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Multi-Scale Stochastic Modeling of Dynamics of a Time-Averaged Variable

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Dynamics of a time-averaged variable depends on the interval of time-averaging. We investigate this dependence by devising a multi-scale stochastic modeling of dynamics. The key concept of modeling is the conditional probability distribution which describes variation of a time-averaged variable between two times and depends on the time interval. The distribution gives us inference of time series with finer resolution from rougher ones, providing both realizations and statistics of the variable.

KEYWORDS: time-averaged variable, coarse-grained dynamics, stochastic modeling, conditional probability

1. Background and Objective

Stochastic modeling of underlying dynamics of a variable from time series data have been developed in various research fields [1–6]. Recently, progress of information technology and big data science promotes acquisition of finer and finer time series data and thus it becomes insufficient to model the dynamics in just a fixed temporal resolution. Moreover, we sometimes still want to know the behavior of the same variable in even finer resolution than actually obtained time series data [2, 3, 6]. Therefore, we need a scaling theory of dynamics, which assumes underlying temporally continuous dynamics of a variable and gives coarse-grained models with different resolutions. Generally, time scaling properties of dynamics depend on the way of coarse-graining: (a) sampling an instantaneous value or (b) taking time-averaged value of a variable [6, 7]. In many cases, values of actual time series data are time-averaged in a certain interval which is usually equal to the resolution of the data. Hence, multi-timescale modeling of time-averaged variable is important, but has not been studied extensively.

For the sake of formulation of the modeling procedure for the dynamics of a time-averaged variable, we here develop a stochastic modeling with one variable as a first step. We show general formalism of the modeling in section 2, and two specific examples of the model are presented in section 3. Finally, conclusion and future directions are presented in section 4.

2. Formalism of Multi-Scale Stochastic Modeling

To formulate a simple standard modeling procedure of dynamics of time-averaged variable, we begin with one dimensional stochastic differential equation:

$$dx = f(x)dt + \sigma dL_t \quad (1)$$

where x is a dynamical variable evolving with time t, and L_t is a stochastic process. The first and second terms in the right side of Eq. (1) represent deterministic and stochastic drives of x respectively, and the function $f(x)$ and the constant σ denotes the intensities of each drive. We assume Eq. (1) to be the underlying continuous dynamics behind some discrete time series data, and its temporal integration is formally described as

$$x(t) = x(0) + \int_0^t f(x(s))ds + \sigma \int_0^t dL_s. \tag{2}$$

The next step is to integrate $x(t)$ of Eq. (2) over the time interval Δt to derive the time-averaged variable $\bar{x}_{\Delta t}$ as

$$\bar{x}_{\Delta t}(t) = \frac{1}{\Delta t}\int_t^{t+\Delta t} x(s)ds. \tag{3}$$

The integration in Eq. (3) can be done in any part of t then the displacement of $\bar{x}_{\Delta t}$ between the interval Δt is

$$\Delta\bar{x}_{\Delta t}(t) = \bar{x}_{\Delta t}(t+\Delta t) - \bar{x}_{\Delta t}(t). \tag{4}$$

We can model the dynamics of the time-averaged variable by the conditional probability density function $p\left(\Delta\bar{x}_{\Delta t}|\bar{x}_{\Delta t}\right)$ which depends on Δt and gives the value of $\Delta\bar{x}_{\Delta t}(t)$ stochastically for each $\bar{x}_{\Delta t}(t)$. Consequently, if any specific form of underlying continuous dynamics is given, the corresponding dynamics of time-averaged variable can be obtained as conditional probability distribution involving scale of coarse-graining. In the next section, we discuss about two specific model examples: linear and nonlinear.

3. Model Examples

3.1 The Simplest Linear Model

When the model has linear $f(x)$ and Wiener process W_t, we can analytically derive the formulae presented in the last section. We set $f(x) = -rx (r > 0)$ then the solution Eq. (2) of $x(t)$ is

$$x(t) = x(0)e^{-rt} + \sigma\int_0^t e^{r(s-t)}dW_s \tag{5}$$

where parameter r represents decaying rate constant. We deal with temporal averaging of Eq. (5) over $t \in [0, \Delta t]$ and $t \in [\Delta t, 2\Delta t]$ to obtain

$$\Delta\bar{x}_{\Delta t}(0) = -(1-e^{-r\Delta t})\bar{x}_{\Delta t}(0) + \frac{\sigma}{r\Delta t}e^{-r\Delta t}\int_0^{\Delta t}(-1+e^{rs})dW_s + \frac{\sigma}{r\Delta t}e^{-r\Delta t}\int_{\Delta t}^{2\Delta t}(1-e^{r(s-2\Delta t)})dW_s \tag{6}$$

