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LOCAL SOLVABILITY OF A FULLY NONLINEAR PARABOLIC EQUATION

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Abstract. This paper is concerned with the existence of local (in time) positive solutions to the Cauchy-Neumann problem in a smooth bounded domain of $\mathbb{R}^N$ for some fully nonlinear parabolic equation involving the *positive part function* $r \in \mathbb{R} \mapsto (r)_+ := r \vee 0$. To show the local solvability, the equation is reformulated as a mixed form of two different sorts of doubly nonlinear evolution equations in order to apply an energy method. Some approximated problems are also introduced and the global (in time) solvability is proved for them with an aid of convex analysis, an energy method and some properties peculiar to the nonlinearity of the equation. Moreover, two types of comparison principles are also established, and based on these, the local existence and the finite time blow-up of positive solutions to the original equation are concluded as the main results of this paper.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. In this paper, we discuss the local (in time) existence and the finite time blow-up of positive solutions of the Cauchy-Neumann problem for a fully nonlinear parabolic equation,

$$\partial_t u = g(u)\left(\lambda^2 \Delta u + u\right)_+, \quad x \in \Omega,\ t > 0, \tag{1.1}$$

$$\partial_\nu u = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.2}$$

$$u = \phi, \quad x \in \Omega,\ t = 0, \tag{1.3}$$

where $\partial_t = \partial/\partial t$, $g(u)$ is a positive continuous function in $(0, +\infty)$, $(s)_+ := s \vee 0$ stands for the positive part of $s \in \mathbb{R}$, $\lambda > 0$ is a fixed constant, Δ is the standard Laplacian, and ∂_ν denotes the normal derivative. The one dimensional version (i.e., $N = 1$ and $\Omega = (0, 1)$) of Equation (1.1) was originally proposed by Barenblatt and Prostokishin in the context of Damage Mechanics as a model of damage accumulation processes taking account of microstructural effects in [2] (see also [3, §2]), where a typical choice of $g(u)$ is given by a power function,

$$g(u) = u^\alpha, \quad \alpha \geq 0. \tag{1.4}$$

One of peculiarities of the problem is found in the *unidirectional evolution* of solutions; more precisely, $u = u(x, t)$ is non-decreasing in time due to the non-negativity of the right-hand side of (1.1). This feature plays a crucial role in the model of damage accumulation as a natural hypothesis on the unidirectional evolution of an internal variable called a *damage factor*. It is also worth mentioning that solutions of (1.1)–(1.3) may blow up in finite time. Indeed, one can obtain a spatially uniform explicit solution that blows up in finite time for the case (1.4) with $\alpha > 0$ (see also the proof of Lemma 5.1 below).

The main purpose of this paper is to prove the local (in time) existence and the finite time blow-up of positive solutions to (1.1)–(1.3) in an $L^2(\Omega)$ framework.

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For the one-dimensional case, Bertsch and Bisegna [3] proved the local existence and the finite time blow-up of classical solutions under the assumptions that

$$\left.\begin{array}{r}\phi \geq \delta \ \text{ in } [0,1], \quad \phi \in C^1([0,1]), \quad \phi' \text{ is Lipschitz continuous in } [0,1], \\ \phi'(0) = \phi'(1) = 0, \quad (\lambda^2\phi'' + \phi)_+ \in C([0,1])\end{array}\right\} \tag{1.5}$$

for some $\delta > 0$ and some structural conditions on $g(u)$. Moreover, they also investigated qualitative properties of blow-up phenomena; in particular, *regional blow-up* phenomena may occur, that is, the blow-up set is an interval of nonzero measure (but not the whole of Ω) under a suitable initial configuration. In order to take account of chemical aggression, Natalini et al [9] extended the one-dimensional model as a system of nonlinear parabolic equations and also studied it in view of numerical analysis. Furthermore, Nitsch [10] proved the local well-posedness of the system and investigated blow-up properties of solutions (see also [11]).

In this paper, we shall treat Equation (1.1) on the basis of an energy method, that is a totally different way from the previous studies. Generally speaking, energy methods are not so effective for fully nonlinear equations, and therefore, such a severe nonlinearity often prevents us to construct solutions in a suitable energy class. We shall reformulate (1.1) as a mixed form of two different sorts of *doubly nonlinear evolution equations*, for which energy methods are more effective, with an aid of convex analysis. In the next section, we actually reformulate Equation (1.1) and state a main result of the paper. Moreover, an outline of a proof is also exhibited. Sections 3–5 are devoted to a proof of the main result. In Appendix §A, we give a brief exposition of the relevant material on convex analysis for the convenience of the reader.

2. Reformulation of the equation and main result

Define a strictly increasing function $\beta \in C^1_{loc}((0,+\infty))$ by

$$\beta(s) = \int_{s_0}^{s} \frac{\mathrm{d}\sigma}{g(\sigma)} + C_0 \quad \text{ for } s > 0 \tag{2.1}$$

for some constants s_0, $C_0 \in \mathbb{R}$ (the following reformulation will not depend on the choice of s_0, C_0, because we shall treat the derivative of β only, i.e., β' or $\partial_t \beta(u)$ for $u = u(x,t)$). Particularly, in the case of (1.4), the function β can be given as

$$\beta(s) = \begin{cases} \dfrac{s^{1-\alpha}}{1-\alpha} & \text{if } \ \alpha \neq 1, \\ \log s & \text{if } \ \alpha = 1 \end{cases} \quad \text{ for } s > 0. \tag{2.2}$$

Then since $\partial_t \beta(u) = \beta'(u)\partial_t u = \partial_t u / g(u)$, Equation (1.1) is equivalently rewritten as

$$\partial_t \beta(u) = \left(\lambda^2 \Delta u + u\right)_+, \quad x \in \Omega, \ t > 0. \tag{2.3}$$

Moreover, let us define the indicator function $I_{[0,+\infty)} : \mathbb{R} \to [0,+\infty]$ over the set $[0,+\infty)$ by

$$I_{[0,+\infty)}(s) := \begin{cases} 0 & \text{if } \ s \geq 0, \\ +\infty & \text{otherwise} \end{cases} \tag{2.4}$$

and let $\partial I_{[0,+\infty)} : \mathbb{R} \to 2^{\mathbb{R}}$ be the subdifferential operator of $I_{[0,+\infty)}$ given by

$$\partial I_{[0,+\infty)}(s) := \{\xi \in \mathbb{R} \colon I_{[0,+\infty)}(\sigma) - I_{[0,+\infty)}(s) \geq \xi(\sigma - s) \ \text{ for all } \ \sigma \in \mathbb{R}\} \quad \text{ for } \ s \in \mathbb{R} \tag{2.5}$$

with domain $D(\partial I_{[0,+\infty)}) = [0,+\infty)$ (see also § Appendix A). One can observe that

$$\partial I_{[0,+\infty)}(s) = \begin{cases} \{0\} & \text{if } \ s > 0, \\ (-\infty, 0] & \text{if } \ s = 0. \end{cases}$$

Hence $s + \partial I_{[0,+\infty)}(s)$ is the (multivalued) inverse function of $(s)_+ = s \vee 0$. Therefore (2.3) is transformed into the inclusion,

$$\partial_t \beta(u) + \partial I_{[0,+\infty)}(\partial_t \beta(u)) \ni \lambda^2 \Delta u + u, \quad x \in \Omega, \ t > 0.$$

Since β is a strictly increasing function, so we observe that $\partial_t \beta(u) > 0$ if and only if $\partial_t u > 0$; therefore, we have

$$\partial I_{[0,+\infty)}(\partial_t \beta(u)) = \partial I_{[0,+\infty)}(\partial_t u). \tag{2.6}$$

Thus (1.1)–(1.3) has been reformulated as the Cauchy-Neumann problem (denoted by (P) below) for

$$\partial_t \beta(u) + \partial I_{[0,+\infty)}(\partial_t u) \ni \lambda^2 \Delta u + u, \quad x \in \Omega, \ t > 0, \tag{2.7}$$

equipped with the Neumann boundary condition (1.2) and the initial condition (1.3).

Equation (2.7) now falls within the scope of an energy method and it can be regarded as a *mixed form* of two sorts of *doubly nonlinear problems*; one is a sort of *nonlinear diffusion equations* (e.g., porous medium/fast diffusion equation) in the form

$$\partial_t \beta(u) = \Delta u, \quad x \in \Omega, \ t > 0, \tag{2.8}$$

and the other one is a sort of *generalized gradient flows*, e.g., the unidirectional heat flow (see [1]),

$$\partial I_{[0,+\infty)}(\partial_t u) = \Delta u, \quad x \in \Omega, \ t > 0. \tag{2.9}$$

The former one has been vigorously studied so far; however, to the best of the author's knowledge, the mixed form such as (2.7) has not yet been fully pursued.

Prior to stating the main result of this paper, let us introduce the following assumptions $(\beta)_\delta$ for general β and a constant $\delta > 0$: There exists a constant $C_\delta > 0$ such that

$$\left.\begin{array}{r} \beta \in C^1([\delta,+\infty)) \quad \text{and} \quad 0 < \beta'(s) \le C_\delta \ \text{ for all } \ s \ge \delta, \\ \beta' \text{ is non-increasing on } [\delta,+\infty), \end{array}\right\} \tag{$(\beta)_\delta$}$$

which implies

$$|\beta(s)| \le C_\delta s + |\beta(\delta)| \ \text{ for all } \ s \ge \delta \tag{2.10}$$

and the strict increase of β on $[\delta,\infty)$. We note that $(\beta)_\delta$ holds true in the case of (1.4) for any $\alpha \ge 0$.

Here and henceforth, $C_{w*}([0,T];L^\infty(\Omega))$ denotes the set of all $L^\infty(\Omega)$-valued weakly star continuous functions on $[0,T]$. Moreover, we refer the reader to Definition 2.3 below for the precise definition of *strong solutions* of (P).

