right) > 0.$$

Thus we conclude that

$$\mu_1\left(-\Delta_r - \mathfrak{c}_m(m-1)\phi^{m-2} + j^2/r^2\right) = \lambda_{i,j} = 0, \tag{46}$$

which implies that $i = 1$ (i.e., $\lambda_{1,j} = 0$). Moreover, one can uniquely determine j satisfying $\lambda_{1,j} = 0$ by the strict monotonicity of the eigenvalues in the following sense: if $j < l$, then

$$\mu_1\big(-\Delta_r - \mathfrak{c}_m(m-1)\phi^{m-2} + j^2/r^2\big) < \mu_1\big(-\Delta_r - \mathfrak{c}_m(m-1)\phi^{m-2} + l^2/r^2\big).$$

Since the limit u_∞ of (u_n) is an eigenfunction corresponding to $\lambda_{1,j}$, it is represented as

$$u_\infty(x) = v_{1,j}(r)(\alpha\cos j\theta + \beta\sin j\theta) \quad \text{for } x = x(r,\theta)$$

with some $(\alpha,\beta) \neq (0,0)$. Since u_∞ is G_k invariant, it is $2\pi/k$ periodic in θ, and hence, j/k must be an integer (i.e., j is a multiple of k).

Repeating the argument above again with k replaced by another $1 \leqslant k' \leqslant K$, we get some $j' \geqslant 1$ such that $\lambda_{1,j'} = 0$ and j'/k' is an integer. Then it holds that $j = j'$ because $\lambda_{1,j} = \lambda_{1,j'} = 0$. Thus j is independent of the choice of k, and therefore, it is a common multiple of all natural numbers up to K. It completes the proof. $\square$

We further need a couple of lemmas. The first one is related to the prime number theorem. We refer the reader to a book of Hardy and Wright [15, Theorem 414] for its proof.

Lemma 6.6 *(Estimates for LCMs). Let $U(n)$ be the least common multiple of all natural numbers up to n. Then there exist constants $A, B > 0$ such that*

$$e^{An} \leqslant U(n) \leqslant e^{Bn} \quad \textit{for all } n \in \mathbb{N},\ n \geqslant 2.$$

The next lemma provides an L^∞-estimate for the positive radial solution ϕ of (6), (7), which will be proved later on.

Lemma 6.7 *(Uniform bound for the positive radial solution). It holds that*

$$\|\phi\|_\infty^{m-2} \leqslant \mathfrak{c}_m^{-1}\pi^{2(m-1)}(b/a)^{m-1}(b-a)^{-2}.$$

We are now in a position to complete the proof of Theorem 4.7 by proving that (B) is impossible if $(b-a)/a$ is small enough.

Proof of Theorem 4.7. Let $(b-a)/a > 0$ be small enough. Recall that K is the largest integer k satisfying (36). Hence K is also large enough. Now, we suppose on the contrary that (B) holds. Let j be the integer determined by Lemma 6.4. Then j is a common multiple of all natural numbers up to K, and hence $j \geqslant U(K)$. Therefore we have $j \geqslant e^{AK}$ by Lemma 6.6.

We next claim that the potential $-\mathfrak{c}_m(m-1)\phi^{m-2} + j^2/r^2$ is positive when the ratio $(b-a)/a$ is small enough. Indeed, since K is the largest integer satisfying (36), we have

$$(K+1)^2\left(\frac{b-a}{\pi a}\right)^2 \geqslant m-2,$$

which is written as

$$a^2(b-a)^{-2} \leqslant \pi^{-2}(m-2)^{-1}(K+1)^2.$$

Since $(b-a)/a > 0$ is small enough, it holds that $a < b < 2a$. By the inequality above together with Lemma 6.7, we have

$$0 \leqslant \mathfrak{c}_m(m-1)\phi^{m-2} \leqslant (m-1)\pi^{2(m-1)}(b/a)^{m-1}(b-a)^{-2} \leqslant Cb^{-2}(K+1)^2$$

with the constant $C := 2^{m+1}(m-1)(m-2)^{-1}\pi^{2(m-2)} > 0$. Thus we find, for all $r \in (a,b)$,

$$-\mathfrak{c}_m(m-1)\phi^{m-2} + \frac{j^2}{r^2} \geqslant b^{-2}\big\{-C(K+1)^2 + e^{2AK}\big\} > 0,$$

provided that K is large enough. Then by (46), we obtain

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$$0<\mu_1(-\Delta_r)<\mu_1\big(-\Delta_r-\mathfrak{c}_m(m-1)\phi^{m-2}+j^2/r^2\big)=0,$$

which is a contradiction. Therefore (B) never holds. Consequently, by Lemma 6.3, we have proved Theorem 4.7. □

We finally prove Lemmas 6.5 and 6.7. For an independent interest, we prove these lemmas for general dimension N. Instead of Lemma 6.7, we prove the next lemma.

