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Anomalous eddy viscosity for two-dimensional turbulence

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We study eddy viscosity for generalized two-dimensional (2D) fluids. The governing equation for generalized 2D fluids is the advection equation of an active scalar q by an incompressible velocity. The relation between q and the stream function ψ is given by $q = -(-\nabla^2)^{\alpha/2}\psi$. Here, α is a real number. When the evolution equation for the generalized enstrophy spectrum $Q_\alpha(k)$ is truncated at a wavenumber k_c, the effect of the truncation of modes with larger wavenumbers than k_c on the dynamics of the generalized enstrophy spectrum with smaller wavenumbers than k_c is investigated. Here, we refer to the effect of the truncation on the dynamics of $Q_\alpha(k)$ with $k < k_c$ as eddy viscosity. Our motivation is to examine whether the eddy viscosity can be represented by normal diffusion. Using an asymptotic analysis of an eddy-damped quasi-normal Markovian (EDQNM) closure approximation equation for the enstrophy spectrum, we show that even if the wavenumbers of interest k are sufficiently smaller than k_c, the eddy viscosity is not asymptotically proportional to $k^2 Q_\alpha(k)$, i.e., a normal diffusion, but to $k^{4-\alpha}Q_\alpha(k)$ for $\alpha > 0$ and $k^4 Q_\alpha(k)$ for $\alpha < 0$, i.e., an anomalous diffusion. This indicates that the eddy viscosity as normal diffusion is asymptotically realized only for $\alpha = 2$ (Navier–Stokes system). The proportionality constant, the eddy viscosity coefficient, is asymptotically negative. These results are confirmed by numerical calculations of the EDQNM closure approximation equation and direct numerical simulations of the governing equation for forced and dissipative generalized 2D fluids. The negative eddy viscosity coefficient is explained using Fjørtoft's theorem and a spreading hypothesis for the spectrum. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4916956]

I. INTRODUCTION

It is well known that the mixing and transport in fluids are enhanced by turbulence. This phenomenon is frequently referred to as turbulent diffusion or eddy viscosity and has been a central topic of turbulence studies.[1] Furthermore, eddy viscosity is also an important concept in numerical simulations of turbulence. Because the nonlinearity of the governing equations of fluids leads to interaction between motions with a spatial scale resolved by the grid points of numerical computation and those with a scale less than the grid interval, appropriate representation of the interaction between them is crucial for numerical simulations of turbulence. The interaction between the resolved-scale motions and the subgrid-scale motions is frequently represented as eddy viscosity which is represented by a simple operator of the form $\nu_{\text{eddy}}\nabla^2$ acting on resolved-scale variables. This is a counterpart of the molecular viscosity, but the eddy viscosity coefficient ν_{eddy} depends on the scale of interest, in contrast to the molecular viscosity coefficient, because the scale separation between the resolved-scale and the subgrid-scale is generally insufficient. However, it

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has been presumed that the subgrid-scale motions asymptotically act as a normal diffusion process on the resolved-scale motions, i.e., the eddy viscosity coefficient ν_{eddy} is independent of the scale of interest when the scale of interest and the subgrid-scale are well separated. This hypothesis motivated Kraichnan[2] to study the eddy viscosity in the wavenumber space for two-dimensional (2D) and three-dimensional (3D) Navier–Stokes (NS) turbulence using a turbulence closure approximation equation. He formulated the eddy viscosity as the effect of the truncation of modes larger than a wavenumber k_{c}, called the cutoff wavenumber, on the evolution of the energy spectrum with wavenumbers less than k_{c}. He showed that, if the scale separation between the wavenumbers of interest and the cutoff wavenumber is sufficient, the effect of truncation on the evolution of the energy spectrum is asymptotically proportional to the square of the local wavenumber and the local energy spectrum, $k^2E(k)$, where $E(k)$ is kinetic energy spectrum, i.e., normal diffusion. Kraichnan's results confirmed by the analysis of direct numerical simulations for 3D NS turbulence.[3,4]

Here, we will refer to modeling of the subgrid-scale motions as eddy viscosity. We study the eddy viscosity for generalized 2D fluids theoretically and numerically. In particular, we examine the eddy viscosity in wavenumber space similar to the study by Kraichnan.[2]

We will show that the eddy viscosity is generally represented not by normal but by anomalous diffusion, even if the separation between the scale of interest and the subgrid-scale is sufficient. In the following, we introduce the generalized 2D fluid system, briefly review studies of it, and explain our aims.