Now, our main result reads,

Theorem 2.1 (Local solvability of (P) and finite time blow-up of positive solutions). *Assume that*

$$\left.\begin{array}{r} \phi \in H^2(\Omega) \cap L^\infty(\Omega), \quad \partial_\nu \phi = 0 \ \text{ on } \partial\Omega, \quad \left(\lambda^2 \Delta\phi + \phi\right)_- \in L^\infty(\Omega), \\ \phi \ge \delta \ \text{ a.e. in } \Omega, \ \text{ for some constant } \delta > 0, \end{array}\right\} \tag{2.11}$$

where $(s)_- := s \wedge 0 \le 0$ is the negative part of $s \in \mathbb{R}$. Moreover, suppose that $(\beta)_\delta$ is satisfied. Then the Cauchy-Neumann problem $(\mathrm{P}) = \{(2.7),(1.2),(1.3)\}$ admits at least one strong solution u on $[0,T_0]$ for some $T_0 > 0$ such that $u \in C_{w*}([0,T_0];L^\infty(\Omega))$ and

$$\|(\lambda^2 \Delta u + u)_-\|_{L^\infty(\Omega\times(0,T_0))} \le \|(\lambda^2 \Delta\phi + \phi)_-\|_{L^\infty(\Omega)}. \tag{2.12}$$

Moreover, let $u \in C_{w*}([0,S];L^\infty(\Omega))$ be a strong solution of (P) on some interval $[0,S]$ for some data $\phi \in L^\infty(\Omega)$ satisfying $\phi \ge \delta$ a.e. in Ω. Let $T_{\max} > 0$ be the supremum of $\tau \ge S > 0$ for which u can be extended onto $[0,\tau]$ such that $u \in C_{w*}([0,\tau];L^\infty(\Omega))$. In addition, define

$$\hat{T}(s) := \int_s^{+\infty} \frac{\beta'(\zeta)}{\zeta}\, \mathrm{d}\zeta \in (0,+\infty] \quad \text{for} \ \ s > 0.$$

Then it follows that

$$T_{\max} \le \hat{T}(\delta).$$

Moreover, if $T_{\max} < +\infty$, then it holds that

$$\lim_{t \nearrow T_{\max}} \|u(t)\|_{L^\infty(\Omega)} = +\infty. \tag{2.13}$$

Remark 2.2. (i) The function $t \mapsto \|u(t)\|_{L^\infty(\Omega)}$ is left-continuous on $[0,T]$. Indeed, since $u \in C_{w*}([0,T]; L^\infty(\Omega))$ and $u(x,t)$ is non-decreasing in $(0,T)$, we observe that

$$\|u(t)\|_{L^\infty(\Omega)} \le \liminf_{s\nearrow t} \|u(s)\|_{L^\infty(\Omega)} \le \limsup_{s\nearrow t} \|u(s)\|_{L^\infty(\Omega)} \le \|u(t)\|_{L^\infty(\Omega)}$$

for each $t \in [0,T]$.

(ii) The local existence time T_0 can be represented as follows:

$$T_0 = \ell(\|\phi\|_{L^\infty(\Omega)} + \|(\lambda^2\Delta\phi + \phi)_-\|_{L^\infty(\Omega)} + C_\delta + 1),$$

where $\ell(\cdot)$ is a positive strictly decreasing function depending only on β' (see Remark 5.2 below for more details).

(iii) In case $g(u) = u^\alpha$ for $\alpha > 0$ (see also (2.2)), one can check $\hat{T}(\delta) < +\infty$ for any $\delta > 0$. Hence $T_{\max}$ is always finite. As for the case $\alpha = 0$, we shall further exhibit a global (in time) existence result under weaker assumptions on initial data ϕ (see Theorem 3.8 and Remark 3.9 below for details).

(iv) This result is still new even for the case $N = 1$. Indeed, the assumption (2.11) is slightly weaker than (1.5) assumed in [3]. Assumption (1.5) requires ϕ' to be Lipschitz continuous on $[0,1]$, and hence, we particularly have $\phi \in W^{2,\infty}(\Omega) \subset H^2(\Omega)$, since every Lipschitz continuous function belongs to $W^{1,\infty}(\Omega)$. On the other hand, the non-increase of β' is assumed in Theorem 2.1 (cf. [3]), and the uniqueness of (general) solutions (cf. Corollary 4.6) and qualitative properties of blow-up solutions are not discussed in this paper.

An outline of a proof is as follows: We first construct global (in time) solutions to the Cauchy-Neumann problem $(\mathrm{P})_\mu$, $\mu > 0$, for the following approximated equations,

$$\mu\partial_t u + \partial_t\beta(u) + \partial I_{[0,+\infty)}(\partial_t u) \ni \lambda^2\Delta u + u, \quad x \in \Omega,\ t > 0 \tag{2.14}$$

along with (1.2) and (1.3) under milder assumptions. Thanks to the additional time derivative of u in the left-hand side, one may expect the existence of global (in time) solutions. Indeed, we construct a global solution $u_\mu = u_\mu(x,t)$, which is unbounded in time, by using a time-discretization technique and an energy method (see Section 3).

Before going on to the limiting procedure as $\mu \to 0$, we establish an $L^\infty(\Omega)$ estimate for $u_\mu(\cdot,t)$ uniform in t on some interval and $\mu \in (0,1)$. To this end, we prove a comparison principle for *strictly increasing* subsolutions and general supersolutions of $(\mathrm{P})_\mu$ (see Section 4). However, one cannot directly apply the principle to u_μ due to the assumption of the strict increase of subsolutions, which is more restrictive than those of usual comparison principles and arising from a peculiar nonlinearity of $(\mathrm{P})_\mu$. To overcome this defect, we introduce an auxiliary subsolution of $(\mathrm{P})_\mu$ which is greater than u_μ and strictly increasing. Constructing an appropriate spatially uniform supersolution of $(\mathrm{P})_\mu$, we derive a uniform estimate for $\|u_\mu(t)\|_{L^\infty(\Omega)}$ locally in time (see Section 5).

Furthermore, we establish uniform estimates and pass to the limit as $\mu \to 0$ in order to prove the local (in time) existence of solutions for the original problem (P). The finite time blow-up of solutions for (P) is also verified by using a comparison principle for (P) (see Theorem 4.5) and a strictly increasing explicit subsolution (see Section 5).

Let us close this section by giving a definition of strong solutions for (P) and $(\mathrm{P})_\mu$ and some remarks. From now on, $C_w([0,T]; H^1(\Omega))$ stands for the space of all weakly continuous functions on $[0,T]$ with values in $H^1(\Omega)$.

Definition 2.3 (Strong solutions of (P) and $(\mathrm{P})_\mu$). *For $T > 0$ and $\mu \ge 0$, a positive function $u \in W^{1,2}(0,T; L^2(\Omega))$ is called a* strong solution (*or a* solution *for short*) *of* $(\mathrm{P})_\mu$ ($=$ (P) *if* $\mu = 0$) *on* $[0,T]$, *if the following* (i)–(iii) *hold true*:

(i) $u \in C_w([0,T]; H^1(\Omega)),\ \ \beta(u) \in W^{1,2}(0,T; L^2(\Omega)),\ \ \Delta u \in L^2(0,T; L^2(\Omega))$,

(ii) $\partial_\nu u = 0$ *on* $\partial\Omega$ *and* $u(\cdot,t) \in H^2(\Omega)$ *for a.e.* $t \in (0,T)$,

(iii) $\partial_t u(x,t) \geq 0$ *for a.e.* $(x,t) \in \Omega \times (0,T)$, *and there exists* $\xi \in L^2(0,T;L^2(\Omega))$ *such that*

$$\mu\partial_t u + \partial_t \beta(u) + \xi = \lambda^2 \Delta u + u, \quad \xi \in \partial I_{[0,+\infty)}(\partial_t u) \tag{2.15}$$

for a.e. $(x,t) \in \Omega \times (0,T)$.

Solutions of $(\mathrm{P})_\mu$ *(or* (P)*) are also denoted by* (u,ξ) *in order to specify the section* ξ *of* $\partial_t I_{[0,+\infty)}(\partial_t u)$ *as in* (2.15).

Furthermore, a solution u *of* $(\mathrm{P})_\mu$ *(or* (P)*) on* $[0,T]$ *is said to be* strictly increasing *if* $\partial_t u(x,t) > 0$ *for a.e.* $(x,t) \in \Omega \times (0,T)$.

Remark 2.4. Let (u,ξ) be a strong solution of $(\mathrm{P})_\mu$ (or (P)) on $[0,T]$. One may obtain a representation of ξ,

$$\xi = (\lambda^2 \Delta u + u)_- \quad \text{for a.e.} \quad (x,t) \in \Omega \times (0,T). \tag{2.16}$$

Indeed, let $(x,t) \in \Omega \times (0,T)$ be such that (2.15) holds there. In case when $\xi(x,t) = 0$, one has $0 \leq \mu\partial_t u + \partial_t \beta(u) = \lambda^2 \Delta u + u$ at (x,t). Hence $(\lambda^2 \Delta u + u)_-(x,t) = 0$. In case when $\xi(x,t) < 0$, noting by (2.6) that $\partial_t u(x,t) = \partial_t \beta(u)(x,t) = 0$, we deduce that $0 > \xi = \lambda^2 \Delta u + u$ at (x,t). Therefore in both cases, (2.16) follows.

Here and henceforth, for simplicity, we use the same notation $I_{[0,+\infty)}$ for the indicator function over $[0,+\infty)$ defined on $\mathbb{R}$ as well as for that over the set $\{u \in L^2(\Omega) \colon u \geq 0 \ \text{ a.e. in } \Omega\}$ defined on $L^2(\Omega)$, unless any confusion may arise. Moreover, the subdifferential operators (in $L^2(\Omega)$) of the both indicator functions are also denoted by $\partial I_{[0,+\infty)}$ (see also Proposition A.2 for their equivalence).

3. Solvability of the approximated problems $(\mathrm{P})_\mu$

In this section, we construct strong solutions for the approximate problems $(\mathrm{P})_\mu$ for $\mu > 0$ under a milder assumption,

$$\phi \in H^2(\Omega), \quad \partial_\nu \phi = 0 \ \text{ on } \partial\Omega, \quad \phi \geq \delta \ \text{ a.e. in } \Omega, \ \text{ for some constant } \delta > 0 \tag{3.1}$$

without assuming (2.11). Then one can remove the singularity of $\beta(s)$ at $s = 0$ (e.g., see (2.2)) by replacing β with a proper non-decreasing smooth function which coincides with β on $[\delta,+\infty)$ without any loss of generality, since solutions u of $(\mathrm{P})_\mu$ are always supposed to be not less than δ under (3.1). Throughout this section, we always assume that $\beta \in C^1([0,\infty))$.

We start with a time-discretization. Let $N \in \mathbb{N}$, $\tau = \tau_N := T/N > 0$ and consider the following discretized problems:

$$\mu \frac{u_{n+1} - u_n}{\tau} + \frac{\beta(u_{n+1}) - \beta(u_n)}{\tau} + \xi_{n+1} = \lambda^2 \Delta u_{n+1} + u_{n+1} \ \text{ in } L^2(\Omega), \tag{3.2}$$

$$\xi_{n+1} \in \partial I_{[0,+\infty)}\left(\frac{u_{n+1} - u_n}{\tau}\right), \quad u_0 = \phi \ \text{ in } L^2(\Omega) \tag{3.3}$$

for $n = 0,1,\ldots,N-1$. We then claim

Lemma 3.1. *For each* $\tau \in (0,\mu)$, *the discretized problems* (3.2), (3.3) *admit solutions* $(u_{n+1},\xi_{n+1}) \in H^2(\Omega) \times L^2(\Omega)$ *for* $n = 0,1,\ldots,N-1$.

