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ABSTRACT: It is old folklore that the violation of Leibniz rule on a lattice is an obstruction for constructing a lattice supersymmetric model. While it is still true for full supersymmetry, we show that a slightly modified form of the Leibniz rule, which we call cyclic Leibniz rule (CLR), is actually a criterion for the existence of partial lattice supersymmetry. In one dimension, we find sets of lattice difference operator and field multiplication smeared over lattice which satisfy the CLR under some natural assumptions such as translational invariance and locality. Thereby we construct a model of supersymmetric lattice quantum mechanics without spoiling locality. The CLR relation is coincident with the condition that the vanishing of the so-called surface term in the construction by lattice Nicolai map. We can construct superfield formalism with arbitrary superpotential. This also enables us to apply safely a localization technique to our model, because the kinetic term and the interaction terms of our model are independently invariant under the supersymmetry transformation. A preliminary attempt in finding a solution for the higher dimensional case is also discussed.

KEYWORDS: lattice, Leibniz rule, supersymmetry.

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Contents

1. Introduction

Supersymmetry (SUSY) is not only a candidate for physics beyond the standard model but also a possible framework (or a part of it) with definite ultraviolet behaviors in quantum field theories. If we can construct supersymmetric theories nonperturbatively, we shall obtain deeper understandings of vacuum property and other nonperturbative behaviors[1, 2, 3]. Although the SUSY algebra is generally incompatible with a discrete lattice space, there is no obstruction to formulate a free SUSY theory on lattice, if we replace an infinitesimal translation with a lattice translation.

Our real difficulty to overcome is Leibniz rule (LR) on lattice in interacting SUSY theories. Unfortunately, the no-go theorem for the rule on lattice has been proved in our previous papers[4, 5]. The essential point of the proof is the incompatibility of the translational invariance, the locality and the LR on lattice. Under a certain circumstances, SLAC type difference operator satisfies the LR on lattice[4, 6] but it is non-local.

As another approach, we could utilize a Nicolai map for constructing a lattice SUSY theory[7]. For supersymmetric quantum mechanics, a lattice model with a Nicolai map was proposed in Ref.[8],

$$S_N = \sum_n \frac{1}{2}\Big((\Delta\phi)_n - W_n(\phi)\Big)^2 + \sum_{m,n} i\bar{\psi}_m \left(\Delta_{mn} + \frac{\partial W_n(\phi)}{\partial\phi_m}\right)\psi_n\,, \qquad (1.1)$$

where Δ_{mn} is a difference operator and $W_n(\phi)$ is an arbitrary function of ϕ_m. The lattice action (1.1) possesses an exact supersymmetry[9, 10] but contains the unconventional term

$$-\sum_n (\Delta\phi)_n W_n(\phi)\,. \tag{1.2}$$

It should be noticed that the above term is absent from the action of supersymmetric quantum mechanics and would vanish in the continuum as a surface term. It cannot be, however, written into the form of total divergences due to the lack of the Leibniz rule on lattice. The author[11] has discussed, in a topological point of view, an off-shell action

$$S_B = \sum_n \frac{1}{2}(B_n)^2 + \sum_n iB_n\Big((\Delta\phi)_n - W_n(\phi)\Big) + \sum_{m,n} i\bar\psi_m\left(\Delta_{mn} + \frac{\partial W_n(\phi)}{\partial\phi_m}\right)\psi_n\,, \tag{1.3}$$

which reduces to (1.1) after eliminating the auxiliary field B_n. The above lattice action was rederived in a superfield formulation[12]. Although the lattice action has an exact fermionic symmetry, it may be regarded as a lattice version of the stochastic action[13] rather than the standard supersymmetric quantum mechanics,

$$S_{\mathrm{SQM}} = \int dt \left\{\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}F^2 + iFW(\phi) + i\bar\psi\left(\frac{d}{dt} + \frac{\partial W(\phi)}{\partial\phi}\right)\psi\right\} \tag{1.4}$$

with a Euclidean time t. Hence, it will be worth looking for a lattice formulation which leads to a lattice version of S_{SQM} and also does not rely on the existence of local Nicolai maps without unwanted terms like (1.2).

In this paper, we propose a novel criterion for the lattice SUSY realization which we call cyclic Leibniz rule (CLR), that is not a genuine LR on lattice but still a natural lattice analog of the Leibniz rule in the continuum theories in the sense that it is reduced to the corresponding relation in the continuum limit. We construct sets of the lattice difference operator and the associated symmetric field product which satisfy the CLR. With these set of difference operator and field product, we formulate the concrete example of lattice SUSY quantum mechanics. Although only a part of SUSY could be realized exactly, yet they have several advantages over the existing approaches. One of the advantages is that our lattice model is free from the "surface" term (1.2) and is just a lattice analog of the standard supersymmetric action (1.4). This property with CLR allows us to construct a superfield formulation, whose realization is different from that given in Ref.[12]. In the action of our model the kinetic term and the interaction terms are separately invariant under SUSY transformation. Because of this property we can apply a localization technique to obtain some exact results in our model. Since our lattice formulation mimics the continuum supersymmetric quantum mechanics and does not rely on the existence of local Nicolai maps, it may shed a new light on the construction of higher-dimensional supersymmetric lattice models.

In section 2, we propose the cyclic Leibniz rule (CLR) as a criterion for the SUSY by using a concrete example, for definiteness, of $D = 1$ $\mathcal{N} = 2$ Wess-Zumino model on lattice. The lattice difference opertors and associated symmetric product is found as a solution

to the CLR. Based on these, in section 3, we construct exactly (half-)supersymmetric quantum mechanics on lattice with arbitrary interactions. The Witten index is exactly calculated by the Nicolai map and the localization technique. Multicomponent cases are discussed in section 4 and we find a local solution which is a natural generalization of one-component case. In section 5, we give an attempt to a higher-dimensional case letting spinor index aside, while we so far could not find any suitable solution of the CLR for euclidean field theories. Section 6 is devoted to conclusion and discussions. In appendies, we find the general solution of CLR with locality, the properties of the associated product, and explicit multi-field product examples based on CLR.

2. A criterion for lattice supersymmetry

In this section, we begin by recapitulating the difficulty in constructing interacting super-symmetric theories on lattice where the important key concept is the Leibniz rule. Our proposal to avoid the difficulty will appear subsequently in this section.

For concreteness, we consider $D = 1$ model, *i.e.* supersymmetric quantum mechanics, which consists of a multiplet

$$\left(\phi_n, \psi_n, F_n\right) \tag{2.1}$$

where ϕ is a real scalar, F is a real auxiliary field and ψ is a complex fermion. Index n stands for a lattice site. na is a coordinate but hereafter we set lattice constant $a = 1$ for simplicity. The $\mathcal{N} = 2$ SUSY transformation is described as

$$
\begin{aligned}
\delta\phi_n &= \epsilon\bar{\psi}_n - \bar{\epsilon}\psi_n, \\
\delta\psi_n &= \epsilon(i(\Delta\phi)_n + F_n), \\
\delta\bar{\psi}_n &= \bar{\epsilon}(-i(\Delta\phi)_n + F_n), \\
\delta F_n &= -\epsilon i(\Delta\bar{\psi})_n - \bar{\epsilon}i(\Delta\psi)_n,
\end{aligned}
\tag{2.2}
$$

where a difference operator Δ is defined as

$$(\Delta A)_n \equiv \sum_m \Delta_{nm} A_m, \quad \sum_m \Delta_{nm} = 0, \tag{2.3}$$

for arbitray lattice field A_n. The second relation states that the difference of a constant field vanishes. The algebra follows from the above definition:

$$[\delta_1, \delta_2]\left(\phi_n, \psi_n, F_n\right) = 2i(\bar{\epsilon}_1\epsilon_2 - \bar{\epsilon}_2\epsilon_1)\left((\Delta\phi)_n, (\Delta\psi)_n, (\Delta F)_n\right). \tag{2.4}$$

The kinetic terms S_0 and the mass term S_m of the free action

$$S_0 = \sum_n \left(\frac{1}{2}(\Delta\phi)_n^2 + i\bar{\psi}_n(\Delta\psi)_n + \frac{1}{2}F_n^2\right), \tag{2.5}$$

$$S_m = im\sum_n \left(F_n\phi_n + \bar{\psi}_n\psi_n\right) \tag{2.6}$$

are invariant under (2.2) provided

$$\Delta_{nm} = -\Delta_{mn} \tag{2.7}$$

which is a generalized symmetric difference operator.

The obstruction for SUSY realization on lattice appears in the interaction parts. For an illustration we consider here cubic interaction such as

$$S_{\text{int}} = i\frac{g}{2} \sum_{\ell mn} (M_{\ell mn} F_\ell \phi_m \phi_n + 2N_{\ell mn} \phi_\ell \bar\psi_m \psi_n), \tag{2.8}$$

where g is a coupling constant, and $M_{\ell mn}$ and $N_{\ell mn}$ are coefficients for fields products. We can impose $M_{\ell mn} = M_{\ell nm}$ without the loss of generality. Under (2.2), the interaction terms are transformed as follows,

$$\begin{aligned}
\delta S_{\text{int}} = i\frac{g}{2} \sum_{\ell mn} \Big(&- 2\bar\epsilon N_{\ell mn} \psi_\ell \bar\psi_m \psi_n + 2\epsilon N_{\ell mn} \bar\psi_\ell \bar\psi_m \psi_n \\
&- 2\bar\epsilon (M_{\ell mn} - N_{m\ell n}) F_\ell \psi_m \phi_n + 2\epsilon (M_{\ell mn} - N_{nm\ell}) F_\ell \bar\psi_m \phi_n \\
&- i\bar\epsilon (M_{\ell mn} (\Delta\psi)_\ell \phi_m \phi_n + 2N_{\ell mn} \phi_\ell (\Delta\phi)_m \psi_n) \\
&- i\epsilon (M_{\ell mn} (\Delta\bar\psi)_\ell \phi_m \phi_n + 2N_{\ell mn} \phi_\ell \bar\psi_m (\Delta\phi)_n) \Big).
\end{aligned} \tag{2.9}$$

In order to accomplish the invariance under (2.2), we must require

$$\begin{aligned}
N_{\ell mn} &= N_{m\ell n}, \\
M_{\ell mn} &= N_{nm\ell}, \\
\sum_k (\Delta_{k\ell} M_{kmn} &+ N_{n\ell k}\Delta_{km} + N_{m\ell k}\Delta_{kn}) = 0.
\end{aligned} \tag{2.10}$$

for ϵ invariance and

$$\begin{aligned}
N_{\ell mn} &= N_{nm\ell}, \\
M_{\ell mn} &= N_{m\ell n}, \\
\sum_k (\Delta_{k\ell} M_{kmn} &+ N_{nk\ell}\Delta_{km} + N_{mk\ell}\Delta_{kn}) = 0,
\end{aligned} \tag{2.11}$$

for $\bar\epsilon$ invariance. These conditions can be combined into a simpler form

$$M_{\ell mn} = M_{\ell nm} = M_{nm\ell} = N_{\ell mn} = N_{\ell nm} = N_{nm\ell}, \tag{2.12}$$

$$\sum_k (\Delta_{k\ell} M_{kmn} + M_{\ell kn}\Delta_{km} + M_{\ell mk}\Delta_{kn}) = 0. \tag{2.13}$$

With (2.7), the relation (2.13) implies a Leibniz rule on lattice

$$\sum_k \Delta_{\ell k} \sum_{m,n} M_{kmn}\phi_m\psi_n = \sum_{k,n} M_{\ell kn}(\Delta\phi)_k\psi_n + \sum_{m,k} M_{\ell mk}\phi_m(\Delta\psi)_k, \tag{2.14}$$

where ϕ, ψ are any lattice fields.

