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Generalization of Abel's mechanical problem: The extended isochronicity condition and the superposition principle

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This paper presents a simple but nontrivial generalization of Abel's mechanical problem, based on the extended isochronicity condition and the superposition principle. There are two primary aims. The first one is to reveal the linear relation between the transit-time T and the travel-length X hidden behind the isochronicity problem that is usually discussed in terms of the nonlinear equation of motion $\frac{d^2X}{dt^2} + \frac{dU}{dX} = 0$ with $U(X)$ being an unknown potential. Second, the isochronicity condition is extended for the possible Abel-transform approach to designing the isochronous trajectories of charged particles in spectrometers and/or accelerators for time-resolving experiments. Our approach is based on the integral formula for the oscillatory motion by Landau and Lifshitz [*Mechanics* (Pergamon, Oxford, 1976), pp. 27–29]. The same formula is used to treat the non-periodic motion that is driven by $U(X)$. Specifically, this unknown potential is determined by the (linear) Abel transform $X(U) \propto A[T(E)]$, where $X(U)$ is the inverse function of $U(X)$, $A = (1/\sqrt{\pi}) \int_0^E dU/\sqrt{E-U}$ is the so-called Abel operator, and $T(E)$ is the prescribed transit-time for a particle with energy E to spend in the region of interest. Based on this Abel-transform approach, we have introduced the extended isochronicity condition: typically, $\tau = T_A(E) + T_N(E)$ where τ is a constant period, $T_A(E)$ is the transit-time in the *Abel type* [A-type] region spanning $X > 0$ and $T_N(E)$ is that in the *Non-Abel type* [N-type] region covering $X < 0$. As for the A-type region in $X > 0$, the unknown inverse function $X_A(U)$ is determined from $T_A(E)$ via the Abel-transform relation $X_A(U) \propto A[T_A(E)]$. In contrast, the N-type region in $X < 0$ does not ensure this linear relation: the region is covered with a predetermined potential $U_N(X)$ of some arbitrary choice, not necessarily obeying the Abel-transform relation. In discussing the isochronicity problem, there has been no attempt of N-type regions that are practically of full use for the charged-particle spectrometers and/or accelerators. In this Abel-transform approach, the superposition principle simplifies the derivation of $X_A(U)$ satisfying the extended isochronicity condition. Although the extended isochronicity condition inevitably discards the low-energy particles, there is no problem for handling accelerated particles because they do not involve the small-amplitude oscillations around the potential minimum. We present analytic examples of $X_A(U)$ that are instructive. In Appendix B, Urabe's criterion is interpreted in the time domain, using the Abel-transform approach. © 2014 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4865996>]

I. INTRODUCTION

Historically, the study of the isochronous oscillation originated from Galileo's legendary identification of a swinging lamp with the isochronous pendulum, which was followed by Huygens' theoretical finding that the isochronicity can be perfected if the pendulum traces the cycloidal trajectory, instead of the circular one. Although this classical problem is as simple as a one-dimensional

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(1D) one, it has attracted a renewed interest in the last century. Since the 1960s, many investigations have been devoted to finding new families of the isochronous potentials $U(X)$, equivalently the isochronous potential fields $g(X) \equiv \frac{dU}{dX}$, which are represented by some analytic or polynomial forms (cf. Ref. 1 and the current references therein). By definition, $U(X)$ is isochronous when all the oscillatory trajectories resulting from the equation of motion

$$\frac{d^2 X}{dt^2} + \frac{dU}{dX} = 0 \quad (1)$$

share the same period, regardless of their amplitudes (equivalently the energies). Of all the theoretical achievements made so far regarding the isochronous potentials, one of the earliest and the most striking is Urabe's criterion.^{2,3} Specifically, $U(X)$ is isochronous only if it satisfies a particular form containing a continuous function $S(Z)$ having the odd-symmetry around $Z = 0$:

$$\frac{dU}{dX} \equiv g[X(U)] = \frac{2\pi}{\tau} \frac{\sqrt{U}}{1 + S(\sqrt{U})}, \quad (2)$$

where τ is a constant period and $X(U)$ is the inverse function of $U(X)$ that is referred to as the inverse potential in this paper. Since S is an arbitrary function of odd symmetry, this criterion accounts for the fact that there exists an infinite number of different isochronous potentials, some of which can be written in analytic or polynomial forms. After 37 years of this somewhat mysterious finding, Robnik and Romanovski proved Urabe's criterion again in a much simpler manner,⁴ using the integral formula in the well-known textbook;⁸ their proof uncovered the direct relation between the odd function S in Urabe's criterion and the spatial distortion added to the isochronous harmonic potential (see Appendix B).

Partly intrigued by this success of the integral approach but more strongly driven by a practical motivation, this paper will shed a new light over the problem of mechanical isochronicity by presenting a simple but nontrivial generalization of Abel's mechanical problem. For this purpose, the rest of this introduction is devoted to reviewing the original work by Abel^{5,6} from a modern viewpoint and subsequently the Abel-transform approach to periodic motion by Landau and Lifshitz.⁸ On these backgrounds, we brief a new theoretical possibility about this well-examined and classical topic, which can be useful for designing the isochronous trajectories of charged particles in practice.

A. Abel's mechanical problem posed and solved in an A-type region

In 1823, Abel posed and solved a specific mechanical problem based on the integral equation⁵⁻⁷ that was intentionally introduced for the first time in history. As shown in Figure 1(a), the problem involves the two-dimensional (2D) motion of a particle sliding down a slope under the gravitational acceleration g without friction. In an up-to-date term, Abel conceived an inverse problem to determine the unknown 2D shape of the slope from the prescribed transit-time T spent for the sliding-down. Specifically, the particle starts from the initial rest position at the height $Y = H_0$ and descend to the final fixed height at $Y = 0$ in the time $T(E)$ that is a given function of the total energy $E = gH_0$. For simplicity, the mass is assumed to be unity throughout this paper. Abel formulated this 2D shape with the arclength $L(U)$ along the slope from the final position at the origin of $X = Y = 0$ to the transit point at the height H where U is the potential energy at H , namely, $U = gH$. Thus, the vertical height Y can be the exact measure of the potential energy U under a constant g .

