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Akagi, Goro

Otani, Mitsuharu

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TIME-DEPENDENT CONSTRAINT PROBLEMS ARISING FROM MACROSCOPIC CRITICAL-STATE MODELS FOR TYPE-II SUPERCONDUCTIVITY AND THEIR APPROXIMATIONS

Dedicated to the memory of Professor Tsutomu Arai

Goro Akagi* and Mitsuharu Ôtani †

Abstract. This paper is concerned with time-dependent constraint problems arising from macroscopic critical-state models for type-II superconductivity as well as their approximate problems associated with p-Laplacian for enough large number p.

In order to derive their solvabilities, an abstract framework of doubly nonlinear evolution equations governed by time-dependent subdifferential operators in reflexive Banach spaces is established and applied to these problems.

Moreover the convergence of solutions for the approximate problems as $p \to +\infty$ is also proved without any explicit error estimates; furthermore a new method of approximation to the critical-state models is proposed to enable us to derive an explicit error estimate as well as the convergence of the corresponding approximate solutions.

1 Introduction

In the early 20th century, several macroscopic models for superconductivity were proposed and some of them succeeded in explaining the structures of various properties of superconductivity: no electric resistance, Meissner effect, quantization of magnetic flux and so on. In particular, F and H. London [18] illustrated the Meissner effect by introducing their macroscopic model, which is the so-called London model. On the other hand, it seems to be difficult to give full explanations to all principal properties of superconductivity only from the macroscopic viewpoint; indeed superconductivity involves several aspects which can be explained only on the microscopic theory, e.g., BCS (Bardeen-Cooper-Schrieffer) theory.

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However, macroscopic models could be still useful to give intuitive explanations even if they can not cover the whole picture of superconductivity. In this paper, we focus on a somewhat intuitive macroscopic model providing the current-voltage law.

In 1960's, Bean [8] and Kim et al [16] proposed macroscopic critical-state models for type-II superconductivity. Their models can give a description for the magnetization of a type-II superconductor placed in a non-stationary external magnetic field. According to their models, type-II superconductivity is characterized by the following (multi-valued) relationship between the electric field $\mathbf{e}$ and the current density $\mathbf{j}$:

$$\text{(B)} \qquad \mathbf{e} \in \begin{cases} \mathbf{0} & \text{if} \quad |\,\mathbf{j}\,| < j_c, \\ [0, +\infty)\,\mathbf{j} & \text{if} \quad |\,\mathbf{j}\,| = j_c, \\ \emptyset & \text{if} \quad |\,\mathbf{j}\,| > j_c. \end{cases}$$

In particular, if $|\,\mathbf{j}\,| < j_c$, then the electric current can flow even if the electric field vanishes. This feature means the non-existence of electric resistance and the existence of persistent current flow in type-II superconductors. Moreover the strength $|\,\mathbf{j}\,|$ of the current density can not exceed the critical value j_c.

In Bean's model, the critical current density is homogeneous, i.e., j_c does not depend on space and time. However, in Kim's model, j_c depends on the magnetic field $\mathbf{H}$, which possibly depends on space and time. On the other hand, it would be natural to consider the case where j_c depends on space and time but not on the magnetic field such as $j_c = j_c(x,t)$, which gives an intermediate model between above two models. In this paper, we consider such a case that $j_c = \gamma(x,t)$ for some given function γ.

On the other hand, (B) has a high nonlinearity, e.g., $\mathbf{e} = \mathbf{e}(\mathbf{j})$ becomes a multi-valued mapping. Therefore the following approximation is often used to simplify the treatment of (B).

$$\text{(B)}_p \qquad \mathbf{e} = \mathbf{k}_p(\mathbf{j}) := \left| \frac{\mathbf{j}}{j_c} \right|^{p-2} \frac{\mathbf{j}}{j_c},$$

where p is an enough large number. This approximation (B)_p is called the power approximation of (B). By using this approximation, several theoretical and numerical results were obtained (see e.g. [5], [6]).

Prigozhin investigated the magnetization of type-II superconductors in non-stationary external magnetic fields for two specific geometrical configurations: a long cylindrical superconductor placed in a parallel external magnetic field, and a thin superconductor film in a perpendicular external magnetic field (see Prigozhin [20, 21, 22, 23], Barrett and Prigozhin [7]) by using the Bean model or its power approximation. We here follows the setting proposed in [7].

Let Ω_0 be a bounded domain of $\mathbf{R}^2$ with smooth boundary $\partial\Omega_0$, and consider the case of a long cylindrical superconductor with the cross-section Ω_0 ("Long cylinder case") and the case of a thin superconductor film with the surface Ω_0 ("Thin film case").

Let $\mathbf{h}_e := (0, 0, h_e)$ be a non-stationary external magnetic field, which is orthogonal to Ω_0. By Faraday's law of electromagnetic induction, the time variation of the external magnetic field induces a current flow, which is lying on Ω_0. Moreover according to Ampare's law, the current flow induces an internal magnetic field which is orthogonal to the current flow.

Long cylinder case: Denote the current density by $\mathbf{j} := (j_1, j_2, 0)$. Moreover let $\mathbf{h} := (0, 0, h)$ and $\mathbf{h_0} := (0, 0, h_0)$ be an internal magnetic field and its initial data respectively.

Thin film case: Let $\mathbf{j} = (j_1, j_2)$ denote the 2-dimensional sheet current density on the film surface Ω_0. We then denote by h and h_0 the stream function of the sheet current density $\mathbf{j}$ and an initial data of h.

In both cases, Ω_0 admits a finite number of holes Ω_i $(i = 1, 2, \ldots, L)$; then we set $\Omega := \cup_{i=0}^{L} \Omega_i$, which becomes simply connected in $\mathbf{R}^2$. For the sake of the continuity of the total magnetic field $h + h_e$ on $\partial\Omega$ and the non-existence of current flow in the holes, the following conditions are imposed.

$$h = 0 \text{ on } \partial\Omega \quad \text{and} \quad |\nabla h| = 0 \text{ in } \Omega_i \ (i = 1, 2, \ldots, L).$$

Hence as in [7], we employ the following function space:

$$(1.\ 1) \qquad X_p := \left\{ u \in W_0^{1,p}(\Omega); |\nabla u(x)| = 0 \text{ for a.e. } x \in \Omega_i \ (i = 1, 2, \ldots, L) \right\}$$

with the norm $|\cdot|_{X_p} := |\nabla \cdot|_{L^p(\Omega)}$ for each $p \in (1, +\infty)$. When Ω_0 has no hole, we set $\Omega = \Omega_0$ and X_p coincides with $W_0^{1,p}(\Omega)$. Just as in [7], we can derive the following variational inequality from the Maxwell system and (B) with $j_c = \gamma(x, t)$.

$$(\mathrm{P}) \begin{cases} b\left(\dfrac{dh}{dt}(t),\ v - h(t)\right) \geq -\left\langle \dfrac{dh_e}{dt}(t), v - h(t) \right\rangle_{X_2} & \forall v \in K^t,\ 0 < t < T, \\ h(0) = h_0, \end{cases}$$

where $K^t := \{u \in X_2(\Omega)\ ;\ |\nabla u(x)| \leq j_c = \gamma(x, t) \text{ for a.e. } x \in \Omega\}$ and $b(\cdot, \cdot)$ is given by

$$b(u, v) := \begin{cases} \displaystyle\int_\Omega u(x)v(x)dx & \text{(Long cylinder case)}, \\ \displaystyle\int_\Omega \int_\Omega \frac{\nabla_x u(x) \cdot \nabla_{x'} v(x')}{4\pi|x - x'|} dx dx' & \text{(Thin film case)}. \end{cases}$$

According to [7], $b(\cdot, \cdot)$ is a symmetric, continuous, coercive bilinear form in $V_0 \times V_0$, where V_0 is defined by

$$V_0 := \begin{cases} L^2(\Omega) & \text{(Long cylinder case)}, \\ H_{00}^{1/2}(\Omega) & \text{(Thin film case)}, \end{cases}$$

where

$$H_{00}^{1/2}(\Omega) := \left\{ \chi \in H^{1/2}(\Omega);\ \tilde{\chi} := \begin{cases} \chi & \text{in } \Omega, \\ 0 & \text{in } \mathbf{R}^2 \setminus \Omega \end{cases} \in H^{1/2}(\mathbf{R}^2) \right\},$$

equipped with the norm

$$|u|_{V_0} := \begin{cases} |u|_{L^2(\Omega)} = \left(\displaystyle\int_\Omega |u(x)|^2 dx \right)^{1/2} & \text{(Long cylinder case)}, \\ |u|_{H_{00}^{1/2}(\Omega)} := \left(|u|^2_{H^{1/2}(\Omega)} + \displaystyle\int_\Omega \frac{|u(x)|^2}{\mathrm{dist}(x, \partial\Omega)} dx \right)^{1/2} & \text{(Thin film case)}. \end{cases}$$

Furthermore, the Maxwell system and the power approximation $(\mathrm{B})_p$ yield the following equation.

$$(\mathrm{P})_p \begin{cases} b\left(\dfrac{dh_p}{dt}(t),\ v\right) + \displaystyle\int_\Omega \left(\frac{1}{\gamma(x,t)}\right)^p |\nabla h_p(x,t)|^{p-2}\nabla h_p(x,t)\cdot\nabla v(x)dx \\ \qquad = -\left\langle \dfrac{dh_e}{dt}(t), v\right\rangle_{X_p} \quad \forall v \in X_p, \quad 0<t<T, \\ h_p(0) = h_0. \end{cases}$$

For the case where j_c is constant, i.e., the Bean model case, Barrett and Prigozhin [7] proved the existence of the unique solution for $(\mathrm{P})_p$ and discussed the convergence of h_p as $p \to +\infty$. Moreover they proved that the limit of h_p gives the unique solution for (P).

On the other hand, for the case where j_c depends on space and time, i.e., $j_c = \gamma(x,t)$, $(\mathrm{P})_p$ and (P) become non-autonomous systems. Because of this, there arise some technical difficulties and the study on this case is not performed yet. The main purpose of this paper is to show the existence of unique solutions h_p and h for $(\mathrm{P})_p$ and (P) respectively and to discuss the convergence of h_p to h as $p \to +\infty$. To this end, it will be shown that $(\mathrm{P})_p$ and (P) are reduced to Cauchy problems for evolution equations governed by two time-dependent subdifferential operators from a reflexive Banach space V into its dual space V^* such as

$$\partial\psi^t\left(\frac{du}{dt}(t)\right) + \partial\varphi^t(u(t)) \ni f(t) \ \text{ in } V^*,\ 0<t<T, \tag{1. 2}$$

where $\partial\psi^t$ and $\partial\varphi^t$ are the subdifferentials of time-dependent proper lower semicontinuous convex functionals ψ^t and φ^t from V into $(-\infty,+\infty]$ respectively and f is a given function from $[0,T]$ into V^* (see Section 3). Hence in order to assure the existence of the unique solution for $(\mathrm{P})_p$ or (P), we shall develop an abstract theory concerning the solvability of Cauchy problem for (1. 2) in Section 2. The scope of our abstract framework developed here is ample enough not only for $(\mathrm{P})_p$ and (P) but also for other types of problems, which will be discussed in forthcoming papers.

Furthermore in Section 5, another way of approximations of (B) is proposed from the viewpoint of convex analysis. As in the case of the power approximation, we derive approximate problems of (P) from the Maxwell system and the new approximation proposed here; moreover we can show the existence, the uniqueness and the convergence of solutions for the approximate problems. The advantage of our new approximations of (B) lies in the fact that it is possible to show the convergence much more easily and estimate the convergence rate of solutions for the approximate problems towards the solutions of (P).

2 Abstract Theory

Before formulating our problem, we review the definition of subdifferentials. Let $\Phi(E)$ be the set of all *proper lower semicontinuous convex functions* ϕ from a topological linear space E into $(-\infty,+\infty]$, where "*proper*" means that the *effective domain* $D(\phi)$ of ϕ defined by $D(\phi) := \{u \in E; \phi(u) < +\infty\}$ is not empty. Define the subdifferential $\partial_E\phi$ of ϕ in E by

$$\partial_E\phi(u) := \{f \in E^*; \phi(v) - \phi(u) \geq \langle f, v-u\rangle_E \text{ for all } v \in D(\phi)\},$$

where $\langle \cdot, \cdot \rangle_E$ denotes the natural duality between E and its dual space E^*, with its domain $D(\partial_E \phi) = \{u \in E; \partial_E \phi(u) \neq \emptyset\}$. We shall write $\partial \phi$ and $\langle \cdot, \cdot \rangle$ simply instead of $\partial_E \phi$ and $\langle \cdot, \cdot \rangle_E$ respectively when no confusion can arise.

Let V and H be a real reflexive Banach space and a real Hilbert space respectively, and let V^* and H^* be their dual spaces respectively. Now suppose that H is identified with H^* and the following embeddings hold true with densely defined continuous injections.

$$V \subset H \equiv H^* \subset V^* \tag{2. 1}$$

Let ψ^t, φ^t belong to $\Phi(V)$ for all $t \in [0, T]$. Our abstract Cauchy problem is then described as follows:

$$(\mathrm{CP}) \begin{cases} \partial \psi^t \left(\dfrac{du}{dt}(t) \right) + \partial \varphi^t(u(t)) \ni f(t) \quad \text{in } V^*, \quad 0 < t < T, \\ u(0) = u_0. \end{cases}$$

For the case where both ψ^t and φ^t are independent of t, such as $\psi^t = \psi$ and $\varphi^t = \varphi$, Arai [2] and Colli [13] studied the existence of strong solutions for (CP) in the Hilbert space setting (i.e., $V = H$) and in the reflexive Banach space setting respectively. As for the time-dependent subdifferential case, Senba [24] extended Arai's result to the case where ψ^t may depend on t and $\partial \varphi^t = \partial \varphi$ is a positive definite self-adjoint operator independent of t. However the case where both functionals φ^t and ψ^t depend on t has not been studied yet even for the case where $V = H$.

