Subdifferential calculus and doubly nonlinear evolution equations in L^p-spaces with variable exponents

Akagi, Goro

Schimperna, Giulio

(Citation)
Journal of functional analysis,267(1):173-213

(Issue Date)
2014-07-01

(Resource Type)
journal article

(Version)
Accepted Manuscript

(URL)
https://hdl.handle.net/20.500.14094/90002706

SUBDIFFERENTIAL CALCULUS AND DOUBLY NONLINEAR EVOLUTION EQUATIONS IN L^p-SPACES WITH VARIABLE EXPONENTS

GORO AKAGI AND GIULIO SCHIMPERNA

Abstract. This paper is concerned with the Cauchy-Dirichlet problem for a doubly nonlinear parabolic equation involving variable exponents and provides some theorems on existence and regularity of strong solutions. In the proof of these results, we also analyze the relations occurring between Lebesgue spaces of space-time variables and Lebesgue-Bochner spaces of vector-valued functions, with a special emphasis on measurability issues and particularly referring to the case of space-dependent variable exponents. Moreover, we establish a chain rule for (possibly nonsmooth) convex functionals defined on variable exponent spaces. Actually, in such a peculiar functional setting the proof of this integration formula is nontrivial and requires a proper reformulation of some basic concepts of convex analysis, like those of resolvent, of Yosida approximation, and of Moreau-Yosida regularization.

1. Introduction

Nonlinear parabolic equations of the form

$$\beta(\partial_t u) - \Delta u = f \quad \text{in } \Omega \times (0,T), \tag{1. 1}$$

with a maximal monotone graph $\beta : \mathbb{R} \to \mathbb{R}$, a domain Ω of $\mathbb{R}^N$, and a given function $f = f(x,t) : \Omega \times (0,T) \to \mathbb{R}$, have been studied in various contexts (see, e.g., [26]). The linear *Laplacian* is often replaced with nonlinear variants such as the so-called *m-Laplacian* Δ_m given by

$$\Delta_m u = \operatorname{div}\ \left(|\nabla u|^{m-2}\nabla u\right), \quad 1 < m < \infty.$$

In that case, the equation above is called a *doubly nonlinear* parabolic equation. Very often, by setting $u(t) := u(\cdot,t)$, such a nonlinear parabolic equation is interpreted as an abstract evolution equation, i.e., an ordinary differential equation in an infinite-dimensional space X. Namely, one has

$$A(u'(t)) + B(u(t)) = f(t) \quad \text{in } X, \quad 0 < t < T, \tag{1. 2}$$

with unknown function $u : (0,T) \to X$, two (possibly nonlinear) operators A, B in X, and $f : (0,T) \to X$. Therefore, it is natural to build the existence and regularity theory for (1. 2) in some class of vector-valued functions, like the Lebesgue-Bochner space $L^p(0,T;X)$.

Indeed, (1. 1) has been studied mostly by following two lines: the first one has been originally developed by Barbu [7], Arai [5] and Senba [24], who analyze (1. 2) in a Hilbert space $L^2(0,T;H)$ (H denoting here a Hilbert space of functions of space variables, like for instance $H = L^2(\Omega)$). Their methods is based on a time differentiation of (1. 2), which transforms it into another (more tractable) type of doubly nonlinear equation, as well as on some peculiar monotonicity condition, which is, roughly speaking, formulated by asking that

$$(Bu - Bv, A(u-v))_H \geq 0, \tag{1. 3}$$

Date: July 11, 2013.
2010 *Mathematics Subject Classification.* Primary: 35K92, 47J35; Secondary: 46E30.
Key words and phrases. Doubly nonlinear evolution equation; variable exponent Lebesgue space; Bochner space; subdifferential.
G. A. is supported by JSPS KAKENHI Grant Number 25400163, and by Hyogo Science and Technology Association, and by Nikko Co. Ltd. Both authors are also supported by the JSPS-CNR bilateral joint research project *VarEvol: Innovative Variational methods for Evolution PDEs*.

where $(\cdot,\cdot)_H$ denotes the inner product of H, along with the homogeneity of A. The other approach has been initiated by Colli-Visintin [12] and Colli [11], and it relies on the assumption of a power growth for the maximal monotone operator $A : X \to X^*$ (here X is a Banach space, for example $X = L^p(\Omega)$), e.g., p-power growth given as

$$|Au|_{X^*}^{p'} \leq C(|u|_X^p + 1), \quad c_0|u|_X^p \leq \langle Au, u\rangle_X + C \tag{1. 4}$$

with constants $c_0 > 0$, $C \geq 0$ and $p' := p/(p-1)$. Particularly, in [11], equation (1. 2) is analyzed in the Banach space $L^p(0,T;X)$. For other results on doubly nonlinear equations of the form (1. 2), the reader is referred to [1, 3, 4, 6, 9, 16, 17, 18, 21, 22, 23, 25] and references therein.

In this paper, we are concerned with the following Cauchy-Dirichlet problem for doubly nonlinear parabolic equations with *variable exponents*:

$$|\partial_t u|^{p(x)-2}\partial_t u - \Delta_{m(x)}u = f(x,t) \quad \text{in } Q := \Omega \times (0,T), \tag{1. 5}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1. 6}$$

$$u(\cdot,0) = u_0 \quad \text{in } \Omega, \tag{1. 7}$$

where $1 < p(x), m(x) < \infty$ are variable exponents and $\Delta_{m(x)}$ stands for the $m(x)$-*Laplacian* given by

$$\Delta_{m(x)}u = \operatorname{div}\left(|\nabla u|^{m(x)-2}\nabla u\right).$$

The constant exponent case, i.e., $p(x) \equiv p$, $m(x) \equiv m$, can be treated within the classical frame mentioned above by making appropriate assumptions. However, the variable exponent case presents a number of peculiarities which do not permit to apply the standard theory. Indeed, due to the x-dependence of variable exponents, it is difficult to check the monotonicity condition (1. 3) used in the first approach (e.g., [5]) without taking extra assumptions (e.g., a smooth dependence of the variable exponent with respect to the space variables). On the other hand, differently from [12] and [11], here the operator $A : u \mapsto |u(\cdot)|^{p(\cdot)-2}u(\cdot)$ with the variable exponent $p(x)$ does not satisfy any mathematically tractable p-growth condition (1. 4), because the growth order is inhomogeneous over Ω. For all these reasons, a proper functional setting for addressing Problem (1. 5)–(1. 7) seems still to be lacking, and, actually, the main aim of this paper consists exactly in the construction of such a framework.

In order to understand which are our main ideas, we have to focus on the boundedness of the operator $\mathcal{A} : u \mapsto |u|^{p(x)-2}u$ from $L^{p(x)}(Q)$ to the dual space $L^{p'(x)}(Q) = (L^{p(x)}(Q))^*$, where $L^{p(x)}(Q)$ and $L^{p'(x)}(Q)$ stand for variable exponent Lebesgue spaces of space-time variables over the set $Q = \Omega \times (0,T)$ with $p'(x) := p(x)/(p(x) - 1)$. Then, it is easy to notice that this boundedness cannot be formulated in terms of any Lebesgue-Bochner space of vector-valued functions with no loss of integrability, due to the presence of variable exponents. To say it in simple words, the identification $L^{p(x)}(Q) \sim L^{p(x)}(0,T;L^{p(x)}(\Omega))$, which is standardly used for constant exponents $p \in [1,\infty)$, turns out to be meaningless in the variable-exponent setting; actually, once one tries to embed $L^{p(x)}(Q)$ into some vector-valued function space of the time variable, a loss of integrability occurs. For this reason, we are obliged to address equation (1. 5) by mainly working in the space $L^{p(x)}(Q)$, which plays a critical role as far as we need to exploit the fine properties of the operator $\mathcal{A}$. This functional setting forces us to pay attention to the different *measures* and *measurability* concepts characterizing Lebesgue spaces of space-time variables and Lebesgue-Bochner spaces of vector-valued functions, particularly in the variable-exponent case. In addition to this, the monotone structure of the nonlinear operators appearing in equation (1. 5) has to be properly managed in the setting of the space $L^{p(x)}(Q)$. Indeed, while for constant exponents any (maximal) monotone operator acting on $L^p(\Omega)$ can be extended to the time-dependent space $L^p(Q)$ (or, equivalently, to $L^p(0,T;L^p(\Omega))$) in a straightforward way, here this procedure is far from being obvious because space and time variables cannot be "decoupled" in the definition of $L^{p(x)}(Q)$. In order to overcome this problem, we have to revise some concepts in the theory of monotone operators and of subdifferentials and adapt them to the variable exponent setting.

In particular, we need to properly modify the notions of Yosida approximation for monotone operators and of Moreau-Yosida regularization for convex functionals. This permits us to prove a chain rule for subdifferentials, extending the classical result [10, Lemme 3.3, p. 76]) to the space $L^{p(x)}(Q)$ (cf. Prop. 4.1 below). This chain rule will play a crucial role in the existence proof for (1. 5)–(1. 7).

It is worth noting that the functional framework and the convex analysis tools developed in the present paper could also be applied to more general classes of doubly nonlinear parabolic equations, like for instance

$$\beta(x, \partial_t u) - \mathrm{div}\ \mathbf{a}(x, \nabla u) = f,$$

with x-dependent maximal monotone graphs $\beta(x, \cdot)$ in $\mathbb{R}$ and $\mathbf{a}(x, \cdot)$ in $\mathbb{R}^N$ under $p(x)$- and $m(x)$-growth conditions on $\beta(x, \cdot)$ and $\mathbf{a}(x, \cdot)$, respectively, at each point $x \in \Omega$.

Prior to stating main results, let us exhibit our basic assumptions (H):

$$m \in \mathcal{P}_{\log}(\Omega), \quad p \in \mathcal{P}(\Omega), \quad 1 < p^-, m^-, p^+, m^+ < \infty, \tag{H1}$$

$$\operatorname*{ess\,inf}_{x \in \Omega} \ (m^*(x) - p(x)) > 0, \tag{H2}$$

$$f \in L^{p'(x)}(Q), \quad u_0 \in W_0^{1,m(x)}(\Omega), \tag{H3}$$

where $\mathcal{P}_{\log}(\Omega)$ (resp., $\mathcal{P}(\Omega)$) stands for the set of log-Hölder continuous (resp., measurable) exponents $1 \leq p(x) \leq \infty$ over Ω, $p^- :=$ ess inf $p(x)$, $p^+ :=$ ess sup $p(x)$, $m^\pm$ are defined analogously for $m(x)$, and $m^*(x) := (Nm(x))/(N - m(x))_+$ (see §2 below for more details).

We are concerned with solutions of (1. 5)–(1. 7) defined in the following sense:

Definition 1.1 (Strong solutions)**.** *We call* $u \in L^{p(x)}(Q)$ *a* strong solution *of* (1. 5)–(1. 7) *in* Q *whenever the following conditions hold true*:

(i) $t \mapsto u(\cdot, t)$ *is continuous with values in* $L^{p(x)}(\Omega)$ *on* $[0, T]$ *and is weakly continuous with values in* $W_0^{1,m(x)}(\Omega)$ *on* $[0, T]$,
(ii) $\partial_t u \in L^{p(x)}(Q)$, $\Delta_{m(x)} u \in L^{p'(x)}(Q)$,
(iii) *equation* (1. 5) *holds for a.e.* $(x, t) \in Q$,
(iv) *the initial condition* (1. 7) *is satisfied for a.e.* $x \in \Omega$.

Now, our result reads

Theorem 1.2 (Existence of strong solutions)**.** *Assume* (H). *Then the Cauchy-Dirichlet problem* (1. 5)–(1. 7) *admits (at least) one strong solution* u *in the sense of Definition* 1.1.

In the case when the forcing term f is more regular, namely

$$t\partial_t f \in L^{p'(x)}(Q), \tag{1. 8}$$

we can also prove parabolic regularization properties of strong solutions:

Theorem 1.3 (Time-regularization of strong solutions)**.** *Assume* (1. 8) *together with* (H)*. Then, the Cauchy-Dirichlet problem* (1. 5)–(1. 7) *admits a strong solution* u *in the sense of Definition* 1.1*, which additionally satisfies*

$$\operatorname*{ess\,sup}_{t \in (\delta, T)} \|\partial_t u(\cdot, t)\|_{L^{p(x)}(\Omega)} < \infty \quad \text{and} \quad \operatorname*{ess\,sup}_{t \in (\delta, T)} \|\Delta_{m(x)} u(\cdot, t)\|_{L^{p'(x)}(\Omega)} < \infty \tag{1. 9}$$

for any $\delta \in (0, T)$.

The remainder of the paper is organized as follows: in Section 2, we summarize some preliminary material on convex analysis and variable exponent Lebesgue and Sobolev spaces to be used later. In Section 3, we reduce (1. 5)–(1. 7) to a doubly nonlinear evolution equation and discuss its representation in a Lebesgue space of space-time variables as well as a pointwise (in time) one. Moreover, we also provide a summary of the relations occurring between Lebesgue spaces of space-time variables and Lebesgue-Bochner spaces of vector-valued functions (see Proposition 3.1). Section 4 is devoted to a proof of Theorem 1.2. Our argument basically relies on a time-discretization and limiting procedure. In particular, a chain-rule for convex functionals in a mixed frame turns out to play a crucial role. To prove this chain-rule, we introduce some modified definitions for *resolvent*, *Yosida approximation* and *Moreau-Yosida regularization*, which, compared to the standard ones, are more suitable for working in the variable-exponent setting. In Section 5, we give a proof of Theorem 1.3 by performing a second energy estimate in a discrete level. Finally, in the Appendix, we present a survey of the theory of Lebesgue-Bochner spaces and see how this theory can be extended to the variable exponent case. In particular, we give a proof of Proposition 3.1 of § 3.1.

Notation. We write $Q = \Omega \times (0,T)$. For vector-valued functions $u : (0,T) \to X$, we denote by u' the X-valued derivative of u in time. For $u : Q \to \mathbb{R}$, the partial derivative of u in time is denoted by $\partial_t u$.

2. Preliminaries

This section is devoted to recall some preliminary results on convex analysis and on Lebesgue and Sobolev spaces with variable exponents.

2.1. **Convex analysis.** Let E be a reflexive Banach space with a norm $|\cdot|_E$, a dual space E^* (with a norm $|\cdot|_{E^*}$), and a duality pairing $\langle\cdot,\cdot\rangle_E$ (or $\langle\cdot,\cdot\rangle$ for short) between E and E^*. Let $\varphi : E \to (-\infty,\infty]$ be a proper (i.e., $\varphi \not\equiv \infty$), lower semicontinuous convex function with the *effective domain*

$$D(\varphi) := \{u \in E \colon \varphi(u) < \infty\}.$$

The *subdifferential operator* $\partial\varphi : E \to 2^{E^*}$ associated with φ is defined by

$$\partial\varphi(u) := \{\xi \in E^* \colon \varphi(v) - \varphi(u) \geq \langle \xi, v-u\rangle \quad \text{for all} \;\; v \in D(\varphi)\},$$

for $u \in D(\varphi)$, with the domain $D(\partial\varphi) := \{u \in D(\varphi) \colon \partial\varphi(u) \neq \emptyset\}$. It is well known that the subdifferential of any convex functional is a maximal monotone operator in $E \times E^*$. Furthermore, φ is said to be *Fréchet differentiable* in E if, for any $u \in E$, there exists $\xi_u \in E^*$ such that

$$\left| \frac{\varphi(u+v) - \varphi(u) - \langle \xi_u, v\rangle}{|v|_E} \right| \to 0, \quad \text{whenever} \quad v \to 0 \;\; \text{strongly in } E.$$

Then $\mathrm{d}\varphi : E \to E^*$, $\mathrm{d}\varphi : u \mapsto \xi_u$ is called a *Fréchet derivative* of φ. In particular, if φ is convex and Fréchet differentiable in E, then $\partial\varphi = \mathrm{d}\varphi$.