Proof. Let $n \in \{0,1,\ldots,N-1\}$ and let $u_n \in D(-\Delta) := \{v \in H^2(\Omega) \colon \partial_\nu v = 0 \ \text{ on } \partial\Omega\}$ be such that $u_n \geq \delta$ a.e. in Ω. Define functionals $J_{n+1} : H^1(\Omega) \to (-\infty,+\infty]$ by

$$\begin{aligned} J_{n+1}(u) := \frac{\mu}{2\tau}\|u\|^2_{L^2(\Omega)} + \frac{1}{\tau}\psi(u) + I_{[0,+\infty)}\left(\frac{u - u_n}{\tau}\right) + \frac{\lambda^2}{2}\|\nabla u\|^2_{L^2(\Omega)} \\ - \frac{1}{2}\|u\|^2_{L^2(\Omega)} - \left\langle \frac{\beta(u_n)}{\tau} + \mu\frac{u_n}{\tau}, u \right\rangle_{H^1(\Omega)} \quad \text{for } u \in H^1(\Omega), \end{aligned}$$

where ψ is a functional of class C^1 defined on $H^1(\Omega)$ by

$$\psi(u) := \int_\Omega \hat{\beta}(u)\, \mathrm{d}x \quad \text{for} \quad u \in H^1(\Omega) \quad \text{with} \quad \hat{\beta}(s) := \int_0^s \beta(\sigma)\, \mathrm{d}\sigma$$

and $\langle \cdot, \cdot \rangle_{H^1(\Omega)}$ stands for the duality pairing between $H^1(\Omega)$ and $(H^1(\Omega))^*$. Then J_{n+1} is well defined on $H^1(\Omega)$, since we see that $\beta(u_n) \in L^2(\Omega)$ by $(\beta)_\delta$ along with the assumptions of u_n. For $\tau \in (0, \mu)$, each functional J_{n+1} is coercive, strictly convex and lower semicontinuous in $H^1(\Omega)$. Hence employing the Direct Method, one can verify that J_{n+1} admits a unique minimizer $u_{n+1} \in H^1(\Omega)$, and then, u_{n+1} solves the Euler-Lagrange equation,

$$\mu \frac{u_{n+1} - u_n}{\tau} + \frac{\beta(u_{n+1}) - \beta(u_n)}{\tau} + \xi_{n+1} = \lambda^2 \Delta u_{n+1} + u_{n+1} \quad \text{in } (H^1(\Omega))^*, \tag{3.4}$$

$$\xi_{n+1} \in \partial_{H^1(\Omega)} I_{[0,+\infty)} \left(\frac{u_{n+1} - u_n}{\tau} \right) \quad \text{in } (H^1(\Omega))^*, \tag{3.5}$$

where $\partial_{H^1(\Omega)} I_{[0,+\infty)}$ denotes the subdifferential operator from $H^1(\Omega)$ to $2^{(H^1(\Omega))^*}$ of the functional $I_{[0,+\infty)}$ restricted onto $H^1(\Omega)$ (see (A.1) of § Appendix A below).

Note that

$$\partial_{H^1(\Omega)} I_{[0,+\infty)} \left(\frac{u - u_n}{\tau} \right) = \partial_{H^1(\Omega)} I_{[\,\cdot\, \geq u_n]}(u),$$

where $\partial_{H^1(\Omega)} I_{[\,\cdot\, \geq u_n]}$ stands for the subdifferential of the indicator function $I_{[\,\cdot\, \geq u_n]}$ over the set $[\,\cdot \geq u_n] := \{u \in H^1(\Omega) \colon u(x) \geq u_n(x) \text{ for a.e. } x \in \Omega\}$. Then one can rewrite (3.4), (3.5) as the variational inequality of obstacle type,

$$\left(\frac{\mu}{\tau} - 1 \right) u_{n+1} - \lambda^2 \Delta u_{n+1} + \partial_{H^1(\Omega)} I_{[\,\cdot\, \geq u_n]}(u_{n+1}) \ni \frac{\mu}{\tau} u_n - \frac{\beta(u_{n+1}) - \beta(u_n)}{\tau} \quad \text{in } (H^1(\Omega))^*. \tag{3.6}$$

Here we shall exploit a regularity theory for variational inequalities of obstacle type. Let $K := \{u \in H^1(\Omega) \colon u \geq \psi \text{ a.e. in } \Omega\}$ for some $\psi \in L^2(\Omega)$ and let $A : H^1(\Omega) \to (H^1(\Omega))^*$ be the homeomorphism defined by

$$\langle Au, \phi \rangle_{H^1(\Omega)} = \gamma \int_\Omega u \phi\, \mathrm{d}x + \lambda^2 \int_\Omega \nabla u \cdot \nabla \phi\, \mathrm{d}x \quad \text{for } u, \phi \in H^1(\Omega)$$

for some $\gamma > 0$ (i.e., $Au = \gamma u - \lambda^2 \Delta u$). Concerning the variational inequality with $f \in (H^1(\Omega))^*$,

$$u \in K, \quad \langle Au, v - u \rangle_{H^1(\Omega)} \geq \langle f, v - u \rangle_{H^1(\Omega)} \quad \text{for all } v \in K, \tag{3.7}$$

which can be equivalently rewritten as

$$Au + \partial_{H^1(\Omega)} I_K(u) \ni f \quad \text{in } (H^1(\Omega))^*$$

(here I_K stands for the indicator function over the set K), we recall the following proposition (see [1] or [7] with a proper modification):

PROPOSITION 3.2 ([1], [7]). *Suppose that $\psi \in W^{2,p}(\Omega)$, $\partial_\nu \psi = 0$ on $\partial\Omega$ and $f \in L^p(\Omega)$ for some $p \geq 2$. Then the unique weak solution u of* (3.7) *belongs to $W^{2,p}(\Omega)$, $\partial_\nu u = 0$ a.e. on $\partial\Omega$, and*

$$f \leq Au \leq f \vee A\psi \quad \text{a.e. in } \Omega,$$

where $A\psi = \gamma\psi - \lambda^2 \Delta\psi$.

Due to the fact that $u_n \in D(-\Delta)$, by applying Proposition 3.2 to (3.6), one can verify that $u_{n+1} \in D(-\Delta)$ and

$$\begin{aligned} &\frac{\mu}{\tau} u_n - \frac{\beta(u_{n+1}) - \beta(u_n)}{\tau} \leq \left(\frac{\mu}{\tau} - 1 \right) u_{n+1} - \lambda^2 \Delta u_{n+1} \\ &\leq \left(\frac{\mu}{\tau} u_n - \frac{\beta(u_{n+1}) - \beta(u_n)}{\tau} \right) \vee \left(\left(\frac{\mu}{\tau} - 1 \right) u_n - \lambda^2 \Delta u_n \right) \quad \text{for a.e. } x \in \Omega, \end{aligned}$$

which implies

$$
\begin{aligned}
0 &\leq \mu \frac{u_{n+1}-u_n}{\tau} + \frac{\beta(u_{n+1})-\beta(u_n)}{\tau} - \lambda^2 \Delta u_{n+1} - u_{n+1} \\
&\leq 0 \vee \left(-\lambda^2 \Delta u_n - u_n + \frac{\beta(u_{n+1})-\beta(u_n)}{\tau} \right) \quad \text{for a.e.} \quad x \in \Omega.
\end{aligned}
$$

Since $u_n \in H^2(\Omega)$ and $\beta(u_n)$, $\beta(u_{n+1}) \in L^2(\Omega)$ (by (2.10)), we can verify from (3.4) that $\xi_{n+1} \in L^2(\Omega)$ and $\xi_{n+1} \in \partial I_{[0,+\infty)}((u_{n+1}-u_n)/\tau)$. Moreover, from the fact that $\xi_{n+1}(x) \neq 0$ only if $u_{n+1}(x) = u_n(x)$, we also deduce that

$$
0 \leq -\xi_{n+1} \leq -\left(\lambda^2 \Delta u_n + u_n\right)_- \quad \text{for a.e.} \quad x \in \Omega, \tag{3.8}
$$

where $(s)_- = s \wedge 0 \leq 0$.

By virtue of (3.1), starting from $n = 0$ and $u_0 = \phi$, one can iteratively obtain solutions $u_{n+1} \in D(-\Delta)$ and $\xi_{n+1} \in L^2(\Omega)$ of (3.2), (3.3) for $n = 0, 1, \ldots, N-1$. □

We next establish a priori estimates for u_n and ξ_n.

LEMMA 3.3. *There exists a constant $C \geq 0$ depending on μ, λ and ϕ such that*

$$
\max_n \left(\|u_n\|_{L^2(\Omega)}^2 + \|\nabla u_n\|_{L^2(\Omega)}^2 \right) + \sum_{n=0}^{N-1} \tau \left\| \frac{u_{n+1}-u_n}{\tau} \right\|_{L^2(\Omega)}^2 \leq C, \tag{3.9}
$$

$$
\max_n \|\beta(u_n)\|_{L^2(\Omega)}^2 \leq C, \tag{3.10}
$$

$$
\sum_{n=0}^{N-1} \tau \left\| \frac{\beta(u_{n+1})-\beta(u_n)}{\tau} \right\|_{L^2(\Omega)}^2 \leq C. \tag{3.11}
$$

Proof. Test (3.2) by $(u_{n+1}-u_n)/\tau$ to get

$$
\begin{aligned}
&\mu \left\| \frac{u_{n+1}-u_n}{\tau} \right\|_{L^2(\Omega)}^2 + \left(\frac{\beta(u_{n+1})-\beta(u_n)}{\tau}, \frac{u_{n+1}-u_n}{\tau} \right)_{L^2(\Omega)} + \left(\xi_{n+1}, \frac{u_{n+1}-u_n}{\tau} \right)_{L^2(\Omega)} \\
&\quad + \frac{\lambda^2}{2\tau} \|\nabla u_{n+1}\|_{L^2(\Omega)}^2 - \frac{\lambda^2}{2\tau} \|\nabla u_n\|_{L^2(\Omega)}^2 \\
&\leq \left(u_{n+1}, \frac{u_{n+1}-u_n}{\tau} \right)_{L^2(\Omega)} \leq \frac{\mu}{2} \left\| \frac{u_{n+1}-u_n}{\tau} \right\|_{L^2(\Omega)}^2 + C \|u_{n+1}\|_{L^2(\Omega)}^2,
\end{aligned} \tag{3.12}
$$

where $(\cdot,\cdot)_{L^2(\Omega)}$ stands for the inner product in $L^2(\Omega)$, for $n = 0, 1, \ldots, N-1$. Moreover, we note that

$$
\frac{\|u_{n+1}\|_{L^2(\Omega)}^2 - \|u_n\|_{L^2(\Omega)}^2}{2\tau} \leq \frac{1}{2} \|u_{n+1}\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| \frac{u_{n+1}-u_n}{\tau} \right\|_{L^2(\Omega)}^2
$$

and

$$
\left(\xi_{n+1}, \frac{u_{n+1}-u_n}{\tau} \right)_{L^2(\Omega)} = 0.
$$

Hence exploiting the monotonicity of β and summing both sides of (3.12) for $n = 0, 1, \ldots, k \in \{2, 3, \ldots, N-1\}$, we derive

$$
\frac{\mu}{2} \|u_{k+1}\|_{L^2(\Omega)}^2 + \frac{\lambda^2}{2} \|\nabla u_{k+1}\|_{L^2(\Omega)}^2 \leq \frac{\mu}{2} \|\phi\|_{L^2(\Omega)}^2 + \frac{\lambda^2}{2} \|\nabla \phi\|_{L^2(\Omega)}^2 + \left(\frac{\mu}{2} + C \right) \sum_{j=0}^{k+1} \tau \|u_j\|_{L^2(\Omega)}^2.
$$

Exploiting the discrete Gronwall inequality, one has

$$
\max_n \left(\|u_n\|_{L^2(\Omega)}^2 + \|\nabla u_n\|_{L^2(\Omega)}^2 \right) \leq C \left(\|\phi\|_{L^2(\Omega)}^2 + \|\nabla \phi\|_{L^2(\Omega)}^2 \right),
$$

which together with (2.10) also yields (3.10). Furthermore, recalling (3.12), we obtain (3.9).