On these definitions, Abel's mechanical problem is posed as follows. Since the speed at height Y is equal to $v = dL/dt = \sqrt{2(E - U)}$ and the initial and the final conditions are given by $L(U = E) = L(E)$ and $L(U = 0) = 0$, respectively, the transit-time $T(E)$ can be related to $L(U)$ with the use of an integral form

$$T(E) = \int_{L(E)}^0 \frac{dL}{v} = \int_0^{L(E)} \frac{dL}{\sqrt{2(E - U)}}.$$

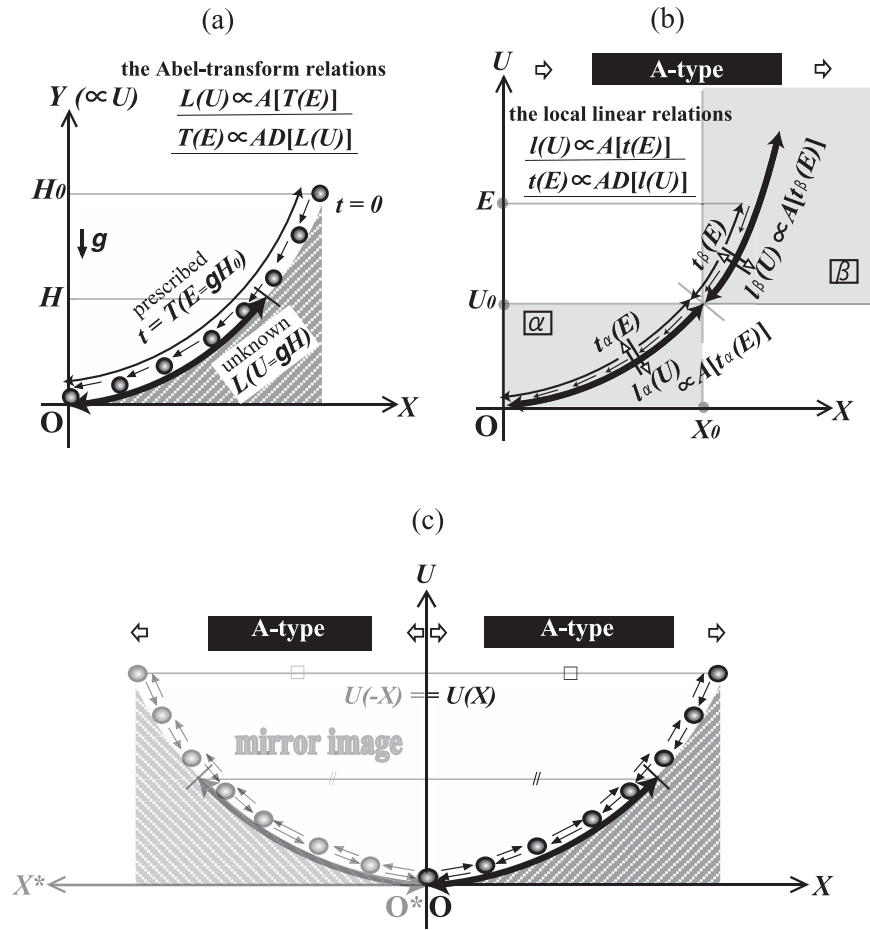


FIG. 1. Abel's mechanical problem to determine the unknown arclength $L(U)$ along the slope from the prescribed transit-time $T(E)$. (a) The original configuration by Abel. (b) The local linear relations between the local transit-time $t(E)$ and the corresponding spatial-length $l(U)$ within an A-type region. (c) The mirror-image assumption of $U(X) = U(-X)$ to treat the periodic motion.

If L is a monotone function of U , we can change the integral variable from L to U to obtain

$$T(U) = \int_0^E dU \frac{1}{\sqrt{2(E-U)}} \left(\frac{dL}{dU} \right) = \sqrt{\frac{\pi}{2}} \cdot AD[L(U)], \quad (3a)$$

where D is the derivative operator and $A = (1/\sqrt{\pi}) \int_0^E dU/\sqrt{E-U}$ is often referred to as the Abel operator [see (2) in Ref. 9]. Thus, $T(E)$ and $L(U)$ are related via an integral transformation that is commonly called the Abel transform now. In Abel's original works,⁵⁻⁷ we find that the solution to (3a) is formally described with A again:

$$L(U) = \frac{\sqrt{2}}{\pi} \cdot \int_0^U dE \frac{T(E)}{\sqrt{U-E}} = \frac{\sqrt{2}}{\pi} \cdot A[T(E)], \quad (3b)$$

where $L(0) = 0$ by definition. Interestingly, $L(U)$ is also given by the Abel transform of $T(E)$ with a different prefactor of $\sqrt{2}/\pi$.

Equations (3a) and (3b) suggest two universal rules: the first is the linear relation between $T(E)$ and $L(U)$ and the other is the symbolic rule concerning A and D , which are explained separately in the following.

Fig. 1(b) illustrates a localized interpretation of the Abel-transform relations (3a) and (3b), which can be used to introduce *Abel type* [A-type] and *Non-Abel type* [N-type] regions as follows.

Definition. If the local transit-time $t(E)$ and the corresponding local spatial-length $l(U)$ are related via the Abel-transform relations (3a) and (3b) anywhere inside a region of interest, it is an A-type region or otherwise a N-type one.

Specifically, let us suppose that the entire half-space in $X \geq 0$ is an A-type region, which is further divided into two regions separated by the energy threshold $U = U_0$ and the spatial boundary $X = X_0$, as shown in Fig. 1(b): the low-energy region α in $0 < U < U_0$ and the high-energy region β in $U > U_0$. Another assumption is that a particle with the energy E spends the local transit-times $t_\alpha(E)$ and $t_\beta(E)$ in the regions α and β , respectively. Since the total transit time is $T(E) = t_\alpha(E) + t_\beta(E)$, Eq. (3b) and the superposition principle give the total arclength $L(U)$ as a sum of two components:

$$L(U) = \frac{\sqrt{2}}{\pi} \cdot A[t_\alpha(E) + t_\beta(E)] = \frac{\sqrt{2}}{\pi} \{A[t_\alpha(E)] + A[t_\beta(E)]\} = l_\alpha(U) + l_\beta(U), \quad (4a)$$

where $l_\alpha(U)$ and $l_\beta(U)$ are the local arclengths within the regions α and β , respectively. If the entire half-space in $X \geq 0$ belongs to A-type, it is interesting to see that the regions α and β should also be A-type because

$$l_\alpha(U) = \frac{\sqrt{2}}{\pi} \cdot A[t_\alpha(E)], \quad (4b)$$

$$l_\beta(U) = \frac{\sqrt{2}}{\pi} \cdot A[t_\beta(E)]. \quad (4c)$$

In Appendix A, these local linear relations are verified by the direct integrations. So we can safely state that the Abel-transform relations (3a) and (3b) are valid in any part of an A-type region.

The second rule is often mentioned in terms of the fractional integrals and the fractional derivatives [see Chapter 6 in Ref. 9]. This rule is illustrated most typically by

$$AAD = \hat{1}, \quad (5)$$

where $\hat{1}$ is the identity operator. This relation suggests that A works like “a half integral operator,” which is proved in Appendix A by the direct integrations. Due to this fractional property, the Abel-transform approach can neatly handle those problems [cf. Chapters 2 and 3 in Ref. 4] that are not tractable with ordinary differential equations of any integer-order.

In his original works,^{6,7} Abel also mentioned the isochronous solution $L_{iso}(U)$ satisfying the ordinary isochronicity condition

$$\tau = T(E) = \text{constant}. \quad (6)$$

Inserting this condition into (3b), we obtain

$$L_{iso}(U) = \frac{\sqrt{2}}{\pi} \cdot A[\tau] = \frac{2\sqrt{2}\tau}{\pi} \sqrt{U}, \quad (7)$$

which reproduces the Huygens’ discovery of a cycloidal pendulum in 1659.