Moreover, the *convex conjugate* $\varphi^* : E^* \to (-\infty,\infty]$ of φ is defined as

$$\varphi^*(\xi) := \sup_{u \in E} \left(\langle \xi, u\rangle - \varphi(u)\right) \quad \text{for} \;\; \xi \in E^*.$$

It particularly holds that $\partial\varphi^* = (\partial\varphi)^{-1}$, that is, $\xi \in \partial\varphi(u)$ if and only if $u \in \partial\varphi^*(\xi)$.

2.2. **Variable exponent Lebesgue and Sobolev spaces.** In this subsection, we briefly review the theory of Lebesgue and Sobolev spaces with variable exponent to be used later. The reader is referred to [13] for a more detailed survey of this field. Let $\mathcal{O}$ be a domain in $\mathbb{R}^N$. We denote by $\mathcal{P}(\mathcal{O})$ the set of all measurable functions $p : \mathcal{O} \to [1,\infty]$. For $p \in \mathcal{P}(\mathcal{O})$, we write

$$p^+ := \operatorname*{ess\,sup}_{x \in \mathcal{O}} \; p(x), \quad p^- := \operatorname*{ess\,inf}_{x \in \mathcal{O}} \; p(x).$$

Throughout this subsection, we assume that $p \in \mathcal{P}(\mathcal{O})$. Then, for $p^+ < +\infty$, the Lebesgue space with a variable exponent $p(x)$ is defined as follows:

$$L^{p(x)}(\mathcal{O}) := \left\{ u : \mathcal{O} \to \mathbb{R} \colon \text{ measurable in } \mathcal{O} \text{ and } \int_{\mathcal{O}} |u(x)|^{p(x)} \mathrm{d}x < \infty \right\}$$

with a Luxemburg-type norm

$$\|u\|_{L^{p(x)}(\mathcal{O})} := \inf \left\{ \lambda > 0 \colon \int_{\mathcal{O}} \left| \frac{u(x)}{\lambda} \right|^{p(x)} \mathrm{d}x \le 1 \right\}.$$

Then $L^{p(x)}(\mathcal{O})$ is a special sort of Musielak-Orlicz space (see [20]) and is sometimes called *Nakano space*. For $p^+ < \infty$, the dual space of $L^{p(x)}(\mathcal{O})$ is identified with $L^{p'(x)}(\mathcal{O})$ with the dual variable exponent $p' \in \mathcal{P}$ given by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \quad \text{for a.e. } x \in \mathcal{O},$$

where we write $1/\infty = 0$. In the case when $p^+ = +\infty$ the above definition can be adapted with minor changes (see, e.g., [13, Chap. 3]).

Hölder's inequalities also hold for variable exponent Lebesgue spaces (cf. [13, Lemma 3.2.20]):

Proposition 2.1 (Hölder's inequality)**.** *For $s, p, q \in \mathcal{P}(\mathcal{O})$, it holds that*

$$\|fg\|_{L^{s(x)}(\mathcal{O})} \le 2\|f\|_{L^{p(x)}(\mathcal{O})}\|g\|_{L^{q(x)}(\mathcal{O})} \quad \text{for all} \ \ f \in L^{p(x)}(\mathcal{O}), \ g \in L^{q(x)}(\mathcal{O}),$$

provided that

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)} \quad \text{for a.e. } x \in \mathcal{O}.$$

In particular, if $\mathcal{O}$ is bounded and $p(x) \le q(x)$ for a.e. $x \in \mathcal{O}$, then $L^{q(x)}(\mathcal{O})$ is continuously embedded in $L^{p(x)}(\mathcal{O})$.

The following proposition plays an important role to establish energy estimates (see, e.g., Theorem 1.3 of [14] for a proof).

Proposition 2.2. *Let $p^+ < \infty$. Then, we have*

$$\sigma^-_{p(\cdot)}(\|w\|_{L^{p(x)}(\mathcal{O})}) \le \int_{\mathcal{O}} |w(x)|^{p(x)} \mathrm{d}x \le \sigma^+_{p(\cdot)}(\|w\|_{L^{p(x)}(\mathcal{O})}) \quad \text{for all} \ \ w \in L^{p(x)}(\mathcal{O})$$

with the strictly increasing functions

$$\sigma^-_{p(\cdot)}(s) := \min\{s^{p^-}, s^{p^+}\}, \quad \sigma^+_{p(\cdot)}(s) := \max\{s^{p^-}, s^{p^+}\} \quad \text{for } s \ge 0.$$

We next define variable exponent Sobolev spaces $W^{1,p(x)}(\mathcal{O})$ as follows:

$$W^{1,p(x)}(\mathcal{O}) := \left\{ u \in L^{p(x)}(\mathcal{O}) \colon \frac{\partial u}{\partial x_i} \in L^{p(x)}(\mathcal{O}) \quad \text{for all} \ \ i = 1, 2, \ldots, N \right\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\mathcal{O})} := \left(\|u\|^2_{L^{p(x)}(\mathcal{O})} + \|\nabla u\|^2_{L^{p(x)}(\mathcal{O})} \right)^{1/2},$$

where $\|\nabla u\|_{L^{p(x)}(\mathcal{O})}$ denotes the $L^{p(x)}(\mathcal{O})$-norm of $|\nabla u|$. Furthermore, let $W^{1,p(x)}_0(\mathcal{O})$ be the closure of $C^\infty_0(\mathcal{O})$ in $W^{1,p(x)}(\mathcal{O})$. Here we note that, usually, the space $W^{1,p(x)}_0(\mathcal{O})$ is defined in a slightly different way for the variable exponent case. However, both definitions are equivalent under the regularity assumption (2. 1) given below.

The following proposition is concerned with the uniform convexity of $L^{p(x)}$- and $W^{1,p(x)}$-spaces.

Proposition 2.3 ([13])**.** *If $p^+ < \infty$, then $L^{p(x)}(\mathcal{O})$ is a separable Banach space. If $p^- > 1$ and $p^+ < \infty$, then $L^{p(x)}(\mathcal{O})$ and $W^{1,p(x)}(\mathcal{O})$ are uniformly convex. Hence they are reflexive.*

Let us exhibit the Poincaré and Sobolev inequalities. To do so, we introduce the *log-Hölder condition*:

$$|p(x) - p(x')| \le \frac{A}{\log(e + 1/|x - x'|)} \quad \text{for all } x, x' \in \mathcal{O} \tag{2. 1}$$

with some constant $A > 0$ (see [13]). This condition is weaker than the Hölder continuity of p over $\overline{\mathcal{O}}$ and it implies $p \in C(\overline{\mathcal{O}})$ and $p^+ < \infty$. We denote by $\mathcal{P}_{\log}(\mathcal{O})$ the set of all $p \in \mathcal{P}(\mathcal{O})$ satisfying the log-Hölder condition (2. 1).

Then the following properties hold:

Proposition 2.4 ([13]). *Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\mathcal{O}$ and let $p \in \mathcal{P}_{\log}(\mathcal{O})$.*

(i) *There exists a constant $C \geq 0$ such that*

$$\|w\|_{L^{p(x)}(\mathcal{O})} \leq C\|\nabla w\|_{L^{p(x)}(\mathcal{O})} \quad \text{for all } \ w \in W_0^{1,p(x)}(\mathcal{O}).$$

In particular, the space $W_0^{1,p(x)}(\mathcal{O})$ has a norm $\|\cdot\|_{1,p(x)}$ given by

$$\|w\|_{1,p(x)} := \|\nabla w\|_{L^{p(x)}(\mathcal{O})} \quad \text{for } \ w \in W_0^{1,p(x)}(\mathcal{O}),$$

which is equivalent to $\|\cdot\|_{W^{1,p(x)}(\mathcal{O})}$.

(ii) *Let $q : \mathcal{O} \to [1,\infty)$ be a measurable and bounded function and suppose that*

$$q(x) \leq p^*(x) := Np(x)/(N - p(x))_+ \quad \text{for a.e. } x \in \mathcal{O},$$

where $(s)_+ := \max\{s, 0\}$ for $s \in \mathbb{R}$. Then $W^{1,p(x)}(\mathcal{O})$ is continuously embedded in $L^{q(x)}(\mathcal{O})$.

In addition, assume that

$$\operatorname*{ess\,inf}_{x\in\mathcal{O}} \big(p^*(x) - q(x)\big) > 0.$$

Then the embedding $W^{1,p(x)}(\mathcal{O}) \hookrightarrow L^{q(x)}(\mathcal{O})$ is compact.

Remark 2.5. In [19], it is proved that the embedding $W_0^{1,p(x)}(\mathcal{O}) \hookrightarrow L^{q(x)}(\mathcal{O})$ is compact when $p^*(x)$ coincides with $q(x)$ on some thin part of $\mathcal{O}$ and the difference between the two variable exponents is appropriately controlled on the other part (see also [15]).

Finally we give a variant of Young's inequality with variable exponents. Let $p \in \mathcal{P}(\mathcal{O})$ with $p^+ < \infty$. For any $\varepsilon > 0$, there exists a constant $C_\varepsilon \geq 0$ independent of x such that

$$ab \leq \varepsilon a^{p(x)} + C_\varepsilon b^{p'(x)} \quad \text{for all } a, b \geq 0 \ \text{ and } \ \text{for a.e. } x \in \mathcal{O}. \tag{2. 2}$$

Indeed, let $\delta \in (0, 1)$ be arbitrarily given. Then, from the standard form of Young's inequality, we have

$$\begin{aligned} ab = (\delta a)\frac{b}{\delta} &\leq \frac{\delta^{p(x)}}{p(x)} a^{p(x)} + \frac{1}{p'(x)\delta^{p'(x)}} b^{p'(x)} \\ &\leq \frac{\delta^{p^-}}{p^-} a^{p(x)} + \frac{1}{(p^+)'\delta^{(p^-)'}} b^{p'(x)}. \end{aligned}$$

For each $\varepsilon > 0$, take a constant $\delta_\varepsilon \in (0, 1)$ such that $\varepsilon \geq \delta_\varepsilon^{p^-}/p^-$. Then (2. 2) follows with a constant $C_\varepsilon := ((p^+)'\delta_\varepsilon^{(p^-)'})^{-1} \geq 0$.

3. Reduction to an abstract evolution equation

3.1. **Setting of spaces and potentials.** We set $V = L^{p(x)}(\Omega)$ and $X = W_0^{1,m(x)}(\Omega)$ with norms $\|u\|_V := \|u\|_{L^{p(x)}(\Omega)}$ and $\|u\|_X := \|\nabla u\|_{L^{m(x)}(\Omega)}$, respectively. Moreover, we write

$$\langle v, u\rangle_V = \int_\Omega u(x)v(x)\,\mathrm{d}x \quad \text{for all } \ u \in V, \ v \in V^* = L^{p'(x)}(\Omega).$$

By assumption (H2) along with Proposition 2.4, it follows that

$$X \hookrightarrow V \quad \text{and} \quad V^* \hookrightarrow X^*$$

where the embeddings are continuous and, in view of [13, Thm. 3.4.12, p. 90], they are also dense.

Define functionals ψ and ϕ on V by

$$\psi(u) := \int_\Omega \frac{1}{p(x)} |u(x)|^{p(x)} \mathrm{d}x \quad \text{for} \quad u \in V$$

and

$$\phi(u) := \begin{cases} \displaystyle\int_\Omega \frac{1}{m(x)} |\nabla u(x)|^{m(x)} \mathrm{d}x & \text{if} \quad u \in X, \\ \infty & \text{if} \quad u \in V \setminus X. \end{cases}$$

Here and henceforth, we use ∂_Ω for the subdifferential in $V = L^{p(x)}(\Omega)$ and ∂_Q for the subdifferential in $L^{p(x)}(Q)$ when any confusion may arise.

Then, ψ is Fréchet differentiable on V and we find that $\partial_\Omega \psi(u) = \mathrm{d}\psi(u) = |u|^{p(x)-2}u$ for any $u \in V$.

On the other hand, ϕ is proper, lower semicontinuous and convex in V. The lower semicontinuity can be proved in a standard way (see Lemma 3.2 of [2]). Moreover, it holds that $\partial_\Omega \phi(u) = -\Delta_{m(x)} u$ with the domain

$$D(\partial_\Omega \phi) = \left\{ u \in X : -\Delta_{m(x)} u \in L^{p'(x)}(\Omega) \right\}$$

incorporating the boundary condition (1. 6) in the sense of traces. Actually, the restriction $\phi|_X$ of ϕ to X is also Fréchet differentiable in X with the representation $\mathrm{d}(\phi|_X)(u) = -\Delta_{m(x)} u$; moreover, $\partial_\Omega \phi(u) = \mathrm{d}(\phi|_X)(u)$ for any $u \in D(\partial_\Omega \phi)$.

The above defined operators permit to reduce (1. 5)–(1. 7) into the following doubly nonlinear evolution equation:

$$\partial_\Omega \psi(u'(t)) + \partial_\Omega \phi(u(t)) = Pf(t) \quad \text{in} \quad V^*, \quad 0 < t < T, \tag{3. 1}$$

$$u(0) = u_0. \tag{3. 2}$$

The operator P represents pointwise in time evaluation for functions of space-time variables and will be more precisely defined later on. Here, we notice that $Pf(t) := f(\cdot, t) \in V^*$ for a.e. $t \in (0, T)$ (this follows from (H3) and Proposition 3.1 below).

As mentioned in the Introduction, we shall work in a mixed frame of Lebesgue-Bochner space $L^p(0, T; L^p(\Omega))$ and Lebesgue space $L^p(Q)$ with $Q = \Omega \times (0, T)$. However, these classes of spaces are originally defined in a different way and their identification is delicate, particularly in the variable-exponent setting, in view of the different types of *measures* involved. In the Appendix we will present a review of the underlying theory by emphasizing the additional difficulties occurring in the variable exponent case. A crucial role will be played by the *pointwise evaluation* operator $P : L^1(Q) \to L^1(0, T; L^1(\Omega))$ (as $|\Omega| < \infty$, L^1 is the largest space), defined by $Pu(t) := u(\cdot, t)$ for $t \in (0, T)$, which permits to pass from Lebesgue functions of space-time variables to Lebesgue-Bochner vector-valued functions. Its properties are summarized in the following proposition, whose proof is postponed to the Appendix, where an extended survey of the properties of P is presented. Here and henceforth, we simply write $Pu(t)$ and $P^{-1}u(x, t)$ instead of $(Pu)(t)$ and $(P^{-1}u)(x, t)$, respectively.

Proposition 3.1. *For any constant exponent $1 \le p < \infty$ and variable one $p(x)$ with $1 \le p^- \le p^+ < \infty$, the following* (i)–(iv) *hold true*:

- (i) *The operator P is a linear, bijective, isometric mapping from $L^p(Q)$ to $L^p(0, T; L^p(\Omega))$. Furthermore, if $u \in L^{p(x)}(Q)$, then $Pu \in L^{p^-}(0, T; L^{p(x)}(\Omega))$.*
- (ii) *The inverse $P^{-1} : L^p(0, T; L^p(\Omega)) \to L^p(Q)$ is well-defined, and for each $u = u(t) \in L^p(0, T; L^p(\Omega))$, it holds that $u(t) = P^{-1}u(\cdot, t)$ for a.e. $t \in (0, T)$.*
- (iii) *If $u \in L^{p(x)}(Q)$ with $\partial_t u \in L^{p(x)}(Q)$, then Pu belongs to the space $W^{1,p^-}(0, T; L^{p(x)}(\Omega))$ and $(Pu)' = P(\partial_t u)$.*

(iv) *If* $u \in W^{1,p}(0,T;L^p(\Omega))$, *then* $\partial_t(P^{-1}u)$ *belongs to* $L^p(Q)$ *and coincides with* $P^{-1}(u')$, *where* u' *denotes the derivative of* $u:(0,T)\to V$ *in time.*

To be more precise (c.f., e.g., (ii) above), for each exponent p (or $p(x)$), we should use a different notation of the operators defined above. However, for simplicity, we shall always write P and P^{-1} regardless of p.