By $(\beta)_\delta$ and the Mean-Value Theorem, we observe that

$$\left|\frac{\beta(u_{n+1})-\beta(u_n)}{\tau}\right| = \frac{1}{\tau}|\beta'((1-\theta_n)u_{n+1}+\theta_n u_n)||u_{n+1}-u_n| \leq C_\delta \left|\frac{u_{n+1}-u_n}{\tau}\right|$$

for some $\theta_n = \theta_n(x) \in (0,1)$, a.e. $x \in \Omega$. Thus (3.11) follows from (3.9). □

We further prove:

LEMMA 3.4. *If $(\lambda^2\Delta\phi+\phi)_- \in L^p(\Omega)$ for some $p \in [2,+\infty]$, it follows that*

$$\|\xi_{n+1}\|_{L^p(\Omega)} \leq \|\xi_n\|_{L^p(\Omega)} \leq \|(\lambda^2\Delta\phi+\phi)_-\|_{L^p(\Omega)} \quad \text{for } n = 1,2,\dots,N-1. \tag{3.13}$$

In particular, (3.13) *follows with $p=2$ from the fact that $\phi \in H^2(\Omega)$ by* (3.1).

Moreover, there exists a constant $C \geq 0$ depending on μ, λ and ϕ such that

$$\sum_{n=0}^{N-1} \tau\|\Delta u_{n+1}\|^2_{L^2(\Omega)} \leq C. \tag{3.14}$$

Proof. By subtraction of equations, we have

$$\mu\left(\frac{u_{n+1}-u_n}{\tau} - \frac{u_n-u_{n-1}}{\tau}\right) + \frac{\beta(u_{n+1})-\beta(u_n)}{\tau} - \frac{\beta(u_n)-\beta(u_{n-1})}{\tau}$$
$$+\xi_{n+1}-\xi_n = \lambda^2\Delta(u_{n+1}-u_n)+u_{n+1}-u_n \tag{3.15}$$

for $n = 1,2,\dots,N-1$. Assume that $(\lambda^2\Delta\phi+\phi)_- \in L^p(\Omega)$ for some $p \in [2,+\infty)$. Let $R>0$ and let $\gamma_R \in C^1(\mathbb{R})$ be a smooth monotone function satisfying

$$\gamma_R(u) = \begin{cases} |u|^{p-2}u & \text{if } |u| \leq R, \\ \operatorname{sgn}(u)(R+1)^{p-1} & \text{if } |u| \geq R+2, \end{cases}$$

where $\operatorname{sgn}(\cdot)$ denotes the sign function. Test (3.15) by $\eta_{n+1} := \gamma_R(\xi_{n+1}) \in L^\infty(\Omega)$. Here we note that η_{n+1} also belongs to $\partial I_{[0,+\infty)}((u_{n+1}-u_n)/\tau)$ and that

$$(-\Delta u, \eta)_{L^2(\Omega)} \geq 0 \quad \text{for all } \eta \in \partial I_{[0,+\infty)}(u) \text{ and } u \in D(-\Delta) \text{ satisfying } u \geq 0$$

by (iii) of Proposition A.2 in Appendix. Furthermore, by definitions of the indicator function $I_{[0,+\infty)}$ and its subdifferential, we observe that

$$\left(\frac{u_{n+1}-u_n}{\tau} - \frac{u_n-u_{n-1}}{\tau}, \eta_{n+1}\right)_{L^2(\Omega)} \geq I_{[0,+\infty)}\left(\frac{u_{n+1}-u_n}{\tau}\right) - I_{[0,+\infty)}\left(\frac{u_n-u_{n-1}}{\tau}\right) = 0,$$

since $\{u_n(x)\}$ is non-decreasing in n a.e. $x \in \Omega$. Noting that

$$\eta_{n+1} \in \partial I_{[0,+\infty)}\left(\frac{u_{n+1}-u_n}{\tau}\right) = \partial I_{[0,+\infty)}\left(\frac{\beta(u_{n+1})-\beta(u_n)}{\tau}\right),$$

one can similarly derive

$$\left(\frac{\beta(u_{n+1})-\beta(u_n)}{\tau} - \frac{\beta(u_n)-\beta(u_{n-1})}{\tau}, \eta_{n+1}\right)_{L^2(\Omega)} \geq 0.$$

Moreover, we also note that

$$(u_{n+1}-u_n, \eta_{n+1})_{L^2(\Omega)} = 0,$$

since $\eta_{n+1}(x) \neq 0$ only if $u_{n+1}(x) = u_n(x)$. Combining all these facts, we have

$$\int_\Omega \hat{\gamma}_R(\xi_{n+1})\,\mathrm{d}x \leq \int_\Omega \hat{\gamma}_R(\xi_n)\,\mathrm{d}x \quad \text{for } n = 1,2,\dots,N-1,$$

where $\hat{\gamma}_R$ stands for the primitive function of γ_R satisfying $\hat{\gamma}_R(0) = 0$. Moreover, letting $R \to +\infty$ and recalling (3.8) with $n=0$, we find by iteration that

$$\|\xi_n\|_{L^p(\Omega)} \leq \|\xi_1\|_{L^p(\Omega)} \leq \|(\lambda^2\Delta\phi+\phi)_-\|_{L^p(\Omega)} \quad \text{for } n = 2,3,\dots,N.$$

As for the case $(\lambda^2\Delta\phi+\phi)_- \in L^\infty(\Omega)$, passing to the limit as $p \to +\infty$ in both sides, we conclude that (3.13) holds true for $p = \infty$.

Finally, (3.14) follows by comparison of both sides in (3.2). $\square$

Now, let us move on to the limiting procedure. To this end, we first introduce the *piecewise forward constant interpolants* $\bar{u}_\tau(t) := u_{n+1}$ and $\bar{\xi}_\tau(t) := \xi_{n+1}$ for $t \in [t_n, t_{n+1})$ and the *piecewise linear interpolants*

$$u_\tau(t) := \frac{t_{n+1}-t}{\tau}u_n + \frac{t-t_n}{\tau}u_{n+1} \quad \text{ if } \ t \in [t_n, t_{n+1}),$$
$$v_\tau(t) := \frac{t_{n+1}-t}{\tau}\beta(u_n) + \frac{t-t_n}{\tau}\beta(u_{n+1}) \quad \text{ if } \ t \in [t_n, t_{n+1})$$

for $n = 0, 1, \ldots, N-1$. Then (3.2) is rewritten as

$$\mu\partial_t u_\tau + \partial_t v_\tau + \bar{\xi}_\tau = \lambda^2\Delta\bar{u}_\tau + \bar{u}_\tau, \quad \bar{\xi}_\tau \in \partial I_{[0,+\infty)}(\partial_t u_\tau). \tag{3.16}$$

From the preceding a priori estimates, we can derive the following convergences by taking a (non-relabeled) subsequence of $\tau \to 0$ (equivalently, $N \to +\infty$):

Lemma 3.5. *It holds, up to a subsequence, that*

$$u_\tau \to u \qquad \textit{weakly star in } L^\infty(0,T;H^1(\Omega)), \tag{3.17}$$
$$\textit{strongly in } C([0,T];L^2(\Omega)), \tag{3.18}$$
$$\bar{u}_\tau \to \bar{u} \qquad \textit{weakly star in } L^\infty(0,T;H^1(\Omega)), \tag{3.19}$$
$$\textit{strongly in } L^\infty(0,T;L^2(\Omega)), \tag{3.20}$$
$$\partial_t u_\tau \to \partial_t u \qquad \textit{weakly in } L^2(0,T;L^2(\Omega)), \tag{3.21}$$
$$v_\tau \to v \qquad \textit{weakly star in } L^\infty(0,T;L^2(\Omega)), \tag{3.22}$$
$$\beta(\bar{u}_\tau) \to \bar{v} \qquad \textit{weakly star in } L^\infty(0,T;L^2(\Omega)), \tag{3.23}$$
$$\partial_t v_\tau \to \partial_t v \qquad \textit{weakly in } L^2(0,T;L^2(\Omega)), \tag{3.24}$$
$$\bar{\xi}_\tau \to \xi \qquad \textit{weakly star in } L^\infty(0,T;L^2(\Omega)), \tag{3.25}$$
$$\Delta\bar{u}_\tau \to \Delta\bar{u} \qquad \textit{weakly in } L^2(0,T;L^2(\Omega)) \tag{3.26}$$

for some $u \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$, $\bar{u} \in L^\infty(0,T;H^1(\Omega))$ *and* $v \in W^{1,2}(0,T;L^2(\Omega))$ *and* $\bar{v}$, $\xi \in L^\infty(0,T;L^2(\Omega))$. *In particular, it follows that* $u(\cdot,0) = \phi$. *Moreover, it holds that* $u = \bar{u}$, $v = \bar{v} = \beta(u)$ *and*

$$\mu\partial_t u + \partial_t\beta(u) + \xi = \lambda^2\Delta u + u. \tag{3.27}$$

Proof. By Lemmas 3.3 and 3.4, we immediately obtain (3.17)–(3.26) except for (3.18) and (3.20). Moreover, (3.18) follows from Ascoli's compactness lemma (see, e.g., [12]) along with (3.9) and Rellich's compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$. Then we also observe $u(\cdot,0) = \phi$. By (3.9), we find that, for $t \in [t_n, t_{n+1})$,

$$\|u_\tau(t) - \bar{u}_\tau(t)\|^2_{L^2(\Omega)} = \left(\frac{t_{n+1}-t}{\tau}\right)^2 \|u_{n+1} - u_n\|^2_{L^2(\Omega)} \overset{(3.9)}{\leq} C\tau \to 0,$$

which gives $u = \bar{u}$ and (3.20). One can similarly derive by (3.11) that $v = \bar{v}$. Since the operator $u \mapsto \beta(u)$ is maximal monotone in $L^2(\Omega)$, thanks to the demiclosedness of maximal monotone operators (see Proposition A.1), one can verify that $v = \beta(u)$. Finally, (3.27) follows from these facts along with (3.2). $\square$

Remark 3.6. Due to $(\beta)_\delta$, i.e., β is Lipschitz continuous in $\mathbb{R}$, and (3.20), one can also directly derive the strong convergence of $\beta(\bar{u}_\tau)$ to $\beta(\bar{u})$ in $L^\infty(0,T;L^2(\Omega))$.

By virtue of the relation $C([0,T];L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \subset C_w([0,T];H^1(\Omega))$ (see [8]), we obtain $u \in C_w([0,T];H^1(\Omega))$.

The next lemma identifies the limit ξ as a section of $\partial I_{[0,+\infty)}(\partial_t u)$.