The generalized 2D fluid system is represented by an advection equation

$$\frac{\partial q}{\partial t} + J(\psi, q) = \mathcal{D} + \mathcal{F}, \tag{1a}$$

$$q = -(-\boldsymbol{\nabla}^2)^{\alpha/2}\psi. \tag{1b}$$

Here, $\psi(\boldsymbol{x}, t)$ is the stream function, $q(\boldsymbol{x}, t)$ is a scalar field advected by the incompressible velocity field $\boldsymbol{v} = \boldsymbol{e}_z \times \boldsymbol{\nabla}\psi$, $\boldsymbol{e}_z$ is a unit vector normal to the plane of motion, J is the 2D Jacobian, α is a real number, $\boldsymbol{\nabla}^2$ is the Laplacian in 2D space, and $\mathcal{D}$ and $\mathcal{F}$ are dissipation and forcing terms, respectively. The generalized 2D fluid system is frequently referred to as α-turbulence system, because (1) includes the parameter α and turbulence governed by (1) has been mainly studied previously. It was originally introduced by Pierrehumbert *et al.*[5] as a tool for studying the effects of spectral non-locality on 2D NS turbulence. This system, which also includes three realizable members of 2D geophysical fluid systems,[5,6] has been actively investigated both theoretically and numerically over the past decade.[7–19,26] When $\alpha = 2$, q is the familiar vorticity, and governing equation (1) reduces to the vorticity equation for a 2D incompressible barotropic fluid (2D NS system for $\mathcal{D} = \nu\boldsymbol{\nabla}^2 q$, where ν is the kinematic viscosity coefficient, and a 2D Euler system for $\mathcal{D} = 0$). For $\alpha > 3$, the velocity induced by a point vortex, which is a delta-functional distribution of q, increases with the distance from the point vortex, as shown by Iwayama and Watanabe.[15] This calls into question the physical relevance of these systems. However, all values of α give well-defined systems. Thus, we examine the systems with all values of α in this study.

The purpose of investigating the generalized 2D fluid system is twofold. The first objective is to understand geophysical 2D fluid systems from a unified perspective. The second is to elucidate the universality/peculiarity of prevailing theories for 2D Euler and NS systems and to develop them further.

Similar to the 2D Euler system, (1) has two quadratic inviscid invariants in the absence of dissipation and forcing, the generalized energy $\mathcal{E}_\alpha$ and the generalized enstrophy $\mathcal{Q}_\alpha$,

$$\mathcal{E}_\alpha \equiv -\frac{1}{2}\overline{\psi q}, \tag{2}$$

$$\mathcal{Q}_\alpha \equiv \frac{1}{2}\overline{q^2}. \tag{3}$$

Here, the overbar denotes a spatial average over the flow domain. For brevity, we will henceforth refer to $\mathcal{E}_\alpha$ and $\mathcal{Q}_\alpha$ as energy and enstrophy, respectively. In addition, we refer to q as generalized vorticity, or simply as vorticity.

The existence of two inviscid invariants leads to the dual energy and enstrophy cascades in turbulence governed by (1), which, for simplicity, will be hereafter referred to as α-turbulence. Past efforts were made to analyze the statistical properties of the energy and enstrophy cascading ranges. In particular, the extent to which the so-called Kraichnan-Leith-Batchelor (KLB) phenomenology[20–22] works well in the energy and enstrophy cascading ranges of forced-dissipative α-turbulence has been actively discussed.[5,7–12,14,15,18,26] KLB phenomenology relies on the spectral locality, i.e., the localness of triad interactions between wavenumbers. However, for $\alpha \geq 2$, the strain rate in the enstrophy cascading range is dominated by non-local triad interactions. Thus, the enstrophy spectrum $Q_\alpha(k)$, which is defined by