Before going into the further analysis, we describe the translational invariant and local difference operator and product-coefficients as holomorphic functions[4]. From translational invariance, these objects depend only on the difference of site indices, so we denote conveniently

$$\Delta(k) \equiv \Delta_{m\,m-k}, \quad M(k,\ell) \equiv M_{m\,m-k\,m-\ell}. \tag{2.15}$$

By transforming them to the momentum representation ($w-$expression), the complex functions

$$\hat{\Delta}(w) \equiv \sum_{n=-\infty}^{\infty} w^n \Delta(n), \quad \hat{M}(w,z) \equiv \sum_{m,n=-\infty}^{\infty} w^m z^n M(m,n) = \hat{M}(z,w) \tag{2.16}$$

are defined. Substituting $w = \exp(ipa)$, $z = \exp(iqa)$, the expression (2.16) exactly corresponds to its Fourier expansion and p, q become momentum variables when p, q are real. The locality of the difference operator is exactly equivalent to the holomorphism of the w-expression, and the holomorphic domain $\mathcal{D}$ is an annulus in the complex w-plane which includes a unit circle around the origin. For the product $\hat{M}(w,z)$, the domain is $\mathcal{D} \otimes \mathcal{D}$ as a two-variables function. Namely, we proved that the locality, *i.e.* long distance behaviors

$$|\Delta_{m\,n}| \leq C \exp\left(-K_0|m-n|\right), \quad |M_{\ell m n}| \leq C_1 \exp\left(-K_1|\ell-m| - K_2|\ell-n|\right) \tag{2.17}$$

with C, C_1, K_0, K_1, $K_2 > 0$ are equivalent to the holomorphism of $\hat{\Delta}(w)$ and $\hat{M}(w,z)$[4]. From the second relation of (2.3), the correspondence $\hat{\Delta}$ has the property,

$$\hat{\Delta}(1) = 0. \tag{2.18}$$

Turning the relation (2.13) into the w-expression, we can find

$$\hat{M}(w,z)\hat{\Delta}(\frac{1}{wz}) + \hat{M}(w,z)\hat{\Delta}(w) + \hat{M}(w,z)\hat{\Delta}(z) = 0. \tag{2.19}$$

The only holomorphic solution around $w = z = 1$ of (2.19) is $\hat{\Delta}(w) \sim \log w$ *i.e.* a SLAC-type difference operator besides a trivial solution with $\hat{M}(w,z) = 0$. The SLAC-type operator is, however, not holomorphic in the annulus $\mathcal{D}$. Thus there is no set of Δ and $M(\neq 0)$ with translational invariance and locality that satisfies (2.13). This is an essence of the no-go theorem for the Leibniz rule on lattice proved in our previous work[4]. So the lack of lattice Leibniz rule is an obstruction for lattice supersymmetry with interaction.

To overcome the above problem, we demand only a subset of supersymmetry. For example, if we impose the invariance only for ϵ in (2.2), then we have the half of the condition, *i.e.* (2.10) without (2.11). Thus, instead of (2.12) and (2.13), we obtain

$$M_{\ell m n} = M_{\ell n m} = N_{n m \ell} = N_{m n \ell}, \tag{2.20}$$

$$\sum_k (\Delta_{k\ell} M_{kmn} + M_{kn\ell}\Delta_{km} + M_{k\ell m}\Delta_{kn}) = 0. \tag{2.21}$$

We call (2.21) as *cyclic Leibniz rule* (CLR). This relation cannot be factorized, like (2.19), into a separate condition for a difference operator, but must be solved by the combination

of a difference operator and a product. It is worth noting that the symmetry in the product $M_{k\ell m}$ of (2.12) is different from that of (2.20); the former is totally-symmetric for the three indices and the latter is only symmetric for the second and the third indices. Although the conventional LR (2.13) seems equivalent to the CLR (2.21) in the naive continuum limit because the limit implies $M_{k\ell m} \to \delta_{k\ell}\delta_{km}$, they clearly have different property for the finite lattice constant. In contrast to (2.14), the CLR can be expressed as

$$\sum_k \Delta_{\ell k} \sum_{m,n} M_{kmn}\phi_m\psi_n = \sum_{n,k} M_{kn\ell}(\Delta\phi)_k\psi_n + \sum_{k,m} M_{k\ell m}\phi_m(\Delta\psi)_k, \qquad (2.22)$$

whose indices are indeed summed in a different way from (2.14). Although the CLR is found in constructing a specific model with cubic interaction, it will turn out to be very useful for the analysis of more general supersymmetric theories in later sections. Instead of the usual Leibniz rule (2.13), we can adopt the CLR as the key rule for the exact lattice supersymmetry. As seen later, we can construct a superfield formalism and find a Nicolai map with vanishing surface terms in interacting cases. This framework with the exact symmetry are a real advantage of our approach .

In w-represenation, the cyclic Leibniz rule (2.21) can be expressed as

$$\hat{M}(w,z)\hat{\Delta}(\frac{1}{wz}) + \hat{M}(z,\frac{1}{wz})\hat{\Delta}(w) + \hat{M}(\frac{1}{wz},w)\hat{\Delta}(z) = 0. \qquad (2.23)$$

We give here an example solution for (2.23) keeping translational invariance and locality

$$\hat{M}(w,z) = \frac{1}{6}\Big(2wz + wz^{-1} + zw^{-1} + 2(wz)^{-1}\Big), \quad \hat{\Delta}(w) = \frac{w - w^{-1}}{2}. \qquad (2.24)$$

This corresponds to a real space expression

$$M_{\ell mn} = \frac{1}{6}(2\delta_{\ell,m-1}\delta_{\ell,n-1} + \delta_{\ell,n-1}\delta_{\ell,m+1} + \delta_{\ell,n+1}\delta_{\ell,m-1} + 2\delta_{\ell,m+1}\delta_{\ell,n+1}),$$
$$\Delta_{mn} = \frac{1}{2}(\delta_{m,n-1} - \delta_{m,n+1}). \qquad (2.25)$$

This simple example has doublers which can be resolved by a supersymmetric Wilson term or a supersymmetric overlap-type mass. We shall see these terms explicitly in the subsection 3.1. More generic other solutions to (2.23) are given in Appendix A.

3. Analysis of supersymmetric lattice quantum mechanics

In the previous section, we have found the important relation, the CLR. Its solution consists of a local difference operator and a symmetric field product smeared over the lattice sites keeping locality. In this section, we explicitly construct general supersymmetric quantum mechanics using the CLR as a guiding principle. And we will analyze the model by localization technique.

3.1 Construction of supersymmetric action

We saw in the previous section that a cubic action can be half-supersymmetric if the product satisfies the CLR. This can be immediately extended to other order terms including quadratic, quartic or higher. An important point is that the CLR for each order guarantees the invariance of each order term independently. In other words, the coupling constant for each order term can be taken independently. This naturally leads us to the superfield formalism with half-supersymmetry to construct generic action. Also we are able to utilize this property to make an exact calculation with localization technique which will be discussed in the next subsection.

Let us first define a 2-body product of lattice fields using the coefficient $M_{\ell m n}$

$$\{\phi, \psi\}_\ell \equiv \sum_{mn} M_{\ell m n} \phi_m \psi_n = (-1)^{\epsilon_\phi \epsilon_\psi} \{\psi, \phi\}_\ell, \tag{3.1}$$

where ϵ_ϕ and ϵ_ψ are the Grassmann-parity for ϕ and ψ. We refer this product as *a smeared symmetric product*, because this is a symmetric product smeared over lattice sites. The locality of the product is guaranteed by the holomorphic property of its w-expression $\hat{M}(z, w)$. We also define an inner product of two lattice fields

$$(\phi, \psi) \equiv \sum_n \phi_n \psi_n. \tag{3.2}$$

Then a simple 3-body summed product of lattice fields is expressed as

$$(\phi, \{\psi, \chi\}) \equiv \sum_{\ell m n} M_{\ell m n} \phi_\ell \psi_m \chi_n. \tag{3.3}$$

With these notations, as well as (2.7), (2.21), the CLR can be rewritten into a compact form

$$(\Delta\phi, \{\psi, \chi\}) + (\Delta\psi, \{\chi, \phi\}) + (\Delta\chi, \{\phi, \psi\}) = 0. \tag{3.4}$$

In the case of three same fields, (3.4) is reduced to

$$(\Delta\phi, \{\phi, \phi\}) = 0. \tag{3.5}$$

The expression (3.5) is equivalent to (3.4) if we take the symmetry of the product M into account. These simple expressions are useful in further SUSY analysis.

Now we construct various terms in the action using superfield formalism. As stated before, the kinetic term is fully supersymmetric with ϵ and $\bar{\epsilon}$, while the interaction term is half-supersymmetric with, say ϵ. Therefore we can use $\mathcal{N} = 2$ superspace for the former, but must use $\mathcal{N} = 1$ for the latter.

An $\mathcal{N} = 2$ supermultiplet for a superspace $(n, \theta, \bar{\theta})$

$$\Xi_n(\theta, \bar{\theta}) = \Phi_n(\theta) - \bar{\theta}\Psi_n(\theta) \tag{3.6}$$

is decomposed into two $\mathcal{N} = 1$ multiplets,

$$\Phi_n(\theta) = \phi_n + \theta\bar{\psi}_n \tag{3.7}$$

which is Grassmann even and

$$\Psi_n(\theta) = \psi_n + \theta F_n \tag{3.8}$$

which is Grassmann odd.