Let $q \in (1, +\infty)$ and introduce the following assumptions on ψ^t.

- (A1) There exist constants $\alpha > 0$ and $C_1 \geq 0$ such that
  $\alpha |u|_V^q \leq \psi^t(u) + C_1$ for all $u \in D(\psi^t)$ and $t \in [0, T]$.
- (A2) There exist a constant $C_2 \geq 0$ and a function $a \in L^1(0, T)$ such that
  $|\eta|_{V^*}^{q'} \leq C_2 \psi^t(u) + a(t)$ for all $[u, \eta] \in \partial \psi^t$ and $t \in [0, T]$.

Now define the mapping $\mathcal{B} : L^q(0, T; V) \to L^{q'}(0, T; V^*)$ as follows: for any $[u, f] \in L^q(0, T; V) \times L^{q'}(0, T; V^*)$,

$$\mathcal{B}u \ni f \quad \Leftrightarrow \quad \partial \psi^t(u(t)) \ni f(t) \qquad \text{for a.e. } t \in (0, T).$$

Then $\mathcal{B}$ is said to be a weakly closed mapping from $L^q(0, T; V)$ into $L^{q'}(0, T; V^*)$ if the following holds true: if $u_n \to u$ weakly in $L^q(0, T; V)$, $\eta_n \in \mathcal{B}(u_n)$ and $\eta_n \to \eta$ weakly in $L^{q'}(0, T; V^*)$, then $\eta \in \mathcal{B}(u)$. Now our third assumption on ψ is as follows.

- (A3) $\mathcal{B}$ is a weakly closed mapping from $L^q(0, T; V)$ into $L^{q'}(0, T; V^*)$.

In order to verify the uniqueness of solutions for (CP), we employ the following (A3)$'$ instead of (A3).

- (A3)$'$ For every $t \in [0, T]$, $D(\partial \varphi^t) \subset D(\partial \psi^t)$, the graph of $\partial \psi^t$ is linear,
  i.e., $\alpha \xi + \beta \eta \in \partial \psi^t(\alpha u + \beta v)$ $\forall [u, \xi], [v, \eta] \in \partial \psi^t$, $\forall \alpha, \beta \in \mathbf{R}$,
  and $\partial \psi^t$ is symmetric, where "symmetric" means $\langle \xi, v \rangle = \langle \eta, u \rangle$
  $\forall [u, \xi], [v, \eta] \in D(\partial \psi^t)$.

Remark 2.1 (1) Suppose that (A1) with $C_1 = 0$ and (A3)$'$ hold. Moreover assume that $\psi^{t_0}(0) = 0$ and $D(\partial\psi^{t_0}) \neq \{0\}$ for some $t_0 \in [0, T]$. Then q must be equal to 2. Indeed, by the definition of $\partial\psi^{t_0}$, we get

$$\psi^{t_0}(u) \;\leq\; \langle \xi, u\rangle \leq |\xi|_{V^*}|u|_V \quad \forall [u, \xi] \in \partial\psi^{t_0}. \tag{2. 2}$$

Hence (A1) with $C_1 = 0$ implies

$$\alpha|u|_V^q \;\leq\; \psi^{t_0}(u) \leq |\xi|_{V^*}|u|_V.$$

Let $\mu \in (0, +\infty)$ and let $[u_0, \xi_0] \in \partial\psi^{t_0}$ be such that $u_0 \neq 0$ and $\xi_0 \neq 0$. Then putting $u = \mu u_0$ and $\xi = \mu\xi_0$, we find that

$$\alpha\mu^q|u_0|_V^q \;\leq\; \mu^2|\xi_0|_{V^*}|u_0|_V \quad \forall \mu \in (0, +\infty),$$

which yields $q = 2$.

(2) The condition (A3) is assured by (A3)$'$. Actually by (A3)$'$, we observe that the graph of $\mathcal{B}$ is linear. Moreover by Proposition 1.1 of [14], $\mathcal{B}$ is demiclosed in $L^q(0, T; V) \times L^{q'}(0, T; V^*)$; hence by Mazur's lemma, $\mathcal{B}$ becomes a weakly closed mapping from $L^q(0, T; V)$ into $L^{q'}(0, T; V^*)$.

As for φ^t, we assume the following condition.

(A4) There exist a Banach space X and a non-decreasing function ℓ defined on $\mathbf{R}$ such that X is compactly embedded in V and $|u|_X \leq \ell(|\varphi^t(u)| + |u|_H)$ for all $u \in D(\partial\varphi^t)$ and $t \in [0, T]$.

From now on, we write $\{\varphi^t\}_{t\in[0,T]} \in \Phi(V, [0, T]; \alpha, \beta, C_0, r)$ for some functions $\alpha, \beta : [0, T] \to \mathbf{R}$ and numbers $C_0 \in \mathbf{R}$, $r \in (1, +\infty)$ if the following (1) and (2) hold true.

(1) $\varphi^t \in \Phi(V)$ for all $t \in [0, T]$.

(2) $\exists\delta > 0,\ \forall t_0 \in [0, T],\ \forall u_0 \in D(\varphi^{t_0}),\ \exists u : I_\delta(t_0) := [t_0 - \delta, t_0 + \delta] \cap [0, T] \to V;$

$$\begin{aligned} |u(t) - u_0|_V &\leq |\alpha(t) - \alpha(t_0)|\{|\varphi^{t_0}(u_0)| + C_0\}^{1/r}, \\ \varphi^t(u(t)) &\leq \varphi^{t_0}(u_0) + |\beta(t) - \beta(t_0)|\{|\varphi^{t_0}(u_0)| + C_0\} \quad \forall t \in I_\delta(t_0). \end{aligned}$$

We introduce the following condition to describe the t-smoothness of functionals.

(A5) There exist functions $\alpha_1 \in W^{1,\rho}(0, T)$ with $\rho = \max\{q, 2\}$, $\alpha_2 \in W^{1,r}(0, T)$, $\beta_i \in W^{1,1}(0, T)$ $(i = 1, 2)$ and constants $C_0 \in \mathbf{R}$, $r \in (1, +\infty)$ such that $\{\varphi^t\}_{t\in[0,T]} \in \Phi(V, [0, T]; \alpha_1, \beta_1, C_0, \rho)$ and $\{\psi^t\}_{t\in[0,T]} \in \Phi(V, [0, T]; \alpha_2, \beta_2, C_0, r)$.

In order to derive the uniqueness of solutions for (CP), we need to assume the following (A5)$'$.

(A5)$'$ There exist functions $\alpha_2, \beta_2 \in W^{1,1}(0, T)$ such that $\{\psi^t\}_{t\in[0,T]} \in \Phi(V, [0, T]; \alpha_2, \beta_2, 0, q)$.

Here and henceforth, we are concerned with strong solutions of (CP) in the following sense.

Definition 2.2 *A function* $u \in C([0,T];V)$ *is said to be a strong solution of* (CP), *if the following conditions are satisfied*:

(a) u *is a* V*-valued absolutely continuous function on* $[0,T]$.

(b) $u(0) = u_0$.

(c) $u(t) \in D(\partial\varphi^t)$, $du(t)/dt \in D(\partial\psi^t)$ *for a.e.* $t \in (0,T)$ *and there exist sections* $g(t) \in \partial\varphi^t(u(t))$ *and* $\eta(t) \in \partial\psi^t(du(t)/dt)$ *such that*

$$\eta(t) + g(t) = f(t) \quad \text{in } V^* \text{ for a.e. } t \in (0,T). \tag{2. 3}$$

(d) *The function* $t \mapsto \varphi^t(u(t))$ *is differentiable for a.e.* $t \in (0,T)$ *and the function* $t \mapsto \psi^t(du(t)/dt)$ *is integrable on* $(0,T)$.

Our main theorem in this section is as follows.

Theorem 2.3 *Let* $q \in (1,+\infty)$ *and suppose that* (A1)-(A5) *are all satisfied. Then for all* $f \in L^{q'}(0,T;V^*)$ *and* $u_0 \in D(\varphi^0)$, (CP) *has at least one strong solution* u *on* $[0,T]$ *satisfying*

$$u \in W^{1,q}(0,T;V), \quad g, \eta \in L^{q'}(0,T;V^*), \tag{2. 4}$$

where $g(t)$ *and* $\eta(t)$ *denote the sections of* $\partial\varphi^t(u(t))$ *and* $\partial\psi^t(du(t)/dt)$ *in* (2. 3) *respectively.*

In particular, suppose that (A1) *with* $C_1 = 0$, (A2) *with* $a \equiv 0$, (A3)$'$ *and* (A5)$'$ *are satisfied and* $\psi^0(0) = 0$. *Then the solution is unique.*

Throughout the present paper, we denote by C non-negative constants, which do not depend on the elements of the corresponding space or set.

2.1 Basic Lemmas

In this subsection, we summarize the relevant materials on subdifferentials which will be used later without their proofs. Let F_E be the duality mapping from a real reflexive Banach space E onto its dual space E^*. Then we can assume that the duality mapping F_E is single-valued without any loss of generality (see [4]). To begin with, we give a definition of the resolvent and the Yosida approximation of a (possibly multi-valued) maximal monotone operator $A : E \to 2^{E^*}$.

Definition 2.4 *Let* A *be a maximal monotone operator from* E *into* E^*. *Then the resolvent* $J_{E,\lambda} : E \to D(A)$ *of* A *is given by* $J_{E,\lambda}u := v_\lambda$ *for all* $u \in E$, *where* v_λ *is a unique solution of the following inclusion*:

$$F_E(v_\lambda - u) + \lambda A v_\lambda \ni 0. \tag{2. 5}$$

Moreover the Yosida approximation $A_\lambda : E \to E^*$ *of* A *is given by*

$$A_\lambda u \ := \ \frac{F_E(u - v_\lambda)}{\lambda}. \tag{2. 6}$$

In particular, if E *is a Hilbert space and* E *is identified with its dual space* E^*, *then we can take* $F_E = I$ *in* (2. 5) *and* (2. 6), *where* I *denotes the identity in* E.

For any $\phi \in \Phi(E)$, it is well known that $\partial_E \phi$ becomes a maximal monotone operator from E into E^*. Moreover the Yosida approximation $(\partial_E \phi)_\lambda$ of $\partial_E \phi$ coincides with the subdifferential of the *Moreau-Yosida regularization* ϕ_λ of ϕ given by

$$\phi_\lambda(u) := \inf_{v \in E} \left\{ \frac{1}{2\lambda} |u - v|_E^2 + \phi(v) \right\} \quad \forall \lambda > 0, \ \forall u \in E. \tag{2. 7}$$

More precisely, the following lemma holds.

Lemma 2.5 *Let $\phi \in \Phi(E)$. Then ϕ_λ is a Gâteaux differentiable convex functional from E into $\mathbf{R}$ (in particular, if E is a Hilbert space, then ϕ_λ becomes Fréchet differentiable). Moreover the infimum in* (2. 7) *is attained by the $J_{E,\lambda}^{\phi} u$, where $J_{E,\lambda}^{\phi}$ denotes the resolvent of $\partial_E \phi$, i.e.,*

$$\begin{aligned} \phi_\lambda(u) &= \frac{1}{2\lambda} |u - J_{E,\lambda}^{\phi} u|_E^2 + \phi(J_{E,\lambda}^{\phi} u) \\ &= \frac{\lambda}{2} |(\partial_E \phi)_\lambda(u)|_{E^*}^2 + \phi(J_{E,\lambda}^{\phi} u). \end{aligned}$$

Furthermore the following (1)-(3) *hold.*

(1) $\partial_E(\phi_\lambda) = (\partial_E \phi)_\lambda$, *where $\partial_E(\phi_\lambda)$ is the subdifferential (Gâteaux derivative) of ϕ_λ.*

(2) $\phi(J_{E,\lambda}^{\phi} u) \le \phi_\lambda(u) \le \phi(u)$ *for all $u \in E$ and $\lambda > 0$.*

(3) $\phi_\lambda(u) \to \phi(u)$ *as $\lambda \to +0$ for all $u \in E$.*

The theory of subdifferentials of time-dependent functionals in the Hilbert space setting has been studied in detail by many authors (see e.g. [15], [19] and [26]). In the following four lemmas, we denote by H a real Hilbert space, which is identified with its dual space; moreover we suppose that $\{\phi^t\}_{t \in [0,T]} \in \Phi(H, [0, T]; \alpha, \beta, C_0, q)$ for some $C_0 \in \mathbf{R}$, $q \in (1, +\infty)$, $\alpha \in W^{1,q}(0, T)$ and $\beta \in W^{1,1}(0, T)$.

The following lemma can be found in Kenmochi [14, Lemma 3.2].

Lemma 2.6 *There exists a constant C such that*

$$\phi^t(u) \ \ge \ -C(|u|_H + 1) \quad \forall u \in H, \ \forall t \in [0, T]. \tag{2. 8}$$

The following lemma is concerned with fundamental estimates relative to the resolvent of $\partial_H \phi^t$ and the Moreau-Yosida regularization of ϕ^t.