Lemma 3.7. *It follows that $\partial_t u \geq 0$ a.e. in $\Omega \times (0,T)$ and $\xi \in \partial I_{[0,+\infty)}(\partial_t u)$.*

Proof. The proof given below basically relies on Minty's trick. However, compared to usual doubly nonlinear evolution equations such as (2.8) and (2.9), some additional difficulty may arise from the coexistence of two sorts of nonlinearities, $\partial_t \beta(u)$ and $\partial I_{[0,+\infty)}(\partial_t u)$. Noting that $\bar{\xi}_\tau \in \partial I_{[0,+\infty)}(\partial_t u_\tau)$, by Lemma 3.5, we first derive

$$\begin{aligned}
\limsup_{\tau\to 0} \int_0^T \left(\bar{\xi}_\tau, \partial_t u_\tau\right)_{L^2(\Omega)} \mathrm{d}t &\overset{(3.16)}{=} \limsup_{\tau\to 0} \int_0^T \left(\bar{u}_\tau + \lambda^2 \Delta \bar{u}_\tau - \mu \partial_t u_\tau - \partial_t v_\tau, \partial_t u_\tau\right)_{L^2(\Omega)} \mathrm{d}t \\
&\leq \lim_{\tau\to 0} \int_0^T (\bar{u}_\tau, \partial_t u_\tau)_{L^2(\Omega)} \mathrm{d}t - \frac{\lambda^2}{2} \liminf_{\tau\to 0} \|\nabla u_\tau(T)\|^2_{L^2(\Omega)} + \frac{\lambda^2}{2} \|\nabla \phi\|^2_{L^2(\Omega)} \\
&\quad - \mu \liminf_{\tau\to 0} \int_0^T \|\partial_t u_\tau\|^2_{L^2(\Omega)} \mathrm{d}t - \liminf_{\tau\to 0} \int_0^T (\partial_t v_\tau, \partial_t u_\tau)_{L^2(\Omega)} \mathrm{d}t \\
&\leq \int_0^T (u, \partial_t u)_{L^2(\Omega)} \mathrm{d}t - \frac{\lambda^2}{2} \|\nabla u(T)\|^2_{L^2(\Omega)} + \frac{\lambda^2}{2} \|\nabla \phi\|^2_{L^2(\Omega)} \\
&\quad - \mu \int_0^T \|\partial_t u\|^2_{L^2(\Omega)} \mathrm{d}t - \liminf_{\tau\to 0} \int_0^T (\partial_t v_\tau, \partial_t u_\tau)_{L^2(\Omega)} \mathrm{d}t. \qquad (3.28)
\end{aligned}$$

Here the last inequality follows from the weak lower semicontinuity of norms and the fact that

$$u_\tau(T) \to u(T) \quad \text{weakly in } H^1(\Omega).$$

Moreover, we emphasize that the last term of (3.28) arises from the mixed double nonlinearity of Equation (2.7).

We claim that

$$\liminf_{\tau\to 0} \int_0^T (\partial_t v_\tau, \partial_t u_\tau)_{L^2(\Omega)} \mathrm{d}t \geq \int_0^T (\partial_t \beta(u), \partial_t u)_{L^2(\Omega)} \mathrm{d}t. \qquad (3.29)$$

To prove this, by using the Mean-Value Theorem, we observe that

$$\begin{aligned}
\int_0^T (\partial_t v_\tau, \partial_t u_\tau)_{L^2(\Omega)} \mathrm{d}t &= \sum_{n=0}^{N-1} \tau \left(\frac{\beta(u_{n+1}) - \beta(u_n)}{\tau}, \frac{u_{n+1} - u_n}{\tau} \right)_{L^2(\Omega)} \\
&= \sum_{n=0}^{N-1} \tau \int_\Omega \beta'(\bar{z}_\tau) \left| \frac{u_{n+1} - u_n}{\tau} \right|^2 \mathrm{d}x
\end{aligned}$$

for some $\bar{z}_\tau = \bar{z}_\tau(x,t) \in (\bar{u}_\tau(x,t-\tau), \bar{u}_\tau(x,t))$. Here, as in the proof of Lemma 3.5, one can verify that

$$\bar{u}_\tau(\cdot - \tau) \to u \quad \text{strongly in } L^\infty(0,T;L^2(\Omega)),$$

which also yields that

$$\bar{z}_\tau \to u \quad \text{strongly in } L^\infty(0,T;L^2(\Omega)).$$

Since $\beta'(s)$ is continuous on $[\delta, +\infty)$ by $(\beta)_\delta$, we have

$$\beta'(\bar{z}_\tau(x,t)) \to \beta'(u(x,t)) \quad \text{for a.e. } (x,t) \in \Omega \times (0,T). \qquad (3.30)$$

Moreover, noting by $(\beta)_\delta$ that

$$\left| \sqrt{\beta'(\bar{z}_\tau(x,t))} \right| \leq C_\delta^{1/2} \quad \text{for a.e. } (x,t) \in \Omega \times (0,T)$$

and applying the Lebesgue dominated convergence theorem, we obtain

$$\sqrt{\beta'(\bar{z}_\tau)} \to \sqrt{\beta'(u)} \quad \text{strongly in } L^q(\Omega \times (0,T))$$

for any $q \in [1,+\infty)$. Here we particularly take $q > 2$.

For any $\varphi \in L^\infty(0,T;L^r(\Omega))$ with $r \in (1,+\infty)$ sufficiently large (e.g., $r = 2q/(q-2) > 2$), it follows that

$$\int_0^T \int_\Omega \varphi \sqrt{\beta'(\bar{z}_\tau)}\, \partial_t u_\tau \ \mathrm{d}x\mathrm{d}t \to \int_0^T \int_\Omega \varphi \sqrt{\beta'(u)}\, \partial_t u \ \mathrm{d}x\mathrm{d}t$$

as $\tau \to 0$. Hence we deduce that

$$\sqrt{\beta'(\bar{z}_\tau)}\, \partial_t u_\tau \to \sqrt{\beta'(u)}\, \partial_t u \quad \text{weakly in } L^1(0,T;L^{r'}(\Omega)),$$

where $r' := r/(r-1) < 2$. Moreover, note by (3.9) that

$$\int_0^T \int_\Omega \beta'(\bar{z}_\tau)|\partial_t u_\tau|^2 \ \mathrm{d}x\mathrm{d}t \le C_\delta \int_0^T \|\partial_t u_\tau\|_{L^2(\Omega)}^2 \mathrm{d}t \le C,$$

which implies

$$\sqrt{\beta'(\bar{z}_\tau)}\, \partial_t u_\tau \to \sqrt{\beta'(u)}\, \partial_t u \quad \text{weakly in } L^2(0,T;L^2(\Omega)).$$

From the weak lower semicontinuity of a norm, we conclude that

$$\liminf_{\tau\to 0} \int_0^T \int_\Omega \beta'(\bar{z}_\tau)|\partial_t u_\tau|^2 \ \mathrm{d}x\mathrm{d}t \ge \int_0^T \int_\Omega \beta'(u)|\partial_t u|^2 \ \mathrm{d}x\mathrm{d}t = \int_0^T (\partial_t \beta(u), \partial_t u)_{L^2(\Omega)} \mathrm{d}t,$$

which implies the desired assertion (3.29).

Now, we obtain

$$\begin{aligned}
\limsup_{\tau\to 0} \int_0^T \left(\bar{\xi}_\tau, \partial_t u_\tau\right)_{L^2(\Omega)} \mathrm{d}t &\le \int_0^T (u, \partial_t u)_{L^2(\Omega)} \mathrm{d}t - \frac{\lambda^2}{2}\|\nabla u(T)\|_{L^2(\Omega)}^2 + \frac{\lambda^2}{2}\|\nabla \phi\|_{L^2(\Omega)}^2 \\
&\quad - \mu \int_0^T \|\partial_t u\|_{L^2(\Omega)}^2 \mathrm{d}t - \int_0^T (\partial_t \beta(u), \partial_t u)_{L^2(\Omega)} \mathrm{d}t \\
&= \int_0^T \left(u + \lambda^2 \Delta u - \mu \partial_t u - \partial_t \beta(u), \partial_t u\right)_{L^2(\Omega)} \mathrm{d}t \\
&\overset{(3.27)}{=} \int_0^T (\xi, \partial_t u)_{L^2(\Omega)} \mathrm{d}t,
\end{aligned}$$

which together with the maximal monotonicity of $\partial I_{[0,+\infty)} : L^2(\Omega) \to L^2(\Omega)$ gives $\partial_t u \ge 0$ a.e. in $\Omega \times (0,T)$ and $\xi \in \partial I_{[0,+\infty)}(\partial_t u)$ (see Proposition A.1). The proof is completed. $\square$

Finally, let us derive an energy inequality to be used later. Recall (3.13) and (3.25). Due to the weak star lower semicontinuity of a norm, we can obtain

$$\begin{cases} \operatorname{ess\,sup}_{t\in(0,T)} \|\xi(t)\|_{L^p(\Omega)} \le \|(\lambda^2 \Delta\phi + \phi)_-\|_{L^p(\Omega)} & \text{if } \ p \in [2,+\infty), \\ \|\xi\|_{L^\infty(\Omega\times(0,T))} \le \|(\lambda^2\Delta\phi + \phi)_-\|_{L^\infty(\Omega)} & \text{if } \ p = \infty, \end{cases}$$

if $(\lambda^2\Delta\phi + \phi)_-$ belongs to $L^p(\Omega)$ for some $p \in [2,+\infty]$. Therefore we conclude that

Theorem 3.8 (Solvability of $(\mathrm{P})_\mu$). *Let $\mu > 0$ and assume (3.1) and $(\beta)_\delta$ for some constant $\delta > 0$. Then the Cauchy-Neumann problem $(\mathrm{P})_\mu = \{(2.14), (1.2), (1.3)\}$ admits at least one strong solution (u,ξ) satisfying*

$$\xi \in L^\infty(0,T;L^p(\Omega)), \quad \operatorname*{ess\,sup}_{t\in(0,T)} \|\xi(t)\|_{L^p(\Omega)} \le \|(\lambda^2\Delta\phi + \phi)_-\|_{L^p(\Omega)} \tag{3.31}$$

with $p = 2$. In addition, if $(\lambda^2\Delta\phi + \phi)_-$ belongs to $L^q(\Omega)$ for some $q \in [2,+\infty)$, then (3.31) holds for any $p \in [2,q]$. If $(\lambda^2\Delta\phi + \phi)_- \in L^\infty(\Omega)$, then ξ belongs to $L^\infty(\Omega\times(0,T))$, and it holds that

$$\|\xi\|_{L^\infty(\Omega\times(0,T))} \le \|(\lambda^2\Delta\phi + \phi)_-\|_{L^\infty(\Omega)}.$$

Remark 3.9. In the case of (1.4) with $\alpha = 0$, by virtue of a scaling argument, Theorem 3.8 also ensures the existence of a global (in time) solution for (P) under the milder assumption (3.1). Furthermore, the non-increase of β' (see $(\beta)_\delta$) is not used in the proof of Theorem 3.8.

4. Comparison principle for strictly increasing subsolutions

We first define the notions of a *subsolution* and a *supersolution* for (2.14) and (2.7), respectively.

Definition 4.1 (Sub- and supersolution). *For $\mu > 0$, a positive function $u \in W^{1,2}(0,T;L^2(\Omega))$ is called a* subsolution *of* (2.14) *if the following* (i)–(iii) *holds true:*

(i) $\beta(u) \in W^{1,2}(0,T;L^2(\Omega))$,
(ii) $u(\cdot,t) \in H^2(\Omega)$ *for a.e.* $t \in (0,T)$,
(iii) $\partial_t u(x,t) \geq 0$ *for a.e.* $(x,t) \in \Omega \times (0,T)$, *and there exist* $\xi(\cdot,t) \in L^2(\Omega)$ *for a.e.* $t \in (0,T)$ *such that*

$$\mu \partial_t u + \partial_t \beta(u) + \xi \leq \lambda^2 \Delta u + u, \quad \xi \in \partial I_{[0,+\infty)}(\partial_t u) \tag{4.1}$$

for a.e. $(x,t) \in \Omega \times (0,T)$.