Lemma 6.8 *(Uniform bound for the positive radial solution). Let ϕ be the unique positive radial solution of (6), (7) for the N-dimensional annulus. Then it holds that*

$$\|\phi\|_\infty^{m-2}\leqslant \mathfrak{c}_m^{-1}\pi^{2(m-1)}(b/a)^{(m-1)(N-1)}(b-a)^{-2}.$$

Proof. We start with estimating the L^m-norm of ϕ. For a radial function $u=u(|x|)$, the Rayleigh quotient $R(u)$ is reduced to

$$R(u)=\omega_N^{(m-2)/m}\int_a^b |u'|^2 r^{N-1}\,dr\left(\int_a^b |u|^m r^{N-1}\,dr\right)^{-2/m}.$$

As in Proposition 2.1 with (10), the unique positive radial solution ϕ minimizes $R(\cdot)$ over $H^1_{0,r}(\Omega)$ (see Notation in Section 1). Hence we have

$$R(\phi)\leqslant R(u)\quad \text{for all } u\in H^1_{0,r}(\Omega)\setminus\{0\}.$$

Even if R is replaced by $\omega_N^{-(m-2)/m}R$, the relation above remains valid. Hence hereafter we define $R(u)$ by

$$R(u):=\int_a^b |u'|^2 r^{N-1}\,dr\left(\int_a^b |u|^m r^{N-1}\,dr\right)^{-2/m}. \tag{47}$$

Choose $u(r)=\sin(\pi(r-a)/(b-a))$ and compute $R(u)$. Then an easy calculation shows

$$\int_a^b |u'|^2 r^{N-1}\,dr\leqslant b^{N-1}\int_a^b |u'|^2\,dr=\frac{\pi^2 b^{N-1}}{2(b-a)}. \tag{48}$$

By the Hölder inequality, we have

$$\frac{b-a}{2}=\int_a^b u^2\,dr\leqslant\left(\int_a^b |u|^m\,dr\right)^{2/m}(b-a)^{(m-2)/m},$$

which implies

$$\int_a^b |u|^m r^{N-1}\,dr\geqslant a^{N-1}\int_a^b |u|^m\,dr\geqslant 2^{-m/2}a^{N-1}(b-a).$$

By these inequalities, we obtain

$$R(u)\leqslant \pi^2 a^{-2(N-1)/m}b^{N-1}(b-a)^{-(m+2)/m}.$$

Since $R(\phi)\leqslant R(u)$, it follows that

$$R(\phi)\leqslant \pi^2 a^{-2(N-1)/m}b^{N-1}(b-a)^{-(m+2)/m}.$$

By (25), we have

$$R(\phi)=\mathfrak{c}_m\left(\int_a^b \phi^m r^{N-1}\,dr\right)^{(m-2)/m}.$$

Combining the two relations above, we find

$$\int_a^b \phi^m r^{N-1}\,dr \leqslant \mathfrak{c}_m^{-m/(m-2)} \pi^{2m/(m-2)} a^{-2(N-1)/(m-2)} b^{m(N-1)/(m-2)} (b-a)^{-(m+2)/(m-2)}. \tag{49}$$

Now, if $\phi'(r)=0$, then $\phi''(r)<0$ by Eq. (17). Hence the critical point of ϕ is unique in (a,b), and we denote it by c. Then $\phi'>0$ in (a,c) and $\phi'<0$ in (c,b). Multiplying (17) by r^{N-1} and integrating it over (c,r), we get

$$r^{N-1}\phi'(r) = -\mathfrak{c}_m \int_c^r \phi(\rho)^{m-1}\rho^{N-1}\,d\rho \quad \text{for all } r\in(c,b),$$

which shows

$$0>\phi'(r) \geqslant -\mathfrak{c}_m \int_c^r \phi(\rho)^{m-1}\,d\rho \quad \text{for all } r\in(c,b). \tag{50}$$