A kinetic term of the action can be written as

$$S_0 = \frac{1}{2} \int d\bar{\theta}d\theta(\bar{D}\Xi(\theta,\bar{\theta}), D\Xi(\theta,\bar{\theta})), \tag{3.9}$$

where supercovariant difference operators are defined as

$$\bar{D}_{mn} \equiv -i\frac{\partial}{\partial\theta}\delta_{mn} - \bar{\theta}\Delta_{mn}, \quad D_{mn} \equiv -i\frac{\partial}{\partial\bar{\theta}}\delta_{mn} - \theta\Delta_{mn}. \tag{3.10}$$

To formulate mass term and Wilson term supersymmetrically, we consider a generic bilinear form

$$(\phi, G\psi) \equiv \sum_{k\ell} \phi_k G_{k\ell}\psi_\ell, \tag{3.11}$$

where G_{mn} is a certain operator on a lattice. The translational invariance implies that G is a function of the difference of its indices $G_{mn} = G(m - n)$, and satisfies $\Delta G = G\Delta$ with translationally-invariant difference operator Δ. Thus we have

$$(\Delta\phi, G\psi) = -(\phi, G\Delta\psi), \tag{3.12}$$

with $\Delta^T = -\Delta$. Then

$$S_2 = i \int d\theta \, (\Psi(\theta), G\Phi(\theta)) \tag{3.13}$$

gives a supersymmetric quadratic action invariant under ϵ-transformation. For example, a simple mass term with Wilson term is given by choosing

$$\hat{G}(w) \equiv \sum_n w^n G(n) = im + ir\frac{2 - w - w^{-1}}{2} \tag{3.14}$$

where m is a bare mass and r is a standard Wilson term parameter. For more generic doubler-suppressing term, we can write

$$\hat{G}(w) = im + i\hat{H}(w, M), \tag{3.15}$$

where $\hat{H}(w, M)$ is a holomorphic function in $\mathcal{D}$ with a parameter $M > 0$ and corresponds to w-expression of a translationally invariant operator $H_{mn}(M)$, i.e. $\hat{H}(w, M) = \sum_m w^m H_{\ell\,\ell+m}(M)$. We further impose that H is symmetric, i.e. $H^T = H$ or $\hat{H}(w) = \hat{H}(1/w)$, throughout this paper. If we need an overlap-type action, we choose $\hat{H}$ to have the poles at $w = 0, \infty$ and the branch points at $w = -1/M, -M$ with $M > 1$, as well as the property $\hat{H}(w = 1, M) = 0$. The explicit form for the overlap-type term[14, 15] is given as

$$\frac{\hat{H}(w, M)}{M + 1} = 1 - \frac{(M + \frac{w+w^{-1}}{2})}{\sqrt{-(\frac{w-w^{-1}}{2})^2 + (M + \frac{w+w^{-1}}{2})^2}} = 1 - \frac{(M + \frac{w+w^{-1}}{2})}{\sqrt{(M + w)(M + w^{-1})}}, \tag{3.16}$$

Note that, together with an explicit solution for the CLR

$$\hat{\Delta}(w) = \frac{(M+1)(w - w^{-1})}{2\sqrt{(M+w)(M+w^{-1})}}, \tag{3.17}$$

$$\hat{M}(w,z) = \frac{1}{6(M+1)}\sqrt{(M+w)(M+w^{-1})}\Big(2wz + wz^{-1} + zw^{-1} + 2(wz)^{-1}\Big), \tag{3.18}$$

the above H satisfies an analog of the Ginsparg-Wilson relation[16, 17],

$$\hat{\Delta}(w)\hat{\Delta}(w^{-1}) + \hat{H}(w,M)\hat{H}(w^{-1},M) = (M+1)\Big(\hat{H}(w,M) + \hat{H}(w^{-1},M)\Big), \tag{3.19}$$

where the Dirac operator is given by $\hat{D}_{Dirac}(w) = i\hat{\Delta}(w) + i\hat{H}(w,M)$.

The quadratic terms including mass and doubler-suppressing term and the cubic interaction term of the supersymmetric action are now obtained as

$$S_2 + S_{\text{int}} = i\int d\theta\Big((\Psi(\theta), G\Phi(\theta)) + g_2(\Psi(\theta), \{\Phi(\theta), \Phi(\theta)\})\Big) \tag{3.20}$$

with a coupling constant g_2 (here subscript 2 stands for 2-body product). The action (3.20) is clearly invariant under SUSY transformation

$$\begin{aligned}
\delta\phi_n &= \epsilon\bar{\psi}_n, \\
\delta\psi_n &= \epsilon(i(\Delta\phi)_n + F_n), \\
\delta\bar{\psi}_n &= 0, \\
\delta F_n &= -i\epsilon(\Delta\bar{\psi})_n.
\end{aligned} \tag{3.21}$$

This transformation can be realized on the superfield as follows,

$$\delta\Xi_n(\theta,\bar{\theta}) = \sum_m \epsilon\Big(\frac{\partial}{\partial\theta}\delta_{nm} + i\bar{\theta}\Delta_{nm}\Big)\Xi_m(\theta,\bar{\theta}) = \epsilon\Big(\frac{\partial}{\partial\theta}\Phi_n(\theta) + \bar{\theta}\frac{\partial}{\partial\theta}\Psi_n(\theta) + i\bar{\theta}(\Delta\Phi)_n(\theta)\Big), \tag{3.22}$$

or equivalently on the $\mathcal{N} = 1$ superfields

$$\begin{aligned}
\delta\Phi_n(\theta) &= \epsilon\frac{\partial}{\partial\theta}\Phi_n(\theta), \\
\delta\Psi_n(\theta) &= \epsilon\frac{\partial}{\partial\theta}\Psi_n(\theta) + i\epsilon(\Delta\Phi)_n(\theta).
\end{aligned} \tag{3.23}$$

Thus the total action

$$\begin{aligned}
S_{\text{t}} &= S_0 + S_2 + S_{\text{int}} \\
&= \frac{1}{2}(\Delta\phi, \Delta\phi) + i(\bar{\psi}, \Delta\psi) + \frac{1}{2}(F, F) \\
&\quad + (F, (im + iH)\phi) + (\bar{\psi}, (im + iH)\psi) + ig_2(F, \{\phi, \phi\}) - 2ig_2(\psi, \{\phi, \bar{\psi}\}) \quad (3.24)
\end{aligned}$$

is also invariant under (3.22) or (3.23),

$$\delta S_{\text{t}} = 0, \tag{3.25}$$

due to the relation (3.12) and the CLR (3.5). It is worth emphasizing here that our formulation keeps locality; the holomorphic property of the w-expression of various operators and product guarantees the locality.

To extend to multi-body supersymmetric interactions, we define the smeared N-body products for a bosonic superfield

$$\{\Phi, \ldots, \Phi\}_n \equiv \sum_{m_1, \ldots, m_N} M_{n;m_1, \ldots, m_N} \Phi_{m_1} \cdots \Phi_{m_N}. \tag{3.26}$$

where $m_1, \ldots, m_N$ are totally symmetric indices (we have been omitting semicolon for $N = 2$, i.e. $M_{n;m_1,m_2} = M_{nm_1m_2}$.)

If this product satisfies the N-body CLR

$$(\Delta\Phi, \{\Phi, \ldots, \Phi\}) = 0, \tag{3.27}$$

then the action

$$\int d\theta \, ig_N \, (\Psi, \{\Phi, \ldots, \Phi\}) \tag{3.28}$$

is invariant under the supersymmetry transformations (3.23) in exactly same way as the 2-body case. It should be noticed that the R-symmetry

$$\theta \to e^{i\alpha}\theta,$$
$$\Phi \to \Phi,$$
$$\Psi \to e^{i\alpha}\Psi, \tag{3.29}$$

allows only the type of interaction terms (3.28), which consists of Φ and Ψ and is necessary for constructing supersymmetric $2N$-body interactions for scalar fields with a coupling constant g_N. Also note that N-body product satisfying the CLR can be constructed in terms of 2-body products which will be discussed in Appendix B.

Before closing this subsection, it may be helpful to understand the exact SUSY invariance in an alternative way. We write the total action S_t as a sum of kinetic term S_0, bare mass term S_m, doubler-suppressing term S_ds and interaction term S_int

$$S_t = S_0 + S_m + S_\mathrm{ds} + S_\mathrm{int}. \tag{3.30}$$

Then each term can be expressed as a supersymmetric transform of something

$$S_0 = \frac{1}{2}\delta'(\psi, (-i\Delta\phi + F)),$$
$$S_m = im\delta'(\psi, \phi),$$
$$S_\mathrm{ds} = i\delta'(\psi, H\phi),$$
$$S_\mathrm{int} = S_3 + S_4 + \cdots = ig_2\delta'(\psi, \{\phi, \phi\}) + ig_3\delta'(\psi, \{\phi, \phi, \phi\}) + \cdots, \tag{3.31}$$

with coupling constants $g_2, g_3, \cdots$. Here we denote the transformation without parameter by δ', i.e. $\delta = \epsilon\delta'$, or explicitly

$$\delta'\phi_n = \bar{\psi}_n,$$
$$\delta'\psi_n = i(\Delta\phi)_n + F_n,$$
$$\delta'\bar{\psi}_n = 0,$$
$$\delta'F_n = -i(\Delta\bar{\psi})_n. \tag{3.32}$$

Due to the nilpotencey of the transformation $\delta'^2 = 0$, the invariance of each term is manifest,

$$\delta' S_0 = \delta' S_m = \delta' S_{\text{ds}} = \delta' S_{\text{int}} = 0. \tag{3.33}$$

3.2 Localization technique and calculation of Witten index

With the action (3.30), we can find a Nicolai map for our system, and apply the localization technique[18, 19] to calculate the Witten index (the partition function with periodic boundary condition for all variables).

Before going into the detail, it would be useful to compare with the previous attempts in the existing literature. In the references [3, 12] the partition function has been calculated in the ultra local limit, where the N-site partition function Z_N reduces to N copies of the one-site one, *i.e.* $Z_N = (Z_1)^N$. This result indicates that every degrees of freedom may contribute to the Witten index that is quite different from the usual understanding of its topological nature. Also it depends on whether the number of lattice sites is even or odd for $Z_1 = -1$ case of which we could not understand the validity. On the other hand, our lattice formulation is suitable to use the localization technique as will be seen below. It will turn out that the Witten index is determined by only zero modes and all fluctuation modes are irrelevant in it.