$$Q_\alpha = \int_0^\infty Q_\alpha(k)\,\mathrm{d}k, \tag{4}$$

$$Q_\alpha(k) \equiv \pi\, k \left\langle |\hat{q}(\boldsymbol{k})|^2 \right\rangle \tag{5}$$

for an infinite flow domain, where $\hat{q}(\boldsymbol{k})$ is the Fourier transform of q with wavenumber vector $\boldsymbol{k}$, $k = |\boldsymbol{k}|$, and the angle bracket denotes average over ensemble or average over an azimuthal angle in the wavenumber space, deviates from the KLB spectrum in the enstrophy cascading range,

$$Q_\alpha(k) \propto k^{-(7-2\alpha)/3}, \tag{6}$$

and shallows to k^{-1} for $\alpha > 2$.[5,10,11] In contrast, Burgess and Shepherd[18] (hereafter denoted as BS13) studied the energy spectrum $E_\alpha(k) \equiv k^{-\alpha} Q_\alpha(k)$ in the energy cascading range and concluded that the strain rate for $\alpha < 4$ there is locally dominated. Furthermore, they showed from an eddy-damped quasi-normal Markovian (EDQNM) closure approximation equation for the energy spectrum that the KLB spectrum in the energy cascading range,

$$E_\alpha(k) \propto k^{-(7-\alpha)/3}, \tag{7}$$

is associated with downscale energy flux for $\alpha > 2.5$ and a vanishing energy flux for $\alpha = 2.5$. On the basis of this, they predicted that KLB spectrum (7) would not be observed for $\alpha \geq 2.5$. Indeed, they showed from low-resolution simulations of forced-dissipative α-turbulence that KLB spectrum (7) is realized for $\alpha < 2.5$ but not for $\alpha \geq 2.5$.

In contrast, Iwayama and Watanabe[19] (hereafter denoted by IW14) studied the enstrophy spectrum in the infrared range ($k \to 0$) of α-turbulence using the EDQNM closure approximation equation for the enstrophy spectrum and direct numerical simulations. They showed that there are three canonical cases of infrared enstrophy spectra. In particular, the evolution from a narrow initial spectrum exhibits a universal infrared spectrum of the form $Q_\alpha(k \to 0) \propto k^5$.

The study by IW14 simultaneously proposed an expression for the eddy viscosity for α-turbulence. They evaluated the contribution of the nonlocal triad $k \ll l \simeq m$ among the triad interactions with $\boldsymbol{k} + \boldsymbol{l} + \boldsymbol{m} = \boldsymbol{0}$ on the dynamics of the enstrophy spectrum by expanding the enstrophy transfer function of the EDQNM closure approximation equation in terms of the small parameter $\epsilon \equiv k/l$. To the leading order in ϵ, the enstrophy transfer function consists of two terms. One is proportional to k^5 for all α and the other is proportional to $k^{4-\alpha} Q_\alpha(k)$ for $\alpha > 0$ and to $k^4 Q_\alpha(k)$ for $\alpha < 0$. The term proportional to k^5 contributes the formation of the infrared spectrum, and the others correspond to the spectral form of the eddy viscosity formulated by Kraichnan.[2] Therefore, the study by IW14 proposed that the asymptotic effect of the truncation on the dynamics of the spectrum with wavenumbers less than k_c depends on the relation between the vorticity and the stream function, i.e., α. Moreover, it implied that the eddy viscosity as the normal diffusion is asymptotically realized only when $\alpha = 2$, i.e., in the NS system. In general, the eddy viscosity is asymptotically anomalous rather than normal diffusion.