Lemma 2.7 *The resolvent $J_{H,\lambda}^t$ of $\partial_H \phi^t$ is a non-expansive mapping in H, i.e.,*

$$|J_{H,\lambda}^t u - J_{H,\lambda}^t v|_H \le |u - v|_H \quad \forall u, v \in H.$$

Moreover it follows that $\partial_H \phi_\lambda^t(u) = (u - J_{H,\lambda}^t u)/\lambda$.

Furthermore there exists a constant C such that

$$\phi_\lambda^t(u) \ \le \ \frac{C}{\lambda}(|u|_H^2 + 1) \quad \forall u \in H, \ \forall \lambda \in (0, 1], \ \forall t \in [0, T], \tag{2. 9}$$

$$|J_{H,\lambda}^t u|_H \ \le \ C\,(|u|_H + 1) \quad \forall u \in H, \ \forall \lambda \in (0, 1], \ \forall t \in [0, T], \tag{2. 10}$$

$$|\phi_\lambda^t(u)| \ \le \ \phi_\lambda^t(u) + C(|u|_H + 1) \quad \forall u \in H, \ \forall \lambda \in (0, 1], \ \forall t \in [0, T], \tag{2. 11}$$

$$|\phi^t(J_{H,\lambda}^t u)| \ \le \ \phi_\lambda^t(u) + C\,(|u|_H + 1) \quad \forall u \in H, \ \forall \lambda \in (0, 1], \ \forall t \in [0, T]. \tag{2. 12}$$

Proof of Lemma 2.7 As for the first two fundamental properties of $J^t_{H,\lambda}$ and $\partial_H\phi^t_\lambda$, we refer to [11] and [15]. From the fact that $\{\phi^t\}_{t\in[0,T]} \in \Phi(H,[0,T];\alpha,\beta,C_0,q)$, we can construct a function $w:[0,T]\to H$ such that for a suitable number $r_0>0$ (see Kenmochi [15, Lemma 1.5.1]),

$$|w(t)|_H \le r_0, \quad \phi^t(w(t)) \le r_0 \quad \forall t\in[0,T].$$

Then by (2. 7), we have

$$\phi^t_\lambda(u) \ \le\ \frac{1}{2\lambda}|u-w(t)|^2_H + \phi^t(w(t)) \le \frac{C}{\lambda}(|u|^2_H+1)$$

for all $u\in H$ and $\lambda\in(0,1]$, which implies (2. 9).

The proof of (2. 10) can be found in Lemma 1.2.1 of [15].

For the case where $\phi^t_\lambda(u)\ge 0$, it is obvious that $\phi^t_\lambda(u)=|\phi^t_\lambda(u)|$; for the case where $\phi^t_\lambda(u)<0$, it follows from (2. 8) that

$$\begin{aligned}\frac{1}{2}\phi^t_\lambda(u)-\frac{1}{2}|\phi^t_\lambda(u)| \ &=\ \phi^t_\lambda(u)\\ &\ge\ \phi^t(J^t_{H,\lambda}u) \ge -C\left(|J^t_{H,\lambda}u|_H+1\right),\end{aligned}$$

which together with (2. 10) implies (2. 11). Moreover repeating the same argument above, we can also verify (2. 12). ∎

The following lemma can be found also in Kenmochi [15, Lemma 1.2.2, 1.2.3].

Lemma 2.8 *For all $u\in L^1(0,T;H)$, the functions $t\mapsto J^t_{H,\lambda}u(t)$ and $t\mapsto \partial_H\phi^t_\lambda(u(t))$ belong to $L^1(0,T;H)$. Moreover the functions $t\mapsto\phi^t_\lambda(u(t))$ and $t\mapsto\phi^t(u(t))$ are measurable on $(0,T)$.*

The next lemma can be proved by the same arguments as in the proof of Proposition 3.1 of [26] with slight modifications.

Lemma 2.9 *There exists a positive number δ such that for all $u\in H$ and $s,t\in[0,T]$ with $|t-s|<\delta$, it follows that*

$$\begin{aligned}&\left|J^t_{H,\lambda}u - J^s_{H,\lambda}u\right|^2_H\\ &\quad\le\ |\alpha(t)-\alpha(s)|\left\{|u-J^t_{H,\lambda}u|_H+|u-J^s_{H,\lambda}u|_H\right\}\\ &\qquad\times\left[\{|\phi^t(J^t_{H,\lambda}u)|+C_0\}^{1/q}+\{|\phi^s(J^s_{H,\lambda}u)|+C_0\}^{1/q}\right]\\ &\qquad+\lambda|\beta(t)-\beta(s)|\left\{|\phi^t(J^t_{H,\lambda}u)|+|\phi^s(J^s_{H,\lambda}u)|+2C_0\right\}.\end{aligned}\tag{2. 13}$$

The following two lemmas are concerned with the differentiability and chain rules for time-dependent functionals in the reflexive Banach space E; they can be verified by modifying the proof of Lemma 2.4 and Proposition 2.6 of [19]. In the rest of this subsection, we suppose that $\{\phi^t\}_{t\in[0,T]}\in\Phi(E,[0,T];\alpha,\beta,C_0,q)$ for some $C_0\in\mathbf{R}$, $q\in(1,+\infty)$, $\alpha\in W^{1,q}(0,T)$ and $\beta\in W^{1,1}(0,T)$.

Lemma 2.10 *Let u be an E-valued absolutely continuous function on $[0,T]$ such that the function $t \mapsto \phi^t(u(t))$ is differentiable and $u(t) \in D(\partial_E \phi^t)$ for a.e. $t \in (0,T)$. Then we have*

$$(2.\ 14) \qquad \begin{aligned} &\left| \frac{d}{dt}\phi^t(u(t)) - \left\langle g(t), \frac{du}{dt}(t) \right\rangle \right| \\ &\quad \leq \ |\dot{\alpha}(t)||g(t)|_{E^*}\{|\phi^t(u(t))| + C_0\}^{1/q} + |\dot{\beta}(t)| \left\{ |\phi^t(u(t))| + C_0 \right\}, \\ &\qquad \forall g \in L^1(0,T;E^*) \text{ satisfying } g(t) \in \partial_E \phi^t(u(t)) \text{ for a.e. } t \in (0,T), \end{aligned}$$

where $\dot{\alpha}(t) := d\alpha(t)/dt$ and $\dot{\beta}(t) := d\beta(t)/dt$.

Lemma 2.11 *Let $u \in W^{1,q}(0,T;E)$ and let $g \in L^{q'}(0,T;E^*)$ be such that $u(t) \in D(\partial_E \phi^t)$ and $g(t) \in \partial_E \phi^t(u(t))$ for a.e. $t \in (0,T)$. Then $u(t) \in D(\phi^t)$ for all $t \in (0,T]$ and the function $t \mapsto \phi^t(u(t))$ is differentiable for a.e. $t \in (0,T)$. Moreover the function $t \mapsto d\phi^t(u(t))/dt$ is integrable on $(0,T)$.*

Now suppose that E and H satisfy (2. 1) with V replaced by E. We then define the extension $\phi_H : H \to (-\infty, +\infty]$ of $\phi \in \Phi(E)$ on H as follows.

$$(2.\ 15) \qquad \phi_H(u) := \begin{cases} \phi(u) & \text{if } u \in E, \\ +\infty & \text{if } u \in H \setminus E. \end{cases}$$

It is obvious that ϕ_H is convex and proper in H. Moreover if ϕ possesses some suitable property, e.g., coerciveness, then ϕ_H becomes lower semicontinuous in H.

Now we suppose that (A4) is satisfied with φ^t and V replaced by ϕ^t and E respectively. Then we can verify that ϕ^t_H becomes lower semicontinuous in H for every $t \in [0,T]$ (see [1]). Moreover it is easily seen that $\{\phi^t_H\}_{t\in[0,T]} \in \Phi(H,[0,T];\alpha,\beta,C_0,q)$, $D(\phi^t_H) = D(\phi^t)$, $D(\partial_H \phi^t_H) \subset D(\partial_E \phi^t)$ and $\partial_H \phi^t_H(u) \subset \partial_E \phi^t(u)$ for all $u \in D(\partial_H \phi^t_H)$ and $t \in [0,T]$.

The next lemma gives an information on the chain rule of the Moreau-Yosida regularization $\phi^t_{H,\lambda}$ of ϕ^t_H, which can be proved by the same arguments with obvious modifications as in the proof of Lemma 2.5 of [19].

Lemma 2.12 *Let u be an H-valued absolutely continuous function on $[0,T]$ such that $u(t) \in E$ for all $t \in [0,T]$. Then the function $t \mapsto \phi^t_{H,\lambda}(u(t))$ is differentiable for a.e. $t \in (0,T)$ and its derivative is integrable on $(0,T)$. Moreover the following inequality holds true.*

$$(2.\ 16) \qquad \begin{aligned} &\left| \frac{d}{dt}\phi^t_{H,\lambda}(u(t)) - \left(\partial_H \phi^t_{H,\lambda}(u(t)), \frac{du}{dt}(t) \right)_H \right| \\ &\quad \leq \ |\dot{\alpha}(t)||\partial_H \phi^t_{H,\lambda}(u(t))|_{E^*}\{|\phi^t_{H,\lambda}(J^t_{H,\lambda}u(t))| + C_0\}^{1/q} \\ &\qquad + |\dot{\beta}(t)|\{|\phi^t_{H,\lambda}(J^t_{H,\lambda}u(t))| + C_0\} \quad \text{for a.e. } t \in (0,T), \end{aligned}$$

where $J^t_{H,\lambda}$ denotes the resolvent of $\partial_H \phi^t_H$.

2.2 Proof of Theorem 2.3

The proof is divided into 4 steps.

Approximation: To construct a strong solution of (CP), we introduce the following approximate problems for (CP) in the Hilbert space H.

$$
(\mathrm{CP})_\lambda \begin{cases} \lambda \dfrac{du_\lambda}{dt}(t) + \eta_\lambda(t) + \partial_H \varphi^t_{H,\lambda}(u_\lambda(t)) = f_\lambda(t) \ \text{ in } H, \quad 0 < t < T, \\ \eta_\lambda(t) \in \partial_H \psi^t_H \left(\dfrac{du_\lambda}{dt}(t) \right), \quad u_\lambda(0) = u_0, \end{cases}
$$

where ψ^t_H (resp. φ^t_H) denotes the extension of ψ^t (resp. φ^t) on H defined as in (2. 15) and (f_λ) denotes a sequence in $C([0,T];H)$ such that $f_\lambda \to f$ strongly in $L^{q'}(0,T;V^*)$ as $\lambda \to +0$. Moreover $\partial_H \varphi^t_{H,\lambda}$ denotes the Yosida approximation of $\partial_H \varphi^t_H$.

Since $(\lambda I + \partial_H \psi^t_H)^{-1}$ and $\partial_H \varphi^t_{H,\lambda}$ become Lipschitz continuous in H with Lipschitz constants $1/\lambda$ and $2/\lambda$ respectively, the mapping $u \mapsto A(t,u) := (\lambda I + \partial_H \psi^t_H)^{-1} \{f_\lambda(t) - \partial_H \varphi^t_{H,\lambda}(u)\}$ becomes Lipschitz continuous with respect to u for each $t \in [0,T]$, where I denotes the identity in H. Moreover from the fact that $(\lambda I + \partial_H \psi^t_H)^{-1} u = J^{\psi^t}_{H,1/\lambda}(u/\lambda)$, we have

$$
A(t,u) \ = \ J^{\psi^t}_{H,1/\lambda} \left(\frac{1}{\lambda} \left\{ f_\lambda(t) - \frac{u - J^t_{H,\lambda} u}{\lambda} \right\} \right),
$$

where $J^{\psi^t}_{H,\lambda}$ and $J^t_{H,\lambda}$ denote the resolvents of $\partial_H \psi^t_H$ and $\partial_H \varphi^t_H$ respectively. Therefore $(\mathrm{CP})_\lambda$ is equivalent to the following:

$$
\begin{cases} \dfrac{du_\lambda}{dt}(t) = A(t, u_\lambda(t)) \ \text{ in } H, \quad 0 < t < T, \\ u_\lambda(0) = u_0. \end{cases}
$$

Now by Lemmas 2.7 and 2.9, we note that

$$
|\partial_H \varphi^t_{H,\lambda}(u)|_H \ = \ \left| \frac{u - J^t_{H,\lambda} u}{\lambda} \right|_H \le \frac{C}{\lambda} \left(|u|_H + 1 \right)
$$

and

$$
\begin{aligned}
&|A(t,u) - A(s,u)|_H \\
&\quad \le \left| J^{\psi^t}_{H,1/\lambda} \left(\frac{f_\lambda(t) - \partial_H \varphi^t_{H,\lambda}(u)}{\lambda} \right) - J^{\psi^s}_{H,1/\lambda} \left(\frac{f_\lambda(t) - \partial_H \varphi^t_{H,\lambda}(u)}{\lambda} \right) \right|_H \\
&\qquad + \frac{|f_\lambda(t) - f_\lambda(s)|_H}{\lambda} + \frac{|J^t_{H,\lambda} u - J^s_{H,\lambda} u|_H}{\lambda^2} \\
&\quad \le C_\lambda \sum_{i=1,2} \{ |\alpha_i(t) - \alpha_i(s)| + |\beta_i(t) - \beta_i(s)| \}^{1/2} + \frac{|f_\lambda(t) - f_\lambda(s)|_H}{\lambda},
\end{aligned}
$$

where C_λ denotes a constant depending on $\lambda, |u|_H$ and $\sup_{t \in [0,T]} |f_\lambda(t)|_H$. Therefore the function $t \mapsto A(t,u)$ belongs to $C([0,T];H)$ for each $u \in H$. Consequently Theorem 1.4 of [9] assures the existence of a unique strong solution u_λ for $(\mathrm{CP})_\lambda$ satisfying $u_\lambda \in C^1([0,T];H)$ and $du_\lambda(t)/dt \in D(\partial_H \psi_H^t)$ for all $t \in [0,T]$. Moreover by Lemma 2.12, the function $t \mapsto \varphi_{H,\lambda}^t(u_\lambda(t))$ is differentiable for a.e. $t \in (0,T)$ and $\sup_{t\in[0,T]} \varphi_{H,\lambda}^t(u_\lambda(t)) < +\infty$. Now letting $g_\lambda(t) := \partial_H \varphi_{H,\lambda}^t(u_\lambda(t))$, we find, by Lemma 2.7,

$$
\begin{aligned}
|g_\lambda(t) - g_\lambda(s)|_H &\le \frac{1}{\lambda}\left(|u_\lambda(t) - u_\lambda(s)|_H + |J_{H,\lambda}^t u_\lambda(t) - J_{H,\lambda}^s u_\lambda(s)|_H\right) \\
&\le \frac{1}{\lambda}\left(2|u_\lambda(t) - u_\lambda(s)|_H + |J_{H,\lambda}^t u_\lambda(t) - J_{H,\lambda}^s u_\lambda(t)|_H\right).
\end{aligned}
$$

Hence by Lemmas 2.7 and 2.9, g_λ belongs to $C([0,T];H)$; therefore $(\mathrm{CP})_\lambda$ implies $\eta_\lambda \in C([0,T];H)$.