On the other hand, using the Hölder inequality and (49), we estimate the right hand side as

$$\begin{aligned}\int_c^r \phi(\rho)^{m-1}\,d\rho &\leqslant \left(\int_c^r \phi(\rho)^m \rho^{N-1}\,d\rho\right)^{(m-1)/m} \left(\int_c^r \rho^{-(m-1)(N-1)}\,d\rho\right)^{1/m}\\ &\leqslant \mathfrak{c}_m^{-(m-1)/(m-2)} \pi^{2(m-1)/(m-2)} (b/a)^{(m-1)(N-1)/(m-2)} (b-a)^{-m/(m-2)}.\end{aligned}$$

Here we have used the fact that $\rho^{-(m-1)(N-1)} \leqslant a^{-(m-1)(N-1)}$. Combine the estimate above with (50), integrate it over (c,b) and note that $\phi(c)=\|\phi\|_\infty$ and $\phi(b)=0$. Then we get

$$\|\phi\|_\infty = \phi(c) \leqslant \mathfrak{c}_m^{-1/(m-2)} \pi^{2(m-1)/(m-2)} (b/a)^{(m-1)(N-1)/(m-2)} (b-a)^{-2/(m-2)},$$

which completes the proof. □

It remains only to prove Lemma 6.5. If Ω is a bounded convex domain in $\mathbb{R}^2$, Lemma 6.5 has been proved by Lin [18]. However, in our case, $\Omega = A_2(a,b)$ is not convex. Using a similar idea to that in [18], we prove the next lemma.

Lemma 6.9 *(Negativity and nonnegativity of eigenvalues). Let μ_1 and μ_2 be the first and second eigenvalues of (44), (45) with*

$$\Delta_r = \frac{d^2}{dr^2} + \frac{N-1}{r}\frac{d}{dr}$$

for the N-dimensional case. Then it holds that

$$\mu_1 < 0 \leqslant \mu_2.$$

Proof. We define the Rayleigh quotient $L(u)$ associated with (44) by

$$L(u) := \int_a^b \left\{|u'|^2 - \mathfrak{c}_m(m-1)\phi^{m-2}u^2\right\} r^{N-1}\,dr \left(\int_a^b u^2 r^{N-1}\,dr\right)^{-1}. \tag{51}$$

Then the first eigenvalue μ_1 is negative, because we note by (25) that

$$\mu_1 = \inf\left\{L(u)\colon u\in H^1_{0,r}(\Omega)\setminus\{0\}\right\} \leqslant L(\phi) < 0.$$

Let us show that $\mu_2 \geqslant 0$. Recall that ϕ is a global minimizer of the Rayleigh quotient $R(u)$ defined by (47) over $H^1_{0,r}(\Omega)\setminus\{0\}$. For $u\in H^1_{0,r}(\Omega)\setminus\{0\}$, we define

$$f(t) := R(\phi+tu) = \int_a^b |\phi'+tu'|^2 r^{N-1}\,dr \left(\int_a^b |\phi+tu|^m r^{N-1}\,dr\right)^{-2/m}.$$

Since $f(t)$ attains its minimum at $t=0$, it follows that $f'(0)=0$ and $f''(0)\geqslant 0$. Putting

$$g(t)=\int_a^b \left|\phi'+tu'\right|^2 r^{N-1}\,dr,\qquad h(t)=\int_a^b |\phi+tu|^m r^{N-1}\,dr,$$

we have

$$f'(t)=g'h^{-2/m}-\frac{2}{m}gh^{-(m+2)/m}h',$$
$$f''(t)=g''h^{-2/m}-\frac{4}{m}g'h^{-(m+2)/m}h'+\frac{2(m+2)}{m^2}gh^{-2(m+1)/m}\left(h'\right)^2-\frac{2}{m}gh^{-(m+2)/m}h''.$$

Moreover, a direct calculation shows that

$$g(0)=\int_a^b \left|\phi'\right|^2 r^{N-1}\,dr,\qquad g'(0)=2\int_a^b \phi'u'r^{N-1}\,dr,$$
$$g''(0)=2\int_a^b \left|u'\right|^2 r^{N-1}\,dr,\qquad h(0)=\int_a^b \phi^m r^{N-1}\,dr,$$
$$h'(0)=m\int_a^b \phi^{m-1}ur^{N-1}\,dr,\qquad h''(0)=m(m-1)\int_a^b \phi^{m-2}u^2r^{N-1}\,dr.$$