According to the central limit theorem and expectation values of $\Delta\bar{x}_{\Delta t}(0)$ and $(\Delta\bar{x}_{\Delta t}(0))^2$, $\Delta\bar{x}_{\Delta t}(0)$ obeys normal distribution with mean $-\tilde{r}(\Delta t)\bar{x}_{\Delta t}(0)\Delta t$ and variance $\tilde{\sigma}^2(\Delta t)\Delta t$ where

$$\tilde{r}(\Delta t) = \frac{1-e^{-r\Delta t}}{\Delta t} \tag{7}$$

$$\tilde{\sigma}^2(\Delta t) = \left(\frac{\sigma}{r\Delta t}\right)^2\left\{1 + e^{-2r\Delta t} - \frac{1-e^{-2r\Delta t}}{r\Delta t}\right\}. \tag{8}$$

Therefore, Eq. (6) can be transformed into the stochastic difference equation as

$$\Delta\bar{x}_{\Delta t}(0) = -\tilde{r}(\Delta t)\bar{x}_{\Delta t}(0)\Delta t + \tilde{\sigma}(\Delta t)\Delta W \tag{9}$$

where ΔW is gaussian white noise with zero mean and variance Δt. Statistical properties of $\Delta\bar{x}_{\Delta t}(t)$ is identical with those of $\Delta\bar{x}_{\Delta t}(0)$ because the integration in Eq. (3) is equivalent at any part of t.

The distribution functions of simulated ensembles of $\Delta\bar{x}_{\Delta t}(0)$ on the several values of $\bar{x}_{\Delta t}(0)$ and Δt coincide the above formulae as shown in Fig. 1 and Fig. 2.

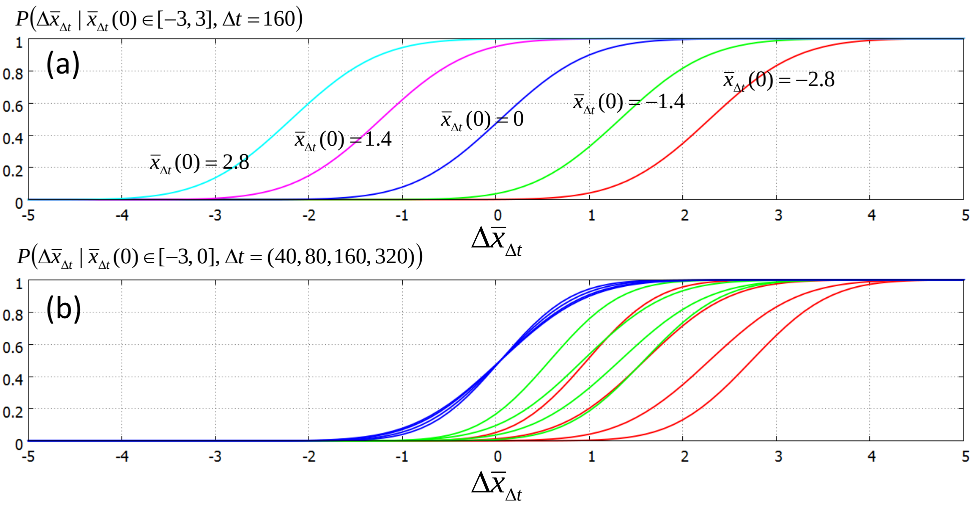

Fig. 1. Cumulative probability distribution $P(\Delta\bar{x}_{\Delta t}|\bar{x}_{\Delta t}) = \int_{-\infty}^{\Delta\bar{x}_{\Delta t}} p(u|\bar{x}_{\Delta t})du$ of variations of the time-averaged variable $\Delta\bar{x}_{\Delta t}$ obtained by numerical simulation of the model in section 3.1 with $r = 0.01$ and $\sigma^2 = 0.02$: (a) The distribution functions with different values of $\bar{x}_{\Delta t}(0)$. (b) The distribution functions with different values of Δt. All of them are normal distributions whose absolute value of mean $|\tilde{r}(\Delta t)\bar{x}_{\Delta t}\Delta t|$ and variance $\tilde{\sigma}^2(\Delta t)$ tend to become larger as Δt increases, which are parametrically controlled as Eq. (7) and Eq. (8).