We now claim that

$$
\psi^t(u) \le \langle \eta, u\rangle + C(|\eta|_{V^*} + 1) \quad \forall [u,\eta] \in \partial\psi^t,\ \forall t \in [0,T]. \tag{2. 17}
$$

Indeed by (A5), we can choose a function $w : [0,T] \to V$ and a constant r_0 such that $|w(t)|_V \le r_0$ and $\psi^t(w(t)) \le r_0$ for all $t \in [0,T]$. Then by the definition of $\partial\psi^t$, we have

$$
\begin{aligned}
\psi^t(u) &\le \psi^t(w(t)) + \langle \eta, u - w(t)\rangle \\
&\le \langle \eta, u\rangle + r_0(|\eta|_{V^*} + 1) \quad \forall [u,\eta] \in \partial\psi^t,\ \forall t \in [0,T].
\end{aligned}
$$

Therefore putting $u = du_\lambda(t)/dt$ and $\eta = \eta_\lambda(t)$ in (2. 17), we have $\sup_{t\in[0,T]} \psi^t(du_\lambda(t)/dt) < +\infty$, which together with (A1) implies $\sup_{t\in[0,T]} |du_\lambda(t)/dt|_V < +\infty$. Hence combining this with the fact that $du_\lambda/dt \in C([0,T];H)$, we conclude that $du_\lambda/dt \in C_w([0,T];V)$, where $C_w([0,T];V)$ denotes the set of all weakly continuous functions from $[0,T]$ into V.

A priori estimates: We first establish the following a priori estimates for $u_\lambda(t)$.

Lemma 2.13 *There exists a constant M_1 such that*

$$
\sup_{t\in[0,T]} |\varphi_{H,\lambda}^t(u_\lambda(t))| \le M_1, \tag{2. 18}
$$

$$
\sup_{t\in[0,T]} |u_\lambda(t)|_V \le M_1, \tag{2. 19}
$$

$$
\lambda \int_0^T \left|\frac{du_\lambda}{dt}(t)\right|_H^2 dt \le M_1, \tag{2. 20}
$$

$$
\int_0^T \left|\frac{du_\lambda}{dt}(t)\right|_V^q dt \le M_1, \tag{2. 21}
$$

$$
\int_0^T \psi^t\left(\frac{du_\lambda}{dt}(t)\right) dt \le M_1 \tag{2. 22}
$$

for all $\lambda \in (0,1]$.

Proof of Lemma 2.13 Multiplying $(\mathrm{CP})_\lambda$ by $du_\lambda(t)/dt$ and recalling (2. 17), we get

$$
\begin{aligned}
&\lambda\left|\frac{du_\lambda}{dt}(t)\right|_H^2 + \psi^t\left(\frac{du_\lambda}{dt}(t)\right) + \left\langle g_\lambda(t), \frac{du_\lambda}{dt}(t)\right\rangle \\
&\quad\le \left\langle f_\lambda(t), \frac{du_\lambda}{dt}(t)\right\rangle + C\left(|\eta_\lambda(t)|_{V^*} + 1\right)
\end{aligned} \tag{2. 23}
$$

for a.e. $t \in (0,T)$, where $g_\lambda(t) := \partial_H \varphi^t_{H,\lambda}(u_\lambda(t))$. Now by (A5) and Lemma 2.12, it follows that

$$
(2.\ 24) \qquad \begin{aligned} &\left| \frac{d}{dt}\varphi^t_{H,\lambda}(u_\lambda(t)) - \left\langle g_\lambda(t), \frac{du_\lambda}{dt}(t) \right\rangle \right| \\ &\quad \le |\dot\alpha_1(t)||g_\lambda(t)|_{V^*} \left\{ |\varphi^t_{H,\lambda}(J^t_{H,\lambda}u_\lambda(t))| + C_0 \right\}^{1/\rho} \\ &\qquad + |\dot\beta_1(t)| \left\{ |\varphi^t_{H,\lambda}(J^t_{H,\lambda}u_\lambda(t))| + C_0 \right\}. \end{aligned}
$$

Moreover by (2. 12) in Lemma 2.7, we notice that

$$
(2.\ 25) \qquad \begin{aligned} |\varphi^t_H(J^t_{H,\lambda}u_\lambda(t))| &\le \varphi^t_{H,\lambda}(u_\lambda(t)) + C(|u_\lambda(t)|_H + 1) \\ &\le |\varphi^t_{H,\lambda}(u_\lambda(t))| + |u_\lambda(t)|^q_V + C, \end{aligned}
$$

where we note that V is continuously embedded in H. Hence combining these inequalities, we get by (A1) and $(\mathrm{CP})_\lambda$,

$$
(2.\ 26) \qquad \begin{aligned} &\lambda \left| \frac{du_\lambda}{dt}(t) \right|^2_H + \frac{\alpha}{2}\left| \frac{du_\lambda}{dt}(t) \right|^q_V + \frac{1}{2}\psi^t\left(\frac{du_\lambda}{dt}(t) \right) + \frac{d}{dt}\varphi^t_{H,\lambda}(u_\lambda(t)) \\ &\le |\dot\alpha_1(t)||g_\lambda(t)|_{V^*} \left\{ |\varphi^t_{H,\lambda}(u_\lambda(t))| + |u_\lambda(t)|^q_V + C \right\}^{1/\rho} \\ &\quad + |\dot\beta_1(t)| \left\{ |\varphi^t_{H,\lambda}(u_\lambda(t))| + |u_\lambda(t)|^q_V + C \right\} \\ &\qquad + |f_\lambda(t)|_{V^*} \left| \frac{du_\lambda}{dt}(t) \right|_V + C\left(|\eta_\lambda(t)|_{V^*} + 1 \right) + \frac{C_1}{2} \\ &\le |\dot\alpha_1(t)| \left\{ |f_\lambda(t)|_{V^*} + \lambda \left| \frac{du_\lambda}{dt}(t) \right|_{V^*} + |\eta_\lambda(t)|_{V^*} \right\} \\ &\qquad \times \left\{ |\varphi^t_{H,\lambda}(u_\lambda(t))| + |u_\lambda(t)|^q_V + C \right\}^{1/\rho} \\ &\quad + |\dot\beta_1(t)| \left\{ |\varphi^t_{H,\lambda}(u_\lambda(t))| + |u_\lambda(t)|^q_V + C \right\} + |f_\lambda(t)|_{V^*} \left| \frac{du_\lambda}{dt}(t) \right|_V \\ &\quad + C\left(|\eta_\lambda(t)|_{V^*} + 1 \right) + \frac{C_1}{2}. \end{aligned}
$$

Therefore by Young's inequality, since $\rho = \max\{q, 2\}$, it follows from (A2) that

$$
(2.\ 27) \qquad \begin{aligned} &\frac{\lambda}{2} \left| \frac{du_\lambda}{dt}(t) \right|^2_H + \frac{\alpha}{2}\left| \frac{du_\lambda}{dt}(t) \right|^q_V + \frac{1}{4}\psi^t\left(\frac{du_\lambda}{dt}(t) \right) + \frac{d}{dt}\varphi^t_{H,\lambda}(u_\lambda(t)) \\ &\le C\left\{ |\dot\alpha_1(t)|^\rho + |\dot\beta_1(t)| + |f_\lambda(t)|^{q'}_{V^*} + |a(t)| + 1 \right\} \\ &\quad + C\left\{ |\dot\alpha_1(t)|^\rho + |\dot\beta_1(t)| \right\} \left\{ |\varphi^t_{H,\lambda}(u_\lambda(t))| + |u_\lambda(t)|^q_V \right\}. \end{aligned}
$$

We here prepare the following three inequalities; the first one is derived immediately from Lemma 2.6.

$$
\begin{aligned} \psi^t\left(\frac{du_\lambda}{dt}(t) \right) = \psi^t_H\left(\frac{du_\lambda}{dt}(t) \right) &\ge -C\left(\left| \frac{du_\lambda}{dt}(t) \right|_H + 1 \right) \\ &\ge -\alpha \left| \frac{du_\lambda}{dt}(t) \right|^q_V - C. \end{aligned}
$$

The second one is as follows.

$$
\begin{aligned}
(2.\ 28) \qquad 2\frac{d}{dt}|u_\lambda(t)|_V^q &= 2q|u_\lambda(t)|_V^{q-1}\frac{d}{dt}|u_\lambda(t)|_V \\
&\le 2q|u_\lambda(t)|_V^{q-1}\left|\frac{du_\lambda}{dt}(t)\right|_V \\
&\le C|u_\lambda(t)|_V^q + \frac{\alpha}{4}\left|\frac{du_\lambda}{dt}(t)\right|_V^q .
\end{aligned}
$$

Furthermore (2. 11) in Lemma 2.7 implies

$$
\begin{aligned}
|\varphi_{H,\lambda}^t(u_\lambda(t))| &\le \varphi_{H,\lambda}^t(u_\lambda(t)) + C(|u_\lambda(t)|_H + 1) \\
&\le \varphi_{H,\lambda}^t(u_\lambda(t)) + |u_\lambda(t)|_V^q + C.
\end{aligned}
$$

Now integrating (2. 27) over $(0,t)$ and combining the inequalities stated above, we have

$$
\begin{aligned}
(2.\ 29) \qquad & \lambda\int_0^t \left|\frac{du_\lambda}{d\tau}(\tau)\right|_H^2 d\tau + |u_\lambda(t)|_V^q + |\varphi_{H,\lambda}^t(u_\lambda(t))| \\
&\le 2|u_0|_V^q + \varphi^0(u_0) + C\left\{\int_0^T |\dot\alpha_1(\tau)|^\rho d\tau + \int_0^T |\dot\beta_1(\tau)|d\tau \right. \\
&\quad \left. + \int_0^T |f_\lambda(\tau)|_{V^*}^{q'} d\tau + \int_0^T |a(\tau)|d\tau + 1\right\} \\
&\quad + C\int_0^t \left\{|\dot\alpha_1(\tau)|^\rho + |\dot\beta_1(\tau)| + 1\right\}\left\{|\varphi_{H,\lambda}^\tau(u_\lambda(\tau))| + |u_\lambda(\tau)|_V^q\right\} d\tau
\end{aligned}
$$

for all $t \in [0,T]$. Thus Gronwall's inequality implies

$$
(2.\ 30) \qquad \sup_{t\in[0,T]} \left\{|u_\lambda(t)|_V^q + |\varphi_{H,\lambda}^t(u_\lambda(t))|\right\} \le C,
$$

which yields (2. 18) and (2. 19). Moreover (2. 20) follows immediately from (2. 29) and (2. 30). Furthermore integrating both sides of (2. 27) over $(0,T)$, by (2. 30) we obtain (2. 21) and (2. 22). ∎

Lemma 2.14 *There exists a constant M_2 such that*

$$
(2.\ 31) \qquad \int_0^T |\eta_\lambda(t)|_{V^*}^{q'} dt \le M_2,
$$

$$
(2.\ 32) \qquad \int_0^T |g_\lambda(t)|_{V^*}^{\sigma} dt \le M_2
$$

for all $\lambda \in (0,1]$, where $\sigma = \min\{2, q'\}$.

Proof of Lemma 2.14 By (A2), we get

$$
\int_0^T |\eta_\lambda(t)|_{V^*}^{q'} dt \le C_2\int_0^T \psi^t\left(\frac{du_\lambda}{dt}(t)\right)dt + \int_0^T a(t)dt,
$$

which together with (2. 22) implies (2. 31).

Since $g_\lambda(t) = f_\lambda(t) - \lambda(du_\lambda(t)/dt) - \eta_\lambda(t)$ and (f_λ) is bounded in $L^{q'}(0,T;V^*)$, (2. 20) and (2. 31) yield (2. 32). ∎

In the following lemma, we provide a priori estimates for $J_{H,\lambda}^t u_\lambda(t)$.