Set $\mathcal{V} := L^{p(x)}(Q)$ with the norm $\|u\|_{\mathcal{V}} := \|u\|_{L^{p(x)}(Q)}$ and $\langle u,v\rangle_{\mathcal{V}} := \iint_Q u(x,t)v(x,t)\,\mathrm{d}x\,\mathrm{d}t$ whenever $u\in\mathcal{V}$ and $v\in\mathcal{V}^* = L^{p'(x)}(Q)$. Define functionals Ψ and Φ on $\mathcal{V}$ as

$$\Psi(u) := \iint_Q \frac{1}{p(x)}|u(x,t)|^{p(x)}\mathrm{d}x\,\mathrm{d}t = \int_0^T \psi(Pu(t))\,\mathrm{d}t,$$

where the latter equality follows from Fubini's lemma and the fact $u\in L^{p(x)}(Q)$, and

$$\Phi(u) := \begin{cases} \displaystyle\int_0^T \phi(Pu(t))\,\mathrm{d}t & \text{if } \ Pu(t)\in X \ \text{ for a.e. } t\in(0,T),\ t\mapsto\phi(Pu(t))\in L^1(0,T), \\ \infty & \text{otherwise} \end{cases}$$

for $u\in\mathcal{V}$. Then Ψ is Fréchet differentiable and convex in $\mathcal{V}$ (hence $D(\Psi)=\mathcal{V}$) and Φ is proper, lower semicontinuous and convex in $\mathcal{V}$ with $D(\Phi)=\{u\in\mathcal{V}\colon Pu(t)\in X \ \text{ for a.e. } t\in(0,T) \text{ and } t\mapsto\phi(Pu(t))\in L^1(0,T)\}$. To prove the lower semicontinuity of Φ on $\mathcal{V}$, it suffices to check it on a larger space $L^1(0,T;V)$ which continuously embeds $\mathcal{V}$ (see (ii) of Lemma 4.5 below) by using the lower semicontinuity of ϕ in V. The details, quite standard, are given in Appendix B below. The subdifferential operator $\partial_Q\Psi:\mathcal{V}\to\mathcal{V}^*$ of Ψ is formulated as

$$\partial_Q\Psi(u) := \left\{\xi\in\mathcal{V}^*\colon \Psi(v)-\Psi(u)\ge\iint_Q \xi(v-u)\,\mathrm{d}x\,\mathrm{d}t \quad \text{for all } \ v\in D(\Psi)\right\}$$

with domain $D(\partial_Q\Psi) := \{u\in D(\Psi)\colon \partial_Q\Psi(u)\neq\emptyset\}$, and $\partial_Q\Phi$ can be also defined analogously.

In the constant exponent case, a similar extension of convex functionals onto Lebesgue-Bochner spaces (e.g., $L^p(0,T;V)$) is a standard issue. On the other hand, in our case $\mathcal{V}=L^{p(x)}(Q)$, the given extensions Φ and Ψ of ϕ and ψ do not correspond to those provided by the standard theory. Correspondingly, some basic properties of these functionals (like, e.g., subdifferentials, or regularizations) need to be properly analyzed.

The following relations will be frequently used in the sequel:

(i) For $u\in\mathcal{V}$ and $\xi\in\mathcal{V}^*$,

$$[u,\xi]\in\partial_Q\Phi \quad \text{if and only if} \quad [Pu(t),P\xi(t)]\in\partial_\Omega\phi \ \text{ for a.e. } t\in(0,T),$$
$$\text{i.e., } -\Delta_{m(x)}u(x,t)=\xi(x,t) \ \text{ for a.e. } (x,t)\in Q.$$

(ii) For $u\in\mathcal{V}$ and $\eta\in\mathcal{V}^*$,

$$[u,\eta]\in\partial_Q\Psi \quad \text{if and only if} \quad [Pu(t),P\eta(t)]\in\partial_\Omega\psi \ \text{ for a.e. } t\in(0,T),$$
$$\text{i.e., } |u(x,t)|^{p(x)-2}u(x,t)=\eta(x,t) \ \text{ for a.e. } (x,t)\in Q.$$

The above properties (i)–(ii) will be proved in the next subsection, where, actually, more general results will be presented. Finally, on account of the previous discussion, we can restate equation (3. 1) in Lebesgue spaces of space-time variables as follows:

$$\partial_Q\Psi(\partial_t(P^{-1}u))+\partial_Q\Phi(P^{-1}u)=f \ \text{ in } \mathcal{V}^*. \tag{3. 3}$$

Then $\hat{u}:=P^{-1}u$ corresponds to a strong solution of (1. 5)–(1. 7) as in Definition 1.1, provided that $\hat{u}$ enjoys sufficient regularity.

3.2. **Representation of subdifferential operators associated with variable exponents.** In this section, we set $V = L^{p(x)}(\Omega)$ with $1 < p^- \le p^+ < \infty$ (then V turns out to be a reflexive and separable Banach space) and let $p'(x)$ be the (pointwise) conjugate exponent of $p(x)$, that is, $p'(x) := p(x)/(p(x)-1)$. We also let $\varphi : V \to [0,\infty]$ be proper (i.e., $D(\varphi) \neq \emptyset$; for simplicity, we assume $D(\varphi) \ni 0$), convex and lower semicontinuous (note that here φ is a *generic* functional, non-necessarily corresponding to the functional ϕ of our equation). Then, we define a functional Φ on $\mathcal{V} := L^{p(x)}(Q)$ by setting

$$\Phi(u) := \begin{cases} \int_0^T \varphi(Pu(t))\,\mathrm{d}t & \text{if } \varphi(Pu(\cdot)) \in L^1(0,T), \\ \infty & \text{otherwise.} \end{cases} \tag{3. 4}$$

Then Φ is proper, lower semicontinuous and convex. As expected, we have

Lemma 3.2. *For $u \in \mathcal{V}$, $\xi \in \mathcal{V}^*$ with $1 < p^- \le p^+ < \infty$, the following property holds*:

$$\xi \in \partial_Q \Phi(u) \quad \text{if and only if} \quad P\xi(t) \in \partial_\Omega \varphi(Pu(t)) \text{ for a.e. } t \in (0,T). \tag{3. 5}$$

Proof. Define the operator $A_{\rm ext} : \mathcal{V} \to \mathcal{V}^*$ as

$$\xi \in A_{\rm ext}(u) \quad \overset{\text{define}}{\Longleftrightarrow} \quad P\xi(t) \in \partial_\Omega \varphi(Pu(t)) \text{ for a.e. } t \in (0,T), \tag{3. 6}$$

where a function $u \in \mathcal{V}$ belongs to the domain $D(A_{\rm ext})$ whenever $Pu(t) \in D(\partial_\Omega \varphi)$ for a.e. $t \in (0,T)$ and, moreover, there exists a function $\xi \in \mathcal{V}^*$ such that $P\xi(t) \in \partial_\Omega\varphi(Pu(t))$ for a.e. $t \in (0,T)$. Then one readily verifies by integration in time and Fubini's lemma that $A_{\rm ext} \subset \partial_Q \Phi$. So it remains to prove that $A_{\rm ext}$ is maximal. To this aim, we observe that the operator

$$Z_\Omega : V \to V^*, \quad (Z_\Omega v)(x) := |v(x)|^{p(x)-2} v(x) \tag{3. 7}$$

is strictly monotone, bounded (cf. Lemma 4.6 below), continuous, and coercive. The same happens, of course, for

$$Z_Q : \mathcal{V} \to \mathcal{V}^*, \quad (Z_Q v)(x,t) := |v(x,t)|^{p(x)-2} v(x,t). \tag{3. 8}$$

Here we also remark that

$$Z_\Omega(Pu(t)) = P(Z_Q(u))(t) \quad \text{for } u \in \mathcal{V}.$$

Properly modifying the proof of [8, Chap. II, Theorem 1.2, p. 39], one can see that a (possibly) multivalued monotone graph A in $V \times V^*$ (resp., $\mathcal{V} \times \mathcal{V}^*$) is maximal if and only if $A + \lambda Z_\Omega$ (resp., $A + \lambda Z_Q$) is surjective for some $\lambda > 0$. In other words, one can use Z_Ω (resp., Z_Q) in place of the *duality mapping* between V and V^* (resp., $\mathcal{V}$ and $\mathcal{V}^*$), which behaves badly with respect to integration in time. Now, we are in position to prove that $A_{\rm ext}$ is maximal in $\mathcal{V} \times \mathcal{V}^*$. Let $f \in \mathcal{V}^*$. Then, by Proposition 3.1, Pf belongs to $L^{(p^+)'}(0,T;V^*)$. Since $\partial_\Omega \varphi$ is maximal monotone in $V \times V^*$, one can uniquely take $u(t) \in D(\partial_\Omega \varphi)$ such that

$$Z_\Omega(u(t)) + \partial_\Omega \varphi(u(t)) \ni Pf(t) \quad \text{for a.e. } t \in (0,T). \tag{3. 9}$$

The next task consists in proving that $u : t \mapsto u(t)$ is strongly measurable with values in V.

Indeed, equation (3. 9) can be rewritten as

$$u(t) = (Z_\Omega + \partial_\Omega \varphi)^{-1}(Pf(t)).$$

Let us now show that $T_\Omega := (Z_\Omega + \partial_\Omega \varphi)^{-1}$ is demicontinuous from V^* into V. Indeed, let $g_n \to g$ strongly in V^* and put $w_n := T_\Omega g_n$ and $w := T_\Omega g$. Then, rewriting (3. 9) with $u(t)$ and $Pf(t)$ replaced, respectively, by w_n and g_n, and multiplying by w_n, we have

$$\begin{aligned} \int_\Omega |w_n(x)|^{p(x)}\mathrm{d}x + \varphi(w_n) &\le \varphi(0) + \langle g_n, w_n \rangle_V \\ &\overset{(2.\,2)}{\le} \varphi(0) + \frac{1}{2}\int_\Omega |w_n(x)|^{p(x)}\mathrm{d}x + C\int_\Omega |g_n(x)|^{p'(x)}\mathrm{d}x. \end{aligned} \tag{3. 10}$$

This relation, together with Proposition 2.2, ensures that w_n is bounded in V uniformly with n. Hence, by subtraction of equations and multiplication by $w_n - w$, we get

$$\langle Z_\Omega w_n - Z_\Omega w, w_n - w\rangle_V \le \langle g_n - g, w_n - w\rangle_V \to 0,$$

which along with the definition of Z_Ω gives

$$\left(|w_n(x)|^{p(x)-2} w_n(x) - |w(x)|^{p(x)-2} w(x)\right)(w_n(x) - w(x)) \to 0 \quad \text{for a.e. } x \in \Omega.$$

Then, for almost all fixed $x \in \Omega$, $w_n(x)$ is uniformly bounded and converges to $w(x)$. The combination of uniform boundedness in V and pointwise convergence implies that (the whole sequence) w_n tends to w weakly in V. Hence, T_Ω is demicontinuous from V^* to V.

Now, let $h_n \in C([0,T]; V^*)$ be such that $h_n \to Pf$ strongly in $L^{(p^+)'}(0,T;V^*)$. Then, from the demicontinuity of T_Ω, we see that $u_n := T_\Omega(h_n(\cdot))$ is weakly continuous with values in V on $[0,T]$, and, hence, u_n is strongly measurable by Pettis' lemma and the separability of V. Moreover, $u_n(t)$ converges to $u(t) = T_\Omega(Pf(t))$ weakly in V for a.e. $t \in (0,T)$, and, therefore, u is also strongly measurable in $(0,T)$ with values in V. Repeating an estimate similar to (3. 10), one finds that

$$\int_0^T \left(\int_\Omega |(u(t))(x)|^{p(x)} \,\mathrm{d}x\right) \mathrm{d}t < \infty, \tag{3. 11}$$

which particularly implies $u \in L^{p^-}(0,T;V) \subset L^1(0,T;L^1(\Omega))$. Furthermore, we get $\hat{u} := P^{-1}u \in \mathcal{V}$ by Fubini's lemma along with (3. 11). Then, by (3. 9), $\hat{u}$ solves $Z_Q(\hat{u}) + A_{\mathrm{ext}}(\hat{u}) \ni f$. In particular, $\hat{u} \in D(A_{\mathrm{ext}})$ since $f - Z_Q(\hat{u}) \in \mathcal{V}^*$. Therefore, A_{ext} is maximal monotone in $\mathcal{V} \times \mathcal{V}^*$. Since the graph of A_{ext} is contained in that of a (maximal) monotone operator $\partial_Q \Phi$, the two operators must coincide, as desired. □

4. Proof of Theorem 1.2

This section is aimed at giving a proof of Theorem 1.2. As mentioned in the Introduction, it is a major difference of this study from the constant exponent case (e.g., [11]) that one has to work in a mixed framework of (generalized) Lebesgue spaces of *space and time variables* and of Lebesgue-Bochner spaces (i.e., vector-valued Lebesgue spaces). Particularly, it is a crucial point how to incorporate chain rules for subdifferentials into such a specific framework. So let us begin with the following proposition:

Proposition 4.1 (Chain rule for subdifferentials in a mixed frame)**.** *Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy $1 < p^- \le p^+ < \infty$. Let $u \in \mathcal{V} := L^{p(x)}(Q)$ be such that $\partial_t u \in \mathcal{V}$. Suppose that there exists $\xi \in \mathcal{V}^* = L^{p'(x)}(Q)$ such that $\xi \in \partial_Q \Phi(u)$, where Φ is given by (3. 4) for a proper lower semicontinuous convex functional φ on $V := L^{p(x)}(\Omega)$. Then, the function $t \mapsto \varphi(Pu(t))$ is absolutely continuous over $[0,T]$. Moreover, for each $t \in (0,T)$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(Pu(t)) = \langle \eta, (Pu)'(t)\rangle_V \quad \text{for all } \eta \in \partial_\Omega \varphi(Pu(t)),$$

whenever Pu and $\varphi(Pu(\cdot))$ are differentiable at t. In particular, for $0 \le s < t \le T$, we have

$$\varphi(Pu(t)) - \varphi(Pu(s)) = \iint_{\Omega \times (s,t)} \xi \partial_\tau u \,\mathrm{d}x\,\mathrm{d}\tau.$$

This chain rule will be exploited at the end of the proof given below for Theorem 1.2, more precisely, for the identification of a limit (see §4.6 for more details).

4.1. **Moreau-Yosida regularizations in variable exponent spaces.** To prove Proposition 4.1, let us introduce a variant of *Moreau-Yosida regularizations* (cf. see [8] for usual ones) for functionals defined on variable exponent spaces. Namely, we set

$$\varphi_\lambda(u) := \min_{v \in V} \left(\int_\Omega \frac{\lambda}{p(x)} \left| \frac{v(x) - u(x)}{\lambda} \right|^{p(x)} \mathrm{d}x + \varphi(v) \right) \quad \text{for } u \in V, \tag{4. 1}$$

for $\varphi : V \to [0, \infty]$, and analogously,

$$\Phi_\lambda(u) := \min_{v \in \mathcal{V}} \left(\iint_Q \frac{\lambda}{p(x)} \left| \frac{v(x,t) - u(x,t)}{\lambda} \right|^{p(x)} \mathrm{d}x\,\mathrm{d}t + \Phi(v) \right) \quad \text{for } u \in \mathcal{V}, \tag{4. 2}$$

for $\Phi : \mathcal{V} \to [0, \infty]$. In the definition of φ_λ and Φ_λ, the position of λ is crucial, particularly in view of Lemma 4.3 below, due to the presence of variable exponents. On the other hand, for constant exponent cases, the position and the power of λ in Moreau-Yosida regularizations would not be a problem even in the L^p-framework, and, in fact, φ_λ is defined in a simpler way.

Moreover, define the *modified resolvent* $J_\lambda : V \to V$ of $A := \partial_\Omega \varphi$ by setting $J_\lambda u := u_\lambda$, which is a unique solution of the equation

$$Z_\Omega \left(\frac{u_\lambda - u}{\lambda} \right) + A(u_\lambda) \ni 0 \quad \text{in } V^* \tag{4. 3}$$

(here Z_Ω is defined as in Lemma 3.2). The *modified Yosida approximation* $A_\lambda : V \to V^*$ of A is given by

$$A_\lambda(u) := Z_\Omega \left(\frac{u - J_\lambda u}{\lambda} \right) \in A(J_\lambda u) \quad \text{for each } u \in V.$$

Then one can prove as in [8] that A_λ is single-valued and monotone, and, furthermore, J_λ and A_λ are demicontinuous. These notions and properties are still available for general maximal monotone operators $A : V \to V^*$. Moreover, analogue properties hold in the frame of $\mathcal{V} = L^{p(x)}(Q)$ as well.