Here, let us make a note on the treatment of the periodic motion. Since the cycloidal solution (7) was mentioned by Abel himself,^{6,7} it is certainly possible to treat the 1D periodic motion, using the Abel-transform approach. The treatments thus based on the integral method usually assume the even-symmetry condition $U(-X) = U(X)$, as shown in Fig. 1(c). However, this mirror-image assumption is never essential to the isochronicity: as proclaimed by Urabe’s criterion^{2,3} in (2), there are infinite numbers of the isochronous solutions that do not obey the even-symmetry condition (see Appendix B). Besides, the isochronous solutions thus obtained on the mirror-image assumption do not necessarily ensure the analytic continuation around the origin $X = 0$: as suggested by Fig. 1(c), the mirror-image assumption holds the continuity of U around $X = 0$ but not that of dU/dX . To mark this possible discontinuity at the origin, this paper will implement the different coordinates of the

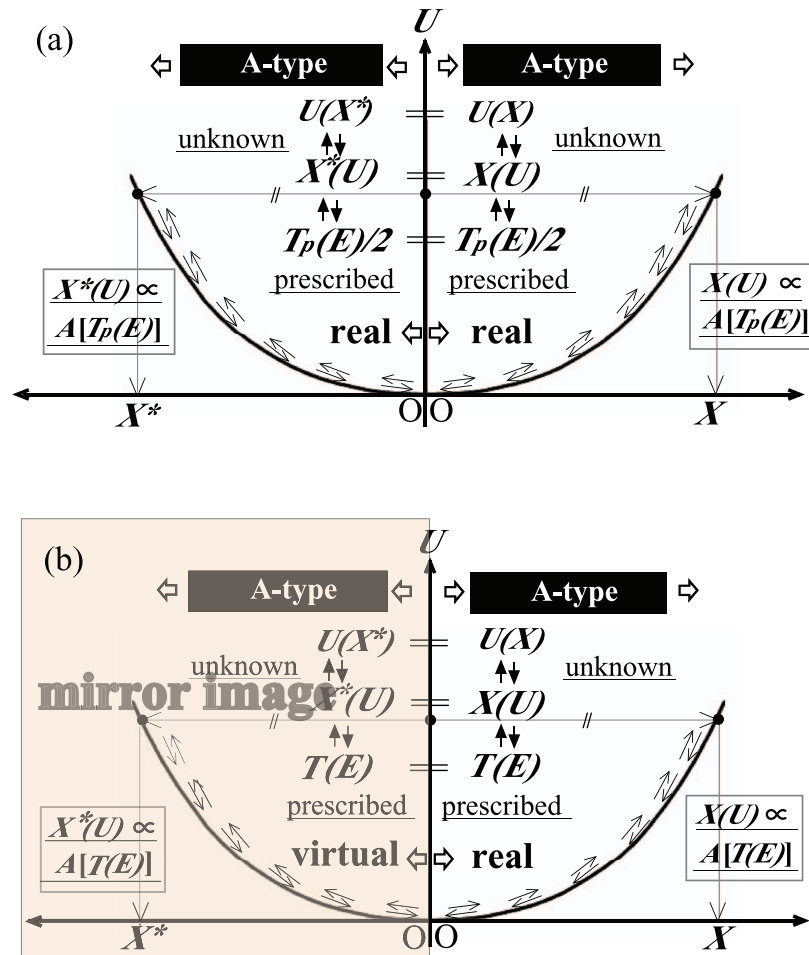


FIG. 2. Abel-transform approach to the one-dimensional oscillation in an unknown potential $U(X)$ under the mirror-image assumption $X^*(U) = X(U)$. (a) The oscillatory motion by Landau and Lifshitz.⁸ (b) The non-periodic motion treated by adding a virtual oscillation in the left half-space where the potential $U(X^* > 0)$ is the conceptual mirror-image of the real one $U(X > 0)$.

opposite directions: $X^* > 0$ for the left half-space and $X > 0$ for the right half-space, as shown in Fig. 2(b). These different coordinates offer another merit as well because the transit-time $T(E)$ and the spatial length L are never negative-valued, which significantly simplifies checking the physical meaningfulness of the results.

B. Abel-transform approach to periodic motion by Landau and Lifshitz

So far, there are several mechanical problems that have been treated with the Abel-transform approach identical to (3a) and (3b) [cf. Chapter 2 in Ref. 4]. Among these examples, Landau and Lifshitz pointed out the most general one: the oscillation in one dimension.⁸ As shown in Figure 2(a), they presented a generalized form of Abel's mechanical problem to determine the one-dimensional potential $U(X)$ from the prescribed period $T_p(E)$ of the oscillation, assuming the even-symmetry condition of $X^*(U) = X(U)$ and the potential minimum at $U(X = 0) = 0$. In their textbook,⁸ we find the intriguing formula (12.8) showing another Abel-transform relation between the period $T_p(E)$ and the inverse potential $X(U)$ that represents the distance from $X = 0$ to the turning-point for the energy $E = U$. As schematically indicated in Figure 2(a), this inverse potential $X(U)$ is uniquely determined under the conditions of the even-symmetry $X(U) = X^*(U)$ as well as the monotonicity of both $X(U)$

and $X^*(U)$ in $U > 0$:

$$X(U) = \frac{1}{2\sqrt{2\pi}} \int_0^U dE \frac{T_p(E)}{\sqrt{E-U}} = \frac{1}{2\sqrt{2\pi}} \cdot A[T_p(E)]. \quad (8a)$$

In a manner identical to (3a), the period $T_p(E)$ is given by the Abel transform having a different prefactor due to the back and forth motion:

$$\begin{aligned} T_p(E) &= 4 \int_0^{X(E)} \frac{dX}{\sqrt{2(U-E)}} \\ &= 2\sqrt{2} \int_0^E dU \frac{X'(U)}{\sqrt{U-E}} = 2\sqrt{2\pi} \cdot AD[X(U)]. \end{aligned} \quad (8b)$$

The uniqueness of $U(X)$ is nontrivial as the inverse potential $X(U)$ is not necessarily increasing in $U > 0$. In the present coordinate system, the inverse potential (8a) is meaningful only if

$$\frac{dX}{dU} > 0 \quad \text{and} \quad \frac{dX^*}{dU} > 0 \quad (U > 0). \quad (9)$$

In Sec. IV, we mention the possible breakdown of this monotonicity condition.

Let us mention a simple application of the formula (8a) to the isochronicity problem. From the ordinary isochronicity condition (6) and the Abel-transform relation (8a), the inverse potential is given by

$$X_{hmp}(U) = \frac{1}{2\sqrt{2\pi}} A[\tau] = \frac{\tau}{\sqrt{2\pi}} \sqrt{U}. \quad (10a)$$

This result represents a harmonic potential

$$U_{hmp}(X) = \frac{2\pi^2}{\tau^2} X^2 = U_{hmp}(X^*), \quad (10b)$$

which is obviously isochronous regardless of the energy E , in a manner analogous to the pendulum motion. Under the conditions of the ordinary isochronicity (6) and the even-symmetry $X(U) = X^*(U)$, the Abel-transform approach uniquely yields the harmonic solution (10a) and (10b) that obviously holds the analytic continuation around the origin of $X = X^* = 0$.

Notably, this smooth harmonic solution is not an automatic outcome of the Abel-transform approach: as suggested by Fig. 1(c), the mirror-image assumption assures the continuity of $U(X)$ but not necessarily that of dU/dX . In fact, the Abel-transform approach can handle the breakdown of the mirror-image assumption, that is, $X^*(U) \neq X(U)$ (see Appendix B). In this paper, we will point out another important flexibility of the Abel-transform approach that certainly extends the ordinary isochronicity condition (6).