Now let us begin by defining the Nicolai map

$$\xi_n = -(\Delta\phi)_n + m\phi_n + (H\phi)_n + g_2\{\phi, \phi\}_n + g_3\{\phi, \phi, \phi\}_n + \cdots. \tag{3.34}$$

With this, the Witten index can be calculated as

$$Z = \int D\phi D\bar{\psi} D\psi DF\, e^{-S_{\text{t}}} = \int D\xi D\bar{\psi} D\psi |\det\eta|^{-1} e^{-\sum_n \frac{1}{2}\xi_n\xi_n - i\sum_{kl}\bar{\psi}_l\eta_{lk}\psi_k}$$

$$= \int D\xi |\det\eta|^{-1} \det\eta\, e^{-\sum_n \frac{1}{2}\xi_n\xi_n}, \tag{3.35}$$

where all variables follow a periodic boundary condition and

$$\eta_{\ell k} \equiv \frac{\partial\xi_k}{\partial\phi_\ell} = \Delta_{\ell k} + m\delta_{\ell k} + (H)_{\ell k} + g_2\frac{\partial}{\partial\phi_\ell}\{\phi, \phi\}_k + g_3\frac{\partial}{\partial\phi_\ell}\{\phi, \phi, \phi\}_k + \cdots. \tag{3.36}$$

The prefactor of (3.35)

$$|\det\eta|^{-1}\det\eta, \tag{3.37}$$

is not generally a constant but $\det\eta$ can vanish for certain cases. Also the path of the integral (3.35) is not definite. Therefore we must carry out more detailed analysis.

In order to control the infrared behavior, we restrict a size of the system to finite N. Also we assume the highest power of the potential in (3.34) is p, for definiteness. We rescale our kinetic and doubler-suppressing term with parameters t and t_{ds} respectively,

$$S_0 \to tS_0, \quad S_{\text{ds}} \to t_{\text{ds}}S_{\text{ds}}. \tag{3.38}$$

Note that $t_{\rm ds}$ looks like a Wilson term parameter for Wilson fermion case. Then we integrate out F_n and obtain a t-scaled partition function,

$$
\begin{aligned}
Z(t,t_{\rm ds}) &\equiv \int D\phi D\bar\psi D\psi DF \, e^{-S(t,t_{\rm ds})} \\
&\equiv \int D\phi D\bar\psi D\psi DF \, e^{-tS_0 - t_{\rm ds}S_{\rm ds} - S_m - S_{\rm int}} \\
&= t^{-N/2} \int D\phi D\bar\psi D\psi \\
&\quad \times \, e^{-\left(tS_0^b + t_{\rm ds}^2 t^{-1}S_{\rm ds}^b + t^{-1}S_{m+{\rm int}}^b + t_{\rm ds}t^{-1}S_{\rm cross}^b\right) - \left(tS_0^f + t_{\rm ds}S_{\rm ds}^f + S_{m+{\rm int}}^f\right)}.
\end{aligned}
\tag{3.39}
$$

Here we denote $S_0^b, S_m^b, S_{\rm ds}^b$ and $S_{\rm int}^b$ as bosonic (without fermion) parts of kinetic, mass, doubler-suppressing and interaction terms, respectively. $S_0^f, S_m^f, S_{\rm ds}^f$ and $S_{\rm int}^f$ are fermionic (with fermion) counterparts. We summarize each part of our action in Table 1. In addition,

Table 1: Bosonic and fermionic parts of the action after integrating of F_n

S_0^b	$S_{\rm ds}^b$	$S_{m+{\rm int}}^b$
$\frac{1}{2}(\Delta\phi, \Delta\phi)$	$\frac{1}{2}(\phi, H^2\phi)$	$\frac{1}{2}(m\phi + g_2\{\phi,\phi\} + \cdots, m\phi + g_2\{\phi,\phi\} + \cdots)$

S_0^f	$S_{\rm ds}^f$	$S_{m+{\rm int}}^f$
$i(\bar\psi, \Delta\psi)$	$i(\bar\psi, H\psi)$	$im(\bar\psi, \psi) - i(\psi, 2g_2\{\phi,\bar\psi\} + \cdots)$

the cross terms $S_{\rm cross}^b$ for doubler-suppressing term and mass plus interaction terms are defined as

$$
S_{\rm cross}^b \equiv (H\phi, \, m\phi + g_2\{\phi,\phi\} + \cdots).
\tag{3.40}
$$

It is clear that this $Z(t,t_{\rm ds})$ is t-independent, because before F_n-integration one can show

$$
\begin{aligned}
\frac{\partial}{\partial t}Z(t,t_{\rm ds}) &= -\int D\phi D\bar\psi D\psi DF \, S_0 \, e^{-tS_0 - S_m - t_{\rm ds}S_{\rm ds} - S_{\rm int}} \\
&= -\int D\phi D\bar\psi D\psi DF \, \delta'(Xe^{-tS_0 - S_m - t_{\rm ds}S_{\rm ds} - S_{\rm int}}) = 0,
\end{aligned}
\tag{3.41}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t_{\rm ds}}Z(t,t_{\rm ds}) &= -\int D\phi D\bar\psi D\psi DF \, S_{\rm ds} \, e^{-tS_0 - S_m - t_{\rm ds}S_{\rm ds} - S_{\rm int}} \\
&= -\int D\phi D\bar\psi D\psi DF \, \delta'(Ye^{-tS_0 - S_m - t_{\rm ds}S_{\rm ds} - S_{\rm int}}) = 0,
\end{aligned}
\tag{3.42}
$$

where $X \equiv \frac{1}{2}\left(\psi, -i\Delta\phi + F\right)$, $Y \equiv \left(\psi, H\phi\right)$. Here we used the invariance of the path-integral measure under δ' and $\delta'(S(t,t_{\rm ds})) = 0$.

In the following calculations of the Witten index, we take two typical limiting cases. One is analogous to the continuum theory in which important modes are only zero-modes. The second case is peculiar to lattice theories, which has doubling modes in addition to zero-modes.[1] Although the two cases are physically and mathematically different, after using localization technique, we can obtain the exactly same results. We first decompose each field into zero mode (script 0), doubler mode (script d) and the other modes:

$$\phi_n = \frac{\phi_0}{\sqrt{N}} + \frac{(-1)^n}{\sqrt{N}}\phi_d + a_n,$$

$$\psi_n = \frac{\psi_0}{\sqrt{N}} + \frac{(-1)^n}{\sqrt{N}}\psi_d + \chi_n,$$

$$\bar{\psi}_n = \frac{\bar{\psi}_0}{\sqrt{N}} + \frac{(-1)^n}{\sqrt{N}}\bar{\psi}_d + \bar{\chi}_n \tag{3.43}$$

Then we rescale each part with t or t_{ds} in such a way that the only important mode survives in the infinitely scaling limit.

$\underline{\text{Case (i)}}$ We make the scaling of each field as

$$\phi_n = \frac{\phi_0}{\sqrt{N}} + \frac{(-1)^n}{\sqrt{N}\sqrt{t_{\mathrm{ds}}}}\phi_d + \frac{1}{\sqrt{t}}a_n,$$

$$\psi_n = \frac{\psi_0}{\sqrt{N}} + \frac{(-1)^n}{\sqrt{N}\sqrt{t_{\mathrm{ds}}}}\psi_d + \frac{1}{\sqrt{t}}\chi_n,$$

$$\bar{\psi}_n = \frac{\bar{\psi}_0}{\sqrt{N}} + \frac{(-1)^n}{\sqrt{N}\sqrt{t_{\mathrm{ds}}}}\bar{\psi}_d + \frac{1}{\sqrt{t}}\bar{\chi}_n, \tag{3.44}$$

then we take $t, t_{\mathrm{ds}} \to \infty$ with a fixed t/t_{ds}. This case is similar to the continuum theory, because the real zero mode (not doubler) is only t, t_{ds}-unscaled. A scalar zero mode ϕ_0 contributes to $t^{-1}S^b_{m+\mathrm{int}} + S^f_{m+\mathrm{int}}$ and a scalar doubler mode ϕ_d contributes to

$$\frac{t^2_{\mathrm{ds}}}{t}S^b_{\mathrm{ds}} = \frac{t_{\mathrm{ds}}}{tN}\sum_m\left(\sum_n H_{mn}(-1)^n\phi_d\right)^2 = \frac{t_{\mathrm{ds}}}{t}(\hat{H}(-1))^2\phi_d^2, \tag{3.45}$$

where $\hat{H}(w) \equiv \sum_n w^n H(n)$ for translational invariant $H_{mn} = H(m-n)$. On the other hand, the fermion zero modes contribute to $S^f_{m+\mathrm{int}}$, and a fermion doubler mode contributes to

$$t_{\mathrm{ds}}S^f_{\mathrm{ds}} = i\frac{1}{N}\sum_{mn}(-1)^m\bar{\psi}_d H_{mn}(-1)^n\psi_d = i\hat{H}(-1)\bar{\psi}_d\psi_d. \tag{3.46}$$

Any non-zero modes can be integrated out in the limit $t, t_{\mathrm{ds}} \to \infty$ with fixed t/t_{ds}, and the t factor of the partition function (3.39) is exactly canceled. $(t_{\mathrm{ds}}/t)S^b_{\mathrm{cross}}$ terms in the action vanish in the limit.

---

[1]Hereafter we assume the number of sites N is even for simplicity. There is no doubler mode for odd N, so that it gives the same result as Case (i) below.

Therefore

$$
\begin{aligned}
Z(t, t_{\mathrm{ds}}) &= \frac{\sqrt{t_{\mathrm{ds}}}}{t} \int d\phi_0 d\phi_d d\bar{\psi}_0 d\psi_0 d\bar{\psi}_d d\psi_d \ e^{-(t_{\mathrm{ds}}^2 t^{-1} S_{\mathrm{ds}}^b + t^{-1} S_{m+\mathrm{int}}^b) - (t_{\mathrm{ds}} S_{\mathrm{ds}}^f + S_{m+\mathrm{int}}^f)} \\
&= \frac{\sqrt{t_{\mathrm{ds}}}}{t} \int d\phi_d \bar{\psi}_d \psi_d \ e^{-t_{\mathrm{ds}}^2 t^{-1} S_{\mathrm{ds}}^b - t_{\mathrm{ds}} S_{\mathrm{ds}}^f} \int d\phi_0 d\bar{\psi}_0 d\psi_0 \ e^{-t^{-1} S_{m+\mathrm{int}}^b - S_{m+\mathrm{int}}^f} \\
&= \frac{\sqrt{t_{\mathrm{ds}}}}{t} \sqrt{\frac{t}{t_{\mathrm{ds}}(\hat{H}(-1))^2}} \, \hat{H}(-1) \sqrt{t} \int d\xi |\eta_0|^{-1} \eta_0 \ e^{-\frac{1}{2}\xi^2},
\end{aligned} \tag{3.47}
$$

where

$$
\xi(\phi_0) \equiv \frac{1}{\sqrt{t}} \left(m\phi_0 + \frac{g_2}{\sqrt{N}} \hat{M}(1,1)\phi_0^2 + \frac{g_3}{N} \hat{M}(1,1,1)\phi_0^3 + \cdots + \frac{g_p}{\sqrt{N}^{p-1}} \hat{M}(1,\cdots,1)\phi_0^p \right),
$$

$$
\eta_0(\phi_0) \equiv m + \frac{2g_2}{\sqrt{N}} \hat{M}(1,1)\phi_0 + \frac{3g_3}{N} \hat{M}(1,1,1)\phi_0^2 + \cdots + \frac{pg_p}{\sqrt{N}^{p-1}} \hat{M}(1,\cdots,1)\phi_0^{p-1} \tag{3.48}
$$

with w-expression of translaionally invariant multi-body product

$$
\hat{M}(w_1,\cdots,w_{n-1}) = \sum_{m_1,\cdots,m_{n-1}} w_1^{m_1} \cdots w_{n-1}^{m_{n-1}} M_{\ell;\ell+m_1,\cdots,\ell+m_{n-1}}. \tag{3.49}
$$

The final expression (3.47) results in $Z(t, t_{\mathrm{ds}}) = 0$ for $p =$ even and $Z(t, t_{\mathrm{ds}}) = \pm 1$ for $p =$ odd (its sign depends on the asymptotic behavior of the potential).