Going back to the modified Moreau-Yosida regularization and following the lines of [8, Theorem II.2.2, p. 57] with the proper adaptations, we can also verify that φ_λ is convex, continuous and Gâteaux differentiable in V. Moreover, the subdifferential (= Gâteaux derivative) $\partial_\Omega \varphi_\lambda$ of φ_λ coincides with the modified Yosida approximation $A_\lambda = (\partial_\Omega \varphi)_\lambda$ of $\partial_\Omega \varphi$. Furthermore, the infimum in (4. 1) is achieved at $v = J_\lambda u$, namely,

$$\varphi_\lambda(u) = \int_\Omega \frac{\lambda}{p(x)} \left| \frac{J_\lambda u(x) - u(x)}{\lambda} \right|^{p(x)} \mathrm{d}x + \varphi(J_\lambda u).$$

Hence we have $D(\varphi_\lambda) = V$ and $\varphi(J_\lambda u) \le \varphi_\lambda(u) \le \varphi(u)$ for any $u \in V$, which implies

$$\varphi_\lambda(u) \to \varphi(u) \quad \text{for all } u \in V, \tag{4. 4}$$

since $J_\lambda u \to u$ strongly in V for $u \in D(\varphi)$. A further notable property is given by the following

Lemma 4.2. *Noting as Φ_λ the Moreau-Yosida regularization of Φ in $\mathcal{V}$ and as φ_λ the Moreau-Yosida regularization of φ in V, we have the relation*

$$\Phi_\lambda(u) = \int_0^T \varphi_\lambda(Pu(t))\,\mathrm{d}t \quad \textit{for all } u \in \mathcal{V}. \tag{4. 5}$$

In particular, for $u \in \mathcal{V}$ and $\xi \in \mathcal{V}^*$, Lemma 3.2 *ensures that*

$$\xi_\lambda = \partial_Q \Phi_\lambda(u) \quad \textit{if and only if} \quad P\xi_\lambda(t) = \partial_\Omega \varphi_\lambda(Pu(t)) \ \textit{ for almost all } t \in (0,T).$$

In other words, also in the variable exponent setting, Moreau-Yosida regularization and integration in time commute.

Proof. Let $u \in \mathcal{V}$. Then, we know that $Pu(t) \in V$ for a.e. $t \in (0,T)$. Moreover, we also have

$$\Phi_\lambda(u) = \iint_Q \frac{\lambda}{p(x)} \left| \frac{u(t,x) - u_\lambda(t,x)}{\lambda} \right|^{p(x)} \mathrm{d}x\,\mathrm{d}t + \Phi(u_\lambda), \tag{4. 6}$$

where $u_\lambda \in D(\partial_Q \Phi)$ satisfies

$$Z_Q\left(\frac{u_\lambda - u}{\lambda}\right) + \partial_Q \Phi(u_\lambda) \ni 0. \tag{4. 7}$$

Analogously, since $Pu(t) \in V$ for a.a. $t \in (0,T)$, there exists $\hat{u}_\lambda(t) := J_\lambda(Pu(t)) \in D(\partial_\Omega \varphi)$, where J_λ stands for the modified resolvent of $\partial_\Omega \varphi$, such that

$$\varphi_\lambda(Pu(t)) = \int_\Omega \frac{\lambda}{p(x)} \left| \frac{u(x,t) - P^{-1}\hat{u}_\lambda(x,t)}{\lambda} \right|^{p(x)} \mathrm{d}x + \varphi(\hat{u}_\lambda(t)). \tag{4. 8}$$

Here $\hat{u}_\lambda(t)$ satisfies, for a.e. $t \in (0,T)$,

$$Z_\Omega\left(\frac{\hat{u}_\lambda(t) - Pu(t)}{\lambda}\right) + \partial_\Omega \varphi(\hat{u}_\lambda(t)) \ni 0. \tag{4. 9}$$

Since $J_\lambda : V \to V$ is demicontinuous, one can prove that $P^{-1}\hat{u}_\lambda \in \mathcal{V}$ proceeding as in Lemma 3.2. Integrating (4. 8) in time and using Fubini's lemma, we obtain

$$\int_0^T \varphi_\lambda(Pu(t))\,\mathrm{d}t = \iint_Q \frac{\lambda}{p(x)} \left| \frac{u(t,x) - P^{-1}\hat{u}_\lambda(t,x)}{\lambda} \right|^{p(x)} \mathrm{d}x\,\mathrm{d}t + \Phi(P^{-1}\hat{u}_\lambda), \tag{4. 10}$$

provided that $\varphi(\hat{u}_\lambda(\cdot)) \in L^1(0,T)$.

Note that, up to this point, the functions u_λ and $\hat{u}_\lambda$ need not be related to each other. However, observing that

$$Z_\Omega\left(\frac{\hat{u}_\lambda(t) - Pu(t)}{\lambda}\right) = Z_\Omega\left(P\left(\frac{P^{-1}\hat{u}_\lambda - u}{\lambda}\right)(t)\right) = P\left(Z_Q\left(\frac{P^{-1}\hat{u}_\lambda - u}{\lambda}\right)\right)(t),$$

by virtue of (4. 9) and Lemma 3.2, we infer that

$$Z_Q\left(\frac{P^{-1}\hat{u}_\lambda - u}{\lambda}\right) + \partial_Q \Phi(P^{-1}\hat{u}_\lambda) \ni 0. \tag{4. 11}$$

Comparing this with (4. 7), we deduce that $P^{-1}\hat{u}_\lambda$ coincides with u_λ, whence we also obtain $\varphi(\hat{u}_\lambda(\cdot)) = \varphi(Pu_\lambda(\cdot)) \in L^1(0,T)$ and the thesis follows from (4. 10). □

4.2. **Proof of Proposition 4.1.** Now, we are in position to prove Proposition 4.1. We first claim that

$$\int_s^t \langle P\xi_\lambda(\tau), (Pu)'(\tau)\rangle_V \,\mathrm{d}\tau = \varphi_\lambda(Pu(t)) - \varphi_\lambda(Pu(s)) \quad \text{for all } \ s,t \in [0,T] \tag{4. 12}$$

for any function $u \in \mathcal{V}$ satisfying $\partial_t u \in \mathcal{V}$ and $\xi_\lambda = \partial_Q \Phi_\lambda(u) \in \mathcal{V}^*$ (see §4.1). Indeed, we deduce from Proposition 3.1 that

$$Pu \in W^{1,p^-}(0,T;V);$$

hence, Pu is absolutely continuous with values in V. Furthermore, since $A_\lambda := \partial_\Omega \varphi_\lambda$ is bounded from V to V^* and $P\xi_\lambda(t) = A_\lambda(Pu(t))$ by Lemma 4.2, we find that $P\xi_\lambda$ belongs to $L^\infty(0,T;V^*)$. Therefore, by using a standard chain-rule for subdifferentials, one can obtain (4. 12).

We next pass to the limit as $\lambda \searrow 0$ in (4. 12). Concerning the left-hand side, we first give the following lemma:

Lemma 4.3. *Let $A : V \to V^*$ be a maximal monotone operator and let A_λ be the modified Yosida approximation of A. Then for any $[u,\eta] \in A$ and $\lambda > 0$, it follows that*

$$\int_\Omega \frac{1}{p'(x)} |A_\lambda u(x)|^{p'(x)} \mathrm{d}x \le \int_\Omega \frac{1}{p'(x)} |\eta(x)|^{p'(x)} \mathrm{d}x.$$

An analogous statement also holds for maximal monotone operators $\mathcal{A} : \mathcal{V} \to \mathcal{V}^*$.

In §4.1, the notions of *resolvent* and *Yosida approximation* (and hence, *modified Moreau-Yosida regularization* as well) were defined in such a way as to let this lemma hold true.

Proof. Let $[u, \eta] \in A$ and observe by the monotonicity of A that $0 \leq \langle \eta - A_\lambda(u), u - J_\lambda u \rangle_V$, where J_λ denotes the resolvent of A. By Young's inequality and the definition of A_λ, we have

$$\begin{aligned}\int_\Omega \left| \frac{u(x) - J_\lambda u(x)}{\lambda} \right|^{p(x)} \mathrm{d}x &\leq \int_\Omega \eta(x) \frac{u(x) - J_\lambda u(x)}{\lambda} \, \mathrm{d}x \\ &\leq \int_\Omega \frac{1}{p'(x)} |\eta(x)|^{p'(x)} \mathrm{d}x + \int_\Omega \frac{1}{p(x)} \left| \frac{u(x) - J_\lambda u(x)}{\lambda} \right|^{p(x)} \mathrm{d}x.\end{aligned}$$

Note that $|A_\lambda u(x)|^{p'(x)} = |Z_\Omega((u - J_\lambda u)/\lambda)(x)|^{p'(x)} = |(u(x) - (J_\lambda u)(x))/\lambda|^{p(x)}$. Then the desired inequality follows. □

By this lemma, we have

$$\iint_Q \frac{1}{p'(x)} |\xi_\lambda(x,t)|^{p'(x)} \mathrm{d}x \, \mathrm{d}t \leq \iint_Q \frac{1}{p'(x)} |\xi(x,t)|^{p'(x)} \mathrm{d}x \, \mathrm{d}t,$$

which along with Proposition 2.2 implies the boundedness of ξ_λ in $\mathcal{V}^*$. Thus we deduce in particular that, for a subsequence (not relabelled) of $\lambda \searrow 0$,

$$\xi_\lambda \to \overline{\xi} \quad \text{weakly in } \mathcal{V}^*.$$

This relation, together with the fact that $(Pu)' = P(\partial_t u)$ (see Proposition 3.1), implies

$$\begin{aligned}\int_s^t \langle P\xi_\lambda(\tau), (Pu)'(\tau) \rangle_V \mathrm{d}\tau &= \iint_{\Omega \times (s,t)} \xi_\lambda(x,\tau) \partial_t u(x,\tau) \, \mathrm{d}x \, \mathrm{d}\tau \\ &\to \iint_{\Omega \times (s,t)} \overline{\xi}(x,\tau) \partial_t u(x,\tau) \, \mathrm{d}x \, \mathrm{d}\tau = \int_s^t \langle P\overline{\xi}(\tau), (Pu)'(\tau) \rangle_V \, \mathrm{d}\tau.\end{aligned}$$

Therefore, noting that $\varphi_\lambda(Pu(t)) \to \varphi(Pu(t))$ by (4. 4), we have

$$\varphi(Pu(t)) - \varphi(Pu(s)) = \int_s^t \left\langle P\overline{\xi}(\tau), (Pu)'(\tau) \right\rangle_V \mathrm{d}\tau \quad \text{for all } \ 0 \leq s < t \leq T,$$

which implies that $t \mapsto \varphi(Pu(t))$ is absolutely continuous on $[0,T]$, since $t \mapsto \langle P\overline{\xi}(t), (Pu)'(t) \rangle_V = \int_\Omega \overline{\xi}(x,t) \partial_t u(x,t) \mathrm{d}x$ is integrable over $(0,T)$ by Fubini's lemma and the fact that $\overline{\xi} \partial_t u \in L^1(Q)$.

Now, let $t \in (0,T)$ be such that Pu and $\varphi(Pu(\cdot))$ are differentiable at t and take $\eta \in \partial_\Omega \varphi(Pu(t))$ arbitrarily. Then by definition of subdifferential, we have, for $h > 0$,

$$\frac{\varphi(Pu(t+h)) - \varphi(Pu(t))}{h} \geq \left\langle \eta, \frac{Pu(t+h) - Pu(t)}{h} \right\rangle_V.$$

Taking the limit $h \to 0_+$ and using the differentiability of $\varphi(Pu(\cdot))$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi(Pu(t)) \geq \langle \eta, (Pu)'(t) \rangle_V.$$

The converse inequality also follows by choosing $h < 0$ and letting $h \to 0_-$. Finally, we remark that $(Pu(t+h) - Pu(t))/h \to (Pu)'(t)$ strongly in V for a.e. $t \in (0,T)$, since Pu belongs to $W^{1,p^-}(0,T;V)$. In particular, substitute $\eta = P\xi(t) \in \partial_\Omega \varphi(Pu(t))$ to get

$$\varphi(Pu(t)) - \varphi(Pu(s)) = \int_s^t \langle P\xi(\tau), (Pu)'(\tau) \rangle_V \mathrm{d}\tau = \iint_{\Omega \times (s,t)} \xi \partial_\tau u \, \mathrm{d}x \, \mathrm{d}\tau.$$

Thus we obtain the desired formula.

4.3. **Time-discretization.** We address system (1. 5)–(1. 7) by means of the following time-discretization scheme:

$$\partial_\Omega \psi\left(\frac{u_{n+1}-u_n}{h}\right)+\partial_\Omega \phi(u_{n+1}) = f_{n+1} \ \text{ in } V^*, \quad n=0,1,\dots,N-1, \tag{4. 13}$$

where $N \in \mathbb{N}$, $h := T/N$, $f_n \in V^*$ is given by

$$f_n := \frac{1}{h}\int_{t_{n-1}}^{t_n} Pf(\theta)\,\mathrm{d}\theta = \frac{1}{h}\int_{t_{n_1}}^{t_n} f(\cdot,\theta)\,\mathrm{d}\theta \tag{4. 14}$$

with $t_n := nh$ (hence $t_0 = 0$ and $t_N = T$) and the prescribed initial data u_0. The existence of solutions $\{u_n\}_{n=1,2,\dots,N}$ for (4. 13) can be proved at each step n by minimizing the functional $J_n : V \to (-\infty,\infty]$ given by

$$J_n(u) := h\psi\left(\frac{u-u_n}{h}\right)+\phi(u)-\langle f_{n+1},u\rangle_V \quad \text{ for } \ u\in V,$$

which is convex and lower semicontinuous in V. Indeed, J_n is coercive in V by the fact that

$$\begin{aligned} J_n(u) &\ge h\psi\left(\frac{u-u_n}{h}\right) - C_n\|u\|_V \\ &= h\int_\Omega \frac{1}{p(x)}\left|\frac{u(x)-u_n(x)}{h}\right|^{p(x)} \mathrm{d}x - C_n\|u\|_V \\ &\ge \frac{h}{p^+}\sigma^-_{p(\cdot)}\left(\left\|\frac{u-u_n}{h}\right\|_V\right) - C_n\|u\|_V, \end{aligned}$$

which is coercive in V by $p^- > 1$ for each fixed n. Therefore, for each $n \in \{0,1,\dots,N-1\}$, one can take a minimizer $u_{n+1} \in D(\phi)$ of J_n. Hence, it holds that

$$\partial_\Omega J_n(u_{n+1}) \ni 0 \ \text{ in } V^*.$$

Since $D(\psi) = V$, by the sum rule for subdifferentials (see, e.g., [8]), we have the representation formula

$$\partial_\Omega J_n(u_{n+1}) = \partial_\Omega\psi\left(\frac{u_{n+1}-u_n}{h}\right)+\partial_\Omega\phi(u_{n+1}) - f_{n+1}.$$

Thus the minimizers $\{u_n\}_{n=1,2,\dots,N}$ solve (4. 13).