Lemma 2.15 *There exists a constant M_3 such that*

$$\sup_{t\in[0,T]} |J^t_{H,\lambda}u_\lambda(t)|_H \leq M_3, \tag{2. 33}$$

$$\sup_{t\in[0,T]} |\varphi^t(J^t_{H,\lambda}u_\lambda(t))| \leq M_3, \tag{2. 34}$$

$$\sup_{t\in[0,T]} |J^t_{H,\lambda}u_\lambda(t)|_X \leq M_3 \tag{2. 35}$$

for all $\lambda \in (0,1]$.

Proof of Lemma 2.15 By (2. 10) in Lemma 2.7, we see

$$|J^t_{H,\lambda}u_\lambda(t)|_H \leq C(|u_\lambda(t)|_H + 1) \quad \forall t \in [0,T].$$

Therefore it follows from (2. 19) that $\sup_{t\in[0,T]} |J^t_{H,\lambda}u_\lambda(t)|_H$ is bounded for all $\lambda \in (0,1]$. Moreover by (2. 18), (2. 19) and (2. 25), we get

$$\begin{aligned} \sup_{t\in[0,T]} |\varphi^t_H(J^t_{H,\lambda}u_\lambda(t))| &\leq \sup_{t\in[0,T]} \left\{ |\varphi^t_{H,\lambda}(u_\lambda(t))| + C(|u_\lambda(t)|_H + 1) \right\} \\ &\leq C. \end{aligned} \tag{2. 36}$$

Furthermore (2. 35) follows immediately from (A4), (2. 33) and (2. 34). ∎

Convergence: From a priori estimates established in Lemmas 2.13-2.15, we can take a sequence (λ_n) in $(0,1]$ such that $\lambda_n \to +0$ as $n \to +\infty$ and the following Lemmas 2.16 and 2.17 hold.

Lemma 2.16 *There exists $u \in W^{1,q}(0,T;V)$ such that*

$$u_{\lambda_n} \to u \quad \textit{weakly in } W^{1,q}(0,T;V), \tag{2. 37}$$

$$\lambda_n \frac{du_{\lambda_n}}{dt} \to 0 \quad \textit{strongly in } L^2(0,T;H), \tag{2. 38}$$

$$J^t_{H,\lambda_n}u_{\lambda_n} \to u \quad \textit{strongly in } C([0,T];V) \tag{2. 39}$$

as $\lambda_n \to 0$.

Proof of Lemma 2.16 We can derive (2. 37) and (2. 38) immediately from (2. 19)-(2. 21). Moreover by (2. 21) and Lemma 2.7, we can easily see

$$\begin{aligned} &|J^t_{H,\lambda}u_\lambda(t) - J^s_{H,\lambda}u_\lambda(s)|_H \\ &\leq |J^t_{H,\lambda}u_\lambda(t) - J^s_{H,\lambda}u_\lambda(t)|_H + |u_\lambda(t) - u_\lambda(s)|_H \\ &\leq |J^t_{H,\lambda}u_\lambda(t) - J^s_{H,\lambda}u_\lambda(t)|_H + C\left(\int_0^T \left|\frac{du_\lambda}{d\tau}(\tau)\right|_V^q d\tau\right)^{1/q} (t-s)^{1/q'} \\ &\leq |J^t_{H,\lambda}u_\lambda(t) - J^s_{H,\lambda}u_\lambda(t)|_H + CM_1^{1/q}(t-s)^{1/q'}. \end{aligned}$$

Hence by Lemmas 2.9, 2.13 and 2.15, the function $t \mapsto J^t_{H,\lambda}u_\lambda(t)$ is equi-continuous in $C([0,T];H)$ for all $\lambda \in (0,1]$. Therefore recalling (2. 35) and using the compact embedding $X \subset V$, by Theorem 5 of [25], we can assure that there exists $v \in C([0,T];V)$ such that

$$J^t_{H,\lambda_n}u_{\lambda_n} \to v \quad \text{strongly in } C([0,T];V). \tag{2. 40}$$

Now by (2. 32), we have

$$\begin{aligned}\int_0^T |u_{\lambda_n}(t) - J^t_{H,\lambda_n} u_{\lambda_n}(t)|^\sigma_{V^*} dt &= \lambda_n^\sigma \int_0^T |g_{\lambda_n}(t)|^\sigma_{V^*} dt \\ &\le \lambda_n^\sigma M_2 \to 0 \quad \text{as } \lambda_n \to 0.\end{aligned}$$

Hence it follows from (2. 37) and (2. 40) that $v = u$, which implies (2. 39). ∎

Lemma 2.17 *There exist $g, \eta \in L^{q'}(0,T;V^*)$ such that*

$$\begin{aligned} &(2.\ 41) & \eta_{\lambda_n} &\to \eta & &\textit{weakly in } L^{q'}(0,T;V^*), \\ &(2.\ 42) & g_{\lambda_n} &\to g & &\textit{weakly in } L^{\sigma}(0,T;V^*)\end{aligned}$$

as $\lambda_n \to 0$. Moreover we have

$$\begin{aligned} &(2.\ 43) & \eta(t) &\in \partial\psi^t\left(\frac{du}{dt}(t)\right), \\ &(2.\ 44) & g(t) &\in \partial\varphi^t(u(t)), \\ &(2.\ 45) & \eta(t) + g(t) &= f(t)\end{aligned}$$

for a.e. $t \in (0,T)$.

Proof of Lemma 2.17 It is easily seen that (2. 31) and (2. 32) imply (2. 41) and (2. 42) respectively. Moreover we can derive (2. 43) from (2. 37), (2. 41) and (A3). Furthermore by (2. 39) and (2. 42), the demiclosedness of subdifferentials and Proposition 1.1 of [14] imply (2. 44). Finally since $f_{\lambda_n} \to f$ strongly in $L^{q'}(0,T;V^*)$, it follows from $(\mathrm{CP})_\lambda$, (2. 38), (2. 41) and (2. 42) that $g = f - \eta \in L^{q'}(0,T;V^*)$. ∎

By Lemma 2.11, the function $t \mapsto \varphi^t(u(t))$ is differentiable for a.e. $t \in (0,T)$. Moreover it follows from (2. 22) and (2. 37) that

$$(2.\ 46) \qquad \int_0^T \psi^t\left(\frac{du}{dt}(t)\right) dt \le \liminf_{\lambda_n \to 0} \int_0^T \psi^t\left(\frac{du_{\lambda_n}}{dt}(t)\right) dt \le M_1,$$

which implies that the function $t \mapsto \psi^t(du(t)/dt)$ is integrable on $(0,T)$. Consequently u becomes a strong solution of (CP).

Uniqueness: Let u_1 and u_2 be strong solutions of (CP). Then there exist $g_i(t) \in \partial\varphi^t(u_i(t))$ and $\eta_i(t) \in \partial\psi^t(du_i(t)/dt)$ $(i = 1, 2)$ such that

$$(2.\ 47) \qquad \eta_1(t) - \eta_2(t) + g_1(t) - g_2(t) = 0 \quad \text{for a.e. } t \in (0,T).$$

Let $w(t) := u_1(t) - u_2(t)$ and let $\xi(t) := \eta_1(t) - \eta_2(t)$. Then multiplying (2. 47) by $w(t)$ and taking account of monotonicity of $\partial\varphi^t$, we get

$$(2.\ 48) \qquad \langle \xi(t), w(t) \rangle \le 0 \quad \text{for a.e. } t \in (0,T).$$

Now since the graph of $\partial\psi^t$ is linear, it follows that $\xi(t) \in \partial\psi^t(dw(t)/dt)$. Moreover we note that $w(t) \in D(\partial\psi^t)$ since $(\mathrm{A3})'$ implies $u_1(t), u_2(t) \in D(\partial\varphi^t) \subset D(\partial\psi^t)$. Therefore by $(\mathrm{A3})'$, we also deduce

$$(2.\ 49) \qquad \left\langle \zeta(t), \frac{dw}{dt}(t) \right\rangle \le 0 \quad \text{for a.e. } t \in (0,T),$$

where $\zeta(t) \in \partial\psi^t(w(t))$. Hence by (A2) with $a \equiv 0$ and (A5)$'$, Lemma 2.10 yields

$$\begin{aligned}\frac{d}{dt}\psi^t(w(t)) &\leq |\dot{\alpha}_2(t)||\zeta(t)|_{V^*}\psi^t(w(t))^{1/q} + |\dot{\beta}_2(t)|\psi^t(w(t)) \\ &\leq \left\{C_2^{1/q'}|\dot{\alpha}_2(t)| + |\dot{\beta}_2(t)|\right\}\psi^t(w(t)).\end{aligned}$$

Integrating this over $(0,t)$, we get

$$\psi^t(w(t)) \leq \psi^0(w(0)) + \int_0^t \left\{C_2^{1/q'}|\dot{\alpha}_2(\tau)| + |\dot{\beta}_2(\tau)|\right\}\psi^\tau(w(\tau))d\tau.$$

Thus Gronwall's inequality implies

$$\psi^t(w(t)) \leq \psi^0(w(0))\exp\left(\int_0^T \left\{C_2^{1/q'}|\dot{\alpha}_2(\tau)| + |\dot{\beta}_2(\tau)|\right\}d\tau\right),$$

which together with the fact that $\psi^0(w(0)) = \psi^0(0) = 0$ yields $\psi^t(w(t)) = 0$ for all $t \in [0,T]$. Consequently (A1) with $C_1 = 0$ implies $w(t) = 0$ for all $t \in [0,T]$, which completes the proof. ∎

3 Existence and Uniqueness of Solutions

In this section, the existence of a unique solution will be verified for $(\mathrm{P})_p$ and (P) by using the preceding abstract theory. We first give a definition of solutions for $(\mathrm{P})_p$ or (P) as follows.

Definition 3.1 *A function $u \in W^{1,2}(0,T;V_0)$ is said to be a solution of $(\mathrm{P})_p$ (resp. (P)), if the following conditions are all satisfied*:

$$\begin{aligned}&\text{(i)} \quad u(t) \in X_p \quad (\textit{resp. } u(t) \in K^t) \quad \textit{for a.e. } t \in (0,T),\\ &\text{(ii)} \quad u(t) \textit{ satisfies } (\mathrm{P})_p \quad (\textit{resp. } (\mathrm{P})) \textit{ for a.e. } t \in (0,T),\end{aligned}$$

where $K^t := \{u \in X_2;\ |\nabla u(x)| \leq \gamma(x,t) \ \textit{ for a.e. } x \in \Omega\}$.

Now we summarize useful properties of the bilinear form $b(\cdot,\cdot)$ (see [7] for more details).

(3. 1) $$b(u,v) = b(v,u) \quad \text{for all } u,v \in V_0,$$

(3. 2) $$C_3|u|_{V_0}^2 \leq b(u,u) \quad \text{for all } u \in V_0,$$

(3. 3) $$|b(u,v)| \leq C_4|u|_{V_0}|v|_{V_0} \quad \text{for all } u,v \in V_0,$$

where $C_i > 0$ $(i = 3,4)$.

In the following two subsections, we shall verify the existence of a unique solution for $(\mathrm{P})_p$ or (P) under the following assumptions on Ω, h_e and γ.

$$\text{(A)} \quad \begin{cases}\Omega \text{ is a bounded domain in } \mathbf{R}^2 \text{ with smooth boundary } \partial\Omega,\\ \dfrac{dh_e}{dt} \in L^2(0,T;V_0^*),\\ \gamma(x,t) = \pi(x)\phi(t),\ \pi \in L^\infty(\Omega),\ \phi \in W^{1,2}(0,T),\\ \gamma(x,t) \geq \delta_0 > 0 \quad \text{for a.e. } x \in \Omega \text{ and all } t \in [0,T].\end{cases}$$

3.1 Existence and Uniqueness of Solutions for $(\mathrm{P})_p$

Theorem 3.2 *Suppose that* (A) *is satisfied and let* $p \in [2, +\infty)$. *Then for any* $h_0 \in X_p$, $(\mathrm{P})_p$ *has a unique solution.*

Set $V = V_0$ and $H = L^2(\Omega)$. Then it is easily seen that V and H satisfy (2. 1). Moreover we define φ_p^t and ψ as follows:

$$(3.\ 4) \qquad \varphi_p^t(u) := \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega \left(\frac{|\nabla u(x)|}{\gamma(x,t)}\right)^p dx & \text{if } u \in X_p, \\ +\infty & \text{if } u \in V \setminus X_p, \end{cases}$$

$$(3.\ 5) \qquad \psi(u) := \frac{1}{2} b(u,u) \quad \text{for all } u \in V.$$

Then since $0 < \delta_0 \leq \gamma(x,t) \leq |\pi|_{L^\infty(\Omega)} |\phi|_{C([0,T])}$ for a.e. $x \in \Omega$ and all $t \in [0,T]$, it follows that $\varphi_p^t \in \Phi(V)$, $D(\partial \varphi_p^t) = X_p$ and $D(\partial\psi) = V$. Moreover $\partial\varphi_p^t(u)$ coincides with

$$-\nabla \cdot \left\{ \left(\frac{1}{\gamma(x,t)} \right)^p |\nabla u(x)|^{p-2} \nabla u(x) \right\}$$

equipped with the homogeneous Dirichlet boundary condition in the distribution sense. Furthermore we can easily verify

$$(3.\ 6) \qquad \langle \partial\psi(u), v \rangle = b(u,v) \quad \forall u, v \in V,$$

which together with (3. 3) implies

$$(3.\ 7) \qquad |\partial\psi(u)|_{V^*} \leq C_4 |u|_V \quad \forall u \in V.$$

Therefore $(\mathrm{P})_p$ is equivalent to the following Cauchy problem $(\mathrm{E})_p$.

$$(\mathrm{E})_p \begin{cases} \partial\psi\left(\dfrac{dh_p}{dt}(t)\right) + \partial\varphi_p^t(h_p(t)) = f(t) & \text{in } V^*, \quad 0 < t < T, \\ h_p(0) = h_0, \end{cases}$$

where $f(t) := -dh_e(t)/dt \in L^2(0,T;V^*)$.