C. The aims and the contents of this paper

This paper is motivated by two primary aims. The first one is to reveal the linear relation between the transit-time T and the travel-length X behind the problem of mechanical isochronicity that is usually discussed in terms of the nonlinear equation of motion (1). Second, the isochronicity condition is modified in such a way so as to illustrate the above-mentioned linear relation, which will possibly enable the application of the Abel-transform approach to designing the isochronous trajectories of charged particles in spectrometers and/or accelerators for time-resolving experiments.

To accomplish these aims, we organize this paper as follows. In Sec. II, the isochronicity condition is modified by introducing the preset N-type region, in which the transit-time $T(E)$ and the inverse potential $X(U)$ are not necessarily related by the Abel transforms. Our idea is to maintain the

total isochronicity by combining the A-type and the N-type regions while discarding the low-energy particles: the extended isochronicity condition [see (12)]. In practice, this extended isochronicity condition was hit upon while examining the possible application of the Abel transforms to designing the isochronous trajectories of charged particles in spectrometers and/or accelerators for time-resolving experiments. Generally, these particles are accelerated and, therefore, do not involve with the small-amplitude oscillations around the bottom of a potential-well. To author's knowledge, all the derived variants of Abel's mechanical problem have implicitly assumed so far that the whole region is A-type: the transit-time and the travel-length are related by the Abel transform. In contrast, the existing charged-particle spectrometers and/or accelerators are composed fully of N-type regions as the application of the Abel-transform approach has never been employed in practice. Based on the extended isochronicity condition, Sec. III describes the derivation of the inverse potential $X_A(U)$ in the A-type region, referred to as the A-type solution, with the use of the superposition principle. Section IV presents the two instructive examples of $X_A(U)$ that are given in analytic forms. Finally, we summarize and conclude the present findings while listing some possible future works. In Appendix B, a time-domain interpretation of Urabe's criterion is presented, using the Abel-transform approach.

II. THE N-TYPE REGION AND THE EXTENDED ISOCHRONICITY CONDITION

In this section, we present a simple but nontrivial generalization of Abel's mechanical problem by introducing the N-type region and the extended isochronicity condition.

First, we formulate the non-periodic motion in general, using the Abel-transform approach. As shown in Figure 2(b), this is accomplished again by the mirror-image assumption of $X^*(U) = X(U)$, but the outcome is interpreted quite differently: the potential $U(X^* > 0)$ in the left half-space is conceptually added to satisfy the even-symmetry condition $X^*(U) = X(U)$. In contrast, the non-periodic motion in the right half-space of $X > 0$ is governed by a real potential $U(X > 0)$. For this real motion, the transit-time $T(E)$ is given by halving the integral formula (8b):

$$T(E) = \int_0^U dU \frac{X'(U)}{2\sqrt{2(U-E)}} = \sqrt{2\pi} \cdot AD[X(U)]. \quad (11a)$$

From the symbolic relation $AAD = \hat{1}$ in (5), the solution is given by

$$X(U) = \frac{1}{\sqrt{2\pi}} \int_0^U dE \frac{T(E)}{\sqrt{E-U}} = \frac{1}{\sqrt{2\pi}} \cdot A[T(E)]. \quad (11b)$$

Next, we introduce the extension of the ordinary isochronicity condition (6). In Figure 2(b), the left half-space represents a virtual region that can be replaced with some real one. Specifically, the left half-space can accommodate a predetermined potential $U_N(X)$ of some arbitrary choice that can be N-type, as shown in Figures 3(a) and 3(b). In these two examples, it is evident that each potential $U_N(X)$ in the N-type region is not given by the Abel transform of the corresponding transit-time $T_N(E)$ because its inverse function $X_N(U)$ is no longer a monotone function of U . Figure 3(a) shows a typical example of the extended isochronicity condition written as

$$\tau = T_A(E) + T_N(E), \quad (12)$$

where τ is a constant period and $T_A(E)$ is the transit-time in the A-type region covering $X > 0$, in which the inverse potential is given by the Abel-transform relation $X_A(U) \propto A[T_A(E)]$. In contrast, $T_N(E)$ is newly introduced to represent the transit-time in a preset N-type region governed by $U_N(X < 0)$ of some arbitrary choice, which does not necessarily hold the linear relation between the transit-time and the inverse potential.

In fact, this extended isochronicity condition was hit upon while examining the possible application of the Abel-transform approach to designing the isochronous trajectories of charged particles in spectrometers and/or accelerators for time-resolving experiments. This attempt was for preparing the shortest pulses of charged particles in the time domain, which will offer the perfect time-resolving power in theory. Interestingly, the existing spectrometers and/or accelerators are built fully of N-type

combination of the A-type and the N-type regions to satisfy the extended isochronicity condition (12) in total.

III. THE SUPERPOSITION PRINCIPLE TO DERIVE THE INVERSE POTENTIAL $X_A(U)$ IN THE A-TYPE REGION

In this section, we show that the superposition principle does simplify the derivation of the inverse potential $X_A(U)$ in the A-type region, which is referred to as the *Abel type* solution in this paper.

Formally, $X_A(U)$ is given simply by applying $(1/\sqrt{2\pi})A$ to (12) and subsequently by separating the individual components with the use of the superposition principle:

$$(1/\sqrt{2\pi})A[\tau] = (1/\sqrt{2\pi})A[T_A(E) + T_N(E)] = (1/\sqrt{2\pi})\{A[T_A(E)] + A[T_N(E)]\}.$$

From the non-periodic formula (11b), the result is

$$2X_{hmp}(U) = X_A(U) + X_N^C(U), \quad (13a)$$

where $X_{hmp}(U)$ is the harmonic potential in (10a) and

$$X_A(U) = (1/\sqrt{2\pi})A[T_A(E)], \quad (13b)$$

$$X_N^C(U) = (1/\sqrt{2\pi})A[T_N(E)]. \quad (13c)$$

In this expression, $X_N^C(U)$ has an intentional superscript C because $X_N^C(U)$ does not mean the inverse potential $X_N(U)$ in the N-type region as we see below.