<u>Case (ii)</u> An alternative way of rescaling of fields is given by

$$
\begin{aligned}
\phi_n &= \frac{1}{\sqrt{N}}\phi_0 + \frac{(-1)^n}{\sqrt{N}}\phi_d + \frac{1}{\sqrt{t}}a_n, \\
\psi_n &= \frac{1}{\sqrt{N}}\psi_0 + \frac{(-1)^n}{\sqrt{N}}\psi_d + \frac{1}{\sqrt{t}}\chi_n, \\
\bar{\psi}_n &= \frac{1}{\sqrt{N}}\bar{\psi}_0 + \frac{(-1)^n}{\sqrt{N}}\bar{\psi}_d + \frac{1}{\sqrt{t}}\bar{\chi}_n.
\end{aligned} \tag{3.50}
$$

Then we take the limit $t \to \infty$ with a fixed t_{ds}. In this case not only zero mode but also doubler mode are t-unscaled. A ϕ_0 contributes to

$$
t^{-1} S_{m+\mathrm{int}}^b, \quad S_{m+\mathrm{int}}^f, \tag{3.51}
$$

and a ϕ_d contributes to

$$
t_{\mathrm{ds}}^2 t^{-1} S_{\mathrm{ds}}^b, \quad t^{-1} S_{m+\mathrm{int}}^b, \quad t_{\mathrm{ds}} t^{-1} S_{\mathrm{cross}}^b, \quad S_{m+\mathrm{int}}^f. \tag{3.52}
$$

Fermion zero modes $\psi_0, \bar{\psi}_0$ contribute to

$$
S_{m+\mathrm{int}}^f, \tag{3.53}
$$

and fermion doubler modes $\psi_d, \bar{\psi}_d$ contribute to

$$
t_{\mathrm{ds}} S_{\mathrm{ds}}^f, \quad S_{m+\mathrm{int}}^f. \tag{3.54}
$$

Any non-zero modes can be integrated out similarly to the case (i) in the limit $t \to \infty$.

Therefore Z is evaluated as

$$
Z(t, t_{\mathrm{ds}}) = \frac{1}{t} \int d\phi_0 d\phi_d d\bar\psi_0 d\psi_0 d\bar\psi_d d\psi_d
$$
$$
\times \ e^{-(t^{-1}t_{\mathrm{ds}}^2 S_{\mathrm{ds}}^b + t^{-1}t_{\mathrm{ds}} S_{\mathrm{cross}}^b + t^{-1} S_{m+\mathrm{int}}^b) - (t_{\mathrm{ds}} S_{\mathrm{ds}}^f + S_{m+\mathrm{int}}^f)}
$$
$$
= \int d\xi_0 d\xi_d | \det \eta(\phi_0(\xi), \phi_d(\xi))|^{-1} \det \eta(\phi_0(\xi), \phi_d(\xi)) e^{-\frac{1}{2}\xi_0^2 - \frac{1}{2}\xi_d^2}, \quad (3.55)
$$

where

$$
\xi_\ell(\phi_0, \phi_d) \equiv \frac{1}{\sqrt{t}} \left(\frac{t_{\mathrm{ds}}}{\sqrt{N}} \sum_n H_{\ell n} (-1)^n \phi_d + \frac{1}{\sqrt{N}} \sum_n m \delta_{\ell n} (\phi_0 + (-1)^n \phi_d) \right.
$$
$$
\left. + \cdots + \frac{g_p}{\sqrt{N^p}} \sum_{n_1, \cdots, n_p} M_{\ell; n_1, n_2, \cdots, n_p} (\phi_0 + (-1)^{n_1} \phi_d) \cdots (\phi_0 + (-1)^{n_p} \phi_d) \right)
$$
$$
\equiv \frac{1}{\sqrt{N}} \left(\xi_0(\phi_0, \phi_d) + (-1)^\ell \xi_d(\phi_0, \phi_d) \right), \quad (3.56)
$$

and

$$
\eta(\phi_0, \phi_d) = \begin{pmatrix} \dfrac{\partial \xi_0(\phi_0, \phi_d)}{\partial \phi_0} & \dfrac{\partial \xi_0(\phi_0, \phi_d)}{\partial \phi_d} \\[2mm] \dfrac{\partial \xi_d(\phi_0, \phi_d)}{\partial \phi_0} & \dfrac{\partial \xi_d(\phi_0, \phi_d)}{\partial \phi_d} \end{pmatrix}. \quad (3.57)
$$

The final expression (3.55) results in $Z(t, t_{\mathrm{ds}}) = 0$ for $p =$even and $Z(t, t_{\mathrm{ds}}) = \pm 1$ for $p =$odd.

In summary, these two limiting procedures give exactly the same result, although the doubler mode survives in the expression (3.55) of the limit (ii). The final results reproduce known results for the supersymmetric quantum mechanics in the continuum. Also there is no subtle dependence on the number of lattice sites.

4. Multicomponent case

In order to consider a higher-dimensional extension with spinorial index or a multi-flavor extension, we need a framework for multi-component fields. A Wilson term or an overlap-type term can be made more realistic in the model with multi-component indices. The CLR for the multi-component case, however, is slightly modified and should be treated with care. In this section we will explain this in a simple situation where each field has a single index which we do not specify with either spinor or flavor at the beginning.

For the fields $(\phi_n^a, \psi_n^a, \bar\psi_n^a, F_n^a)$ with single index a, we consider the following supersymmetry transformation

$$
\begin{aligned}
\delta \phi_n^a &= \epsilon \bar\psi_n^a & &= \epsilon \delta' \phi_n^a, \\
\delta \psi_n^a &= \epsilon (i(\Delta\phi\Gamma)_n^a + F_n^a) & &= \epsilon \delta' \psi_n^a, \\
\delta \bar\psi_n^a &= 0 & &= \epsilon \delta' \bar\psi_n^a, \\
\delta F_n^a &= -i\epsilon (\Delta\bar\psi\Gamma)_n^a & &= \epsilon \delta' F_n^a,
\end{aligned} \quad (4.1)
$$

where Γ is a certain constant matrix on the index with property $\Gamma = \Gamma^T$, $\Gamma^2 = 1$. We can easily obtain a free action including mass and doubler-suppressing terms invariant under the above transformation,

$$S_0 = \frac{1}{2}(\Delta\phi^a, \Delta\phi^a) + i(\bar{\psi}^a, (\Gamma)_{ab}\Delta\psi^b) + \frac{1}{2}(F^a, F^a)$$
$$= -\frac{i}{2}\delta'(\psi^a, (\Delta\phi\Gamma)^a) + \frac{1}{2}\delta'(\psi^a, F^a), \tag{4.2}$$
$$S_m = im(\phi^a, F^a) + im(\bar{\psi}^a, \psi^a) = im\delta'(\phi^a, \psi^a), \tag{4.3}$$
$$S_{\mathrm{ds}} = i(\phi^a, H_{ab}F^b) + i(\bar{\psi}^a, H_{ab}\psi^b) = i\delta'(\phi^a, H_{ab}\psi^b), \tag{4.4}$$

where H_{ab} is a multi-component extension of H in the previous sections and satisfies $\sum_m (H_{ab})_{mn} = \sum_m H_{ab}(m-n) = \hat{H}_{ab}(1) = 0$ and $((H\Gamma\Delta)_{ab})_{mn} = -((H\Gamma\Delta)_{ba})_{nm}$. The interaction term

$$S_{\mathrm{int}} = ig_2\Big((F^a, \{\phi,\phi\}^a) - 2(\psi^a, \{\phi,\bar{\psi}\}^a)\Big) = ig_2\,\delta'(\psi^a, \{\phi,\phi\}^a), \tag{4.5}$$

is also invariant under (4.1) where we used $\{\phi,\psi\}_k^a \equiv \sum_{b,c,m,n} M_{kmn}^{abc}\phi_m^b\psi_n^c$ and

$$\sum_a ((\Delta\phi\Gamma)^a, \{\phi,\phi\}^a) = 0. \tag{4.6}$$

The translational invariance implies that the Δ and M only depend on the difference of site indices, as before,

$$\Delta_{mn} = \Delta(m-n), \ M_{kmn}^{abc} = M^{abc}(k-m, k-n). \tag{4.7}$$

Their $w-$expressions are

$$\hat{\Delta}(w) \equiv \sum_m w^m \Delta(m), \ \hat{M}^{abc}(w, z) \equiv \sum_{mn} w^m z^n M^{abc}(m, n). \tag{4.8}$$

Using the combined quantity

$$P^{abc}(w, z) \equiv \hat{\Delta}(1/(wz))\Gamma^{ad}\hat{M}^{dbc}(w, z), \tag{4.9}$$

the multicomponent CLR is expressed as

$$P^{abc}(w, z) + P^{bca}(z, 1/(wz)) + P^{cab}(1/(wz), w) = 0,$$
$$P^{abc}(w, 1/w) = 0, \qquad P^{abc}(w, z) = P^{acb}(z, w). \tag{4.10}$$

Although solutions for the relation (4.10) are complicated in general, a simple and special solution is given by

$$P^{abc}(w, z) = f^{abc}P(w, z), \tag{4.11}$$

where f^{abc} is a totally symmetric constant and $P(w, z) = \hat{\Delta}(1/(wz))\hat{M}(w, z)$ is a solution of single component CLR (2.23). This means

$$\hat{M}^{abc}(w, z) = \Gamma^{ad}f^{dbc}\hat{M}(w, z). \tag{4.12}$$

We comment here that when the above indices a, b, c are somehow interpreted as spinorial indices (like dimensional reduction to 1-dim.), the Wilson and overlap-type[14, 15] fermions appear in analogous way with higher dimension. The Wilson fermion case is realized by a doubler-suppressing term $\bar{\psi} H \psi$ where

$$H_{mn}^{ab} = \delta^{ab} ir \frac{2 - \delta_{m,n-1} - \delta_{m,n+1}}{2}, \qquad \hat{H}(w)^{ab} = r \frac{2 - w - w^{-1}}{2} \delta^{ab}. \qquad (4.13)$$

The Wilson-Dirac operator is written as

$$i\hat{D}_{\text{WD}}(w) = \frac{w - w^{-1}}{2} \Gamma + r \frac{2 - w - w^{-1}}{2}, \qquad (4.14)$$

and the bosonic inverse propagator with the doubler-suppressing term is

$$\hat{D}_B(w) = \left(\frac{w - w^{-1}}{2} \right)^2 - r^2 \left(\frac{2 - w - w^{-1}}{2} \right)^2 = -\gamma_5 \hat{D}_{\text{WD}}(w) \gamma_5 \hat{D}_{\text{WD}}(w), \qquad (4.15)$$

where γ_5 is a matrix satisfying $\{\gamma_5, \Gamma\} = 0$ and $\gamma_5^2 = 1$.