A positive function $u \in W^{1,2}(0,T;L^2(\Omega))$ is called a supersolution *of* (2.14) *if the conditions* (i)–(iii) *hold with the inequality of* (4.1) *replaced by the inverse one, i.e.,* $\mu\partial_t u + \partial_t \beta(u) + \xi \geq \lambda^2 \Delta u + u$ *in* $\Omega \times (0,T)$.

A positive function $u \in C([0,T];L^2(\Omega))$ is called a subsolution *of* (2.7) *if the conditions* (i)–(iii) *are satisfied with $\mu = 0$. The notion of a* supersolution *of* (2.7) *is also analogously defined.*

Let us next state a comparison principle for (2.14).

Theorem 4.2 (Comparison principle for (2.14)). *Let $\mu > 0$ be fixed and assume that*

$$\beta \text{ is strictly increasing in } (0,\infty). \tag{4.2}$$

Let u be a subsolution of (2.14) *and let v be a supersolution of* (2.14) *such that $u(x,0) \leq v(x,0)$ for a.e. $x \in \Omega$ and $\partial_\nu u \leq \partial_\nu v$ for a.e. $(x,t) \in \partial\Omega \times (0,T)$. Suppose that*

$$\partial_t u > 0 \quad \text{a.e. in } \ \Omega \times (0,T). \tag{4.3}$$

Then it holds that $u \leq v$ for a.e. $x \in \Omega$ and all $t \in [0,T]$.

Proof. Set $w = u - v$. By subtraction of equations, we have

$$\mu \partial_t w + \partial_t \beta(u) - \partial_t \beta(v) + \xi - \eta \leq \lambda^2 \Delta w + w, \quad \xi \in \partial I_{[0,+\infty)}(\partial_t u), \quad \eta \in \partial I_{[0,+\infty)}(\partial_t v).$$

By assumption, we find that $\xi \equiv 0$. Test it by $z := \mathrm{sgn}(w) \vee 0 = \mathrm{sgn}(\beta(u) - \beta(v)) \vee 0 \geq 0$. Then we have

$$\mu \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega (w)_+ \mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\beta(u) - \beta(v)\right)_+ \mathrm{d}x + \int_\Omega -\eta z \ \mathrm{d}x \leq \int_\Omega (w)_+ \mathrm{d}x.$$

Here we used the fact that $(-\Delta w, z)_{L^2(\Omega)} \geq 0$ by $\partial_\nu w \leq 0$ and the monotonicity of the mapping $s \mapsto \mathrm{sgn}(s) \vee 0$. Noting that $-\eta z \geq 0$, $w(x,0) \leq 0$ and $\beta(u(x,0)) - \beta(v(x,0)) \leq 0$ for a.e. $x \in \Omega$, we have

$$\mu \int_\Omega (w)_+(x,t) \ \mathrm{d}x + \int_\Omega \left(\beta(u) - \beta(v)\right)_+ (x,t) \ \mathrm{d}x \leq \int_0^t \int_\Omega (w)_+(x,s) \ \mathrm{d}x\mathrm{d}s.$$

By exploiting Gronwall's inequality, we deduce that

$$\mu \int_\Omega (w)_+(x,t) \ \mathrm{d}x \equiv 0 \quad \text{for all } \ t \in [0,T],$$

which concludes that $u \leq v$ for a.e. $x \in \Omega$ and all $t \in [0,T]$. □

One can immediately obtain the following corollaries:

Corollary 4.3. *Let $\mu > 0$ and assume in addition to* (4.2) *that $\phi \in L^2(\Omega)$ satisfies $\phi \geq 0$ a.e. in Ω. Then strictly increasing solutions $u = u(x,t)$ of $(\mathrm{P})_\mu$ are unique.*

COROLLARY 4.4. *Let $\mu > 0$ and assume that $(\beta)_\delta$ holds and $\phi \in L^2(\Omega)$ satisfies $\phi \geq \delta$ a.e. in Ω for some constant $\delta > 0$. Then any solution $u = u(x,t)$ of $(\mathrm{P})_\mu$ on $[0,+\infty)$ diverges to $+\infty$ as $t \to +\infty$.*

Proof. Let z be a solution of the Cauchy problem,

$$\mu z'(t) + \frac{\mathrm{d}}{\mathrm{d}t}\beta(z(t)) = z(t) \quad \text{for} \quad t > 0, \qquad z(0) = z_0 := \delta.$$

Define a function Φ_μ by

$$\Phi_\mu(z) = \int_{z_0}^{z} \frac{\mu + \beta'(\zeta)}{\zeta}\,\mathrm{d}\zeta \qquad \text{for} \quad z \geq z_0 > 0.$$

Then one can observe that

$$\Phi_\mu(z) < \infty,\ \Phi'_\mu(z) > 0 \ \text{ for all } z \in [z_0,+\infty), \qquad \lim_{z\to+\infty} \Phi_\mu(z) \geq \lim_{z\to+\infty} \int_{z_0}^{z} \frac{\mu}{\zeta}\,\mathrm{d}\zeta = +\infty.$$

Hence the inverse function $\Phi_\mu^{-1} : [0,+\infty) \to [z_0,+\infty)$ of $t = \Phi_\mu(z)$ exists, and moreover, $\Phi_\mu^{-1}(t)$ is strictly increasing on $[0,+\infty)$, $\Phi_\mu^{-1}(0) = z_0$, and $\Phi_\mu^{-1}(t) \to +\infty$ as $t \to +\infty$.

Now, $z(t)$ is explicitly written as

$$z(t) = \Phi_\mu^{-1}(t) \quad \text{for} \quad t \geq 0.$$

Moreover, since $\partial_t z > 0$ and $z(t)$ is uniform in Ω, z becomes a strictly increasing (sub)solution of $(\mathrm{P})_\mu$ on $[0,+\infty)$. Hence applying Theorem 4.2, we have

$$u(x,t) \geq z(t) \quad \text{for a.e.} \quad x \in \Omega \quad \text{and all} \quad t \geq 0,$$

which implies the desired conclusion. □

The next theorem provides a comparison principle for (2.7), which will be used to verify the blow-up in finite time of positive solutions for (P) in Section 5.

THEOREM 4.5 (Comparison principle for (2.7)). *Assume $(\beta)_\delta$ for some constant $\delta > 0$ and let u be a subsolution of (2.7) and let v be a supersolution of (2.7) such that $\delta \leq u(x,0) \leq v(x,0)$ for a.e. $x \in \Omega$ and $\partial_\nu u \leq \partial_\nu v$ for a.e. $(x,t) \in \partial\Omega \times (0,T)$. Suppose that*

$$\partial_t u > 0 \quad \text{a.e. in} \ \ \Omega \times (0,T) \quad \text{and} \quad \|u\|_{L^\infty(\Omega\times(0,T))} \vee \|v\|_{L^\infty(\Omega\times(0,T))} \leq M \tag{4.4}$$

for some constant $M \geq 0$. Then it holds that $u \leq v$ for a.e. $x \in \Omega$ and all $t \in [0,T]$.

Proof. Repeating the same argument as in the proof of Theorem 4.2, we have

$$\int_\Omega (\beta(u) - \beta(v))_+ (x,t)\,\mathrm{d}x \leq \int_0^t \int_\Omega (w)_+(x,s)\,\mathrm{d}x\mathrm{d}s \qquad \text{for all} \quad t \in [0,T],$$

where $w = u - v$. Here letting z be a function defined as in the proof of Theorem 4.2, we observe that

$$(\beta(u) - \beta(v))\,z = \int_0^1 \beta'\left((1-\theta)v + \theta u\right)(w)_+\mathrm{d}\theta \geq c_M (w)_+$$

for some constant $c_M > 0$, which is given as the minimum of β' on $[\delta, M]$ by $(\beta)_\delta$. Thus it yields

$$(w)_+ \leq \frac{1}{c_M}\left(\beta(u) - \beta(v)\right) z = \frac{1}{c_M}\left(\beta(u) - \beta(v)\right)_+.$$

Therefore we obtain

$$\int_\Omega (\beta(u) - \beta(v))_+ (x,t)\,\mathrm{d}x \leq \frac{1}{c_M}\int_0^t \int_\Omega (\beta(u) - \beta(v))_+ (x,s)\,\mathrm{d}x\mathrm{d}s \qquad \text{for all} \quad t \in [0,T],$$

which implies

$$\int_\Omega (\beta(u) - \beta(v))_+ (x,t)\,\mathrm{d}x \equiv 0 \qquad \text{for all} \quad t \in [0,T].$$

Thus $u \le v$ for a.e. $x \in \Omega$ and all $t \in [0,T]$. $\square$

We close this section by an immediate consequence of the theorem stated above.

COROLLARY 4.6. *Assume that* $(\beta)_\delta$ *holds and* $\phi \in L^2(\Omega)$ *satisfies* $\phi \ge \delta$ *a.e. in* Ω *for some constant* $\delta > 0$. *Then bounded strictly increasing solutions of* (P) *are unique.*

5. Local solvability of (P) and finite time blow-up of solutions

This section is devoted to proving Theorem 2.1. Throughout this section, we assume (2.11) and also suppose that $\beta \in C^1([0,\infty))$ without any loss of generality as in §3.

To construct a solution of (P), we employ strong solutions (u_μ, ξ_μ) of $(\mathrm{P})_\mu$ on $[0,T]$ constructed in §3 as approximated solutions for (P). However, in contrast with $(\mathrm{P})_\mu$ for $\mu > 0$, solutions of (P) may blow up in finite time (see an explicit solution blowing up in finite time given by (5.2) below). So it is a key step to establish a local (in time) estimate for u_μ. To this end, we shall apply the comparison principle (see Theorem 4.2) to u_μ (as a subsolution) and an explicit supersolution. However, in Theorem 4.2, subsolutions are supposed to be strictly increasing (see (4.3)), and this assumption is somewhat restrictive to directly apply the principle to u_μ, whose time derivative may vanish. Hence it is crucial how to apply the principle and obtain a local (in time) uniform estimate of u_μ.