Actually, $X_N^C(U)$ can be interpreted as follows. If $T_N(E) = 0$ or there is no N-type region in $X < 0$, Eqs. (13a)–(13c) yield $X_A(U) = 2X_{hmp}(U)$: a half cycle of an isochronous harmonic oscillation in $X > 0$, which is bounced back by a potential-cliff of an infinite height standing at the origin $X = 0$. Next, we suppose a finite N-type region satisfying $T_N(E) > 0$. In this case, the extended isochronicity condition (12) demands that the transit-time $T_A(E)$ in $X > 0$ must be decreased, to a certain extent. Evidently, this decrease in time is achieved if the distance $X_A(U)$, spanning from $X = 0$ to the turning-point for the energy E , becomes somewhat shorter than $X_{hmp}(U)$ [see Figure 4] and so the particle will spend

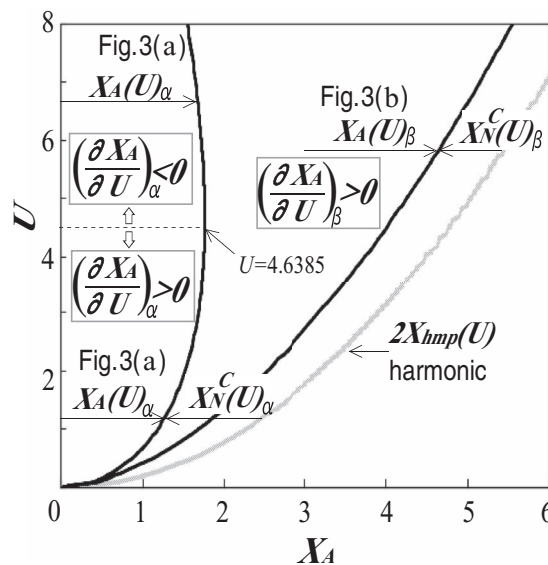


FIG. 4. $X_A(U)$ for the examples in Figure 3 with $|U_0| = 2|X_0| = |F_0|/2 = 1$ and $\tau = 5$. The subscripts α and β specify the examples in Figs. 3(a) and 3(b), respectively. In the case of Fig. 3(b), $X_A(U) = X_{1A}(U) + X_{2A}(U)$ by definition.

less time in $X > 0$. In short, $X_n^C(U)$ represents the exact extent of this shortening to compensate $T_N(U) > 0$. Thus, $X_N^C(U)$ exhibits its presence in $X > 0$, whereas the inverse function $X_N(U)$ does in $X < 0$.

To end this section, we mention a general trend that $X_N^C(U)$ becomes longer as $T_N(E)$ is increased. The reason is that the Abel-transform relation $X_N^C(U) \propto A[T_N(E)]$ contains the positive-valued kernel $1/\sqrt{E - U}$ in A . As a result, $X_A(U)$ tends to resemble the harmonic potential $2X_{hmp}(U)$ if $T(E) \ll \tau$ [see $X_{A-}(U)_\beta$ in Figure 4]. In contrast, if $T_A(E)$ becomes excessively large, there should be a chance that the monotonicity condition (9) ensuring the meaningful solution $X_A(U)$ is broken down [see $X_{A-}(U)_\alpha$ in Figure 4]. In Sec. IV, this breakdown and the further utility of the superposition principle are shown with analytic examples of $X_A(U)$.

IV. ANALYTIC EXAMPLES OF THE A-TYPE SOLUTION $X_A(U)$

In this section, we present two analytic examples of $X_A(U)$ for the N-type regions in Figure 3. These examples are composed exclusively of the free-motion and the uniform-acceleration parts, which are commonly used in the existing charged-particle spectrometers and/or accelerators.

A. $X_A(U)$ for the unbounded N-type region [Figure 3(a)]

First, we consider the example in Figure 3(a). In this example, the transit-time $T_N(E)$ in the preset N-type region is further decomposed into the two parts, i.e., the free-motion part $T_{free}(E)$ spent in $-|X_0| < X < 0$ and the uniform-acceleration part $T_{acc}(E)$ in $X < -|X_0|$ that is governed by a constant field $F_0 = -(dU_N/dX)$. With the use of the superposition principle again, $X_N^C(U)$ is given by a sum of the two components:

$$X_N^C(U)_\alpha = X_{free}^C(U) + X_{acc}^C(U)_\alpha, \quad (14a)$$

where

$$X_{free}^C(U) = (1/\sqrt{2\pi})A[T_{free}(E)], \quad (14b)$$

$$X_{acc}^C(U)_\alpha = (1/\sqrt{2\pi})A[T_{acc}(E)_\alpha]. \quad (14c)$$

In the rest of this section, we use the additional subscripts α and β , if needed, to specify the examples in Figures 3(a) and 3(b), respectively.

As for the free-motion part, it is held at a constant potential of $-|U_0|$, and the speed is kept at $\sqrt{2(E + |U_0|)}$. Therefore, the transit-time is given by

$$T_{free}(E) = \frac{2|X_0|}{\sqrt{2(E + |U_0|)}}, \quad (15a)$$

where the factor 2 in the denominator is due to the back and forth motion. As a result,

$$\begin{aligned} X_{free}^C(U) &= \frac{1}{2\pi} \int_0^U dE \frac{2|X_0|}{\sqrt{E + |U_0|}\sqrt{U - E}} \\ &= \frac{2|X_0|}{\pi} \left[-\arctan \left(\frac{\sqrt{U - E}}{\sqrt{E + |U_0|}} \right) \right]_{E=0}^{E=U} = \frac{2|X_0|}{\pi} \arctan \left(\sqrt{U/|U_0|} \right). \end{aligned} \quad (15b)$$

$X_{free}^C(U)$ is by no means the inverse function of the field-free potential $U_{free}(U) = -|U_0|$ in $X < 0$.

Next we consider the uniform-acceleration in $X < -|X_0|$. Since the speed is increased from 0 to $\sqrt{2(E + |U_0|)}$ at a constant rate, the transit-time for the reciprocal motion is

$$T_{acc}(E)_\alpha = \frac{2\sqrt{2(E + |U_0|)}}{|F_0|}. \quad (16a)$$

For the analytic evaluation of $A[T_{acc}(E)_\alpha]$, we note a useful formula

$$\begin{aligned}\Phi_{\{U_p\}}(E) &= \int dE \frac{\sqrt{E+U_p}}{\sqrt{U-E}} \\ &= -\sqrt{U-E}\sqrt{E+U_p} - (U+U_p) \arctan\left(\frac{\sqrt{U-E}}{\sqrt{E+U_p}}\right),\end{aligned}$$

where U_p is a constant. Using this expression and (16a), we obtain

$$\begin{aligned}X_{acc}^C(U)_\alpha &= \frac{1}{\sqrt{2}\pi} \int_0^U dE \frac{2\sqrt{2(E+|U_0|)}}{|F_0|\sqrt{U-E}} = \frac{2}{\pi|F_0|} [\Phi_{\{U_0\}}(E)]_{E=0}^{E=U} \\ &= \frac{2}{\pi|F_0|} \left\{ \sqrt{U|U_0|} + (U+|U_0|) \arctan(\sqrt{U/|U_0|}) \right\}.\end{aligned}\quad (16b)$$

Apparently, $X_{acc}^C(U)_\alpha$ does not represent the inverse function of the real constant-force potential $U_{acc}(X) = -|F_0|(X - |X_0|) - |U_0|$ in $X < 0$.

Since $X_{acc}^C(U)_\alpha$ and $X_{free}^C(U)$ are thus given in analytic forms, the same is true for $X_N^C(U)$ in (14a)–(14c) and for $X_A(U)$ in (13a)–(13c). From this example, it is readily seen that $X_A(U)$ can be described in an analytic manner if the preset N-type region is composed exclusively of the free-motion and/or the uniform-acceleration (deceleration) parts.