By (3. 1)-(3. 3) and (3. 7), the next lemma follows immediately.

Lemma 3.3 (A1), (A2) *and* (A3)$'$ *hold true with* $q = 2$, $C_1 = 0$ *and* $a \equiv 0$.

Now we proceed to the proof of Theorem 3.2.

Proof of Theorem 3.2 We first check (A4). To do this, noting that

$$\frac{1}{p}\left(|\pi|_{L^\infty(\Omega)}|\phi|_{C([0,T])}\right)^{-p} \int_\Omega |\nabla u(x)|^p dx \leq \varphi_p^t(u) \quad \forall u \in D(\varphi_p^t),\ \forall t \in [0,T],$$

and that $W_0^{1,p}(\Omega)$ is compactly embedded in V, we can deduce that (A4) holds true with $X = W_0^{1,p}(\Omega)$ and $\varphi^t = \varphi_p^t$ respectively.

We next derive (A5) and (A5)$'$ from (3. 4) and (3. 5). Let $t_0 \in [0,T]$ and $u_0 \in D(\varphi_p^{t_0})$ be fixed. Moreover define the function $u : [0,T] \to V$ by

$$u(t) := \frac{\phi(t)}{\phi(t_0)} u_0 \quad \text{for all } t \in [0,T].$$

Since $\gamma(x,t) = \pi(x)\phi(t)$, we find

$$\nabla u(t) = \frac{\phi(t)}{\phi(t_0)} \nabla u_0 = \frac{\gamma(x,t)}{\gamma(x,t_0)} \nabla u_0, \tag{3. 8}$$

which implies $u(t) \in D(\varphi_p^t)$ for all $t \in [0,T]$. Hence since $|u|_V \le C|\nabla u|_H$ for all $u \in X_2$ and $p^{1/p} \le e^{1/e}$, it follows that

$$\begin{aligned}
|u(t) - u_0|_V &= \left| \frac{\phi(t)}{\phi(t_0)} - 1 \right| |u_0|_V \\
&\le \frac{C}{\delta_0} |\pi|_{L^\infty(\Omega)} |\phi(t) - \phi(t_0)| |\nabla u_0|_H \\
&\le \frac{C}{\delta_0} |\pi|^2_{L^\infty(\Omega)} |\phi|_{C([0,T])} |\phi(t) - \phi(t_0)| |\Omega|^{(p-2)/(2p)} \left\{ p\varphi_p^t(u_0) \right\}^{1/p} \\
&\le \frac{Ce^{1/e}}{\delta_0} |\pi|^2_{L^\infty(\Omega)} |\phi|_{C([0,T])} |\phi(t) - \phi(t_0)| (|\Omega| + 1)^{1/2} \\
&\quad \times \left\{ \varphi_p^t(u_0) + 1 \right\}^{1/2} \\
&\le |\alpha_0(t) - \alpha_0(t_0)| \left\{ \varphi_p^t(u_0) + 1 \right\}^{1/2},
\end{aligned} \tag{3. 9}$$

where α_0 is given by

$$\alpha_0(t) = \frac{Ce^{1/e}}{\delta_0} |\pi|^2_{L^\infty(\Omega)} |\phi|_{C([0,T])} (|\Omega| + 1)^{1/2} \phi(t) \in W^{1,2}(0,T).$$

Moreover it follows from (3. 8) that

$$\varphi_p^t(u(t)) = \frac{1}{p} \int_\Omega \left(\frac{|\nabla u(x,t)|}{\gamma(x,t)} \right)^p dx = \frac{1}{p} \int_\Omega \left(\frac{|\nabla u_0(x)|}{\gamma(x,t_0)} \right)^p dx = \varphi_p^{t_0}(u_0). \tag{3. 10}$$

Hence, since ψ is independent of t-variable, putting $C_0 = 1$, $\alpha_1 = \alpha_0, \alpha_2 = \beta_1 = \beta_2 \equiv 0$, $\psi^t = \psi$ and $\varphi^t = \varphi_p^t$, we find that (A5) and (A5)$'$ are satisfied.

Therefore using Lemma 3.3 and applying Theorem 2.3 to $(\mathrm{E})_p$, we can assure the existence of a unique strong solution for $(\mathrm{E})_p$. ∎

3.2 Existence and Uniqueness of Solutions for (P)

Theorem 3.4 *Suppose that* (A) *is satisfied. Then for any* $h_0 \in K^0 = \{u \in X_2;\ |\nabla u(x)| \le \gamma(x,0)$ *for a.e.* $x \in \Omega\}$, (P) *has a unique solution.*

Proof of Theorem 3.4 Define

$$\varphi_\infty^t(u) := \begin{cases} 0 & \text{if } u \in K^t, \\ +\infty & \text{if } u \in V \setminus K^t. \end{cases}$$

Then it is easy to see that $\varphi_\infty^t \in \Phi(V)$, $D(\varphi_\infty^t) = K^t$ for all $t \in [0,T]$ and that (P) is equivalent to the following Cauchy problem.

$$\text{(E)} \begin{cases} \partial\psi\left(\dfrac{dh}{dt}(t)\right) + \partial\varphi_\infty^t(h(t)) \ni f(t) \quad \text{in } V^*, \quad 0 < t < T, \\ h(0) = h_0. \end{cases}$$

Let $t \in [0,T]$ be fixed and let $u \in D(\varphi_\infty^t) = K^t$. It then follows that

$$|\nabla u(x)| \ \leq \ \gamma(x,t) \leq |\pi|_{L^\infty(\Omega)}|\phi|_{C([0,T])} \quad \text{for a.e. } x \in \Omega.$$

Hence since $H_0^1(\Omega)$ is embedded in V compactly, (A4) holds true with X and φ^t replaced by $H_0^1(\Omega)$ and φ_∞^t respectively. Moreover just as in the proof of Theorem 3.2, we can deduce that (A5) and (A5)$'$ are satisfied with φ^t and ψ^t replaced by φ_∞^t and ψ respectively. Therefore Lemma 3.3 and Theorem 2.3 ensure the existence of a unique strong solution for (E). ∎

4 Convergence of h_p as $p \to +\infty$

In this section, we investigate the convergence of h_p as $p \to +\infty$. Our result reads.

Theorem 4.1 *Suppose that* (A) *is satisfied. Let* $h_0 \in K^0$ *and let* h_p *be a unique solution of* $(\mathrm{P})_p$ *for every* $p \in [2, +\infty)$. *Then* h_p *converges to* h *as* $p \to +\infty$ *in the following sense*:

$$(4.\ 1) \qquad \begin{cases} h_p \to h & \text{strongly in } C([0,T];V_0), \\ & \text{weakly star in } L^\infty(0,T;X_2), \\ & \text{weakly in } W^{1,2}(0,T;V_0) \end{cases}$$

and h *is a unique solution of* (P). *Moreover the following inequality holds.*

$$(4.\ 2) \qquad \sup_{t\in[0,T]} |h_p(t) - h(t)|_{V_0} \ \leq \ C\left\{\frac{1}{p} + \inf_{v\in\mathcal{K}} \|h_p - v\|_{L^2(0,T;V_0)}\right\}^{1/2},$$

where $\mathcal{K} := \{v \in L^2(0,T;V_0);\ v(t) \in K^t \text{ for a.e. } t \in (0,T)\}$.

Proof of Theorem 4.1 Recalling that h_p and h become strong solutions of $(\mathrm{E})_p$ and (E) respectively, we establish the following a priori estimates.

Lemma 4.2 *There exists a constant* M_4 *such that*

$$(4.\ 3) \qquad \sup_{t\in[0,T]} \varphi_p^t(h_p(t)) \ \leq \ M_4,$$

$$(4.\ 4) \qquad \int_0^T \left|\frac{dh_p}{d\tau}(\tau)\right|_V^2 d\tau \ \leq \ M_4,$$

$$(4.\ 5) \qquad \int_0^T \left|\partial\psi\left(\frac{dh_p}{d\tau}(\tau)\right)\right|_{V^*}^2 d\tau \ \leq \ M_4.$$

Proof of Lemma 4.2 Multiplying $(\mathrm{E})_p$ by $dh_p(t)/dt$, we get by (3. 2)

$$C_3\left|\frac{dh_p}{dt}(t)\right|_V^2+\left\langle \partial\varphi_p^t(h_p(t)),\frac{dh_p}{dt}(t)\right\rangle \ \le\ \left\langle f(t),\frac{dh_p}{dt}(t)\right\rangle .$$

By Lemma 2.10, it follows from (3. 9) and (3. 10) that

$$\begin{aligned}\frac{d}{dt}\varphi_p^t(h_p(t)) \ &\le\ \left\langle \partial\varphi_p^t(h_p(t)),\frac{dh_p}{dt}(t)\right\rangle \\ &\qquad +|\dot\alpha_0(t)||\partial\varphi_p^t(h_p(t))|_{V^*}\left\{\varphi_p^t(h_p(t))+1\right\}^{1/2}.\end{aligned}$$

Therefore (3. 7) implies

$$\begin{aligned}&C_3\left|\frac{dh_p}{dt}(t)\right|_V^2+\frac{d}{dt}\varphi_p^t(h_p(t))\\ \le\ & |\dot\alpha_0(t)||\partial\varphi_p^t(h_p(t))|_{V^*}\left\{\varphi_p^t(h_p(t))+1\right\}^{1/2}+|f(t)|_{V^*}\left|\frac{dh_p}{dt}(t)\right|_V\\ \le\ & \frac{C_3}{2}\left|\frac{dh_p}{dt}(t)\right|_V^2+C\left\{|f(t)|_{V_*}^2+|\dot\alpha_0(t)|^2+|\dot\alpha_0(t)|^2\varphi_p^t(h_p(t))\right\},\end{aligned}$$

since $\partial\varphi_p^t(h_p(t))=f(t)-\partial\psi(dh_p(t)/dt)$. Integrating this over $(0,t)$, we obtain

$$\begin{aligned}&\frac{C_3}{2}\int_0^t\left|\frac{dh_p}{d\tau}(\tau)\right|_V^2d\tau+\varphi_p^t(h_p(t))\\ \le\ & \varphi_p^0(h_0)+C\left\{\int_0^T|f(\tau)|_{V_*}^2d\tau+\int_0^T|\dot\alpha_0(\tau)|^2d\tau\right\}\\ &+C\int_0^t|\dot\alpha_0(\tau)|^2\varphi_p^\tau(h_p(\tau))d\tau.\end{aligned}\tag{4. 6}$$

From the fact that $h_0\in K^0$, we notice

$$\varphi_p^0(h_0)\ =\ \frac{1}{p}\int_\Omega\left(\frac{|\nabla h_0(x)|}{\gamma(x,0)}\right)^p dx\le|\Omega|.$$

Thus by Gronwall's inequality, (4. 6) implies (4. 3). Moreover (4. 4) follows immediately from (4. 3) and (4. 6). Furthermore (3. 7) and (4. 4) yield (4. 5). ∎

By the definition of φ_p^t, it follows from (4. 3) that

$$\sup_{t\in[0,T]}\left\{\frac{1}{p}\int_\Omega\left(\frac{|\nabla h_p(x,t)|}{\gamma(x,t)}\right)^p dx\right\}\ \le\ M_4.\tag{4. 7}$$

Hence we obtain

$$\begin{aligned}&\frac{1}{|\pi|_{L^\infty(\Omega)}|\phi|_{C([0,T])}}\left(\int_\Omega|\nabla h_p(x,t)|^q dx\right)^{1/q}\\ \le\ & \left\{\int_\Omega\left(\frac{|\nabla h_p(x,t)|}{\gamma(x,t)}\right)^q dx\right\}^{1/q}\\ \le\ & \left\{\int_\Omega\left(\frac{|\nabla h_p(x,t)|}{\gamma(x,t)}\right)^p dx\right\}^{1/p}|\Omega|^{(p-q)/(pq)}\le(pM_4)^{1/p}|\Omega|^{(p-q)/(pq)}\end{aligned}\tag{4. 8}$$

for each $q \in [2,p]$. In particular, if $q = 2$, then

$$\sup_{t\in[0,T]} |h_p(t)|_{X_2} \text{ is bounded for all } p \in [2,+\infty). \tag{4. 9}$$

Consequently we can take a sequence (p_n) such that $p_n \to +\infty$ as $n \to +\infty$ and the following lemma holds.

Lemma 4.3 *There exists $h \in L^\infty(0,T;X_2) \cap W^{1,2}(0,T;V)$ such that*

$$\begin{aligned} h_{p_n} \to h \qquad & \textit{weakly in } W^{1,2}(0,T;V), \\ & \textit{weakly star in } L^\infty(0,T;X_2), \\ & \textit{strongly in } C([0,T];V) \end{aligned}$$

as $p_n \to +\infty$ and h is a unique solution of (E).