LEMMA 5.1. *For each* $M > \|\phi\|_{L^\infty(\Omega)}$, *there exists* $T_M > 0$ *independent of* $\mu > 0$ *such that*

$$\delta \le u_\mu(x,t) \le M \quad \textit{for a.e. } x \in \Omega \ \textit{ and all } \ t \in [0, T_M].$$

Proof. We start with constructing an explicit supersolution of (2.14). Let $z = z(t)$ be a solution of the Cauchy problem of the following ODE:

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta(z(t)) = z(t), \quad z(0) = z_0 > 0. \tag{5.1}$$

Here as in the proof of Corollary 4.4, define a function $\Phi : [z_0, \infty) \to [0, \hat{T}(z_0))$ by

$$\Phi(z) := \int_{z_0}^{z} \frac{\beta'(\zeta)}{\zeta}\,\mathrm{d}\zeta, \quad \hat{T}(z_0) := \int_{z_0}^{\infty} \frac{\beta'(\zeta)}{\zeta} \in (0,\infty].$$

Then z is explicitly given by

$$z(t) = \Phi^{-1}(t) \ \text{ for } \ t \in [0, \hat{T}(z_0)). \tag{5.2}$$

Here we remark that z solves (P) with $\phi \equiv z_0$. We find that $z'(t) > 0$ for all t and $z(t) \to +\infty$ as $t \nearrow \hat{T}(z_0)$. Moreover, since z is constant in Ω and $\partial I_{[0,+\infty)}(z'(t)) = \{0\}$ for $t > 0$, it follows that

$$\mu\partial_t z + \partial_t\beta(z) + \partial I_{[0,+\infty)}(\partial_t z) \ge \lambda^2\Delta z + z \ \text{ in } \Omega \times (0, \hat{T}(z_0)), \quad \partial_\nu z = 0 \ \text{ on } \partial\Omega \times (0, \hat{T}(z_0)).$$

Hence z is a supersolution of (2.14) on $[0,S]$ for any $S \in (0, \hat{T}(z_0))$.

Let (u_μ, ξ_μ) be a strong solution of $(\mathrm{P})_\mu$ and define a positive function $\hat{u}_\mu \in W^{1,2}(0,T; L^2(\Omega))$ by

$$\hat{u}_\mu := u_\mu + t + \kappa,$$

where $\kappa \ge 0$ is a constant to be determined later. Then we observe that $\partial_\nu \hat{u}_\mu = 0$ on $\partial\Omega \times (0,T)$ and $\partial_t \hat{u}_\mu = \partial_t u_\mu + 1 > 0$ in $\Omega \times (0,T)$, which implies $\partial I_{[0,+\infty)}(\partial_t \hat{u}_\mu) = \{0\}$. Moreover, we note that

$$\partial_t\beta(\hat{u}_\mu) = \beta'(\hat{u}_\mu)\partial_t\hat{u}_\mu \le \beta'(u_\mu)(\partial_t u_\mu + 1) \le \beta'(u_\mu)\partial_t u_\mu + C_\delta$$

by the non-increase and the boundedness of β' (see $(\beta)_\delta$). Therefore we have

$$\begin{aligned} \mu\partial_t\hat{u}_\mu + \partial_t\beta(\hat{u}_\mu) &\le \mu\partial_t u_\mu + \beta'(u_\mu)\partial_t u_\mu + \mu + C_\delta \\ &= \lambda^2\Delta u_\mu + u_\mu - \xi_\mu + \mu + C_\delta \\ &\le \lambda^2\Delta\hat{u}_\mu + \hat{u}_\mu - \left(\kappa - \|\xi_\mu\|_{L^\infty(\Omega\times(0,T))} - \mu - C_\delta\right). \end{aligned}$$

Here we recall by Theorem 3.8 and (2.11) that

$$\|\xi_\mu\|_{L^\infty(\Omega\times(0,T))} \le \|(\lambda^2\Delta\phi+\phi)_-\|_{L^\infty(\Omega)}.$$

Hence choosing $\kappa = \|(\lambda^2\Delta\phi+\phi)_-\|_{L^\infty(\Omega)} + 1 + C_\delta$, we conclude that, for any $\mu\in(0,1)$, $\hat{u}_\mu$ is a subsolution of (2.14) such that $\partial_t\hat{u}_\mu > 0$ in $\Omega\times(0,T)$.

Let us take $z_0 := \|\phi\|_{L^\infty(\Omega)} + \kappa \ge \hat{u}_\mu(\cdot,0)$ and apply Theorem 4.2 to the strictly increasing subsolution $\hat{u}_\mu$ and the supersolution z. Then we obtain

$$\hat{u}_\mu(x,t) \le z(t) \quad \text{for all } \ t\in[0,\hat{T}(z_0)),$$

which implies

$$u_\mu(x,t) \le z(t) - t - \kappa \quad \text{for all } \ t\in[0,\hat{T}(z_0)).$$

Therefore for each $M > \|\phi\|_{L^\infty(\Omega)}$, one can choose $T_M \in (0,\hat{T}(z_0))$ such that $z(T_M) = M+\kappa$, and hence, $u_\mu(x,t)\le M$ for all $t\in[0,T_M]$. $\square$

Now, fix $M = \|\phi\|_{L^\infty(\Omega)} + 1$ and take $T_0 := T_M > 0$ such that

$$\sup_{t\in[0,T_0]} \|u_\mu(t)\|_{L^\infty(\Omega)} \le M. \tag{5.3}$$

Remark 5.2. One may also obtain

$$T_0 = \ell(\|\phi\|_{L^\infty(\Omega)} + \|(\lambda^2\Delta\phi+\phi)_-\|_{L^\infty(\Omega)} + C_\delta + 1) \tag{5.4}$$

with a strictly decreasing positive function $\ell(\cdot)$ depending only on the choice of β'. Indeed, let z be a solution of (5.1) with $z(0)=z_0$ and let $\ell=\ell(z_0)>0$ be such that $z(\ell(z_0)) = z_0+1$. Then by (5.1),

$$\ell(z_0) = \int_{z_0}^{z_0+1} \frac{\beta'(\zeta)}{\zeta}\,\mathrm{d}\zeta,$$

which implies

$$\ell'(z_0) = \frac{\beta'(z_0+1)}{z_0+1} - \frac{\beta'(z_0)}{z_0}.$$

Since β' is non-increasing and positive (see $(\beta)_\delta$), ℓ is strictly decreasing in z_0. Hence by the choice of T_0 in (5.3), putting $z_0 = \|\phi\|_{L^\infty(\Omega)} + \kappa$, we obtain (5.4).

Then we are in position to derive uniform estimates.

Lemma 5.3. *There exist constants $C\ge 0$ and $c_M>0$ independent of $\mu>0$ such that*

$$\mu\int_0^{T_0}\|\partial_t u_\mu\|^2_{L^2(\Omega)}\mathrm{d}t + c_M\int_0^{T_0}\|\partial_t u_\mu\|^2_{L^2(\Omega)}\mathrm{d}t + \frac{\lambda^2}{2}\sup_{t\in[0,T_0]}\|\nabla u_\mu(t)\|^2_{L^2(\Omega)} \le C, \tag{5.5}$$

$$\sup_{t\in[0,T_0]}\|\beta(u_\mu(\cdot,t))\|_{L^\infty(\Omega)} \le C, \tag{5.6}$$

$$\|\xi_\mu\|_{L^\infty(\Omega\times(0,T_0))} \le \|(\lambda^2\Delta\phi+\phi)_-\|_{L^\infty(\Omega)}, \tag{5.7}$$

$$\int_0^{T_0}\|\partial_t\beta(u_\mu)\|^2_{L^2(\Omega)}\mathrm{d}t \le C, \tag{5.8}$$

$$\int_0^{T_0}\|\Delta u_\mu\|^2_{L^2(\Omega)}\mathrm{d}t \le C. \tag{5.9}$$

Proof. Test (2.14) by $\partial_t u_\mu$ to get

$$\mu\|\partial_t u_\mu\|^2_{L^2(\Omega)} + (\partial_t\beta(u_\mu),\partial_t u_\mu)_{L^2(\Omega)} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\lambda^2}{2}\|\nabla u_\mu\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_\mu\|^2_{L^2(\Omega)}\right) = 0, \tag{5.10}$$

from the fact that $\xi_\mu\partial_t u_\mu = 0$ by $\xi_\mu\in\partial I_{[0,+\infty)}(\partial_t u_\mu)$. Here by virtue of (5.3), we find that

$$(\partial_t\beta(u_\mu),\partial_t u_\mu)_{L^2(\Omega)} = \int_\Omega \beta'(u_\mu)|\partial_t u_\mu|^2\,\mathrm{d}x \ge c_M\int_\Omega|\partial_t u_\mu|^2\,\mathrm{d}x,$$

where $c_M := \inf_{s\in[\delta,M]} \beta'(s)$ $(= \beta'(M)) > 0$ by $(\beta)_\delta$. Hence integrating both sides of (5.10) over $(0,t)$, we deduce that

$$\begin{aligned}\mu \int_0^t \|\partial_\tau u_\mu\|^2_{L^2(\Omega)} \mathrm{d}\tau + c_M \int_0^t \|\partial_\tau u_\mu\|^2_{L^2(\Omega)} \mathrm{d}\tau + \frac{\lambda^2}{2}\|\nabla u_\mu(t)\|^2_{L^2(\Omega)} \\ \leq \frac{1}{2}\|u_\mu(t)\|^2_{L^2(\Omega)} + \frac{\lambda^2}{2}\|\nabla\phi\|^2_{L^2(\Omega)} \leq \frac{M^2|\Omega|}{2} + \frac{\lambda^2}{2}\|\nabla\phi\|^2_{L^2(\Omega)}\end{aligned}$$

for any $t \in [0,T_0]$. Thus (5.5) follows. Moreover, it follows from (2.10) and (5.3) that

$$\sup_{t\in[0,T_0]} \|\beta(u_\mu(\cdot,t))\|_{L^\infty(\Omega)} \leq C_\delta M + |\beta(\delta)|,$$

which gives (5.6). Moreover, (5.7) has already been derived by Theorem 3.8 and (2.11). Estimate (5.8) follows from (5.5) and the fact that

$$|\partial_t \beta(u_\mu)| = |\beta'(u_\mu)||\partial_t u_\mu| \leq C_\delta |\partial_t u_\mu|.$$

Finally, by comparison, we get (5.9). □

Let us proceed to passing to the limit as $\mu \to 0$.

Lemma 5.4. *By taking a (non-relabeled) subsequence of $\mu \to 0$, one can derive*

$$\begin{aligned}
\mu\partial_t u_\mu &\to 0 && \text{strongly in } L^2(0,T_0;L^2(\Omega)), && (5.11)\\
u_\mu &\to u && \text{weakly star in } L^\infty(0,T_0;H^1(\Omega)), && (5.12)\\
& && \text{weakly star in } L^\infty(\Omega\times(0,T_0)), && (5.13)\\
& && \text{weakly in } W^{1,2}(0,T_0;L^2(\Omega)), && (5.14)\\
& && \text{strongly in } C([0,T_0];L^2(\Omega)), && (5.15)\\
\xi_\mu &\to \xi && \text{weakly star in } L^\infty(\Omega\times(0,T_0)), && (5.16)\\
\beta(u_\mu) &\to v && \text{weakly star in } L^\infty(\Omega\times(0,T_0)), && (5.17)\\
\partial_t\beta(u_\mu) &\to \partial_t v && \text{weakly in } L^2(0,T_0;L^2(\Omega)), && (5.18)\\
\Delta u_\mu &\to \Delta u && \text{weakly in } L^2(0,T_0;L^2(\Omega)) && (5.19)
\end{aligned}$$

for some $u \in W^{1,2}(0,T_0;L^2(\Omega)) \cap L^\infty(0,T_0;H^1(\Omega)) \cap L^\infty(\Omega\times(0,T_0))$, $v \in W^{1,2}(0,T_0;L^2(\Omega)) \cap L^\infty(\Omega\times(0,T_0))$ and $\xi \in L^\infty(\Omega\times(0,T_0))$. In particular, one has $u(0) = \phi$. Moreover, it holds that $v = \beta(u)$ and $\partial_t\beta(u) + \xi = \lambda^2\Delta u + u$.