Figure 4 shows a graph of $X_A(U)_\alpha$ for the example in Figure 3(a) with the specific setting of $|U_0| = 2|X_0| = |F_0|/2 = 1$ and $\tau = 5$. From this graph, we find that the existence of a meaningful solution is nontrivial. In the high-energy region above the numerically obtained threshold $U = 4.6385$, $(dX_A/dU)_\alpha < 0$ and the condition (9) for the meaningful solution is certainly broken down. Nevertheless, $X_A(U)_\alpha$ still satisfies the extended isochronicity condition (12) for the particle with $E < 4.6385$ if we discard the meaningless high-energy part in $U > 4.6385$.

The reason for this meaningless result is attributed to the unbounded increase of $T_{acc}(E)_\alpha$ and $X_{acc}^C(U)_\alpha$: from (16a) and (16b), $T_{acc}(E \rightarrow +\infty)_\alpha \rightarrow +\infty$ and $X_{acc}^C(U \rightarrow +\infty)_\alpha \rightarrow +\infty$. If E is sufficiently large, we must have $T_A(E) < 0$, and this negative-valued transit-time cannot be realized by any potential in the A-type region.

B. $X_A(U)$ for the bounded N-type region [Figure 3(b)]

In Figure 3(b), we show another example with $T_N(E)$ and $X_N^C(U)$ being bounded: the central preset N-type region in $U < 0$ is spatially bounded by the adjacent A-type regions while the parameters are kept at $|U_0| = 2|X_0| = |F_0|/2 = 1$ and $\tau = 5$ again. As shown in Fig. 3(b), we use the different coordinates X_1 and X_2 to distinguish the two A-type regions: the region 1 in $X_1 > 0$ and the region 2 in $X_2 > 0$, respectively. Since the constant force F_0 in the region 1 increases the speed from $\sqrt{2E}$ to $\sqrt{2(E+|U_0|)}$ at a constant rate, the transit-time $T_{acc}(E)_\beta$ for the reciprocal motion is given by

$$T_{acc}(E)_\beta = 2 \times \frac{\sqrt{2(E+|U_0|)} - \sqrt{2E}}{|F_0|} = \frac{4|U_0|}{|F_0| [\sqrt{2(E+|U_0|)} + \sqrt{2E}]}. \quad (17a)$$

$T_{acc}(E)_\beta$ is bounded because $T_{acc}(E \rightarrow +\infty)_\beta \rightarrow +0$. From (16b), we can obtain the inverse potential $X_{acc}^C(U)_\beta$ compensating $T_{acc}(E)_\beta$ again in an analytic form:

$$\begin{aligned}X_{acc}^C(U)_\beta &= X_{acc}^C(U)_\alpha - 2 [\Phi_{\{0\}}(E)]_{E=0}^{E=U} = X_{acc}^C(U)_\alpha - \frac{U}{|F_0|} \\ &= \frac{|U_0|}{|F_0|} + \frac{2}{\pi|F_0|} \left\{ \sqrt{|U_0|U} - U \arctan(\sqrt{|U_0|/U}) \right\}.\end{aligned}\quad (17b)$$

This derivation uses the familiar formulas such as $\arctan(x) = \pi/2 - \arctan(1/x)$ and $\lim_{x \rightarrow +\infty} \arctan(x) = \pi/2$. $X_{acc}(U)_\beta$ is also bounded as $X_{acc}^C(U \rightarrow +\infty)_\beta \rightarrow |U_0|/|F_0|$.

With these bounded results of $T_{acc}(E)_\beta$ and $X_{acc}^C(U)_\beta$, the isochronicity problem concerning Figure 3(b) can be posed as follows. First, we define the quantities concerned. As for the temporal quantities, $T_{1A}(E)$ and $T_{2A}(E)$ are the transit-times in the region 1 and the region 2, respectively. With these denotations, the extended isochronicity condition for Figure 3(b) is given by

$$T_A(E) = T_{1A}(E) + T_{2A}(E) = \tau - T_{free}(E) - T_{acc}(E)_\beta, \quad (18a)$$

where $T_A(E)$ represents the overall transit-time spent in all of the A-type regions. As for the spatial quantities, Figure 3(b) shows that $X_{1A}(U)$ and $X_{2A}(U)$ are the distances to the turning-points for the energy $E = U$ in the region 1 and the region 2, respectively. By applying the Abel operator $(1/\sqrt{2\pi})A$ to the above condition (18a) and making use of the superposition principle again, we obtain the A-type solution $X_A(U)_\beta$ for Figure 3(b):

$$X_A(U)_\beta = X_{1A}(U) + X_{2A}(U) = 2X_{hmp}(U) - X_{free}^C(U) - X_{acc}^C(U)_\beta, \quad (18b)$$

where

$$X_A(U)_\beta = \frac{1}{\sqrt{2\pi}}A[T_A(E)], \quad (18c)$$

$$X_{1A}(U) = \frac{1}{\sqrt{2\pi}}A[T_{1A}(E)], \quad (18d)$$

$$X_{2A}(U) = \frac{1}{\sqrt{2\pi}}A[T_{2A}(E)]. \quad (18e)$$

Compared with (13a)–(13c), we formally have the same Abel-transform relation in (18c) while the definitions of $T_A(E)$ and $X_A(U)$ are modified, though.

Figure 4 shows the graph of $X_A(U)_\beta$ in addition to that of $X_A(U)_\alpha$. As expected from the above discussion, $X_A(U)_\beta$ is monotonically increasing in $U > 0$, which makes a marked contrast to the behavior of $X_A(U)_\alpha$. In consequence, $X_A(U)_\beta$ is meaningful in the entire region of $U > 0$. Although the A-type solution $X_A(U)$ can be automatically computed with the use of the Abel transform, the monotonicity as well as the meaningfulness of $X_A(U)$ depend much on the spatial boundedness of the A-type regions concerned.

The further decomposition of $X_A(U)_\beta$ into $X_{1A}(U)$ and $X_{2A}(U)$ definitely remains unsettled, as the similar indeterminacy of an unknown potential $U(X)$ was already stated by Landau and Lifshitz⁸ for the oscillatory motion if $U(X) \neq U(-X)$.

V. FINAL REMARKS

We have shown a simple but nontrivial generalization of Abel's mechanical problem, based on the extended isochronicity condition and the superposition principle.

As for the extended isochronicity condition, the idea is certainly original because the A-type and the newly introduced N-type regions are combined for the first time. In the A-type region, the transit-time $T_A(E)$ and the inverse potential $X_A(U)$ are related via the (linear) Abel-transform relations. In contrast, the N-type region does not necessarily hold these linear relations as its built-in potential $U_N(X)$ is of some arbitrary choice, which does not ensure the uniqueness of the inverse potential $X_N(U)$. To author's knowledge, no theoretical works have attempted at the possibility of N-type regions in considering the isochronicity problem. In practice, however, the existing charged-particle spectrometers and/or accelerators are composed fully of the N-type regions to achieve the isochronous trajectories of charged particles. As shown by the examples in Sec. IV, the extended isochronicity condition yields nontrivial results as well, even though the preset N-type region is simply the combinations of free-motion and uniform-acceleration (deceleration) parts. By the superposition principle, we have found that the A-type solutions $X_A(U)$ can be described in analytic forms that contain the nontrivial expressions like $\arctan(\sqrt{U/|U_0|})$.