The present paper performs detailed analysis of the results of IW14 related to the eddy viscosity and confirms the analysis by executing both numerical calculations of the EDQNM approximation equation and direct numerical simulations of (1). In particular, we focus the asymptotic wavenumber dependence of the eddy viscosity. The rest of the paper is organized as follows. In Sec. II, we formulate the eddy viscosity for generalized 2D fluids. In Sec. III, we perform analysis of the results of IW14 related to the eddy viscosity. In Sec. IV, we present the results of numerical calculations of the EDQNM approximation equation and direct numerical simulations of (1) and confirm the analysis in Sec. III. We show that the eddy viscosity is asymptotically anomalous diffusion, and the asymptotic value of the eddy viscosity coefficient is negative. In Sec. V, we interpret the negative value of the asymptotic eddy viscosity coefficient in terms of the energy and enstrophy transfers using Fjørtoft's theorem[18,23] and a spreading hypothesis for the spectrum. This is an alternative explanation of the negative eddy viscosity coefficient by Kraichnan.[2] Finally, we summarize the results in Sec. VI.

II. FORMULATION

In this study, we consider fluid motions in an unbounded domain with appropriate decaying conditions at infinity or a square domain of the side length L with periodic boundary conditions.

The starting point of our discussion is the evolution equation for the enstrophy spectrum, which is governed by

$$\left(\frac{\partial}{\partial t} + 2\nu_p k^{2p}\right) Q_\alpha(k) = T^Q_\alpha(k) + \hat{\mathcal{F}}^Q(k), \tag{8}$$

where $T^Q_\alpha(k)$ is the enstrophy transfer function and $\hat{\mathcal{F}}^Q(k)$ corresponds to the forcing. We assume the damping term $\mathcal{D}$ of (1a) as a hyper-viscosity term of order p with the hyper-viscosity coefficient ν_p. The enstrophy transfer function $T^Q_\alpha(k)$ is written in terms of the triad enstrophy transfer function $T^Q_\alpha(k,l,m)$ as

$$T^Q_\alpha(k) = \frac{1}{2}\int_0^\infty \mathrm{d}m \int_{|m-k|}^{m+k} \mathrm{d}l\, T^Q_\alpha(k,l,m). \tag{9}$$

Note that the triad enstrophy transfer function satisfies a detailed balance

$$T^Q_\alpha(k,l,m) + T^Q_\alpha(l,m,k) + T^Q_\alpha(m,k,l) = 0 \tag{10}$$

for the wavenumbers, which forms the triangle $\boldsymbol{k} + \boldsymbol{l} + \boldsymbol{m} = \boldsymbol{0}$ and is symmetric with respect to l and m, i.e.,

$$T^Q_\alpha(k,l,m) = T^Q_\alpha(k,m,l). \tag{11}$$

We introduce the cutoff wavenumber k_{c}, which is the reciprocal of the dividing length of the resolved and subgrid scales. We furthermore divide the wavenumber space into two regions, $k < k_{\mathrm{c}}$ and $k \geq k_{\mathrm{c}}$. The former and the latter correspond to the resolved and subgrid scales, respectively. Then, the enstrophy transfer function for wavenumbers $k < k_{\mathrm{c}}$ can be divided into two parts,

$$T^Q_\alpha(k) = T^{Q(<)}_\alpha(k|k_{\mathrm{c}}) + T^{Q(>)}_\alpha(k|k_{\mathrm{c}}), \tag{12a}$$

$$T^{Q(<)}_\alpha(k|k_{\mathrm{c}}) = \int_{k/2}^{k_{\mathrm{c}}} \mathrm{d}m \int_{|m-k|}^{m} \mathrm{d}l\, T^Q_\alpha(k,l,m), \tag{12b}$$

$$T^{Q(>)}_\alpha(k|k_{\mathrm{c}}) = \int_{k_{\mathrm{c}}}^{\infty} \mathrm{d}m \int_{m-k}^{m} \mathrm{d}l\, T^Q_\alpha(k,l,m). \tag{12c}$$

Here, we have used relation (11). $T^{Q(<)}_\alpha(k|k_{\mathrm{c}})$ represents the enstrophy transfer due to the interactions among modes with wavenumbers less than k_{c}. In contrast, $T^{Q(>)}_\alpha(k|k_{\mathrm{c}})$ is the enstrophy transfer due to the interactions in which modes with wavenumbers larger than k_{c} intervene.