For the overlap-type fermion, we take

$$\hat{\Delta}(w) = \frac{w - w^{-1}}{2} K_{\text{OD}}(w), \qquad (4.16)$$

$$\hat{M}^{abc}(w, z) = \frac{f^{abc}}{6 K_{\text{OD}}(w)} \left(2wz + wz^{-1} + zw^{-1} + 2(wz)^{-1} \right), \qquad (4.17)$$

where

$$K_{\text{OD}}(w) = \frac{M + 1}{\sqrt{(M + w)(M + w^{-1})}}. \qquad (4.18)$$

The overlap-Dirac operator is written as

$$i\hat{D}_{\text{OD}} = \Gamma \hat{\Delta}(w) + \hat{H}(w). \qquad (4.19)$$

The Ginsparg-Wilson relation[16, 17] is written as

$$\hat{D}_{\text{OD}} \gamma_5 + \gamma_5 \hat{D}_{\text{OD}} = \hat{D}_{\text{OD}} \gamma_5 \hat{D}_{\text{OD}}. \qquad (4.20)$$

This relation leads us to

$$\hat{\Delta}(w) \hat{\Delta}(w^{-1}) + \hat{H}^2(w) = 2i\hat{H}(w). \qquad (4.21)$$

Then, the bosonic inverse propagator with the doubler-suppressing term is expressed as

$$\hat{D}_B(w) = - \left(\frac{(1 - w^2) K_{\text{OD}}(w)}{2w} \right) + \hat{H}^2(w) = 2i\hat{H}(w). \qquad (4.22)$$

It is a remarkable fact that the inverse propagator of a bosonic variable is determind by the fermion doubler's mass term due to the supersymmetry.

5. Higher-dimensional case

In the previous section we consider a multi-component extension which will give a starting point for higher dimensional case. In this section, we consider a contrary case, namely two dimensional extension of the CLR without spinor index. Although it is not a real higher dimensional extension, we can readily see the difficulty which may exist in the real case. Real higher dimensional extension will be discussed elsewhere.

Let us denote a two dimensional lattice sites by $\boldsymbol{m} \equiv (m_1, m_2)$ and complex coordinates for two-variables w-expression by $\boldsymbol{w} \equiv (w_1, w_2)$. Then we use the following simplified notations:

$$\boldsymbol{wz} \equiv (w_1 z_1, w_2 z_2) \quad \text{for} \quad \boldsymbol{w} \equiv (w_1, w_2) \quad \text{and} \quad \boldsymbol{z} = (z_1, z_2), \tag{5.1}$$

also we write

$$\boldsymbol{1} \equiv (1, 1) \quad \text{and} \quad \boldsymbol{w}^{-1} \equiv (w_1^{-1}, w_2^{-1}). \tag{5.2}$$

Now we consider the following CLR for any bosonic fields

$$(\boldsymbol{\Delta}\Phi, \{\Psi, \Xi\}) + (\boldsymbol{\Delta}\Psi, \{\Xi, \Phi\}) + (\boldsymbol{\Delta}\Xi, \{\Phi, \Psi\}) = 0, \tag{5.3}$$

where the two-dimensional difference operator and the symmetric product are defined by

$$\boldsymbol{\Delta} \equiv (\Delta_1, \Delta_2), \qquad \{\Psi, \Xi\}_{\boldsymbol{k}} \equiv \sum_{\boldsymbol{mn}} M_{\boldsymbol{kmn}} \Psi_{\boldsymbol{m}} \Xi_{\boldsymbol{n}}. \tag{5.4}$$

Here Δ_1 and Δ_2 operate the first and second site index (m_1, m_2) of the field respectively. By imposing the translational invariance,

$$\boldsymbol{\Delta}_{\boldsymbol{mn}} = \boldsymbol{\Delta}(\boldsymbol{m} - \boldsymbol{n}), \qquad M_{\boldsymbol{kmn}} = M(\boldsymbol{k} - \boldsymbol{m}, \boldsymbol{k} - \boldsymbol{n}), \tag{5.5}$$

two holomorphic functions can be defined as two-variables w-expression

$$\hat{\boldsymbol{\Delta}}(\boldsymbol{w}) \equiv \sum_{\boldsymbol{n}} \boldsymbol{w}^{\boldsymbol{n}} \boldsymbol{\Delta}(\boldsymbol{n}), \quad \hat{\boldsymbol{\Delta}}(\boldsymbol{1}) = 0, \tag{5.6}$$

$$\hat{M}(\boldsymbol{w}, \boldsymbol{z}) \equiv \sum_{\boldsymbol{m}, \boldsymbol{n}} \boldsymbol{w}^{\boldsymbol{m}} \boldsymbol{z}^{\boldsymbol{n}} M(\boldsymbol{m}, \boldsymbol{n}) = \hat{M}(\boldsymbol{z}, \boldsymbol{w}), \tag{5.7}$$

where simplified notation $\boldsymbol{w}^{\boldsymbol{m}} \equiv w_1^{m_1} w_2^{m_2}$ is used. Then, the $w-$expression of the CLR can be obtained as

$$\hat{\boldsymbol{\Delta}}(\boldsymbol{w}^{-1}\boldsymbol{z}^{-1})\hat{M}(\boldsymbol{w}, \boldsymbol{z}) + \hat{\boldsymbol{\Delta}}(\boldsymbol{w})\hat{M}(\boldsymbol{z}, \boldsymbol{w}^{-1}\boldsymbol{z}^{-1}) + \hat{\boldsymbol{\Delta}}(\boldsymbol{z})\hat{M}(\boldsymbol{w}^{-1}\boldsymbol{z}^{-1}, \boldsymbol{w}) = 0. \tag{5.8}$$

Setting $\boldsymbol{z} = \boldsymbol{1}$, we obtain

$$\hat{\boldsymbol{\Delta}}(\boldsymbol{w}^{-1}) = -\frac{\hat{M}(\boldsymbol{1}, \boldsymbol{w}^{-1})}{\hat{M}(\boldsymbol{1}, \boldsymbol{w})} \hat{\boldsymbol{\Delta}}(\boldsymbol{w}), \tag{5.9}$$

and

$$\hat{\boldsymbol{\Delta}}(\boldsymbol{wz}) = \frac{\hat{M}(\boldsymbol{1}, \boldsymbol{wz})}{\hat{M}(\boldsymbol{w}, \boldsymbol{z})\hat{M}(\boldsymbol{1}, (\boldsymbol{wz})^{-1})} \Big(\hat{M}(\boldsymbol{z}, (\boldsymbol{wz})^{-1})\hat{\boldsymbol{\Delta}}(\boldsymbol{w}) + \hat{M}(\boldsymbol{w}, (\boldsymbol{wz})^{-1})\hat{\boldsymbol{\Delta}}(\boldsymbol{z}) \Big). \tag{5.10}$$

From eq.(5.10), we have

$$\hat{\boldsymbol{\Delta}}(\boldsymbol{w}^2) = 2\frac{\hat{M}(\boldsymbol{1}, \boldsymbol{w}^2)}{\hat{M}(\boldsymbol{w}, \boldsymbol{w})\hat{M}(\boldsymbol{1}, (\boldsymbol{w})^{-2})}\hat{M}(\boldsymbol{w}, (\boldsymbol{w})^{-2})\hat{\boldsymbol{\Delta}}(\boldsymbol{w}). \qquad (5.11)$$

By the use of the Weierstrass preparation theorem[20] on multi-variable analytic functions, the above difference operator can be parametrized as

$$\hat{\boldsymbol{\Delta}}(\boldsymbol{w}) = \boldsymbol{K}(\boldsymbol{w})\Big(w_1 - 1 + (w_2 - 1)u(w_2)\Big), \qquad (5.12)$$

where $\boldsymbol{K}$ and u are nonzero and holomorphic near $\boldsymbol{w} \sim \boldsymbol{1}$.

   In the following, we shall prove the CLR (5.8) has only a solution $\hat{\boldsymbol{\Delta}}(\boldsymbol{w}) \propto w_1 w_2 - 1$ with the local difference operator $\boldsymbol{\Delta}$ and the local product coefficient M. Since also in this two-dimensional case, the holomorphism is equivalent to the locality, we discuss the relation (5.8) based on a theory of multi-variable holomorphic functions.