Although our success owes much to the integral formula for the periodic motion in one-dimension, given by Landau and Lifshitz,⁸ our further treatment of the non-periodic motion in general has opened the way for the extended isochronicity condition. Besides, we have revealed the

linear nature underlying Abel's mechanical problem for the first time. This is exhibited particularly by the use of the superposition principle in deriving the Abel type solution $X_A(U)$.

Finally, we end up this paper by listing some interesting subjects for the future works.

Isochronicity problem based on Lebesgue integrals: The existence and the uniqueness of the Abel-transformed solution are proved on the ground of Lebesgue integrals (see Appendix 1A in Ref. 9). If the same basis is applied to the isochronicity problem, then, this will encourage the treatment of non-analytic solutions that are of practical uses in charged-particle spectrometers and/or accelerators.

Solutions beyond one-dimension: For the practical applications to charged-particle spectrometers and/or accelerators, the theoretical results presently obtained in one-dimension are very desirable but not sufficient yet: the extended isochronicity condition can be made satisfied for the particles traveling along the central axis of an experimental machine, but this is not necessarily the case for the off-axis particles. If the isochronicity is seriously broken for the particles traveling along the off-axis trajectories, the result will be the degradation of the time-resolution. For this practical purpose, some solutions beyond one-dimension are strongly needed.

Continuity of the potentials at the boundaries: Since the whole 1D space is divided into A-type and N-type regions, the continuity of the resultant inverse potentials must be met, to a certain extent, at all the boundaries between different regions. In the possible applications for the charged-particle spectrometers and/or accelerators, this continuity is vital because $U(X)$ are created in the vacuum with electrostatic means. Specifically, these potentials must satisfy the electrostatic Laplace equation, even when meshed electrodes are used at the boundaries. As a result, the continuity of the potentials is automatically imposed. In mathematical physics, similar requirements will arise in the search for the isochronous potentials of analytic or polynomial forms with the use of the Abel-transform approach. Some systematic method is needed for this subject, too.

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APPENDIX A: A SIMPLE ACCOUNT OF THE LOCAL LINEAR RELATIONS (4b) AND (4c)

We present a simple account of the local linear relations (4b) and (4c), as depicted in Fig. 1(b), which have been used to introduce A-type and N-type regions.

Our goal can be stated as follows: if the total Abel-transform relation (4a) is valid within the entire half-space of $X \geq 0$, so are the local Abel-transform relations of $l_\alpha(U) = (\sqrt{2}/\pi)A[t_\alpha(E)]$ in (4b) and $l_\beta(U) = (\sqrt{2}/\pi)A[t_\beta(E)]$ in (4c). Here, we will verify only the linear relation (4b) in the region α because the corresponding relation (4c) in the region β is automatically given by subtraction (4b) from (4a).

For verifying (4b), our approach is simply the direct integrations. Since the region α contains the origin $X = 0$ and l_α must be a monotone function of U , like $L(U)$, the local transit-time $t_\alpha(E)$ can be computed in a manner similar to (3a):

$$t_\alpha(E) = \int_0^{l_\alpha(E)} \frac{dl_\alpha}{\sqrt{2\{E - U\}}} = \sqrt{\frac{\pi}{2}} \cdot AD[l_\alpha(U)]. \quad (\text{A1})$$

As a result, we have a double-integral form:

$$AAD[l_\alpha(U)] = (\sqrt{2}/\pi) \cdot A[t_\alpha(E)] = (\sqrt{2}/\pi) \int_0^U \frac{dE}{\sqrt{U-E}} \int_0^{l_\alpha(E)} \frac{dl_\alpha}{\sqrt{2\{E-U(l_\alpha)\}}}.$$

If the order of integration is changeable due to Fubini's theorem, we obtain

$$AAD[l_\alpha(U)] = \int_0^{l_\alpha(U)} \frac{dl_\alpha}{\pi} \int_{U(l_\alpha)}^U \frac{dE}{\sqrt{U-E}\sqrt{\{E-U(l_\alpha)\}}} = \int_0^{l_\alpha(U)} dl_\alpha = l_\alpha(U). \quad (\text{A2})$$

Notably, this result makes a proof of $AAD = \hat{1}$ in (5) as well.

From the above reasoning about the two regions of α and β , it is readily seen that the local Abel-transform relation is still valid even if the number of the divisions is increased from two, one by one, with the use of mathematical induction. So this paper has employed this simple rule to distinguish A-type and N-type regions: if the Abel-transform relation between the local transit-time and the local spatial-length is valid anywhere inside a region of interest, it is Abel-type or otherwise N-type. Obviously, the same rule is true for $X(U)$ as well.

APPENDIX B: TIME-DOMAIN INTERPRETATION OF URABE'S CRITERION

In this appendix, we present a time-domain interpretation of Urabe's criterion with the use of the Abel transform, which will make a substantial supplement to the space-domain interpretation.¹⁰ In plain words, we describe a simple reason why Urabe's criterion contains a function S of odd symmetry but does not permit a function R of even symmetry. We also mention a possibility of $R \neq 0$, which becomes acceptable under the extended isochronicity condition.

Figure 5 shows the coordinates X_1 and X_2 used for the present illustration. Compared with x_1 and x_2 in Ref. 4, the direction of X_1 is reversed. This definition is more suitable for discussing the symmetry because both X_1 and X_2 are never negative-valued and completely indistinguishable to each other. First, we assume that the half-spaces in $X_1 > 0$ and in $X_2 > 0$ are both A-type: the local transit-time $t(E)$ and the local inverse potential $x(U)$ are related by the Abel transform in each half-space. This assumption means that the origin of $X_1 = X_2 = 0$ must be the minimum point of the potential U because the inverse potentials are monotonically increasing both in $X_1 > 0$ and in $X_2 > 0$, according to the condition (9) of the meaningful solution.

Let us start with the isochronous potential of the highest symmetry by assuming that the same constant transit-time $\tau/2$ is spent in each half-space. From (10a), the Abel transform of this constant transit-time is reduced to be the harmonic solution $X_{hmp}(U)$. Since $U_{hmp} \propto X^2$ in (10b), this solution ensures the analytic continuation around $X_1 = X_2 = 0$.

Next, we will distort this harmonic potential to construct non-harmonic isochronous solutions. Specifically, we first introduce a distortion $\Delta T(E)$ in the time domain and, then, convert it into a space-domain distortion $\Delta X(U)$, using the Abel-transform approach. Since $\Delta X(U)$ is always represented by [an odd function + an even function] in considering the symmetry around the origin of $X_1 = X_2 = 0$, the superposition principle allows us to separate the two cases: $\Delta X(U)$ of odd symmetry and that of even symmetry.