We next introduce interpolants of the minimizers defined by

$$u_N(t) := \frac{t_{n+1}-t}{h}u_n + \frac{t-t_n}{h}u_{n+1} \tag{4. 15}$$

and

$$\overline{u}_N(t) \equiv u_{n+1} \tag{4. 16}$$

for $t \in (t_n, t_{n+1}]$ and $n = 0,1,\dots,N-1$. Then $u_N \in W^{1,\infty}(0,T;V)$ and $\overline{u}_N \in L^\infty(0,T;V)$, and they satisfy

$$\partial_\Omega\psi(u_N'(t)) + \partial_\Omega\phi(\overline{u}_N(t)) = \overline{f}_N(t) \ \text{ in } V^*, \quad \text{for a.e. } \ t\in(0,T), \quad u_N(0)=u_0, \tag{4. 17}$$

where $\overline{f}_N$ is a piecewise constant interpolant of $\{f_n\}_{n=0,1,\dots,N}$ defined as in (4. 16). Equivalently, by Proposition 3.1 and Lemma 3.2,

$$\partial_Q\Psi(\partial_t(P^{-1}u_N)) + \partial_Q\Phi(P^{-1}\overline{u}_N) = P^{-1}\overline{f}_N \ \text{ in } \mathcal{V}^*, \quad u_N(0)=u_0. \tag{4. 18}$$

By convexity, we also find that $u_N(t)$ and $\overline{u}_N(t)$ belong to $D(\phi)$ for all $t \in [0,T]$.

4.4. **Lemmas for a priori estimates.** We first give a lemma on the boundedness and the convergence of the piecewise constant interpolant $\overline{f}_N$ of $\{f_n\}_{n=0,1,\dots,N}$.

Lemma 4.4. *It holds that*

$$\iint_Q |P^{-1}\overline{f}_N|^{p'(x)} \mathrm{d}x\,\mathrm{d}t \leq \iint_Q |f|^{p'(x)} \mathrm{d}x\,\mathrm{d}t. \tag{4. 19}$$

Moreover, $P^{-1}\overline{f}_N \to f$ *strongly in* $L^{p'(x)}(Q)$ *as* $N \to \infty$.

Proof. By using Jensen's inequality, we have, for all $t \in (t_{n-1}, t_n)$,

$$|P^{-1}\overline{f}_N(x,t)|^{p'(x)} = \left| \frac{1}{h} \int_{t_{n-1}}^{t_n} f(x,t) \mathrm{d}t \right|^{p'(x)} \leq \frac{1}{h} \int_{t_{n-1}}^{t_n} |f(x,t)|^{p'(x)} \mathrm{d}t,$$

which implies (4. 19). The convergence of $P^{-1}\overline{f}_N$ can be proved in a standard way (see, e.g., [21, Lemma 8.7, p. 208], where a similar argument is performed in a Lebesgue-Bochner space setting). However, for the convenience of the reader, let us give a brief sketch of proof. Let $\varepsilon > 0$ be any small number. Then one can take a smooth approximation $f_\varepsilon \in C_0^\infty(Q)$ by $p(\cdot) \in \mathcal{P}(\Omega)$ and $p^+ < \infty$ (see [13, Theorem 3.4.12, p. 90]) such that $\|f_\varepsilon - f\|_{L^{p'(x)}(Q)} < \varepsilon/3$ and $\|P^{-1}\overline{f}_N - P^{-1}\overline{(f_\varepsilon)}_N\|_{L^{p'(x)}(Q)} < \varepsilon/3$, where $\overline{(f_\varepsilon)}_N \in L^\infty(0,T;V^*)$ denotes a piecewise constant interpolant for f_ε given in a similar way to $\overline{f}_N$ for f. Indeed, the latter inequality can be checked from the former one by observing that, thanks to (4. 19),

$$\iint_Q \left| P^{-1}\overline{f}_N - P^{-1}\overline{(f_\varepsilon)}_N \right|^{p'(x)} \mathrm{d}x\,\mathrm{d}t = \iint_Q \left| P^{-1}\overline{(f - f_\varepsilon)}_N \right|^{p'(x)} \mathrm{d}x\,\mathrm{d}t \leq \iint_Q |f - f_\varepsilon|^{p'(x)} \mathrm{d}x\,\mathrm{d}t$$

and using Proposition 2.2.

On the other hand, since f_ε is uniformly continuous on $\overline{Q}$, there exists a modulus of continuity ω_ε for f_ε. Then by Proposition 2.1 it holds that

$$\begin{aligned} \left\| P^{-1}\overline{(f_\varepsilon)}_N - f_\varepsilon \right\|_{L^{p'(x)}(Q)} &\leq 2\|1\|_{L^{p'(x)}(Q)} \left\| P^{-1}\overline{(f_\varepsilon)}_N - f_\varepsilon \right\|_{L^\infty(Q)} \\ &\leq 2(|Q|+1)\omega_\varepsilon(h) \to 0 \end{aligned}$$

as $h \to 0$ (equivalently, $N \to \infty$). Here we also used $\|1\|_{L^{p'(x)}(Q)} \leq (|Q|+1)$. Actually, we have $\iint_Q (1/\lambda)^{p'(x)} \mathrm{d}x\,\mathrm{d}t \leq 1$ with $\lambda = |Q|+1$. Therefore, we can take $N_\varepsilon \in \mathbb{N}$ such that $\|P^{-1}\overline{(f_\varepsilon)}_N - f_\varepsilon\|_{L^{p'(x)}(Q)} < \varepsilon/3$ for any $N \geq N_\varepsilon$. Consequently, it holds that $\|P^{-1}\overline{f}_N - f\|_{L^{p'(x)}(Q)} < \varepsilon$ for any $N \geq N_\varepsilon$. ☐

The following lemma provides continuous embeddings between variable exponent Lebesgue spaces and Lebesgue-Bochner spaces through the mappings P, P^{-1}.

Lemma 4.5. *The following* (i) *and* (ii) *are satisfied*:

(i) $\|P^{-1}u\|_{L^{p(x)}(Q)} \leq C\|u\|_{L^{p^+}(0,T;V)}$ *for all* $u \in L^{p^+}(0,T;V)$.
(ii) $\|Pu\|_{L^{p^-}(0,T;V)} \leq C\|u\|_{L^{p(x)}(Q)}$ *for all* $u \in L^{p(x)}(Q)$.

Proof. By Proposition 3.1, each $u \in L^{p^+}(0,T;V) \subset L^1(0,T;L^1(\Omega))$ has a unique representative $P^{-1}u \in L^1(Q)$. Then (i) follows. Indeed, for each $u \in L^{p^+}(0,T;V)$, by Fubini's lemma and Proposition 2.2 we get

$$\iint_Q |P^{-1}u(x,t)|^{p(x)} \mathrm{d}x\,\mathrm{d}t \leq \int_0^T (\|u(t)\|_V + 1)^{p^+} \mathrm{d}t,$$

which implies $P^{-1}u \in L^{p(x)}(Q)$. Using Proposition 2.2 again, we also note that

$$\sigma^-_{p(\cdot)}\left(\|P^{-1}u\|_{L^{p(x)}(Q)}\right) \le \iint_Q |P^{-1}u(x,t)|^{p(x)} \mathrm{d}x\,\mathrm{d}t.$$

Therefore (i) follows. As for (ii), see Proposition A.3 for a proof. □

We next derive the boundedness of $\partial_Q \Psi : \mathcal{V} \to \mathcal{V}^*$.

Lemma 4.6. *It holds that*

$$\left\| |v|^{p(x)-2}v \right\|_{\mathcal{V}^*} \le \left(\|v\|_{\mathcal{V}} + 1\right)^{p^+-1} \quad \textit{for all} \;\; v \in \mathcal{V}. \tag{4. 20}$$

Proof. Set $\lambda = (\|v\|_{\mathcal{V}} + 1)^{p^+-1} \ge 1$. Let us estimate the following:

$$\iint_Q \left| \frac{|v|^{p(x)-2}v}{\lambda} \right|^{p'(x)} \mathrm{d}x\,\mathrm{d}t = \iint_Q \frac{|v|^{p(x)}}{\lambda^{p'(x)}} \mathrm{d}x\,\mathrm{d}t.$$

Then we note that

$$\lambda^{p'(x)} = (\|v\|_{\mathcal{V}} + 1)^{p'(x)(p^+-1)} \ge (\|v\|_{\mathcal{V}} + 1)^{p(x)} \ge \|v\|_{\mathcal{V}}^{p(x)}.$$

Thus we get

$$\iint_Q \left| \frac{|v|^{p(x)-2}v}{\lambda} \right|^{p'(x)} \mathrm{d}x\,\mathrm{d}t \le \iint_Q \left| \frac{v}{\|v\|_{\mathcal{V}}} \right|^{p(x)} \mathrm{d}x\,\mathrm{d}t \le 1,$$

which implies the assertion. □

4.5. **A priori estimates.** We are now in position to derive a priori estimates. Let us first test (4. 13) by $(u_{n+1} - u_n)/h$ to get

$$\int_\Omega \left| \frac{u_{n+1} - u_n}{h} \right|^{p(x)} \mathrm{d}x + \left\langle \partial_\Omega \phi(u_{n+1}), \frac{u_{n+1} - u_n}{h} \right\rangle_V = \left\langle f_{n+1}, \frac{u_{n+1} - u_n}{h} \right\rangle_V,$$

which, together with Propositions 2.1 and 2.2 and Inequality (2. 2), implies

$$\begin{aligned} &\int_\Omega \left| \frac{u_{n+1} - u_n}{h} \right|^{p(x)} \mathrm{d}x + \frac{\phi(u_{n+1}) - \phi(u_n)}{h} \\ &\le C \int_\Omega |f_{n+1}|^{p'(x)} \mathrm{d}x + \frac{1}{2} \int_\Omega \left| \frac{u_{n+1} - u_n}{h} \right|^{p(x)} \mathrm{d}x. \end{aligned}$$

Hence

$$\frac{1}{2} \int_\Omega \left| \frac{u_{n+1} - u_n}{h} \right|^{p(x)} \mathrm{d}x + \frac{\phi(u_{n+1}) - \phi(u_n)}{h} \le C \int_\Omega |f_{n+1}|^{p'(x)} \mathrm{d}x. \tag{4. 21}$$

Multiplying this by h and summing it up from $n = 0$ to $m \in \{0, 1, \ldots, N-1\}$, we have

$$\frac{1}{2} \sum_{n=0}^{m} h \int_\Omega \left| \frac{u_{n+1} - u_n}{h} \right|^{p(x)} \mathrm{d}x + \phi(u_{m+1}) \le \phi(u_0) + C \iint_Q |P^{-1}\overline{f}_N|^{p'(x)} \mathrm{d}x\,\mathrm{d}t. \tag{4. 22}$$

Thus by Proposition 3.1 and Lemma 4.4 we obtain

$$\iint_Q \left|\partial_t (P^{-1}u_N)\right|^{p(x)} \mathrm{d}x\,\mathrm{d}t + \sup_{t \in (0,T]} \phi(\overline{u}_N(t)) \le C, \tag{4. 23}$$

which together with Proposition 2.2 also gives

$$\sup_{t \in (0,T]} |\overline{u}_N(t)|_X + \sup_{t \in [0,T]} |u_N(t)|_X \le C. \tag{4. 24}$$

One can also deduce from (4. 23) and Proposition 2.2 that

$$\left\|\partial_t (P^{-1}u_N)\right\|_{\mathcal{V}} \le C. \tag{4. 25}$$

Thus by (4. 25) along with Lemma 4.6, one can get

$$\left\|\partial_Q \Psi(\partial_t(P^{-1}u_N))\right\|_{\mathcal{V}^*} \le C. \tag{4. 26}$$

By comparison of terms in (4. 18) together with the boundedness of $P^{-1}\overline{f}_N$ in $\mathcal{V}^*$ (see Lemma 4.4) again, we get

$$\left\|\partial_Q \Phi(P^{-1}\overline{u}_N)\right\|_{\mathcal{V}^*} \le C. \tag{4. 27}$$

4.6. **Convergence.** Recall that $X = W_0^{1,m(x)}(\Omega)$ is compactly embedded in $V = L^{p(x)}(\Omega)$ by (H2). Hence by virtue of (4. 24) and (4. 25) together with Proposition 3.1 (particularly, $\partial_t(P^{-1}u_N) = P^{-1}(u_N'))$ and (ii) of Lemma 4.5, Aubin-Lions's compactness lemma ensures the precompactness of (u_N) in $C([0,T];V)$. Thus we obtain

$$u_N \to u \quad \text{strongly in } C([0,T];V) \text{ and weakly in } W^{1,p^-}(0,T;V), \tag{4. 28}$$

$$\text{weakly star in } L^\infty(0,T;X), \tag{4. 29}$$

with $u \in W^{1,p^-}(0,T;V) \cap L^\infty(0,T;X)$. Furthermore, for each $t \in (t_n, t_{n+1})$, noting that

$$\begin{aligned}\left|P^{-1}u_N(x,t) - P^{-1}\overline{u}_N(x,t)\right| &= \left|\frac{t_{n+1}-t}{h}u_n(x) + \frac{t-t_n}{h}u_{n+1}(x) - u_{n+1}(x)\right| \\ &= (t_{n+1}-t)\left|\frac{u_{n+1}(x)-u_n(x)}{h}\right| \le h\left|\frac{u_{n+1}(x)-u_n(x)}{h}\right|,\end{aligned}$$

one has

$$\begin{aligned}\int_\Omega |P^{-1}u_N(x,t) - P^{-1}\overline{u}_N(x,t)|^{p(x)}\mathrm{d}x &\le \int_\Omega h^{p(x)}\left|\frac{u_{n+1}(x)-u_n(x)}{h}\right|^{p(x)}\mathrm{d}x \\ &\overset{(4.\ 22)}{\le} 2h^{p^- -1}\left(\phi(u_0) + C\iint_Q |f|^{p'(x)}\mathrm{d}x\,\mathrm{d}t\right) \\ &\to 0 \quad \text{uniformly for } t \in (0,T) \quad \text{as } h \to 0.\end{aligned}$$

Here we also used Lemma 4.4. Thus it follows that

$$u_N - \overline{u}_N \to 0 \quad \text{strongly in } L^\infty(0,T;V).$$

Combining this relation with (4. 28), we obtain

$$\overline{u}_N \to u \quad \text{strongly in } L^\infty(0,T;V). \tag{4. 30}$$

Furthermore, it also holds by (4. 24) that

$$\overline{u}_N \to u \quad \text{weakly star in } L^\infty(0,T;X), \tag{4. 31}$$

which, together with the fact that $u \in C([0,T];V)$ and $X \hookrightarrow V$, implies $u \in C_w([0,T];X)$, the space of weakly continuous functions with values in X.

By (4. 30) along with Lemma 4.5, we infer that

$$P^{-1}\overline{u}_N \to P^{-1}u \quad \text{strongly in } \mathcal{V}. \tag{4. 32}$$

We set

$$\hat{u} := P^{-1}u \in \mathcal{V}.$$

By (4. 25)–(4. 27), one can also take $\xi, \eta \in \mathcal{V}^*$ such that

$$\partial_t(P^{-1}u_N) \to \partial_t\hat{u} \quad \text{weakly in } \mathcal{V}, \tag{4. 33}$$

$$\partial_Q\Phi(P^{-1}\overline{u}_N) \to \xi \quad \text{weakly in } \mathcal{V}^*, \tag{4. 34}$$

$$\partial_Q\Psi(\partial_t(P^{-1}u_N)) \to \eta \quad \text{weakly in } \mathcal{V}^*. \tag{4. 35}$$

Hence, we have in particular $\partial_t\hat{u} \in \mathcal{V}$. Moreover, thanks also to Lemma 4.4, we can take the limit $n \to \infty$ in (4. 18) to obtain $\eta + \xi = f$ in $\mathcal{V}^*$. By the maximal monotonicity in $\mathcal{V} \times \mathcal{V}^*$ of the subdifferential operator $\partial_Q\Phi$, we derive $[\hat{u}, \xi] \in \partial_Q\Phi$ from (4. 34) and (4. 32).