   Using eqs. (5.10),(5.11) repeatedly, we obtain the following recursion formula,

$$\hat{\boldsymbol{\Delta}}(\boldsymbol{w}^n) = nF(\boldsymbol{w}, n)\hat{\boldsymbol{\Delta}}(\boldsymbol{w}), \qquad (5.13)$$

where F is a holomorphic function of $\boldsymbol{w}$ and becomes unity in the naive continuum limit, $\boldsymbol{w} \to \boldsymbol{1}$. From the parametrization (5.12), the recursion formula is rewritten by

$$\begin{aligned}
w_1^n - 1 + (w_2^n - 1)u(w_2^n) &= \frac{nF(\boldsymbol{w}, n)K(\boldsymbol{w})}{K(\boldsymbol{w}^n)}\Big(w_1 - 1 + (w_2 - 1)u(w_2)\Big) \\
&= \prod_{k=1}^{n}\Big(w_1 - w^{(k)}(w_2^n)\Big), \qquad (5.14)
\end{aligned}$$

where

$$w^{(k)}(w_2^n) \equiv \omega_n^k\Big(1 - (w_2^n - 1)u(w_2^n)\Big)^{\frac{1}{n}}, \qquad \omega_n = e^{\frac{2i\pi}{n}}. \qquad (5.15)$$

From the uniquness of the factorization near the naive continuum limit $\boldsymbol{1}$,

$$w_1 - w^{(n)}(w_2^n) = w_1 - 1 + (w_2 - 1)u(w_2) \qquad (5.16)$$

because any considered function is holomorphic around the limit. The equation (5.16) can be exactly solved as

$$u(w_2) = \frac{1 - w_2^{-u(1)}}{w_2 - 1}. \qquad (5.17)$$

Imposing the single-valued condition on $u(w_2)$, $u(1)$ must be an integer, ℓ. The resulting expression is

$$\hat{\boldsymbol{\Delta}}(\boldsymbol{w}) = \boldsymbol{K}(\boldsymbol{w})(w_1 - \frac{1}{w_2^\ell}) = \frac{\boldsymbol{K}(\boldsymbol{w})}{w_2^\ell}(w_1 w_2^\ell - 1). \qquad (5.18)$$

Instead of w_1, we can argue the same thing about w_2 of $\hat{\boldsymbol{\Delta}}(\boldsymbol{w})$. Since both cases hold, ℓ must be unity and we can obtain a general form of two-dimensional difference operator,

$$\hat{\boldsymbol{\Delta}}(\boldsymbol{w}) = \boldsymbol{K}(\boldsymbol{w})(w_1 - \frac{1}{w_2}) = \frac{\boldsymbol{K}(\boldsymbol{w})}{w_2}(w_1 w_2 - 1). \qquad (5.19)$$

This general form (5.19) is a trivial extension of one-dimensional difference operators and the general solution of CLR can be found easily by substitution: $w \to w_1 w_2$, $z \to z_1 z_2$ in one-dimensional case. This solution is, however, unsuitable for euclidean field theory because, with this, the kinetic term is always written as a form of $w_1 w_2$ or, in the momentum representation, $p_1 + p_2$.

6. Conclusion and discussion

When we naively tried to realize the full supersymmetry on lattice, we were required to realize a Leibniz rule (LR) on lattice. The no-go theorem, however, prevented us from proceeding further. In this sense, the theorem is indeed an origin of the difficulty in lattice supersymmetry [4, 21, 22]. On the contrary, realizing exact half-supersymmetry leads us a new criterion, *i.e.* the cyclic Leibniz rule (CLR). The important thing is that the CLR actually has solutions which are realized by the set of local difference operator and local symmetric field products. Unlike the LR, the CLR do not contradict with the locality and the translational invariance on lattice. Yet in the naive continuum limit, CLR cannot be distinguished with LR. Because of these features we consider the CLR may be an important key ingredient for further constructing supersymmetric lattice models beyond the quantum mechanics model discussed in this paper.

Using a difference operator and field products which satisfy the CLR, we can easily construct exact half-supersymmetric action with superfield formalism. We can also find the Nicolai map without "surface term". Since each term of our superspace action is independently invariant under supersymmetry, we can introduce deformation parameters with no difficulty. This enable us to utilize a localization technique to compute, for instance, the Witten index of our supersymmetric quantum mechanics models, which reproduces the known results in the continuum.

In order to construct supersymmetric field theoretical model, we have to adapt spinor index for higher dimensional CLR. In this paper, as a first step, we consider two-dimensional CLR without spinor index, in which we find a difficulty that the difference operator has a particular form unsuitable for the kinetic term. We hope to report a further analysis which properly includes spinor index elsewhere.

A. General solution of the CLR and the properties of smeared symmetric product

A general difference operator is written as a function of the difference of site index

$$\Delta_{mn} = \Delta(m-n), \tag{A.1}$$

by the translational invariance. The momentum representation (w-expression) is written as a holomorphic function,

$$\hat{\Delta}(w) \equiv \sum_{n=-\infty}^{\infty} w^n \Delta(n), \tag{A.2}$$

where $w = \exp(ipa)$ in terms of the momentum p dual to the coordinate na. This is holomorphic in the domain $\mathcal{D}$ which is an annulus including a unit circle around the origin. From (2.3), the constraint

$$\hat{\Delta}(1) = 0 \tag{A.3}$$

is imposed on the difference operator. For the product coefficient between lattice fields, we can also write in the translational invariant form

$$M_{\ell mn} = M(\ell - m, \ell - n). \tag{A.4}$$

The momentum representation is written as a two-variables holomorphic function,

$$\hat{M}(w,z) \equiv \sum_{m,n=-\infty}^{\infty} w^m z^n M(m,n) = \hat{M}(z,w), \tag{A.5}$$

where $w = exp(ipa)$, $z = exp(iqa)$ and the holomorphic domain is $\mathcal{D} \otimes \mathcal{D}$. Then, our cyclic Leibniz rule (2.21) can be expressed as

$$\hat{M}(w,z)\hat{\Delta}(\frac{1}{wz}) + \hat{M}(z,\frac{1}{wz})\hat{\Delta}(w) + \hat{M}(\frac{1}{wz},w)\hat{\Delta}(z) = 0. \tag{A.6}$$

The general solution for this equation is

$$\hat{M}(w,z)\hat{\Delta}(\frac{1}{wz}) = f(w,z) + f(z,1) + f(wz,\frac{1}{wz}) + f(z,w) + f(w,1) + f(wz,\frac{1}{wz})$$
$$- f(z,\frac{1}{wz}) - f(\frac{1}{wz},1) - f(\frac{1}{w},w) - f(w,\frac{1}{wz}) - f(\frac{1}{wz},1) - f(\frac{1}{z},z),$$

where $f(w,z)$ is an arbitrary two-variable holomorphic function. By taking $f(w,z) = F(w) + F(z)$, the special solution is found

$$\hat{M}(w,z)\hat{\Delta}(\frac{1}{wz}) = F(w) - F(\frac{1}{w}) + F(z) - F(\frac{1}{z}) - 2F(\frac{1}{wz}) + 2F(wz). \tag{A.7}$$

If we choose $F(w) = \frac{w^2}{12}$, then

$$\hat{M}(w,z)\hat{\Delta}(\frac{1}{wz}) = \frac{2wz + z^{-1}w + wz^{-1} + 2(wz)^{-1}}{6} \frac{wz - (wz)^{-1}}{2}. \tag{A.8}$$

This corresponds to the momentum representation of (2.25).

Note that we cannot impose an associative law on our symmetric product. Namely, the product (A.4) cannot satisfy the associative law

$$\{\phi, \{\psi, \chi\}\} = \{\{\phi, \psi\}, \chi\}. \tag{A.9}$$

If we imposed (A.9), then

$$\sum_\ell M_{\ell mn} M_{j\ell k} = \sum_\ell M_{jm\ell} M_{\ell nk} \tag{A.10}$$

or in w-expression,

$$\hat{M}(w, z)\hat{M}(wz, x) = \hat{M}(z, x)\hat{M}(xz, w) \tag{A.11}$$

would hold which implied $\hat{M}(w, z) = 1$ after local field redefinition[4], while the constant $\hat{M}$ was excluded from our general solution(A.7) .

Another property of the symmetric product (A.4) may come out for a case where there exists a unit element I which satisfies

$$\{\phi, I\} = \{I, \phi\} = \phi. \tag{A.12}$$

Since I_n can be taken as a unit constant by a local field redefinition, combining with the translational invariance, the above implies

$$\hat{M}(w, 1) = \hat{M}(1, w) = 1 \tag{A.13}$$

for any w. Using the solution (A.7), we can find an example satisfying (A.13),

$$\hat{M}(w, z) = 1 + \frac{1}{12wz}(w - 1)(z - 1)(w + z - 2 - 2wz). \tag{A.14}$$

B. Multi-body products and the CLR

In this appendix we describe how multi-body smeared symmetric products satisfying CLR can be constructed from 2-body ones.

B.1 Construction

N-body smeared symmetric product $\{\phi^{(1)}, \phi^{(2)}, \cdots, \phi^{(N)}\}$ is defined as

$$\{\phi^{(1)}, \phi^{(2)}, \cdots, \phi^{(N)}\}_n = \sum_{n_1, \cdots, n_N} M_{n; n_1 \cdots n_N} \phi^{(1)}_{n_1} \phi^{(2)}_{n_2} \cdots \phi^{(N)}_{n_N}, \tag{B.1}$$

with the coefficient $M_{n; n_1 \cdots n_N}$ whose last N indices are totally symmetric. The CLR relation for this can be expressed as

$$(\Delta\phi, \{\phi, \cdots, \phi\}) = 0 \tag{B.2}$$

for a single field ϕ, or

$$\sum_{\{c(i) \,:\, \text{cyclic perm.}\}} \left(\Delta\phi^{c(1)}, \{\phi^{c(2)}, \cdots, \phi^{c(N+1)}\}\right) = 0 \tag{B.3}$$

for $N+1$ fields ϕ^i $(i = 1, 2, \cdots, N+1)$, where the sum is taken over all possible cyclic permutations $\{c(i) \,|\, i = 1, \cdots, N+1\}$.