As for the odd-symmetry case, we have significant flexibility in selecting the time-domain distortion $\Delta T_o(E)$. This is readily seen by denoting

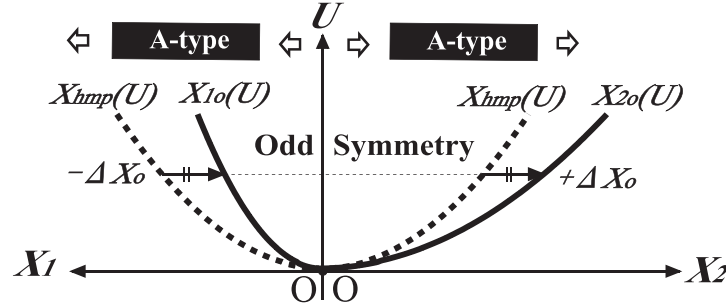
$$T_{1o}(E) = \tau/2 - \Delta T_o(E), \quad (\text{B1a})$$

$$T_{2o}(E) = \tau/2 + \Delta T_o(E), \quad (\text{B1b})$$

where $T_{1o}(E)$ and $T_{2o}(E)$ are the modified transit-times in X_1 and X_2 , respectively. Since $T_{1o}(E) + T_{2o}(E) = \tau$, the total isochronicity is automatically maintained. With the use of the Abel transforms, this set of time-domain distortions can be converted into a corresponding set of space-domain distortions of odd symmetry represented by $\Delta X_o(U) \propto A[\Delta T_o(E)]$. Specifically,

$$(a) \quad \int_0^{\sqrt{U}} S(\sqrt{U}) \cdot d(\sqrt{U}) \propto \Delta X_o(U) \propto A[\Delta T_o(E)]$$

$$\begin{array}{c} \{X_{hmp}(U) - \Delta X_o(U)\} + \{X_{hmp}(U) + \Delta X_o(U)\} = 2 X_{hmp}(U) \\ \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \\ \{\tau/2 - \Delta T_o(E)\} + \{\tau/2 + \Delta T_o(E)\} = \tau \end{array}$$



$$(b) \quad \int_0^{\sqrt{U}} R(\sqrt{U}) \cdot d(\sqrt{U}) \propto \Delta X_e(U) \propto A[\Delta T_e(E)]$$

$$\begin{array}{c} \{X_{hmp}(U) - \Delta X_e(U)\} + 2\Delta X_e(U) + \{X_{hmp}(U) - \Delta X_e(U)\} = 2 X_{hmp}(U) \\ \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \\ \{\tau/2 - \Delta T_e(E)\} + 2\Delta T_e(E) + \{\tau/2 - \Delta T_e(E)\} = \tau \end{array}$$

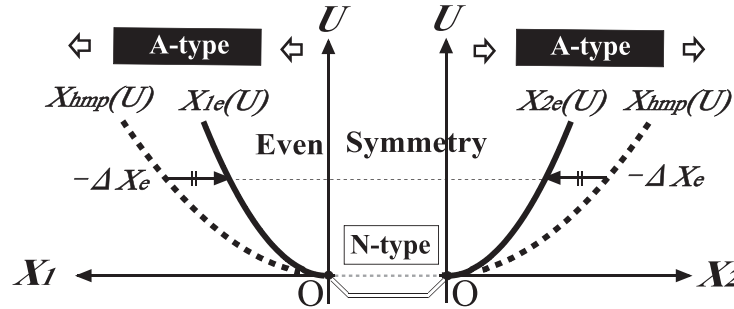


FIG. 5. Time-domain interpretation of Urabe's criterion. (a) The odd-symmetry case. (b) The even-symmetry case.

we obtain

$$X_{1o}(U) = X_{hmp}(U) - \Delta X_o(U), \quad (B2a)$$

$$X_{2o}(U) = X_{hmp}(U) + \Delta X_o(U), \quad (B2b)$$

where $X_{1o}(U)$ and $X_{2o}(U)$ are the distorted inverse potentials in X_1 and X_2 , respectively. Here, we note that $\Delta X_o(0) \equiv 0$ because $\Delta X_o(0)$ is always canceled simply by shifting the origin along the X_1 and the X_2 axes. In other words, the origin can remain to be the minimum point of U even after the distortion is added, as shown in Fig. 5(a). Notably, this property means the isolation of the right and the left half-spaces because $X_{1o}(U) \geq 0$ and $X_{2o}(U) \geq 0$ in $U \geq 0$, because of the monotonicity condition (9).

Carefully comparing $X_{1o}(U)$ and $X_{2o}(U)$ with (8) and (9) in Ref. 4, we find that $\Delta X_o(U)$ is equal to $f(2U)$ in Ref. 4. As a result, the function S of odd symmetry in Urabe's criterion is given by

$$S = \frac{2\pi}{\tau} \sqrt{2U} f'(2U) = \frac{2\pi}{\tau} \sqrt{2U} \Delta X'_o(U). \quad (B3)$$

From this expression, $\Delta X'_o(U) \propto S/\sqrt{U}$ and the further integration in U from 0 to U yields

$$\Delta X_o(U) - \Delta X_o(0) \propto \int_0^U (S/\sqrt{U}) \cdot dU \propto \int_0^{\sqrt{U}} S(\sqrt{U}) \cdot d(\sqrt{U}).$$

Since $\Delta X_o(0) \equiv 0$, the integral form equivalent to (B3) is given by

$$\int_0^{\sqrt{U}} S(\sqrt{U}) \cdot d(\sqrt{U}) \propto \Delta X_o(U) \propto A[\Delta T_o(E)]. \quad (\text{B4})$$

In the present coordinate system, the symmetry of the derivative $\Delta X'_o(U)$ must be the same as that of $\Delta X_o(U)$. Thus, the odd-symmetry function S in Urabe's criterion originates from the time-domain distortion ΔT_o that causes the odd-symmetry distortions in the space domain. The point about the present coordinate system is that the origin of $X_1 = X_2 = 0$ apparently becomes a singular point. So the smooth connection between $X_{1o}(U)$ and $X_{2o}(U)$ across the origin depends on the behavior of $\Delta T_o(E)$ around $E = 0$. Further studies are needed to examine this subject.

Similarly, the even-symmetry function R arises from the time-domain distortion ΔT_e that causes the even-symmetry distortions in the space domain:

$$T_{1e}(E) = \tau/2 - \Delta T_e(E), \quad (\text{B5a})$$

$$T_{2e}(E) = \tau/2 - \Delta T_e(E), \quad (\text{B5b})$$

where $T_{1e}(E)$ and $T_{2e}(E)$ are the distorted transit-times in X_1 and X_2 , respectively. From the ordinary isochronicity condition $T_{1e}(E) + T_{2e}(E) = \tau$, $\Delta T_e(E)$ must be zero. Within the framework of Lebesgue integrals, the uniqueness and the existence of the Abel-transformed solutions are strictly certified (see Appendix 1A in Ref. 9) and so $\Delta X_e(U) = A[\Delta T_e(E)] = 0$, which eventually leads to $\int_0^{\sqrt{U}} R(\sqrt{U}) \cdot d(\sqrt{U}) \propto \Delta X_e(U) = 0$. If the whole region belongs to A-type, then the non-zero R of even symmetry does not appear in Urabe's criterion.

Finally, we mention a possibility, presumably the unique one, to make $R \neq 0$: the introduction of the extended isochronicity condition. As Fig. 5(b) shows, $\Delta T_e(E)$ can be regarded as a time-domain component to compensate for the transit-time spent in the central N-type region. In this case, $\Delta X_e(U) (\propto A[\Delta T_e(E)] \neq 0)$ represents the spatial distortion to shorten the inverse potentials in the A-type regions. Obviously, the isochronicity is broken for $E < 0$ and the origins of $X_1 = 0$ and $X_2 = 0$ are spatially separated, though.

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