We finally claim that

$$[\partial_t \hat{u}, \eta] \in \partial_Q \Psi. \tag{4. 36}$$

We can prove this fact by using the chain rule established in Proposition 4.1 as well as a monotonicity argument. Indeed, a standard chain rule and a simple calculation yield

$$\begin{aligned}\iint_Q \partial_Q \Psi \left(\partial_t(P^{-1}u_N)\right) \partial_t(P^{-1}u_N) \, \mathrm{d}x \, \mathrm{d}t &\stackrel{(4.\,18)}{=} \iint_Q \left(P^{-1}\overline{f}_N - \partial_Q \Phi(P^{-1}\overline{u}_N)\right) \partial_t(P^{-1}u_N) \, \mathrm{d}x \, \mathrm{d}t \\ &\le \iint_Q \left(P^{-1}\overline{f}_N\right) \partial_t(P^{-1}u_N) \, \mathrm{d}x \, \mathrm{d}t - \phi(u_N(T)) + \phi(u_0).\end{aligned}$$

Hence, by virtue of the strong convergence of $P^{-1}\overline{f}_N$ to f in $\mathcal{V}^*$ (see Lemma 4.4), we infer that

$$\begin{aligned}&\limsup_{n\to\infty} \iint_Q \partial_Q \Psi \left(\partial_t(P^{-1}u_N)\right) \partial_t(P^{-1}u_N) \, \mathrm{d}x \, \mathrm{d}t \\ &\le \iint_Q f \partial_t \hat{u} \, \mathrm{d}x \, \mathrm{d}t - \phi(u(T)) + \phi(u_0),\end{aligned}$$

since $\liminf_{N\to\infty} \phi(u_N(T)) \ge \phi(u(T))$ by (4. 28). Furthermore, recall that $\partial_t \hat{u} \in \mathcal{V}$, $\xi \in \mathcal{V}^*$, $[\hat{u}, \xi] \in \partial_Q \Phi$ and exploit the chain rule in Proposition 4.1 to deduce that $t \mapsto \phi(u(t))$ is absolutely continuous on $[0, T]$, and, moreover, it holds that

$$\begin{aligned}\limsup_{n\to\infty} \left\langle \partial_Q \Psi \left(\partial_t(P^{-1}u_N)\right), \partial_t(P^{-1}u_N)\right\rangle_{\mathcal{V}} &\le \iint_Q f \partial_t \hat{u} \, \mathrm{d}x \, \mathrm{d}t - \phi(P\hat{u}(T)) + \phi(P\hat{u}(0)) \\ &= \langle f - \xi, \partial_t \hat{u}\rangle_{\mathcal{V}} = \langle \eta, \partial_t \hat{u}\rangle_{\mathcal{V}},\end{aligned}$$

which together with the maximal monotonicity of $\partial_Q \Phi$ in $\mathcal{V} \times \mathcal{V}^*$ implies that $[\partial_t \hat{u}, \eta] \in \partial_Q \Phi$. Thus $\hat{u}$ is a strong solution of (1. 5)–(1. 7) (see §3.1). This completes the proof.

5. Proof of Theorem 1.3

In this section, we prove Theorem 1.3 under the additional regularity assumption (1. 8). To this end, we shall derive the second energy inequality. Write (4. 13) for the couple of indexes n and $n-1$ (for $n \in \{1, 2, \ldots, N-1\}$) and then take the difference. It follows that

$$\partial_\Omega \psi \left(\frac{u_{n+1} - u_n}{h}\right) - \partial_\Omega \psi \left(\frac{u_n - u_{n-1}}{h}\right) + \partial_\Omega \phi(u_{n+1}) - \partial_\Omega \phi(u_n) = f_{n+1} - f_n.$$

By the monotonicity of $\partial_\Omega \phi$, the multiplication of the above by $u_{n+1} - u_n$ yields

$$\left\langle \partial_\Omega \psi \left(\frac{u_{n+1} - u_n}{h}\right) - \partial_\Omega \psi \left(\frac{u_n - u_{n-1}}{h}\right), u_{n+1} - u_n \right\rangle_V \le \langle f_{n+1} - f_n, u_{n+1} - u_n\rangle_V. \tag{5. 1}$$

Here the left-hand side can be transformed as follows:

$$\begin{aligned}&\left\langle \partial_\Omega \psi \left(\frac{u_{n+1} - u_n}{h}\right) - \partial_\Omega \psi \left(\frac{u_n - u_{n-1}}{h}\right), u_{n+1} - u_n \right\rangle_V \\ &\ge h \left[\psi^* \left(\partial_\Omega \psi \left(\frac{u_{n+1} - u_n}{h}\right)\right) - \psi^* \left(\partial_\Omega \psi \left(\frac{u_n - u_{n-1}}{h}\right)\right)\right],\end{aligned} \tag{5. 2}$$

where we also used that $\partial_\Omega \psi^* = (\partial_\Omega \psi)^{-1}$. Hence, multiplying (5. 1) by $(n-1)$ and using (5. 2), we have

$$\begin{aligned}(n-1)h &\left[\psi^* \left(\partial_\Omega \psi \left(\frac{u_{n+1} - u_n}{h}\right)\right) - \psi^* \left(\partial_\Omega \psi \left(\frac{u_n - u_{n-1}}{h}\right)\right)\right] \\ &\le (n-1)\langle f_{n+1} - f_n, u_{n+1} - u_n\rangle_V.\end{aligned}$$

Summing it up from $n = 2$ to $m \in \mathbb{N}$, we observe by a simple calculation with $m \in \{3, 4, \dots, N-1\}$ that

$$
\begin{aligned}
&(m-1)h\psi^*\left(\partial_\Omega\psi\left(\frac{u_{m+1}-u_m}{h}\right)\right)\\
&\le \sum_{n=2}^{m} h\psi^*\left(\partial_\Omega\psi\left(\frac{u_n-u_{n-1}}{h}\right)\right) + \sum_{n=2}^{m}(n-1)h^2\left\langle \frac{f_{n+1}-f_n}{h}, \frac{u_{n+1}-u_n}{h}\right\rangle_V . \qquad (5.\ 3)
\end{aligned}
$$

Then, we have to control the right-hand side. Firstly we notice that

$$
\sum_{n=2}^{m} h\psi^*\left(\partial_\Omega\psi\left(\frac{u_n-u_{n-1}}{h}\right)\right) \le \int_0^T \psi^*\left(\partial_\Omega\psi(u_N'(t))\right)\mathrm{d}t \le C
$$

from (4. 23) together with the fact that

$$
\psi^*\left(\partial_\Omega\psi(v)\right) = \int_\Omega \frac{1}{p'(x)}\left||v(x)|^{p(x)-2}v(x)\right|^{p'(x)}\mathrm{d}x = \int_\Omega \frac{1}{p'(x)}|v(x)|^{p(x)}\mathrm{d}x
$$

for any $v \in V$. Moreover, the following lemma holds:

Lemma 5.1. *It holds that*

$$
\begin{aligned}
&\sum_{n=2}^{m}(n-1)h^2\left\langle \frac{f_{n+1}-f_n}{h}, \frac{u_{n+1}-u_n}{h}\right\rangle_V\\
&\le \iint_Q \frac{2}{p'(x)}|t\partial_t f|^{p'(x)}\mathrm{d}x\,\mathrm{d}t + \iint_Q \frac{1}{p(x)}\left|\partial_t(P^{-1}u_N)\right|^{p(x)}\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Proof. By a simple computation,

$$
\begin{aligned}
&\sum_{n=2}^{m}(n-1)h^2\left\langle \frac{f_{n+1}-f_n}{h}, \frac{u_{n+1}-u_n}{h}\right\rangle_V\\
&\le \sum_{n=2}^{m} h\int_\Omega \frac{1}{p'(x)}\left|(n-1)\left(f_{n+1}(x)-f_n(x)\right)\right|^{p'(x)}\mathrm{d}x + \iint_Q \frac{1}{p(x)}\left|\partial_t(P^{-1}u_N)(x,t)\right|^{p(x)}\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

for $t \in (t_n, t_{n+1})$. Here we further observe, by Jensen's inequality, that

$$
\begin{aligned}
\left|(n-1)\left(f_{n+1}(x)-f_n(x)\right)\right|^{p'(x)} &= \left|\frac{1}{h}\int_{t_n}^{t_{n+1}}(n-1)\left(f(x,\theta)-f(x,\theta-h)\right)\mathrm{d}\theta\right|^{p'(x)}\\
&\le \frac{1}{h}\int_{t_n}^{t_{n+1}}\left|(n-1)\left(f(x,\theta)-f(x,\theta-h)\right)\right|^{p'(x)}\mathrm{d}\theta\\
&= \frac{1}{h}\int_{t_n}^{t_{n+1}}\left|\int_{\theta-h}^{\theta}(n-1)\partial_s f(x,s)\mathrm{d}s\right|^{p'(x)}\mathrm{d}\theta\\
&\le \frac{1}{h^2}\int_{t_n}^{t_{n+1}}\int_{\theta-h}^{\theta}\left|(n-1)h\partial_s f(x,s)\right|^{p'(x)}\mathrm{d}s\,\mathrm{d}\theta. \qquad (5.\ 4)
\end{aligned}
$$

Since $(n-1)h \le s$ for any $s \in (\theta-h, \theta)$ and $\theta \in (t_n, t_{n+1})$, we infer that

$$
\begin{aligned}
\frac{1}{h^2}\int_{t_n}^{t_{n+1}}\int_{\theta-h}^{\theta}\left|(n-1)h\partial_s f(x,s)\right|^{p'(x)}\mathrm{d}s\,\mathrm{d}\theta &\le \frac{1}{h^2}\int_{t_n}^{t_{n+1}}\int_{\theta-h}^{\theta}\left|s\partial_s f(x,s)\right|^{p'(x)}\mathrm{d}s\,\mathrm{d}\theta\\
&\le \frac{1}{h}\int_{(n-1)h}^{(n+1)h}\left|s\partial_s f(x,s)\right|^{p'(x)}\mathrm{d}s. \qquad (5.\ 5)
\end{aligned}
$$

Collecting (5. 4) and (5. 5) and exploiting Fubini's lemma, we obtain

$$\begin{aligned}
&\sum_{n=2}^{m} h \int_\Omega \frac{1}{p'(x)} \left|(n-1)\left(f_{n+1}(x) - f_n(x)\right)\right|^{p'(x)} \mathrm{d}x \\
&\leq \sum_{n=2}^{m} \int_\Omega \frac{1}{p'(x)} \left(\int_{(n-1)h}^{(n+1)h} |s\partial_s f(x,s)|^{p'(x)} \, \mathrm{d}s \right) \mathrm{d}x \\
&\leq \iint_Q \frac{2}{p'(x)} |s\partial_s f(x,s)|^{p'(x)} \, \mathrm{d}x \, \mathrm{d}s.
\end{aligned}$$

Thus the proof is completed. □

Let $\delta \in (0,T)$ be fixed. Then we claim that one can choose $N \in \mathbb{N}$ so large (equivalently, $h > 0$ so small) that

$$\operatorname*{ess\,sup}_{t \in (\delta,T)} \ \psi^*\left(\partial_\Omega \psi(u_N'(t))\right) \leq C/\delta \tag{5. 6}$$

for some $C > 0$ independent of N and δ. Indeed, we note that

$$(m-1)h\psi^*\left(\partial_\Omega \psi\left(\frac{u_{m+1}-u_m}{h}\right)\right) \geq \frac{m-1}{m+1} t\psi^*\left(\partial_\Omega \psi(u_N'(t))\right) \quad \text{for all } \ t \in (mh,(m+1)h).$$

Hence it follows from (5. 3) together with Lemma 5.1 and Proposition 2.2 that

$$t\psi^*\left(\partial_\Omega \psi(u_N'(t))\right) \leq \frac{m+1}{m-1} \ell\left(\|t\partial_t f\|_{\mathcal{V}^*} + \left\|\partial_t(P^{-1}u_N)\right\|_{\mathcal{V}} \right) \tag{5. 7}$$

with a non-decreasing function ℓ in $\mathbb{R}$ for all $t \in (mh,(m+1)h)$ and $m \in \{3,4,\ldots,N-1\}$.

Now, let $\delta \in (0,T)$ be fixed. For $h \in (0,\delta/3]$, one can take $m_\delta \in \{3,4,\ldots,N-1\}$ such that $m_\delta h \leq \delta \leq (m_\delta+1)h$, which implies

$$\frac{\delta}{h} - 1 \leq m_\delta \leq \frac{\delta}{h} \quad \text{ and } \quad \frac{m_\delta+1}{m_\delta-1} \leq \frac{\delta/h+1}{\delta/h-2} \to 1 \quad \text{ as } \ h \to 0.$$

Moreover, due to the monotonicity of the function $r \mapsto (r+1)/(r-1)$, we note that

$$\frac{m+1}{m-1} \leq \frac{m_\delta+1}{m_\delta-1} \quad \text{ for any } \ m \geq m_\delta.$$

Hence, observing that $m_\delta h \leq \delta$, we conclude that

$$\begin{aligned}
\operatorname*{ess\,sup}_{t \in (\delta,T)} \ t\psi^*\left(\partial_\Omega \psi(u_N'(t))\right) &\leq \operatorname*{ess\,sup}_{t \in (m_\delta h,T)} \ t\psi^*\left(\partial_\Omega \psi(u_N'(t))\right) \\
&\overset{(5.\,7)}{\leq} 2\ell\left(\|t\partial_t f\|_{\mathcal{V}^*} + \left\|\partial_t(P^{-1}u_N)\right\|_{\mathcal{V}} \right)
\end{aligned}$$

for $0 < h \ll 1$. By (4. 25) and assumption (1. 8), we obtain (5. 6). Moreover, we also infer that

$$\sup_{t \in [\delta,T]} \|\partial_\Omega \psi\left(u_N'(t)\right)\|_{V^*} \leq \frac{C}{\delta}. \tag{5. 8}$$

By a comparison of terms in (4. 17) along with Lemma 5.2 given below, we further get

$$\sup_{t \in [\delta,T]} \|\partial_\Omega \phi\left(\overline{u}_N(t)\right)\|_{V^*} \leq \frac{C}{\delta}. \tag{5. 9}$$

Lemma 5.2. *Let $f \in L^{p'(x)}(Q)$ and assume (1. 8). Then it follows that*

$$\sup_{t \in [0,T]} \|t\overline{f}_N(t)\|_{V^*} \leq \ell\left(\iint_Q |f|^{p'(x)} \mathrm{d}x \, \mathrm{d}t + \iint_Q |t\partial_t f|^{p'(x)} \mathrm{d}x \, \mathrm{d}t \right)$$

with a nondecreasing function ℓ in $\mathbb{R}$.

Proof. For $t \in [t_n, t_{n+1})$, we see by Jensen's inequality that

$$\begin{aligned}
&\int_\Omega |tP^{-1}\overline{f}_N(t,x)|^{p'(x)} \mathrm{d}x \\
&\le \int_\Omega |t_{n+1} f_{n+1}(x)|^{p'(x)} \mathrm{d}x \\
&= \int_\Omega \left| \frac{t_{n+1}}{h} \int_{t_n}^{t_{n+1}} f(x,\tau)\,\mathrm{d}\tau \right|^{p'(x)} \mathrm{d}x \\
&= \int_\Omega \left| \frac{t_{n+1}}{h} \int_{t_n}^{t_{n+1}} \frac{1}{\tau} P^{-1} \left\{ \int_0^\tau \frac{\mathrm{d}}{\mathrm{d}s} (sPf)\,\mathrm{d}s \right\} \mathrm{d}\tau \right|^{p'(x)} \mathrm{d}x \\
&\le \sigma^+_{p(\cdot)} \left(\frac{t_{n+1}}{t_n} T \right) \frac{2^{(p^-)'-1}}{T} \iint_Q \left(|f(x,s)|^{p'(x)} + |s\partial_s f(x,s)|^{p'(x)} \right) \mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Here we used the facts that $sPf(s)|_{s=0} = 0$, which will be checked below, and that $\partial_s(sf) = f + s\partial_s f$. Consequently, one can derive the desired inequality by noting that $t_{n+1}/t_n \le 2$.