Proof of Lemma 4.3 By (4. 4) and (4. 9), there exists $h \in L^\infty(0,T;X_2) \cap W^{1,2}(0,T;V)$ such that

$$h_{p_n} \to h \qquad \text{weakly in } W^{1,2}(0,T;V), \tag{4. 10}$$

$$h_{p_n} \to h \qquad \text{weakly star in } L^\infty(0,T;X_2). \tag{4. 11}$$

Now by Mazur's lemma, it follows from (4. 5) and (4. 10) that

$$\partial\psi\left(\frac{dh_{p_n}}{dt}(\cdot)\right) \to \partial\psi\left(\frac{dh}{dt}(\cdot)\right) \qquad \text{weakly in } L^2(0,T;V^*). \tag{4. 12}$$

Moreover, since (h_p) is bounded in $L^\infty(0,T;X_2) \cap W^{1,2}(0,T;V)$ and X_2 is embedded in V densely and compactly, the Aubin-Lions type compactness lemma (see e.g. [25, Corollary 4]) implies

$$h_{p_n} \to h \qquad \text{strongly in } C([0,T];V). \tag{4. 13}$$

Finally we claim that h is a unique strong solution of (E). To this end, we first show that $h(t) \in K^t$ for a.e. $t \in (0,T)$. By (4. 8), we can take a subsequence (p_{n_q}) of (p_n) such that $\nabla h_{p_{n_q}} \to \nabla h$ weakly star in $L^\infty(0,T;L^q(\Omega))$, which together with the fact that $0 < \delta_0 \leq \gamma$ implies $\nabla h_{p_{n_q}}/\gamma \to \nabla h/\gamma$ weakly star in $L^\infty(0,T;L^q(\Omega))$ for each $q < +\infty$. For simplicity, we use the same letter p_n for p_{n_q}. Hence it follows that

$$\begin{aligned} &\left\{\int_0^T \int_\Omega \left(\frac{|\nabla h(x,t)|}{\gamma(x,t)}\right)^q dxdt\right\}^{1/q} \\ &\leq \liminf_{p_n\to+\infty} \left\{\int_0^T \int_\Omega \left(\frac{|\nabla h_{p_n}(x,t)|}{\gamma(x,t)}\right)^q dxdt\right\}^{1/q} \\ &\leq \liminf_{p_n\to+\infty} \left\{\int_0^T \int_\Omega \left(\frac{|\nabla h_{p_n}(x,t)|}{\gamma(x,t)}\right)^{p_n} dxdt\right\}^{1/p_n} (T|\Omega|)^{(p_n-q)/(p_nq)} \\ &\leq \lim_{p_n\to+\infty} (Tp_nM_4)^{1/p_n} (T|\Omega|)^{(p_n-q)/(p_nq)} = (T|\Omega|)^{1/q} \to 1 \qquad \text{as } q \to +\infty. \end{aligned}$$

Therefore we can verify that

$$\frac{|\nabla h(x,t)|}{\gamma(x,t)} \leq 1 \quad \text{for a.e. } (x,t) \in \Omega \times (0,T), \tag{4. 14}$$

which implies that $h(t) \in K^t$ for a.e. $t \in (0,T)$.

For any $w \in L^2(0,T;V)$ satisfying $w(t) \in K^t$ for a.e. $t \in (0,T)$, by $(\mathrm{E})_{p_n}$ and the definition of subdifferentials, it follows that

$$\begin{aligned}
&\int_0^T \left\langle \partial\psi\left(\frac{dh_{p_n}}{d\tau}(\tau)\right) - f(\tau), h_{p_n}(\tau) - w(\tau)\right\rangle d\tau \\
&= \int_0^T \left\langle -\partial\varphi^{\tau}_{p_n}(h_{p_n}(\tau)), h_{p_n}(\tau) - w(\tau)\right\rangle d\tau \\
&\leq \int_0^T \varphi^{\tau}_{p_n}(w(\tau))d\tau - \int_0^T \varphi^{\tau}_{p_n}(h_{p_n}(\tau))d\tau \\
&\leq \frac{1}{p_n}\int_0^T \int_\Omega \left(\frac{|\nabla w(x,\tau)|}{\gamma(x,\tau)}\right)^{p_n} dx d\tau \\
&\leq \frac{T}{p_n}|\Omega| \to 0 \quad \text{as } p_n \to +\infty,
\end{aligned} \tag{4. 15}$$

since $|\nabla w(x,\tau)| \leq \gamma(x,\tau)$ for a.e. $(x,\tau) \in \Omega \times (0,T)$. By (4. 12) and (4. 13), we deduce

$$\int_0^T \left\langle \partial\psi\left(\frac{dh}{d\tau}(\tau)\right) - f(\tau), h(\tau) - w(\tau)\right\rangle d\tau \leq 0,$$

which together with the fact that $\varphi^t_\infty(h(t)) = \varphi^t_\infty(w(t)) = 0$ for a.e. $t \in (0,T)$ implies

$$\begin{aligned}
&\int_0^T \left\langle \partial\psi\left(\frac{dh}{d\tau}(\tau)\right) - f(\tau), h(\tau) - w(\tau)\right\rangle d\tau \\
&\leq 0 = \int_0^T \varphi^t_\infty(w(\tau))d\tau - \int_0^T \varphi^t_\infty(h(\tau))d\tau.
\end{aligned} \tag{4. 16}$$

Hence by Proposition 1.1 in [14], it follows that

$$f(t) - \partial\psi\left(\frac{dh}{dt}(t)\right) \in \partial\varphi^t_\infty(h(t)) \text{ in } V^*, \quad \text{for a.e. } t \in (0,T).$$

Therefore we can deduce that h is a unique strong solution of (E). ∎

Since the limit is unique, we have the same conclusion as in Lemma 4.3 for an arbitrary sequence (p_n) satisfying $p_n \to +\infty$. Finally we verify (4. 2). To do this, recalling (4. 15) with T, w and p_n replaced by t, h and p respectively, we obtain

$$\int_0^t \left\langle \partial\psi\left(\frac{dh_p}{d\tau}(\tau)\right) - f(\tau), h_p(\tau) - h(\tau)\right\rangle d\tau \leq \frac{T}{p}|\Omega|. \tag{4. 17}$$

Moreover for all $w \in L^2(0,T;V)$ satisfying $w(t) \in K^t$ for a.e. $t \in (0,T)$, we have

$$\int_0^t \left\langle \partial\psi\left(\frac{dh_p}{d\tau}(\tau)\right) - f(\tau), h_p(\tau) - h(\tau)\right\rangle d\tau$$

$$
\begin{aligned}
&\geq \frac{C_3}{2}|h_p(t)-h(t)|_V^2 + \int_0^t \left\langle \partial\psi\left(\frac{dh}{d\tau}(\tau)\right) - f(\tau), h_p(\tau)-h(\tau)\right\rangle d\tau \\
&= \frac{C_3}{2}|h_p(t)-h(t)|_V^2 - \int_0^t \langle g_\infty(\tau), h_p(\tau)-w(\tau)\rangle\, d\tau \\
&\qquad + \int_0^t \langle g_\infty(\tau), h(\tau)-w(\tau)\rangle\, d\tau \\
&\geq \frac{C_3}{2}|h_p(t)-h(t)|_V^2 - \int_0^t |g_\infty(\tau)|_{V^*}|h_p(\tau)-w(\tau)|_V d\tau,
\end{aligned}
$$

where $g_\infty(t) = f(t) - \partial\psi(dh(t)/dt) \in \partial\varphi_\infty^t(h(t))$. Therefore it follows that

$$
\begin{aligned}
(4.\ 18)\qquad &\frac{C_3}{2}|h_p(t)-h(t)|_V^2 \\
&\leq \frac{T}{p}|\Omega| + \int_0^T |g_\infty(\tau)|_{V^*}|h_p(\tau)-w(\tau)|_V d\tau \quad \text{for all } t\in[0,T],
\end{aligned}
$$

which implies (4. 2). ▮

5 Another Approximation and Its Convergence

In this section, we introduce another approximation of (B) from the view point of convex analysis. Our new approximation is characterized through the Moreau-Yosida regularization of φ_∞^t in X_2. More precisely, define

$$
(5.\ 1)\qquad \varphi_{X_2,\lambda}^t(u) := \inf_{v\in X_2}\left\{\frac{1}{2\lambda}|\nabla u - \nabla v|_{L^2(\Omega)}^2 + \varphi_\infty^t(v)\right\}.
$$

Then by Lemma 2.5, we have

$$
(5.\ 2)\qquad \varphi_{X_2,\lambda}^t(u) = \frac{1}{2\lambda}|\nabla u - \nabla J_{X_2,\lambda}^t u|_{L^2(\Omega)}^2,
$$

where $J_{X_2,\lambda}^t$ denotes the resolvent of $\partial_{X_2}(\varphi_\infty^t|_{X_2})$. We here remark that $J_{X_2,\lambda}^t u$ is the unique minimizer of the functional of v appearing in the right-hand side of (5. 1). Now the extension φ_λ^t of $\varphi_{X_2,\lambda}^t$ to V_0 is defined as follows.

$$
\varphi_\lambda^t(u) := \begin{cases} \varphi_{X_2,\lambda}^t(u) & \text{if } u\in X_2, \\ +\infty & \text{if } u \in V_0\setminus X_2. \end{cases}
$$

We also follow the same setting for V and H as in §3. Now by (5. 2), we obtain

$$
\begin{aligned}
(5.\ 3)\qquad |u|_{X_2} = |\nabla u|_H &\leq \sqrt{2\lambda\varphi_{X_2,\lambda}^t(u)} + |\nabla J_{X_2,\lambda}^t u|_H \\
&\leq \sqrt{2\lambda\varphi_{X_2,\lambda}^t(u)} + |\pi|_H|\phi|_{C([0,T])} \\
&\qquad\qquad \forall u\in D(\varphi_\lambda^t),\ \forall t\in[0,T],
\end{aligned}
$$

where we used the fact that

$$
|\nabla J_{X_2,\lambda}^t u(x)| \leq \gamma(x,t) \leq |\pi(x)||\phi|_{C([0,T])} \quad \text{for a.e. } x\in\Omega.
$$

Hence (5. 3) implies that $\varphi^t_{X_2,\lambda}$ is coercive in X_2. Hence from the fact that $\varphi^t_{X_2,\lambda} \in \Phi(X_2)$, we can derive $\varphi^t_\lambda \in \Phi(V)$ for all $t \in [0,T]$. Moreover it follows that $D(\varphi^t_\lambda) = D(\partial\varphi^t_\lambda) = X_2$ for all $t \in [0,T]$.

Now we introduce the following Cauchy problem as our new approximation to (E).

$$
(\mathrm{E})_\lambda \begin{cases} \partial\psi\left(\dfrac{dh_\lambda}{dt}(t)\right) + \partial\varphi^t_\lambda(h_\lambda(t)) = f(t) \quad \text{in } V^*, \quad 0 < t < T, \\ h_\lambda(0) = h_0. \end{cases}
$$

As for the existence of a unique strong solution for $(\mathrm{E})_\lambda$, we have

Theorem 5.1 *Suppose that* (A) *is satisfied and let* $\lambda \in (0,1]$. *Then for any* $h_0 \in X_2$, $(\mathrm{E})_\lambda$ *has a unique strong solution.*

Proof of Theorem 5.1 Since $H^1_0(\Omega)$ is compactly embedded in V, (5. 3) implies (A4) with X and φ^t replaced by $H^1_0(\Omega)$ and φ^t_λ respectively.

Moreover just as in the proof of Theorem 3.2, we can show that (A5) and (A5)$'$ are satisfied with φ^t and ψ^t replaced by φ^t_λ and ψ respectively. Indeed, let $t_0 \in [0,T]$ and $u_0 \in D(\varphi^{t_0}_\lambda) = X_2$ be fixed. Define $u(t) := \{\phi(t)/\phi(t_0)\}u_0$ for all $t \in [0,T]$. Then just as in (3. 9), it follows from (5. 3) that

$$
\begin{aligned}
|u(t) - u_0|_V &\le \left|\frac{\phi(t)}{\phi(t_0)} - 1\right| |u_0|_V \\
&\le \left|\frac{\phi(t)}{\phi(t_0)} - 1\right| C\left\{\sqrt{2\lambda\varphi^{t_0}_\lambda(u_0)} + |\pi|_H|\phi|_{C([0,T])}\right\} \\
&\le C(|\phi|_{C([0,T])}, |\pi|_{L^\infty(\Omega)}, |\Omega|, \delta_0)|\phi(t) - \phi(t_0)|\left\{\varphi^{t_0}_\lambda(u_0) + 1\right\}^{1/2},
\end{aligned}
$$

where $C(|\phi|_{C([0,T])}, |\pi|_{L^\infty(\Omega)}, |\Omega|, \delta_0)$ denotes a constant depending only on $|\phi|_{C([0,T])}$, $|\pi|_{L^\infty(\Omega)}$, $|\Omega|$ and δ_0. Moreover we have

$$
\begin{aligned}
\varphi^t_\lambda(u(t)) &= \inf_{v \in X_2}\left\{\frac{1}{2\lambda}\left|\frac{\phi(t)}{\phi(t_0)}\nabla u_0 - \nabla v\right|^2_H + \varphi^t_\infty(v)\right\} \\
&= \inf_{w \in X_2}\left\{\left(\frac{\phi(t)}{\phi(t_0)}\right)^2 \frac{1}{2\lambda}|\nabla u_0 - \nabla w|^2_H + \varphi^{t_0}_\infty(w)\right\},
\end{aligned}
$$

where we put $w = \{\phi(t_0)/\phi(t)\}v$. Hence it follows that

$$
\begin{aligned}
\varphi^t_\lambda(u(t)) &= \inf_{w \in X_2}\left[\frac{1}{2\lambda}|\nabla u_0 - \nabla w|^2_H + \varphi^{t_0}_\infty(w) + \left\{\left(\frac{\phi(t)}{\phi(t_0)}\right)^2 - 1\right\}\frac{1}{2\lambda}|\nabla u_0 - \nabla w|^2_H\right] \\
&\le \varphi^{t_0}_\lambda(u_0) + \left\{\left(\frac{\phi(t)}{\phi(t_0)}\right)^2 - 1\right\}\varphi^{t_0}_\lambda(u_0) \\
&\le \varphi^{t_0}_\lambda(u_0) + \frac{2}{\delta_0^2}|\phi|_{C([0,T])}|\pi|^2_{L^\infty(\Omega)}|\phi(t) - \phi(t_0)|\varphi^{t_0}_\lambda(u_0).
\end{aligned}
$$

Therefore setting

$$\gamma_0(t) := C(|\phi|_{C([0,T])}, |\pi|_{L^\infty(\Omega)}, |\Omega|, \delta_0)\phi(t) \in W^{1,2}(0,T),$$
$$\beta_0(t) := \frac{2}{\delta_0^2}|\phi|_{C([0,T])}|\pi|^2_{L^\infty(\Omega)}\phi(t) \in W^{1,2}(0,T),$$

we conclude that $\{\varphi_\lambda^t\}_{t\in[0,T]}$ belongs to $\Phi(V,[0,T];\gamma_0,\beta_0,1,2)$. Hence by Lemma 3.3 and Theorem 2.3, we can verify the existence of a unique strong solution for $(\mathrm{E})_\lambda$. ∎

As for the convergence of h_λ as $\lambda \to +0$, our result is stated as follows.