By (25), we have $g(0)=\mathfrak{c}_m h(0)$. Since $f'(0)=0$, we get $g'(0)=(2/m)\mathfrak{c}_m h'(0)$. Using these relations, we have

$$\begin{aligned}f''(0)&=2\int_a^b \left|u'\right|^2 r^{N-1}\,dr\left(\int_a^b \phi^m r^{N-1}\,dr\right)^{-2/m}\\&\quad+2(m-2)\mathfrak{c}_m\left(\int_a^b \phi^{m-1}ur^{N-1}\,dr\right)^2\left(\int_a^b \phi^m r^{N-1}\,dr\right)^{-(m+2)/m}\\&\quad-2(m-1)\mathfrak{c}_m\left(\int_a^b \phi^{m-2}u^2r^{N-1}\,dr\right)\left(\int_a^b \phi^m r^{N-1}\,dr\right)^{-2/m}.\end{aligned}$$

If u is orthogonal to ϕ^{m-1} in $L^2(\Omega)$, i.e.,

$$\int_a^b \phi^{m-1}ur^{N-1}\,dr=0,$$

then we recall $f''(0)\geqslant 0$ to get

$$\int_a^b \left|u'\right|^2 r^{N-1}\,dr-\mathfrak{c}_m(m-1)\int_a^b \phi^{m-2}u^2r^{N-1}\,dr\geqslant 0,$$

which implies that $L(u)\geqslant 0$ if $u\perp\phi^{m-1}$. Applying the minimax principle of the second eigenvalue μ_2, we have

$$\mu_2\geqslant\inf\left\{L(u):\ u\in H^1_{0,r}(\Omega),\ u\perp\phi^{m-1}\right\}\geqslant 0.$$

This is our desired conclusion. □

Lemma 6.10 *(Positivity of the second eigenvalue). Under the same setting as in Lemma 6.9, it follows that $\mu_2>0$. Hence $\mu_1<0<\mu_2$.*

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Proof. To the contrary, we assume that $\mu_2 = 0$. Let u be an eigenfunction corresponding to the second eigenvalue $\mu_2 = 0$, that is,

$$u'' + \frac{N-1}{r}u' + \mathfrak{c}_m(m-1)\phi^{m-2}u = 0 \quad \text{in } (a,b), \tag{52}$$

$$u(a) = u(b) = 0. \tag{53}$$

By virtue of the oscillation theory for regular Sturm–Liouville problems, the second eigenfunction u has exactly one zero in (a,b). Hence one can assume that $u'(a) > 0$ and $u'(b) > 0$ after replacing u by $-u$ if necessary. Using (17) and (52), we have

$$\int_a^b \left\{u\left(r^{N-1}\phi'\right)' - \phi\left(r^{N-1}u'\right)'\right\} dr = \mathfrak{c}_m(m-2)\int_a^b \phi^{m-1}ur^{N-1}\,dr.$$

Since the left hand side is zero due to the integration by parts as well as the boundary conditions, it yields that

$$\int_a^b \phi^{m-1}ur^{N-1}\,dr = 0. \tag{54}$$

Put $\psi(r) = r\phi'(r)$. Applying $r(d/dr) + 2$ to both sides of (17), we obtain

$$\psi'' + \frac{N-1}{r}\psi' + \mathfrak{c}_m(m-1)\phi^{m-2}\psi = -2\mathfrak{c}_m\phi^{m-1}. \tag{55}$$

We rewrite (52) and (55) as

$$\left(r^{N-1}u'\right)' = -\mathfrak{c}_m(m-1)\phi^{m-2}ur^{N-1},$$
$$\left(r^{N-1}\psi'\right)' = -2\mathfrak{c}_m\phi^{m-1}r^{N-1} - \mathfrak{c}_m(m-1)\phi^{m-2}\psi r^{N-1}.$$

Multiplying the first equation by ψ and the second one by u and subtracting them, we get

$$\left(r^{N-1}u'\right)'\psi - \left(r^{N-1}\psi'\right)'u = 2\mathfrak{c}_m\phi^{m-1}ur^{N-1}.$$

Integrating both sides over (a,b) and using the integration by parts with (53), we obtain

$$\left[r^{N-1}u'\psi\right]_a^b = \int_a^b \left\{\left(r^{N-1}u'\right)'\psi - \left(r^{N-1}\psi'\right)'u\right\} dr$$
$$= 2\mathfrak{c}_m\int_a^b \phi^{m-1}ur^{N-1}\,dr = 0,$$

where we have used (54). On the other hand, the left hand side is equal to

$$\left[r^{N-1}u'\psi\right]_a^b = b^{N-1}u'(b)\psi(b) - a^{N-1}u'(a)\psi(a) < 0,$$

because $u'(a) > 0$, $u'(b) > 0$, $\psi(a) = a\phi'(a) > 0$ and $\psi(b) = b\phi'(b) < 0$. Hence a contradiction occurs. Therefore it holds that $\mu_2 > 0$. □

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