We finally prove that $tPf(t)|_{t=0} = 0$. By virtue of (1. 8), we find that tPf belongs to $W^{1,(p^+)'}(0,T;V^*)$ by $\partial_t(tf) = f + t\partial_t f \in \mathcal{V}^*$. Hence tPf is continuous with values in V^* on $[0,T]$. In particular, $tPf(t)$ converges to some $g_0 \in V^*$ strongly in V^* as $t \to 0_+$. Hence,

$$\|tPf(t)\|_{V^*} \ge \frac{1}{2}\|g_0\|_{V^*} \quad \text{for } \ 0 < t \ll 1,$$

which implies $\|Pf(t)\|_{V^*} \ge \frac{1}{2t}\|g_0\|_{V^*}$. Since $\|Pf(\cdot)\|_{V^*} \in L^1(0,T)$, we conclude that $g_0 = 0$. □

By (5. 8) and (5. 9), for any $\delta \in (0,T)$, up to a subsequence, it holds that

$$\partial_\Omega \phi(\overline{u}_N(t))|_{(\delta,T)} \to P\xi|_{(\delta,T)} \quad \text{weakly star in } L^\infty(\delta,T;V^*), \tag{5. 10}$$

$$\partial_\Omega \psi(u'_N(t))|_{(\delta,T)} \to P\eta|_{(\delta,T)} \quad \text{weakly star in } L^\infty(\delta,T;V^*). \tag{5. 11}$$

In particular, $P\xi, P\eta \in L^\infty(\delta,T;V^*)$ for any $\delta \in (0,T)$. Moreover, (5. 6) also yields that $u' \in L^\infty(\delta,T;V)$ for any $\delta \in (0,T)$.

Appendix A. Identification between Lebesgue and Bochner spaces

We often identify the Lebesgue-Bochner space $L^p(0,T;L^p(\Omega))$ with the Lebesgue space $L^p(Q)$ for $Q = \Omega \times (0,T)$. For instance, a function $u = u(x,t) \in L^p(Q)$ with space-time variables $(x,t) \in Q = \Omega \times (0,T)$ corresponds to an $L^p(\Omega)$-valued function $Pu \in L^p(0,T;L^p(\Omega))$ through the mapping $P : L^p(Q) \to L^p(0,T;L^p(\Omega))$ given by

$$(Pu)(t) := u(\cdot,t). \tag{A. 1}$$

This mapping is well-defined and turns out to be linear, bijective and isometric (see Proposition A.3). Conversely, one may expect that each $L^p(\Omega)$-valued function $u = u(t) \in L^p(0,T;L^p(\Omega))$ could be identified with a function $Mu \in L^p(Q)$, where M is defined by the relation

$$(Mu)(x,t) := (u(t))\,(x). \tag{A. 2}$$

However, checking that M is well-defined with values in $L^p(Q)$ is somehow delicate due to the different *measures* characterizing the domain and the target space of the map. For instance, it is known that $L^\infty(Q)$ does not coincide with $L^\infty(0,T;L^\infty(\Omega))$ (see, e.g., [21, Example 1.42, p. 24]) because of the difference between the *Lebesgue measurability* of functions in Q and the *strong measurability* of $L^p(\Omega)$-valued functions over $(0,T)$ in Bochner's sense (and also a lack of Pettis' theorem for $L^\infty(\Omega)$).

On the other hand, for $1 \le p < \infty$, the two classes of spaces can be rigorously identified. In this section, we revise the measure-theoretic arguments leading to this identification and show that, with a limited effort, also the case of (measurable) variable exponents can be covered. The exposition will mainly follow the lines of the standard theory of vector-valued L^p-spaces, so we do not claim any particular originality here. However, the extension to the variable exponent case, although not difficult, seems to be new and this is the main reason why we decided to include this part. Throughout this section, we denote by $\mathcal{L}^N$ the N-dimensional Lebesgue measure.

Remark A.1. To be precise, the *equivalence* in Lebesgue space $L^p(Q)$ and Lebesgue-Bochner space $L^p(0,T;L^p(\Omega))$ is to be interpreted as follows:

(i) The equivalence "$u_1 = u_2$ in $L^p(Q)$" means that there exists an $\mathcal{L}^{N+1}$-measurable subset $\hat{Q}$ of Q such that $u_1(x,t) = u_2(x,t)$ for all $(x,t) \in \hat{Q}$ and $\mathcal{L}^{N+1}(Q \setminus \hat{Q}) = 0$.

(ii) On the other hand, "$u_1 = u_2$ in $L^p(0,T;L^p(\Omega))$" has to be intended as $u_1(t) = u_2(t)$ in $L^p(\Omega)$ for all $t \in I$, where I is an $\mathcal{L}^1$-measurable subset of $(0,T)$ satisfying $\mathcal{L}^1((0,T)\setminus I) = 0$. Moreover, "$u_1(t) = u_2(t)$ in $L^p(\Omega)$" means that $(u_1(t))(x) = (u_2(t))(x)$ for all $x \in \Omega_t$ with some $\mathcal{L}^N$-measurable subset Ω_t of Ω satisfying $\mathcal{L}^N(\Omega \setminus \Omega_t) = 0$.

We begin with the following lemma regarding the behavior of the operator M in the class of simple functions.

Lemma A.2. *The operator M given by* (A. 2) *is well-defined for the class of simple functions with values in $L^p(\Omega)$. Moreover, it holds that $M = P^{-1}$ in that class.*

Proof. Let $v : (0,T) \to L^p(\Omega)$ be given by

$$v(t) = \sum_{j\in J} f_j \chi_{I_j}(t), \tag{A. 3}$$

with a finite set J, disjoint subintervals $\{I_j\}_{j\in J}$ each with positive measure, characteristics functions χ_{I_j} over I_j, and $\{f_j\}_{j\in J} \subset L^p(\Omega) \setminus \{0\}$. Define $\hat{v} : Q \to \mathbb{R}$ by

$$\hat{v}(x,t) := \sum_{j\in J} f_j(x)\chi_{I_j}(t).$$

Then $\hat{v}$ is Lebesgue-measurable over Q. Indeed, for any $a \in \mathbb{R}$, the set

$$\{(x,t) \in Q \colon \hat{v} \le a\} = \bigcup_{j\in J} \{x \in \Omega \colon f_j(x) \le a\} \times I_j$$

is Lebesgue-measurable in Q, since so are $\{x \in \Omega \colon f_j(x) \le a\}$ and I_j in Ω and $(0,T)$, respectively.

Even if v is identified with some other simple function $v_0 : (0,T) \to L^p(\Omega)$ as in (ii) of Remark A.1, $\hat{v}$ coincides with $\hat{v}_0 := (v_0(t))(x)$ a.e. in Q. Actually, by assumption $\hat{v}(x,t) = \hat{v}_0(x,t)$ for all $t \in I$ and $x \in \Omega_t$ with some $I \subset (0,T)$ and a family $\Omega_t \subset \Omega$ with full measure. Set $\hat{Q} := \{(x,t) \in Q \colon \hat{v}(x,t) = \hat{v}_0(x,t)\}$. Since $\hat{v}$ and $\hat{v}_0$ are Lebesgue measurable, $\hat{Q}$ is $\mathcal{L}^{N+1}$-measurable. Set $Z_t := \{x \in \Omega \colon (x,t) \in Q \setminus \hat{Q}\}$ for each $t \in (0,T)$. Then for all $t \in I$, we find that $Z_t \subset \Omega \setminus \Omega_t$; hence $\mathcal{L}^N(Z_t) = 0$. Hence by Fubini's lemma, we see that

$$\mathcal{L}^{N+1}(Q \setminus \hat{Q}) = \int_0^T \mathcal{L}^N(Z_t)\,\mathrm{d}t \le \int_I \mathcal{L}^N(Z_t)\,\mathrm{d}t + \mathcal{L}^1((0,T)\setminus I)\mathcal{L}^N(\Omega) = 0.$$

Hence M is well-defined, and from the definition we immediately get the relation $M = P^{-1}$. This completes the proof. $\square$

The following proposition enables us to identify $L^p(0,T;L^p(\Omega))$ with $L^p(Q)$.

Proposition A.3 (Relations between Lebesgue and Bochner spaces)**.** *Let Ω be a* (*possibly unbounded and non-smooth*) *domain of $\mathbb{R}^N$. Let $Q = \Omega \times (0,T)$ with $T > 0$.*

(i) *If* $p^+ < \infty$*, then the mapping* P *(given as in* (A. 1)*) is well-defined from* $L^{p(x)}(Q)$ *into* $L^{p^-}(0,T;L^{p(x)}(\Omega))$*. Moreover,* P *is linear, injective and continuous, that is, there exists a constant* $C \geq 0$ *such that*

$$\|Pu\|_{L^{p^-}(0,T;L^{p(x)}(\Omega))} \leq C\|u\|_{L^{p(x)}(Q)} \quad \textit{for all} \;\; u \in L^{p(x)}(Q).$$

In particular, if $p(x) \equiv p < \infty$*, then* $P : L^p(Q) \to L^p(0,T;L^p(\Omega))$ *is isometric, i.e.,* $\|Pu\|_{L^p(0,T;L^p(\Omega))} = \|u\|_{L^p(Q)}$ *for all* $u \in L^p(Q)$*.*

(ii) *For each* $u \in L^p(0,T;L^p(\Omega))$*, there exists a unique representative* $\hat{u} \in L^p(Q)$ *of* u*, i.e.,* $\hat{u}$ *is the unique function in* $L^p(Q)$ *such that* $u = P\hat{u}$*. Define a mapping* $R : L^p(0,T;L^p(\Omega)) \to L^p(Q)$ *by* $Ru := \hat{u}$*. Then* R *is linear, bijective and isometric. Furthermore,* $P(Ru) = u$ *for all* $u \in L^p(0,T;L^p(\Omega))$ *and* $R(Pu) = u$ *for all* $u \in L^p(Q)$*; hence,* $R = P^{-1}$*. It also holds that*

$$u(t) = Ru(\cdot,t) \quad \textit{for a.e.} \;\; t \in (0,T)$$

for every $u \in L^p(0,T;L^p(\Omega))$*.*

Proof. We first verify (i). Let $u \in L^{p(x)}(Q)$. Since u is measurable and $(x,t) \mapsto |u(x,t)|^{p(x)}$ is integrable over Q, by Fubini's lemma it holds that $u(\cdot,t) \in L^{p(x)}(\Omega)$ for a.e. $t \in (0,T)$. Let us claim that $u(\cdot,t)$ can be uniquely determined in the sense of (ii) of Remark A.1, although u has to be intended as an equivalence class of $L^{p(x)}(Q)$. Indeed, let $u_1, u_2 \in L^{p(x)}(Q)$ satisfy $u_1 = u_2$ in $L^{p(x)}(Q)$, that is, $u_1(x,t) = u_2(x,t)$ for all $(x,t) \in \hat{Q}$ with some subset $\hat{Q}$ of Q satisfying $\mathcal{L}^{N+1}(Q \setminus \hat{Q}) = 0$. Set $Z_t := \{x \in \Omega\colon (x,t) \in Q \setminus \hat{Q}\}$ for each $t \in (0,T)$. Then by Fubini's lemma, $\mathcal{L}^N(Z_t) = 0$ for all $t \in I$ with a subset $I \subset (0,T)$ satisfying $\mathcal{L}^1((0,T) \setminus I) = 0$. Then $u_1(\cdot,t) = u_2(\cdot,t)$ a.e. in Ω for all $t \in I$. Thus $u(\cdot,t)$ is uniquely determined as a vector-valued function.

Now, let us define an $L^{p(x)}(\Omega)$-valued function $Pu : (0,T) \to L^{p(x)}(\Omega)$ by $(Pu)(t) := u(\cdot,t)$. We next verify the strong measurability of Pu in $(0,T)$. For any $v \in L^{p'(x)}(\Omega) = (L^{p(x)}(\Omega))^*$, it follows by Proposition 2.1 that $uv \in L^1(Q)$. Therefore we observe by Fubini's lemma that

$$\langle v, (Pu)(t) \rangle_{L^{p(x)}(\Omega)} = \int_\Omega v(x)u(x,t)\,\mathrm{d}x \in L^1(0,T).$$

Hence $Pu : (0,T) \to L^{p(x)}(\Omega)$ is weakly measurable thanks to the arbitrariness of v. Since $L^{p(x)}(\Omega)$ is separable by $p^+ < \infty$, Pettis' theorem ensures that Pu is also strongly measurable. Moreover, by Proposition 2.2 it follows that

$$\int_0^T \sigma^-_{p(\cdot)}\left(\|(Pu)(t)\|_{L^{p(x)}(\Omega)}\right)\mathrm{d}t \leq \int_0^T \left(\int_\Omega |u(x,t)|^{p(x)}\mathrm{d}x\right)\mathrm{d}t = \iint_Q |u(x,t)|^{p(x)}\mathrm{d}x\,\mathrm{d}t < \infty.$$

Let us define the (measurable) set $\mathcal{T} := \left\{t \in (0,T) :\ \|Pu(t)\|_{L^{p(x)}(\Omega)} \leq 1\right\}$. Then it follows that

$$\begin{aligned}\int_0^T \|Pu(t)\|^{p^-}_{L^{p(x)}(\Omega)}\mathrm{d}t &= \int_{\mathcal{T}} \|Pu(t)\|^{p^-}_{L^{p(x)}(\Omega)}\mathrm{d}t + \int_{(0,T)\setminus\mathcal{T}} \sigma^-_{p(\cdot)}\left(\|Pu(t)\|_{L^{p(x)}(\Omega)}\right)\mathrm{d}t \\ &\leq T + \iint_Q |u(x,t)|^{p(x)}\mathrm{d}x\,\mathrm{d}t < \infty,\end{aligned}$$

which implies $Pu \in L^{p^-}(0,T;L^{p(x)}(\Omega))$. Hence, we obtain that P is a well-defined operator from $L^{p(x)}(Q)$ into $L^{p^-}(0,T;L^{p(x)}(\Omega))$.

The linearity of P follows immediately from its definition. Let us next check the injectivity of P. Let $u_1, u_2 \in L^{p(x)}(Q)$ satisfy $Pu_1 = Pu_2$. Then one can take $I \subset (0,T)$ and a family $(\Omega_t)_{t\in I}$ as in (ii) of Remark A.1 such that $((Pu_1)(t))(x) = ((Pu_2)(t))(x)$ for all $(x,t) \in \{(x,t)\colon x \in \Omega_t,\ t \in I\}$. Since $((Pu_1)(t))(x) - ((Pu_2)(t))(x) = u_1(x,t) - u_2(x,t)$ are Lebesgue measurable over Q, the set $\hat{Q} = \{(x,t) \in Q\colon u_1(x,t) = u_2(x,t)\}$ is $\mathcal{L}^{N+1}$-measurable. Thus we infer that $u_1(x,t) = u_2(x,t)$

for all $(x,t) \in \hat{Q}$; moreover, by Fubini's lemma, the measure $\mathcal{L}^{N+1}(Q \setminus \hat{Q})$ of the complement is zero, since $\mathcal{L}^1((0,T) \setminus I) = 0$ and $\mathcal{L}^N(\Omega \setminus \Omega_t) = 0$ for all $t \in I$. Hence P is injective.

In particular, if $p(x) \equiv p$, then, by Fubini's lemma, we see that

$$\|u\|^p_{L^p(Q)} = \iint_Q |u(x,t)|^p \mathrm{d}x\,\mathrm{d}t = \int_0^T \left(\int_\Omega |u(x,t)|^p \mathrm{d}x \right) \mathrm{d}t = \|Pu\|^p_{L^p(0,T;L^p(\Omega))}$$

for all $u \in L^p(Q)$.