Theorem 5.2 *Suppose that* (A) *is satisfied. Let $h_0 \in K^0$ and let h_λ be a unique strong solution of* $(\mathrm{E})_\lambda$ *for every $\lambda \in (0,1]$. Then $h_\lambda \to h$ as $\lambda \to +0$ in the following sense*:

$$h_\lambda \to h \qquad \textit{strongly in } C([0,T];V_0),$$
$$\textit{weakly star in } L^\infty(0,T;X_2),$$
$$\textit{weakly in } W^{1,2}(0,T;V_0)$$

and h is a unique strong solution of (E). *Moreover the following error estimate holds.*

$$\sup_{t\in[0,T]} |h_\lambda(t)-h(t)|_{V_0} \le C\sqrt{\lambda}. \tag{5. 4}$$

Proof of Theorem 5.2 Multiplying $(\mathrm{E})_\lambda - (\mathrm{E})$ by $h_\lambda(t)-h(t)$, we get

$$\left\langle \partial\psi\left(\frac{d}{dt}\{h_\lambda(t)-h(t)\}\right), h_\lambda(t)-h(t)\right\rangle + \left\langle \partial\varphi_\lambda^t(h_\lambda(t)) - g_\infty(t), h_\lambda(t)-h(t)\right\rangle = 0, \tag{5. 5}$$

where $g_\infty(t) = f(t) - \partial\psi(dh(t)/dt) \in \partial\varphi_\infty^t(h(t))$. We then observe

$$\partial\varphi_\infty^t(h(t)) \subset \partial_{X_2}(\varphi_\infty^t|_{X_2})(h(t)),$$
$$\partial\varphi_\lambda^t(h_\lambda(t)) = \partial_{X_2}\varphi^t_{X_2,\lambda}(h_\lambda(t)) = \frac{F_{X_2}(h_\lambda(t) - J^t_{X_2,\lambda}h_\lambda(t))}{\lambda} \in \partial_{X_2}(\varphi_\infty^t|_{X_2})(J^t_{X_2,\lambda}h_\lambda(t)),$$

where F_{X_2} denotes the duality mapping from X_2 into its dual space X_2^*. Hence we have

$$\begin{aligned}
&\left\langle \partial\varphi_\lambda^t(h_\lambda(t)) - g_\infty(t), h_\lambda(t)-h(t)\right\rangle \\
&= \left\langle \partial_{X_2}\varphi^t_{X_2,\lambda}(h_\lambda(t)) - g_\infty(t), \lambda F_{X_2}^{-1}\left(\partial_{X_2}\varphi^t_{X_2,\lambda}(h_\lambda(t))\right)\right\rangle_{X_2} \\
&\quad + \left\langle \partial_{X_2}\varphi^t_{X_2,\lambda}(h_\lambda(t)) - g_\infty(t), J^t_{X_2,\lambda}h_\lambda(t) - h(t)\right\rangle_{X_2} \\
&\ge \lambda|\partial_{X_2}\varphi^t_{X_2,\lambda}(h_\lambda(t))|^2_{X_2^*} - \lambda\left\langle g_\infty(t), F_{X_2}^{-1}\left(\partial_{X_2}\varphi^t_{X_2,\lambda}(h_\lambda(t))\right)\right\rangle_{X_2} \\
&\ge \frac{\lambda}{2}|\partial_{X_2}\varphi^t_{X_2,\lambda}(h_\lambda(t))|^2_{X_2^*} - \frac{\lambda}{2}|g_\infty(t)|^2_{X_2^*}.
\end{aligned} \tag{5. 6}$$

Therefore it follows from (5. 5) and (5. 6) that

$$\frac{1}{2}\frac{d}{dt}b\left(h_\lambda(t)-h(t), h_\lambda(t)-h(t)\right) \le \frac{\lambda}{2}|g_\infty(t)|^2_{X_2^*} \quad \text{for a.e. } t \in (0,T).$$

Integrating this over $(0,t)$, we get by (3. 2),

$$\frac{C_3}{2}|h_\lambda(t)-h(t)|_V^2 \ \le\ \frac{\lambda}{2}\int_0^T |g_\infty(\tau)|^2_{X_2^*}d\tau \quad \text{for all } t\in[0,T],$$

which implies (5. 4) and that $h_\lambda \to h$ strongly in $C([0,T];V)$ as $\lambda \to 0$.

Repeating the same argument as in the proof of Lemmas 4.2 and 4.3, we can also verify

$$h_\lambda \to h \qquad \begin{array}{l}\text{weakly in } W^{1,2}(0,T;V),\\ \text{weakly star in } L^\infty(0,T;X_2) \quad \text{as } \lambda\to 0. \ \blacksquare\end{array}$$

Remark 5.3 In Section 4, we could not estimate the error $h_p - h$ explicitly; indeed there is no knowing how to evaluate the term $\inf_{v\in\mathcal{K}}\|h_p - v\|_{L^2(0,T;V_0)}$ in (4. 2). However our new approximation enables us to control the error $h_\lambda - h$. More precisely, the error $|h_\lambda(t)-h(t)|_{V_0}$ is estimated above by $C\sqrt{\lambda}$ for all $t\in[0,T]$ in Theorem 5.2. In this sense, φ_λ^t gives a better approximation of φ_∞^t than φ_p^t.

Finally we discuss the representation of our approximation as a current-voltage law. For simplicity, we suppose that $j_c=\gamma\equiv 1$ in the rest of this section; moreover we denote $\varphi_p^t, \varphi_\infty^t$ and $\varphi_{X_2,\lambda}^t$ simply by $\varphi_p, \varphi_\infty$ and $\varphi_{X_2,\lambda}$ respectively. We first remark that the leading term of $(\mathrm{P})_p$ (or $(\mathrm{E})_p$), which is derived from $(\mathrm{B})_p$, can be described as follows:

$$\langle \partial_{X_p}(\varphi_p|_{X_p})(u),\eta\rangle_{X_p} \ =\ \int_\Omega \mathbf{k}_p(\nabla u(x))\cdot\nabla\eta(x)dx \quad \forall u,\eta\in X_p,$$

where $\mathbf{k}_p(\mathbf{v}) = |\mathbf{v}|^{p-2}\mathbf{v}$ for all $\mathbf{v}\in\mathbf{R}^2$. On account of the above remark, we should find a function $\mathbf{k}_\lambda:\mathbf{R}^2\to\mathbf{R}^2$ such that

$$\langle \partial_{X_2}\varphi_{X_2,\lambda}(u),\eta\rangle_{X_2} \ =\ \int_\Omega \mathbf{k}_\lambda(\nabla u(x))\cdot\nabla\eta(x)dx \quad \forall u,\eta\in X_2. \tag{5. 7}$$

We here simplify the problem above as follows: let $\Omega=\Omega_0 := \{x\in\mathbf{R}^2; |x|<R\}$ and find a function $\mathbf{k}_\lambda:\mathbf{R}^2\to\mathbf{R}^2$ such that (5. 7) is satisfied with X_2 and $\varphi_{X_2,\lambda}$ replaced by $X_{rad} := \{u\in H_0^1(\Omega); \exists\tilde{u}\in H^1(0,R) \text{ s.t. } u(x)=\tilde{u}(|x|)\ \forall x\in\overline{\Omega} \text{ and } \tilde{u}(R)=0\}$ and ϕ_λ respectively, where ϕ_λ is defined by (5. 1) with X_2 and φ_∞^t replaced by X_{rad} and φ_∞ respectively.

Now let $u\in X_{rad}$ be fixed. Then there exists a function $\tilde{u}\in H^1(0,R)$ such that $u(x) = \tilde{u}(|x|)$ for all $x\in\overline{\Omega}$ and $\tilde{u}(R)=0$. Moreover we see

$$\nabla u(x) \ =\ (\cos\theta,\sin\theta)^T\tilde{u}'(|x|), \quad |\nabla u(x)| = |\tilde{u}'(|x|)|,$$

where $\theta=\tan^{-1}(y/x)$ and $\tilde{u}'$ denotes the derivative of $\tilde{u}$. Now define

$$\rho(r) \ :=\ \begin{cases} 1 & \text{if } r>1,\\ r & \text{if } |r|\le 1,\\ -1 & \text{if } r<-1\end{cases}$$

and

$$\tilde{v}(r) := -\int_r^R \rho(\tilde{u}'(\xi))d\xi. \tag{5. 8}$$

Then it is easily seen that $\tilde{v} \in W^{1,\infty}(0,R)$ and $\tilde{v}(R) = 0$. Moreover letting $v(x) := \tilde{v}(|x|)$ for all $x \in \overline{\Omega}$, we observe

$$\nabla v(x) = (\cos\theta, \sin\theta)^T \rho(\tilde{u}'(|x|)) \quad \text{for a.e. } x \in \Omega, \tag{5. 9}$$

which implies

$$\begin{aligned} |\nabla u(x) - \nabla v(x)| &= \left|(\cos\theta, \sin\theta)^T \{\tilde{u}'(|x|) - \rho(\tilde{u}'(|x|))\}\right| \\ &= \min_{\mathbf{v}\in\mathbf{R}^2, |\mathbf{v}|\le 1} |\nabla u(x) - \mathbf{v}| \quad \text{for a.e. } x \in \Omega. \end{aligned}$$

Furthermore it follows from (5. 8) and (5. 9) that $v \in K := \{u \in X_{rad}; |\nabla u(x)| \le 1 \text{ for a.e. } x \in \Omega\}$. Therefore noting that

$$\left| \min_{\mathbf{v}\in\mathbf{R}^2, |\mathbf{v}|\le 1} |\nabla u(\cdot) - \mathbf{v}| \right|_{L^2(\Omega)} \le \inf_{w\in K} |\nabla u - \nabla w|_{L^2(\Omega)},$$

we have

$$|\nabla u - \nabla v|_{L^2(\Omega)} = \inf_{w\in K} |\nabla u - \nabla w|_{L^2(\Omega)},$$

which yields

$$\begin{aligned} \phi_\lambda(u) &= \inf_{w\in X_{rad}} \left\{ \frac{1}{2\lambda} |\nabla u - \nabla w|^2_{L^2(\Omega)} + \varphi_\infty(w) \right\} \\ &= \inf_{w\in K} \left\{ \frac{1}{2\lambda} |\nabla u - \nabla w|^2_{L^2(\Omega)} \right\} = \frac{1}{2\lambda} |\nabla u - \nabla v|^2_{L^2(\Omega)}. \end{aligned}$$

Hence it follows from (5. 2) that $J_\lambda u = v$, where J_λ denotes the resolvent of $\partial_{X_{rad}}(\varphi_\infty|_{X_{rad}})$. Therefore by Definition 2.4, it follows that

$$\partial_{X_{rad}} \phi_\lambda(u) = \frac{F_{X_{rad}}(u - J_\lambda u)}{\lambda} = \frac{F_{X_{rad}}(u - v)}{\lambda},$$

where $F_{X_{rad}}$ denotes the duality mapping from X_{rad} into its dual. Therefore since $F_{X_{rad}} = -\Delta$, we have for all $w \in X_{rad}$,

$$\begin{aligned} \langle \partial_{X_{rad}} \phi_\lambda(u), w \rangle_{X_{rad}} &= \int_\Omega \left(\frac{\nabla u(x) - \nabla v(x)}{\lambda} \right) \cdot \nabla w(x) dx \\ &= \int_\Omega \mathbf{k}_\lambda(\nabla u(x)) \cdot \nabla w(x) dx, \end{aligned}$$

where $\mathbf{k}_\lambda$ is given by

$$\mathbf{k}_\lambda(\mathbf{v}) := \begin{cases} \dfrac{\mathbf{v}}{|\mathbf{v}|} \left(\dfrac{|\mathbf{v}| - \rho(|\mathbf{v}|)}{\lambda} \right) & \text{if } \mathbf{v} \neq \mathbf{0}, \\ \mathbf{0} & \text{if } \mathbf{v} = \mathbf{0}. \end{cases}$$

Consequently our approximation can be characterized by the following current-voltage law.

$$\mathbf{e} = \mathbf{k}_\lambda(\mathbf{j}).$$

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Goro Akagi and Mitsuharu Ôtani
Department of Applied Physics,
School of Science and Engineering, Waseda University
3-4-1, Okubo, Shinjuku-ku, Tokyo, 169-8555 Japan