N-body symmetric product can be constructed from 2-body product by the following steps:

1. Write down a tree diagram (without loops) with $N+1$ external lines consists only of trivalent vertices. (See Fig. 1 for an $N = 4$ example.)
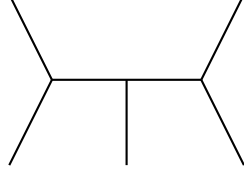

Figure 1: Example of a tree diagram from which one can construct 4-body product

2. Choose one of the external lines and assign it "0". Then number the rest of external lines from 1 to N in arbitrary order. (See Fig. 2 for counter-clockwise order examples)

3. For each vertex mark a dot on a line, among three lines attached to the vertex, which is connected to the portion containing "0"-line. (See Fig. 2) Now this diagram expresses one of 4-body products composed of three 2-body products. Each vertex corresponds to 2-body product coefficient M_{nml} and a dot distinguishes first index. Indices of internal lines (lines connecting vertices) are summed. Thus, for example, Fig. 2(a) corresponds to $\sum_{k,l} M_{nn_1 k} M_{kn_2 l} M_{ln_3 n_4}$ which gives a product of fields $\{\phi^{(1)}, \{\phi^{(2)}, \{\phi^{(3)}, \phi^{(4)}\}\}\}$.
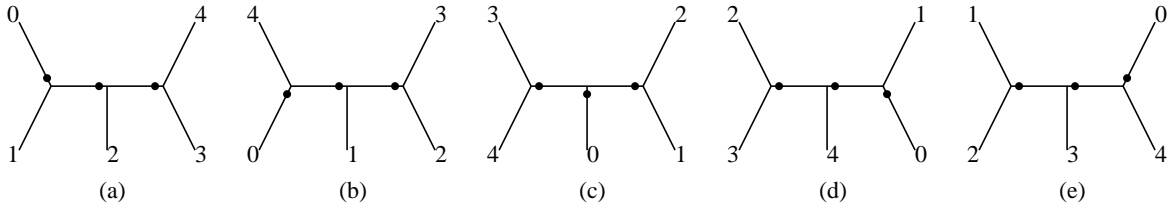

Figure 2: Constructing 4-body product made of 2-body products

4. For all possible choices of "0"-line and any permutatios of "1" through "N", repeat the steps 2 and 3. Then sum up all combined products corresponding to these diagrams with normalization factor $1/(N+1)!$. For $N = 4$ case, all possible choices of "0"-line obtained from Fig. 1 are shown in Fig. 2, and the final expression is

$$M_{n;n_1 n_2 n_3 n_4} = \frac{1}{5!} \sum_{\text{perm.}} \sum_{k,l} \left(M_{nn_{p(1)}k} M_{kn_{p(2)}l} M_{ln_{p(3)}n_{p(4)}} + M_{kn_{p(1)}l} M_{ln_{p(2)}n_{p(3)}} M_{nkn_{p(4)}} \right)$$

$$+ M_{nkl}M_{kn_{p(1)}n_{p(2)}}M_{ln_{p(3)}n_{p(4)}} + M_{nn_{p(1)}k}M_{ln_{p(2)}n_{p(3)}}M_{kln_{p(4)}}$$

$$+ M_{ln_{p(1)}n_{p(2)}}M_{kln_{p(3)}}M_{nkn_{p(4)}}\Bigg) . \tag{B.4}$$

This gives 4-body field product as

$$\{\phi^{(1)},\phi^{(2)},\phi^{(3)},\phi^{(4)}\} = \frac{1}{5!}\sum_{\text{perm.}}\Bigg(\{\phi^{(p(1))},\{\phi^{(p(2))},\{\phi^{(p(3))},\phi^{(p(4))}\}\}\}$$

$$+\{\{\phi^{(p(1))},\{\phi^{(p(2))},\phi^{(p(3))}\}\},\phi^{(p(4))}\} + \{\{\phi^{(p(1))},\phi^{(p(2))}\},\{\phi^{(p(3))},\phi^{(p(4))}\}\}$$

$$+\{\phi^{(p(1))},\{\{\phi^{(p(2))},\phi^{(p(3))}\},\phi^{(p(4))}\}\} + \{\{\{\phi^{(p(1))},\phi^{(p(2))}\},\phi^{(p(3))}\},\phi^{(p(4))}\}\Bigg). \tag{B.5}$$

Thus we obtain one of possible definitions of multi-body symmetric product for each tree diagram we start from. Although tree diagram for $N = 3$ or 4 case is unique, there are a finite number of independent tree diagrams for higher N. Each of these diagrams gives independent product on a lattice, while they may have the same continuum limit.

Note that a class of multi-body symmetric products can be alternatively obtained by a simple equation

$$\{\phi,\phi,\cdots,\phi\} \equiv \frac{1}{N+1}\sum_{k=0}^{N}\{P_{N_k}(\phi),P_k(\phi)\}. \tag{B.6}$$

Here the field $P_n(\phi)$ is defined in a recursive way using 2-body product

$$P_n(\phi) = \{P_{n-1}(\phi),\phi\}, \qquad P_1(\phi) = \phi, \qquad P_0(\phi) = I \tag{B.7}$$

where I is an identity field with respect to the 2-body product ($\phi = \{I,\phi\}$).

B.2 Proof of multi-body CLR

Here we show that the above constructed products satisfy the multi-body CLR. A graphical representation is convenient for this purpose. As in the previous subsection, coefficient of 2-body product M_{nml} is expressed by the trivalent vertex with a dot on the first index. Difference operator Δ_{nm} is now expressed by a line segment with an arrow:

$$M_{nml} \quad n\text{---}\!\!\bullet\!\diagup^{l}_{m} \qquad\qquad \Delta_{nm} \quad n\text{---}\!\!\!\overleftarrow{}\!\!\!\text{---}m \tag{B.8}$$

Anti-symmetry of Δ ($= -\Delta^T$) is represented as a graphical relation:

$$n\text{---}\!\!\!\overleftarrow{}\!\!\!\text{---}m \;=\; -\; n\text{---}\!\!\!\overrightarrow{}\!\!\!\text{---}m \tag{B.9}$$

Then CLR for 2-body product $\sum_{\text{cyclic}}\left(\Delta\phi^{c(1)},\{\phi^{c(2)},\phi^{c(3)}\}\right) = 0$ is graphically represented by Fig. 3.
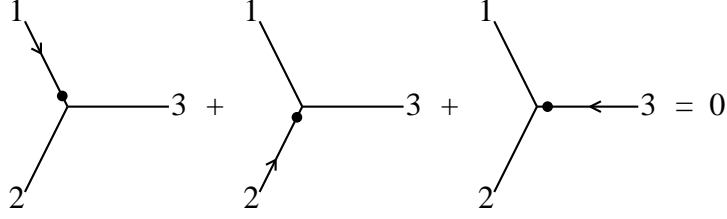
Figure 3: CLR for 2-body products

Let us begin by looking at $N = 4$ example. What we want to show is the relation

$$\sum_{c:\text{cyclic perms.}} \left(\Delta\phi^{c(1)}, \{\phi^{c(2)}, \phi^{c(3)}, \phi^{c(4)}, \phi^{c(5)}\} \right) = 0. \tag{B.10}$$

Substituting the definition of 4-body product (B.5) into the left-hand side of (B.10), we have 600 terms in total (5! kinds of terms times 5 cyclic permutations). These can be rearranged into 5! groups, one of which made of five graphs shown in Fig. 4 and the others are its permutations with respect to external lines. Therefor it is enough to argue the cancellations of graphs in one group. Actually contributions of five graphs in Fig. 4 add up to zero by the CLR for 2-body product (Fig. 3) and the anti-symmetry of Δ. For example,
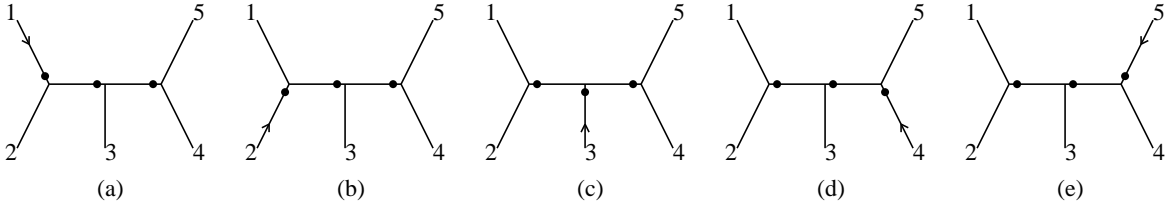

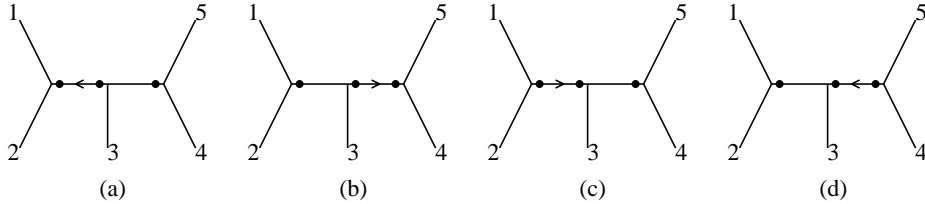**Figure 4:** Vanishing set of graphs in CLR for 4-body products

Figure 5: Canceling graphs: (a)+(c)=0 and (b)+(d)=0 due to $\Delta + \Delta^T = 0$

first two graphs in Fig. 4 add up to become the graph Fig. 5(a) with minus sign by the 2-body CLR, while last two graphs in Fig. 4 do become the graph Fig. 5(b) again with minus

sign. Remaining Fig. 4(c), however, can be rewritten into a sum of the graphs Fig. 5(c) and (d) with minus signs. Therefor they cancel out altogether due to anti-symmetry of Δ.

We can generalize the above arguments to arbitrary N case by induction. Suppose we already have the M-body CLR for M less than N. Consider a tree graph with $N+1 = K+L$ external lines (See left hand side of Fig. 6). We can find an internal line which separates the graph into two parts each of which has K and L external lines respectively (See right hand side of Fig. 6), where $1 < K < N$ and $1 < L < N$. To examine the CLR, we attach
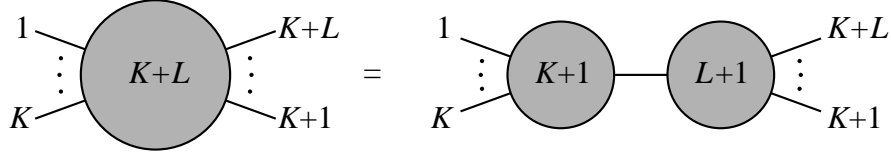

Figure 6: Factorization into products with smaller numbers

Δ at one of external lines and sum up each contributions. Then the contributions of the graphs attached Δ at an external line with number 1 though K are turned out to be that of Fig. 7(a) with minus sign by the K-body CLR, while those with number $K + 1$ through $K + L$ be that of Fig. 7(b) with minus sign by the L-body CLR. As was the case of $N = 4$, Fig. 7(a) and (b) cancel each other by the anti-symmetry of Δ. Thus N-body CLR is satisfied.
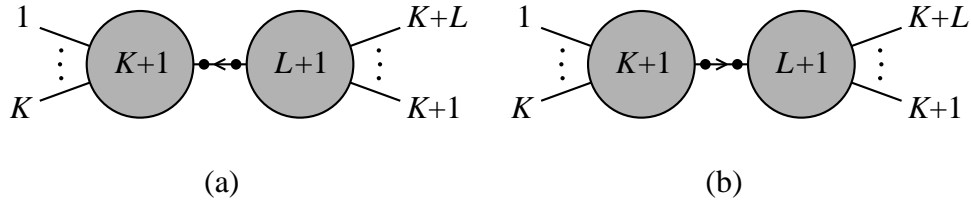

(a)   (b)

Figure 7: Canceling graphs: (a)+(b)=0 due to $\Delta + \Delta^T = 0$

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