Proof. The weak (star) convergences follow immediately from the uniform estimates established so far. Since $\sqrt{\mu}\partial_t u_\mu$ is bounded in $L^2(0,T_0;L^2(\Omega))$, (5.11) follows. Moreover, (5.15) can be verified by using Ascoli's compactness lemma (see, e.g., [12]) along with the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$. Then one can also assure the initial condition $u(0) = \phi$. Finally, the demiclosedness of maximal monotone operators together with (5.15) and (5.17) yields $v = \beta(u)$, and hence, $\partial_t\beta(u) + \xi = \lambda^2\Delta u + u$. □

We also deduce that $u \in C_w([0,T_0];H^1(\Omega))$, since u belongs to $C([0,T_0];L^2(\Omega)) \cap L^\infty(0,T_0;H^1(\Omega))$. Furthermore, by (5.3) and (5.15), one has

$$u_\mu(t) \to u(t) \quad \text{weakly star in } L^\infty(\Omega) \text{ for all } t \in [0,T_0].$$

It follows that

$$\|u(t)\|_{L^\infty(\Omega)} \leq \liminf_{\mu\to 0} \|u_\mu(t)\|_{L^\infty(\Omega)} \leq M \quad \text{for all } t \in [0,T_0],$$

which along with the fact that $u \in C([0,T_0];L^2(\Omega))$ implies the continuity of $t \mapsto u(t)$ in the weak star topology of $L^\infty(\Omega)$, that is, $u \in C_{w*}([0,T_0];L^\infty(\Omega))$. Finally, we identify the limit ξ of ξ_μ as $\mu \to 0$.

LEMMA 5.5. *It holds that $\partial_t u \geq 0$ a.e. in $\Omega \times (0, T_0)$ and $\xi \in \partial I_{[0,+\infty)}(\partial_t u)$.*

Proof. We use Minty's trick again to prove this lemma. Observe that

$$\begin{aligned}\int_0^{T_0} (\xi_\mu, \partial_t u_\mu)_{L^2(\Omega)} \mathrm{d}t &= \int_0^{T_0} \left(\lambda^2 \Delta u_\mu + u_\mu - \mu \partial_t u_\mu - \partial_t \beta(u_\mu), \partial_t u_\mu\right)_{L^2(\Omega)} \mathrm{d}t \\ &= -\frac{\lambda^2}{2} \|\nabla u_\mu(T_0)\|^2_{L^2(\Omega)} + \frac{\lambda^2}{2} \|\nabla \phi\|^2_{L^2(\Omega)} + \int_0^{T_0} (u_\mu, \partial_t u_\mu)_{L^2(\Omega)} \mathrm{d}t \\ &\quad - \mu \int_0^{T_0} \|\partial_t u_\mu\|^2_{L^2(\Omega)} \mathrm{d}t - \int_0^{T_0} (\partial_t \beta(u_\mu), \partial_t u_\mu)_{L^2(\Omega)} \mathrm{d}t.\end{aligned}$$

Here we also remark that

$$\liminf_{\mu \to 0} \int_0^{T_0} (\partial_t \beta(u_\mu), \partial_t u_\mu)_{L^2(\Omega)} \mathrm{d}t \geq \int_0^{T_0} (\partial_t \beta(u), \partial_t u)_{L^2(\Omega)} \mathrm{d}t$$

by verifying that $\sqrt{\beta'(u_\mu)} \partial_t u_\mu \to \sqrt{\beta'(u)} \partial_t u$ weakly in $L^2(0, T_0; L^2(\Omega))$ (see the proof of Lemma 3.7). Finally, from (5.14), (5.15) and the weak lower semicontinuity of norms, we conclude that

$$\begin{aligned}\limsup_{\mu \to 0} \int_0^{T_0} (\xi_\mu, \partial_t u_\mu)_{L^2(\Omega)} \mathrm{d}t &\leq -\frac{\lambda^2}{2} \|\nabla u(T_0)\|^2_{L^2(\Omega)} + \frac{\lambda^2}{2} \|\nabla \phi\|^2_{L^2(\Omega)} + \int_0^{T_0} (u, \partial_t u)_{L^2(\Omega)} \mathrm{d}t \\ &\quad - \int_0^{T_0} (\partial_t \beta(u), \partial_t u)_{L^2(\Omega)} \mathrm{d}t \\ &= \int_0^{T_0} (\xi, \partial_t u)_{L^2(\Omega)} \mathrm{d}t,\end{aligned}$$

whence follows $\partial_t u \geq 0$ a.e. in $\Omega \times (0, T_0)$ and $\xi \in \partial I_{[0,+\infty)}(\partial_t u)$. This completes the proof. $\square$

By Remark 2.4 along with (5.7) and (5.16), one can also derive the inequality (2.12). Thus we have proved the existence of a local (in time) strong solution for (P).

Let us finally verify the finite time blow-up of any solution $u = u(x, t)$ to (P) for any data $\phi \in L^\infty(\Omega)$ satisfying $\phi \geq \delta > 0$ a.e. in Ω. To this end, suppose on the contrary that $T_{\max} > \hat{T}(\delta)$, i.e., for some $T_\delta > \hat{T}(\delta)$, one can extend u onto $[0, T_\delta]$ such that

$$M_\delta := \sup_{t \in [0, T_\delta]} \|\bar{u}(t)\|_{L^\infty(\Omega)} = \|\bar{u}(T_\delta)\|_{L^\infty(\Omega)} < +\infty, \tag{5.20}$$

where $\bar{u}$ stands for the extended solution. We recall the explicit solution $z = z(t)$ on $[0, \hat{T}(\delta))$ of the ODE (5.1) with $z_0 = \delta > 0$. Then z is a strictly increasing (sub)solution of (P) on $[0, \hat{T}(\delta))$. For each $M > \delta$, one can take $\tau_M \in (0, \hat{T}(\delta))$ such that $z(t) < M$ for all $t \in [0, \tau_M)$ and $z(\tau_M) = M$, since $z(t)$ diverges to $+\infty$ as $t \to \hat{T}(\delta)$. Set $Q := M \vee M_\delta < +\infty$. We observe that

$$|z(t)| \leq Q \quad \text{and} \quad \|\bar{u}(t)\|_{L^\infty(\Omega)} \leq Q \quad \text{for a.e.} \quad t \in (0, \tau_M).$$

Due to Theorem 4.5, we have

$$z(t) \leq \bar{u}(x, t) \quad \text{for a.e.} \quad x \in \Omega \quad \text{and all} \quad t \in [0, \tau_M].$$

Now, letting $M \to +\infty$, we infer that $\tau_M \nearrow \hat{T}(\delta)$, and therefore,

$$z(t) \leq \bar{u}(x, t) \quad \text{for a.e.} \quad x \in \Omega \quad \text{and} \quad t \in [0, \hat{T}(\delta)).$$

Hence we obtain

$$\lim_{t \nearrow \hat{T}(\delta)} \operatorname*{ess\,inf}_{x \in \Omega} |\bar{u}(x, t)| = +\infty,$$

which contradicts (5.20). Thus we obtain $T_{\max} \leq \hat{T}(\delta)$. From the definition of $T_{\max}$ and the local existence part, one can prove (2.13) by recalling (5.4) and (2.12). Thus we have proved Theorem 2.1. $\square$

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Appendix A. Convex analysis

Here we briefly recall several notions and propositions related to convex analysis for the convenience of the reader.

Let X be a Banach space with norm $\|\cdot\|_X$ and let $\varphi : X \to (-\infty, +\infty]$ be a proper (i.e., $\varphi \not\equiv +\infty$), lower semicontinuous convex functional with *effective domain* $D(\varphi) := \{u \in X : \varphi(u) < +\infty\}$. The *subdifferential operator* $\partial_X \varphi : X \to 2^{X^*}$ (or simply denoted by $\partial\varphi$) is formulated as

$$\partial\varphi(u) := \{\xi \in X^* : \varphi(v) - \varphi(u) \geq \langle \xi, v - u\rangle_X \quad \text{for all} \quad v \in X\}, \tag{A.1}$$

where $\langle\cdot,\cdot\rangle_X$ is the duality pairing between X and X^*, with domain $D(\partial\varphi) := \{u \in D(\varphi) : \partial\varphi(u) \neq \emptyset\}$. It is well known that $\partial\varphi$ has a maximal monotone graph in $X \times X^*$. In particular, if X is a Hilbert space whose dual space is identified with itself (e.g., $X = L^2(\Omega)$), then $\langle\cdot,\cdot\rangle_X$ can be replaced by an inner product $(\cdot,\cdot)_X$. Furthermore, if ψ is a convex functional on X of class C^1 (in the sense of Fréchet derivatives), then

$$\partial(\varphi + \psi) = \partial\varphi + \psi',$$

where $\psi' : X \to X^*$ is the Fréchet derivative of ψ.

The following proposition is useful to identify the (weak) limit of a sequence in the graph of a nonlinear maximal monotone operator.

Proposition A.1 (Demiclosedness of maximal monotone operators). *Let $A : X \to X^*$ be a (possibly multivalued) maximal monotone operator. Let $[u_n, \xi_n]$ be in the graph of A such that $u_n \to u$ weakly in X and $\xi_n \to \xi$ weakly in X^*. Suppose that*

$$\limsup_{n\to+\infty} \langle \xi_n, u_n\rangle_X \leq \langle \xi, u\rangle_X.$$

Then $[u,\xi]$ belongs to the graph of A, and moreover, it holds that

$$\lim_{n\to+\infty} \langle \xi_n, u_n\rangle_X = \langle \xi, u\rangle_X.$$

Let Ω be a bounded domain in $\mathbb{R}^N$. Let $u \in L^2(\Omega)$ and let α be a maximal monotone graph in $\mathbb{R}^2$. Since every maximal monotone graph in $\mathbb{R}^2$ becomes cyclic monotone, one can take a proper lower semicontinuous convex potential $\theta : \mathbb{R} \to (-\infty, +\infty]$ such that $\partial\theta = \alpha$.

Proposition A.2 ([6, 5]). *Define $\Theta : L^2(\Omega) \to (-\infty, +\infty]$ by*

$$\Theta(u) := \begin{cases} \displaystyle\int_\Omega \theta(u(x))\,\mathrm{d}x & \text{if } u \in L^2(\Omega) \text{ and } \theta(u(\cdot)) \in L^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then the following properties are all satisfied:

(i) *Θ is proper, lower semicontinuous and convex in $L^2(\Omega)$.*
(ii) *For all $f, u \in L^2(\Omega)$, it follows that $f \in \partial_{L^2(\Omega)}\Theta(u)$ if and only if $f(x) \in \partial_{\mathbb{R}}\theta(u(x))$ for a.e. $x \in \Omega$.*

(iii) *Assume that* $u \in D(-\Delta) = \{u \in H^2(\Omega) \colon \partial_\nu u = 0 \text{ a.e. on } \partial\Omega\}$. *It then holds that*

$$-\int_\Omega \Delta u(x)\eta(x)\,\mathrm{d}x \geq 0 \quad \text{for any section } \eta \in \partial_{L^2(\Omega)}\Theta(u).$$

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