We next prove (ii). For each $u \in L^p(0,T;L^p(\Omega))$, one can take a sequence (u_n) of $L^p(\Omega)$-valued simple functions such that

$$u_n \to u \quad \text{strongly in } L^p(0,T;L^p(\Omega)). \tag{A. 4}$$

Hence (u_n) forms a Cauchy sequence in $L^p(0,T;L^p(\Omega))$. Recalling Lemma A.2 and using Fubini's lemma, we have

$$\begin{aligned}\|Mu_n - Mu_m\|^p_{L^p(Q)} &= \iint_Q |(Mu_n)(x,t) - (Mu_m)(x,t)|^p \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \left(\int_\Omega |(Mu_n)(x,t) - (Mu_m)(x,t)|^p \,\mathrm{d}x \right) \mathrm{d}t \\ &= \|u_n - u_m\|^p_{L^p(0,T;L^p(\Omega))} \to 0 \quad \text{as } n,m \to \infty.\end{aligned}$$

Thus (Mu_n) forms a Cauchy sequence in $L^p(Q)$. Hence Mu_n converges to some element $\hat{u}$ strongly in $L^p(Q)$. Moreover, we obtain that $u = P\hat{u}$ in $L^p(0,T;L^p(\Omega))$ by observing that

$$\begin{aligned}&\left(\int_0^T \|u(t) - (P\hat{u})(t)\|^p_{L^p(\Omega)} \,\mathrm{d}t \right)^{1/p} \\ &\leq \left(\int_0^T \|u(t) - u_n(t)\|^p_{L^p(\Omega)} \mathrm{d}t \right)^{1/p} + \left(\int_0^T \|u_n(t) - (P\hat{u})(t)\|^p_{L^p(\Omega)} \,\mathrm{d}t \right)^{1/p} \\ &= \left(\int_0^T \|u(t) - u_n(t)\|^p_{L^p(\Omega)} \mathrm{d}t \right)^{1/p} + \left(\iint_Q |(Mu_n)(x,t) - \hat{u}(x,t)|^p \mathrm{d}x\,\mathrm{d}t \right)^{1/p} \to 0\end{aligned}$$

as $n \to \infty$. Here we used the fact that $u_n = P(Mu_n)$. Thus we get $u = P\hat{u}$, which also implies

$$\|\hat{u}\|_{L^p(Q)} \stackrel{\text{(i)}}{=} \|P\hat{u}\|_{L^p(0,T;L^p(\Omega))} = \|u\|_{L^p(0,T;L^p(\Omega))}.$$

Furthermore, let $\hat{u}_1, \hat{u}_2 \in L^p(Q)$ be representatives of $u \in L^p(0,T;L^p(\Omega))$, that is, $P\hat{u}_1 = u = P\hat{u}_2$ in $L^p(0,T;L^p(\Omega))$. Then it follows from (i) that $\|\hat{u}_1 - \hat{u}_2\|_{L^p(Q)} = \|P\hat{u}_1 - P\hat{u}_2\|_{L^p(0,T;L^p(\Omega))} = 0$, which implies $\hat{u}_1 = \hat{u}_2$ a.e. in Q. Hence the representative $\hat{u} \in L^p(Q)$ of u is uniquely determined.

Thus one can define the mapping $R : L^p(0,T;L^p(\Omega)) \to L^p(Q)$ by setting $Ru = \hat{u}$ for each $u \in L^p(0,T;L^p(\Omega))$. Then, from the above, one can immediately find that $P(Ru) = u$ for all $u \in L^p(0,T;L^p(\Omega))$ (hence, R is injective), and R is isometric, that is,

$$\|Ru\|_{L^p(Q)} = \|u\|_{L^p(0,T;L^p(\Omega))} \quad \text{for all } u \in L^p(0,T;L^p(\Omega)). \tag{A. 5}$$

We also claim that R is linear. Indeed, for $u_1, u_2 \in L^p(0,T;L^p(\Omega))$ and $\alpha, \beta \in \mathbb{R}$, let us take simple functions $u_{1,n}, u_{2,n} : (0,T) \to L^p(\Omega)$ such that $u_{i,n} \to u_i$ strongly in $L^p(0,T;L^p(\Omega))$ for $i = 1,2$. Then observing by the linearity of M that

$$R(\alpha u_1 + \beta u_2) := L^p(Q)\text{-}\lim \left(M(\alpha u_{1,n} + \beta u_{2,n}) \right) = \alpha R u_1 + \beta R u_2,$$

we conclude that R is linear. We next claim that $R(Pu) = u$ for all $u \in L^p(Q)$. Indeed, let $u \in L^p(Q)$. Then Pu belongs to $L^p(0,T;L^p(\Omega))$ by (i); hence, there exist simple functions $(Pu)_n$ such that $(Pu)_n \to Pu$ strongly in $L^p(0,T;L^p(\Omega))$. One can prove that $R((Pu)_n) = M((Pu)_n)$. We observe by (i) and Lemma A.2 that $M((Pu)_n) \to u$ strongly in $L^p(Q)$. On the other hand, $R((Pu)_n)$ converges to $R(Pu)$ strongly in $L^p(0,T;L^p(\Omega))$ by (A. 5). Thus we get $R(Pu) = u$ for all $u \in L^p(Q)$. Furthermore, the procedure also yields that R is surjective. Thus we conclude that R and $P = R^{-1}$ are bijective.

Finally, recall that $P(Ru) = u$ for all $u \in L^p(0,T;L^p(\Omega))$ to obtain $u(t) = P(Ru)(t) = Ru(\cdot,t)$ for a.e. $t \in (0,T)$. The proof is completed. $\square$

The next proposition states, roughly speaking, that the operators R, P behave well with respect to differentiation in time.

Proposition A.4. *Let Ω be a (possibly unbounded and nonsmooth) domain of $\mathbb{R}^N$ and let $T > 0$. Then, for every $u \in W^{1,p}(0,T;L^p(\Omega))$, we have*

$$\partial_t(Ru) = R(u') \in L^p(Q).$$

Moreover, if $u \in L^{p(x)}(Q)$ and $\partial_t u \in L^{p(x)}(Q)$, then $Pu \in W^{1,p^-}(0,T;L^{p(x)}(\Omega))$ and

$$(Pu)' = P(\partial_t u) \in L^{p^-}(0,T;L^{p(x)}(\Omega)).$$

Proof. Let $u \in W^{1,p}(0,T;L^p(\Omega))$. Then $u' \in L^p(0,T;L^p(\Omega))$ and

$$\int_0^T u'(t)\rho(t)\,\mathrm{d}t = -\int_0^T u(t)\rho'(t)\,\mathrm{d}t \quad \forall \rho \in C_0^\infty(0,T)$$

(see, e.g., [8] for vector-valued distributions). Moreover, for any $\phi \in C_0^\infty(Q)$, we see that

$$\int_0^T \langle (P\phi)(t), u'(t)\rangle_{L^p(\Omega)}\,\mathrm{d}t = -\int_0^T \langle (P\phi)'(t), u(t)\rangle_{L^p(\Omega)}\,\mathrm{d}t.$$

Since u and u' belong to $L^p(0,T;L^p(\Omega))$, using Proposition A.3 and the fact that $(P\phi)' = P(\partial_t\phi)$ (indeed, one can prove this just by integration by parts), the above equality can be rewritten as

$$\iint_Q \phi R(u')\,\mathrm{d}x\,\mathrm{d}t = -\iint_Q (Ru)\partial_t\phi\,\mathrm{d}x\,\mathrm{d}t \quad \forall \phi \in C_0^\infty(Q),$$

which implies $\partial_t(Ru) = R(u') \in L^p(Q)$.

If $u \in L^{p(x)}(Q)$ and $\partial_t u \in L^{p(x)}(Q)$, then by standard integration by parts in Sobolev spaces we have

$$\iint_Q \phi\partial_t u\,\mathrm{d}x\,\mathrm{d}t = -\iint_Q u\partial_t\phi\,\mathrm{d}x\,\mathrm{d}t \quad \text{for all } \phi \in C_0^\infty(Q).$$

Let $\rho \in C_0^\infty(0,T)$ and $w \in C_0^\infty(\Omega)$. Then, since $\phi(x,t) = \rho(t)w(x) \in C_0^\infty(Q)$, using Fubini's lemma we see that

$$\begin{aligned}
\left\langle w, \int_0^T Pu(t)\rho'(t)\mathrm{d}t\right\rangle_{L^{p(x)}(\Omega)} &= \int_0^T \langle \rho'(t)w, Pu(t)\rangle_{L^{p(x)}(\Omega)}\,\mathrm{d}t \\
&= \iint_Q \rho'(t)w(x)u(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&= -\iint_Q \rho(t)w(x)\partial_t u(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&= -\left\langle w, \int_0^T \rho(t)P(\partial_t u)(t)\,\mathrm{d}t\right\rangle_{L^{p(x)}(\Omega)}.
\end{aligned}$$

From the fact that $C_0^\infty(\Omega)$ is dense in $L^{p'(x)}(\Omega)$ (see Theorem 3.4.12 of [13]), it follows that

$$\int_0^T Pu(t)\rho'(t)\,\mathrm{d}t = -\int_0^T \rho(t)P(\partial_t u)(t)\,\mathrm{d}t.$$

Thus $(Pu)' = P(\partial_t u)$. $\square$

Appendix B. Lower semicontinuity of Φ on $L^{p(x)}(Q)$

Let $u_n \in \mathcal{V} = L^{p(x)}(Q)$ be such that $\Phi(u_n) \leq a$ with an arbitrary $a \in \mathbb{R}$ and $u_n \to u$ strongly in $\mathcal{V}$. Then by Fatou's lemma along with the fact that $\sup_{n\in\mathbb{N}} \Phi(u_n) \leq a$, we see that $\liminf_{n\to\infty} \phi(Pu_n(\cdot)) \in L^1(0,T)$ and

$$\int_0^T \liminf_{n\to\infty} \phi(Pu_n(t))\ \mathrm{d}t \leq \liminf_{n\to\infty} \int_0^T \phi(Pu_n(t))\ \mathrm{d}t \leq a. \tag{B. 1}$$

By (ii) of Lemma 4.5, $Pu_n \to Pu$ strongly in $L^1(0,T;V)$. Hence it follows from the lower semicontinuity of ϕ in V that

$$\phi(Pu(t)) \leq \liminf_{n\to\infty} \phi(Pu_n(t)) < \infty \quad \text{ for a.e. } t \in (0,T), \tag{B. 2}$$

which particularly means that $Pu(t) \in D(\phi)$ for a.e. $t \in (0,T)$.

On the other hand, we claim that $\phi(Pu(\cdot))$ is Lebesgue measurable in $(0,T)$. Indeed, since Pu is strongly measurable with values in V in $(0,T)$, one can take step functions $s_n : (0,T) \to V$ such that $s_n(t) \to Pu(t)$ strongly in V for a.e. $t \in (0,T)$. Let ϕ_λ be the Moreau-Yosida regularization of ϕ in V (in a standard sense). Then $\phi_\lambda(s_n(\cdot))$ is also a step function in Ω; moreover, $\phi_\lambda(s_n(t)) \to \phi_\lambda(Pu(t))$ for a.e. $t \in (0,T)$. Hence $\phi_\lambda(Pu(\cdot))$ is Lebesgue measurable in $(0,T)$. Furthermore, since $\phi_\lambda(Pu(t)) \to \phi(Pu(t)) < \infty$ for a.e. $t \in (0,T)$, we deduce that $\phi(Pu(\cdot))$ is also measurable.

Recall (B. 2) and the measurability of $\phi(Pu(\cdot))$ to get

$$\int_0^T \phi(Pu(t))\ \mathrm{d}t \leq \int_0^T \liminf_{n\to\infty} \phi(Pu_n(t))\ \mathrm{d}t.$$

Thus it follows from (B. 1) that

$$\Phi(u) = \int_0^T \phi(Pu(t))\ \mathrm{d}t \leq \liminf_{n\to\infty} \int_0^T \phi(Pu_n(t))\ \mathrm{d}t \leq a.$$

Therefore, Φ is lower semicontinuous in $\mathcal{V}$. □

References

[1] Aizicovici, S. and Yan, Q., Convergence theorems for abstract doubly nonlinear differential equations, Panamer. Math. J. **7** (1997), 1–17.

[2] Akagi, G. and Matsuura, K., Nonlinear diffusion equations driven by the $p(\cdot)$-Laplacian, Nonlinear Differential Equations and Applications (NoDEA) **20** (2013), 37–64.

[3] Akagi, G. and Ôtani, M., Time-dependent constraint problems arising from macroscopic critical-state models for type-II superconductivity and their approximations, Adv. Math. Sci. Appl. **14** (2004), 683–712.

[4] Ambrosio, L., Gigli, N. and Savaré, G., *Gradient flows in metric spaces and in the space of probability measures*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2005.

[5] Arai, T., On the existence of the solution for $\partial\phi(u'(t)) + \partial\psi(u(t)) \ni f(t)$, J. Fac. Sci. Univ. Tokyo Sec. IA Math. **26** (1979), 75–96.

[6] Aso, M., Frémond, M. and Kenmochi, N., Phase change problems with temperature dependent constraints for the volume fraction velocities, Nonlinear Anal. **60** (2005), 1003–1023.

[7] Barbu, V., Existence theorems for a class of two point boundary problems, J. Differential Equations **17** (1975), 236–257.

[8] Barbu, V., *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff, Leyden, 1976.

[9] Blanchard, D., Damlamian, A. and Ghidouche, H., A nonlinear system for phase change with dissipation, Differential Integral Equations **2** (1989), 344–362.

[10] Brézis, H., *Opérateurs maximaux monotones et sémi-groupes de contractions dans les éspaces de Hilbert*, Math Studies, Vol.5, North-Holland, Amsterdam/New York, 1973.

[11] Colli, P., On some doubly nonlinear evolution equations in Banach spaces, Japan J. Indust. Appl. Math. **9** (1992), 181–203.

[12] Colli, P. and Visintin, A., On a class of doubly nonlinear evolution equations, Comm. Partial Differential Equations **15** (1990), 737–756.

[13] Diening, L., Harjulehto, P., Hästö, P. and Růžička, M., *Lebesgue and Sobolev spaces with variable exponents,* Lecture Notes in Mathematics, vol. 2017, Springer-Verlag, Heidelberg, 2011.

[14] Fan, X. and Zhao, D., On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl. **263** (2001), 424–446.

[15] Kurata, K. and Shioji, N., Compact embedding from $W_0^{1,2}(\Omega)$ to $L^{q(x)}(\Omega)$ and its application to nonlinear elliptic boundary value problem with variable critical exponent, J. Math. Anal. Appl. **339** (2008), 1386–1394.
[16] Mielke, A. and Theil, F., On rate-independent hysteresis models, NoDEA, Nonlinear Differential Equations Appl. **11** (2004), 151–189.
[17] Mielke, A. and Rossi, R., Existence and uniqueness results for a class of rate-independent hysteresis problems, Math. Models Methods Appl. Sci. **17** (2007), 81–123.
[18] Mielke, A., Rossi, R. and Savaré, G., Nonsmooth analysis of doubly nonlinear evolution equations, Calc. Var. Partial Differential Equations **46** (2013), 253–310.
[19] Mizuta, Y., Ohno, T., Shimomura T. and Shioji, N., Compact embeddings for Sobolev spaces of variable exponents and existence of solutions for nonlinear elliptic problems involving the $p(x)$-Laplacian and its critical exponent, Ann. Acad. Sci. Fenn. Math. **35** (2010), 115–130.
[20] Musielak, J., *Orlicz spaces and modular spaces,* Lecture Notes in Mathematics, vol. 1034. Springer-Verlag, Berlin, 1983.
[21] Roubíček, T., *Nonlinear partial differential equations with applications*, International Series of Numerical Mathematics, Vol. 153. Birkhäuser Verlag, Basel, 2005.
[22] Schimperna, G., Segatti, A. and Stefanelli, U., Well-posedness and long-time behavior for a class of doubly nonlinear equations, Discrete Contin. Dyn. Syst. **18** (2007), 15–38.
[23] Segatti, A., Global attractor for a class of doubly nonlinear abstract evolution equations, Discrete Contin. Dyn. Syst. **14** (2006), 801–820.
[24] Senba, T., On some nonlinear evolution equation, Funkcial Ekvac. **29** (1986), 243–257.
[25] Shirakawa, K., Large time behavior for doubly nonlinear systems generated by subdifferentials, Adv. Math. Sci. Appl. **10** (2000), 417–442.
[26] Visintin, A., *Models of phase transitions*, Progress in Nonlinear Differential Equations and Their Applications, vol. 28, Birkhäuser, 1996.

(Goro Akagi) Graduate School of System Informatics, Kobe University, 1-1 Rokkodai-cho, Nada-ku, Kobe 657-8501, Japan.

E-mail address: `akagi@port.kobe-u.ac.jp`

(Giulio Schimperna) Dipartimento di Matematica, Università di Pavia, Via Ferrata, 1, I-27100 Pavia, Italy.

E-mail address: `giusch04@unipv.it`