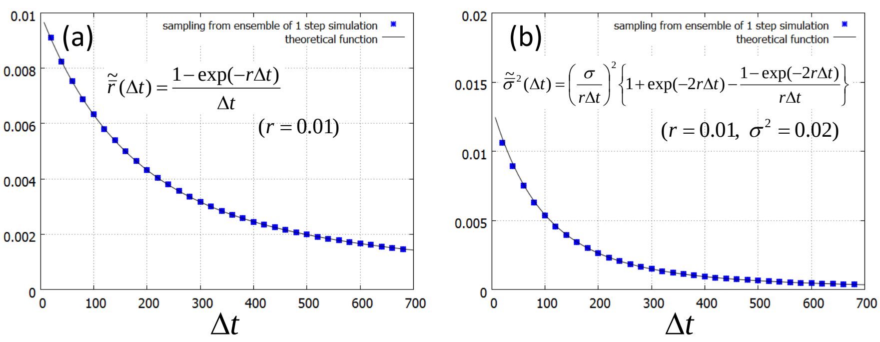

Fig. 2. Δt dependence of parameters (a)$\tilde{r}(\Delta t)$ and (b)$\tilde{\sigma}^2(\Delta t)$ of the distributions in Fig. 1. Simulated blue plots show exact match to theoretical black lines given as Eq. (7) and Eq. (8).

3.2 Nonlinear Double-Well Potential Model

Stochastic differential equation with double-well potential force is often used for a phenomenological model of bi-stability and phase transition of some system [2, 5]. We here adopt the following model as an example of nonlinear dynamics and operate coarse-graining.

$$dx = -\frac{dU(x)}{dx}dt + \sigma dW_t \tag{10}$$

$$U(x) = \frac{r}{8w^2}x^2(x^2 - 2w^2). \tag{11}$$

$U(x)$ is the potential function with two minima at $x = \pm w$ and one maximum at $x = 0$, and $r > 0$ represents decaying rate around the bottom of the wells. We examine this model numerically due to the difficulty in analytical treatment. In this case, the simulated ensembles of $\Delta\bar{x}_{\Delta t}(0)$ on the several values of $\bar{x}_{\Delta t}(0)$ and Δt show asymmetric and complex distributions(Fig. 3).

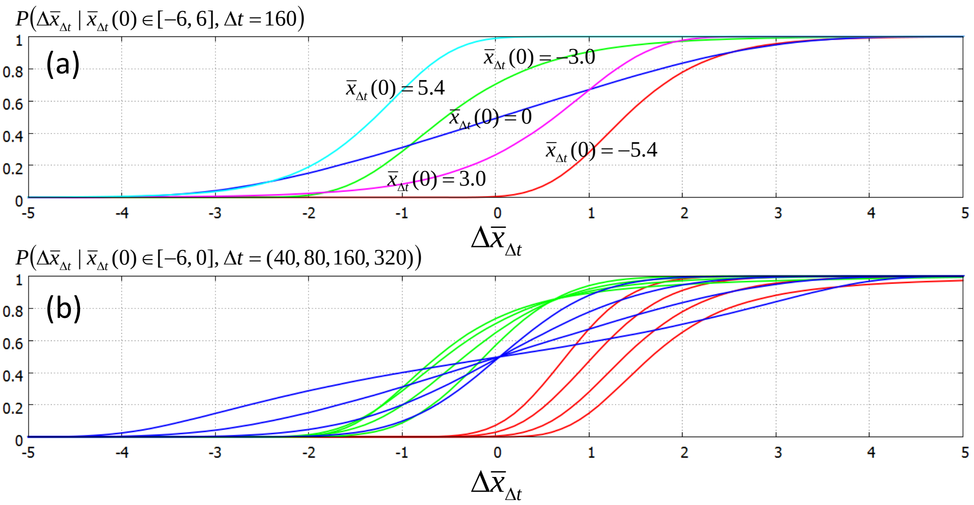

Fig. 3. Cumulative probability distribution $P(\Delta\bar{x}_{\Delta t}|\bar{x}_{\Delta t}) = \int_{-\infty}^{\Delta\bar{x}_{\Delta t}} p(u|\bar{x}_{\Delta t})du$ of variations of the time-averaged variable $\Delta\bar{x}_{\Delta t}$ obtained by numerical simulation of the model in section 3.2 with $r = 0.01$, $w = 4$ and $\sigma^2 = 0.02$: (a) The distribution functions with different values of $\bar{x}_{\Delta t}(0)$. (b) The distribution functions with different values of Δt. All of them shows asymmetric and complex distributions, but similar to the distributions in Fig 1, absolute value of mean and variance of the distributions tend to become larger as Δt increases.

4. Conclusion

We formulate multi-scale stochastic modeling of dynamics of a time-averaged variable using conditional probability distribution function. The distribution can describe both realizations and statistical properties of the variable and represents time scaling of the coarse-grained dynamics. The obtained conditional distributions are normal distributions when the model is linear whereas there appear asymmetric and complex distributions when the model is nonlinear. Time scaling of the distribution obtained in this theory will provide method of inference of dynamics with finer temporal resolution in various problems. Fitting proper functions to the simulated distributions will be our future work for practical applications of the model.

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