According to IW14, we propose that the evolution equations for the enstrophy spectrum in the wavenumber ranges $k < k_{\mathrm{c}}$ for $\alpha > 0$ and $\alpha < 0$ are given by

$$\left[\frac{\partial}{\partial t} + 2\left\{\nu_p k^{2p} + \nu_{\mathrm{T}}(k|k_{\mathrm{c}})k^{4-\alpha}\right\}\right] Q_\alpha(k) = T^{Q(<)}_\alpha(k|k_{\mathrm{c}}) + \hat{\mathcal{F}}^Q(k), \tag{13a}$$

$$\nu_{\mathrm{T}}(k|k_{\mathrm{c}}) = -\frac{T^{Q(>)}_\alpha(k|k_{\mathrm{c}})}{2k^{4-\alpha}Q_\alpha(k)} \tag{13b}$$

and

$$\left[\frac{\partial}{\partial t} + 2\left\{\nu_p k^{2p} + \nu_{\mathrm{T}}(k|k_{\mathrm{c}})k^4\right\}\right] Q_\alpha(k) = T_\alpha^{Q(<)}(k|k_{\mathrm{c}}) + \hat{\mathcal{F}}^Q(k), \tag{14a}$$

$$\nu_{\mathrm{T}}(k|k_{\mathrm{c}}) = -\frac{T_\alpha^{Q(>)}(k|k_{\mathrm{c}})}{2k^4 Q_\alpha(k)}, \tag{14b}$$

respectively. We have not used any approximation for the transformation of (8) into forms (13) or (14). That is, the forms of (13) and (14) are independent of the approximations of the statistical theory of turbulence.

III. EDDY VISCOSITY DERIVED BY THE EDQNM EQUATION

In this section, we perform detailed analysis of the results of IW14 related to the eddy viscosity and calculate the eddy viscosity coefficients defined by (13b) and (14b) for a self-similar spectrum using the EDQNM approximation equation.[19]

In the EDQNM approximation equation, the triad enstrophy transfer function is given by

$$\begin{aligned} T_\alpha^Q(k,l,m) = \frac{2k^{\alpha+2}}{\pi l m}\,\theta_{klm}\, \Bigg\{ & 2a_{klm}\frac{k}{(lm)^\alpha}\, Q_\alpha(l)\, Q_\alpha(m) \\ & - b_{klm}\frac{l}{(mk)^\alpha}\, Q_\alpha(m)\, Q_\alpha(k) \\ & - b_{kml}\frac{m}{(kl)^\alpha}\, Q_\alpha(k)\, Q_\alpha(l) \Bigg\}, \end{aligned} \tag{15}$$

where

$$a_{klm} = \frac{b_{klm} + b_{kml}}{2}, \tag{16a}$$

$$b_{klm} = 2\,\frac{(l^\alpha - m^\alpha)(k^\alpha - m^\alpha)\sqrt{1-x^2}}{k^{\alpha+2}(lm)^{\alpha-2}}, \tag{16b}$$

$$b_{kml} = 2\,\frac{(m^\alpha - l^\alpha)(k^\alpha - l^\alpha)\sqrt{1-x^2}}{k^{\alpha+2}(lm)^{\alpha-2}} \tag{16c}$$

are geometric coefficients and x refers to the cosine of the interior angle of the triangle $\boldsymbol{k} + \boldsymbol{l} + \boldsymbol{m} = \boldsymbol{0}$ facing the side k, i.e.,

$$x = \frac{\boldsymbol{l}\cdot\boldsymbol{m}}{lm}. \tag{17}$$

The function θ_{klm} is the relaxation time of the third-order moments associated with the triad (k,l,m). It is expressed as

$$\theta_{klm} = \frac{1}{\mu_{klm}}, \tag{18a}$$

$$\mu_{klm} = \mu_k + \mu_p + \mu_q, \tag{18b}$$

$$\mu_k = \mu\left\{k^{5-2\alpha} Q_\alpha(k)\right\}^{1/2}. \tag{18c}$$

The proportionality constant μ in (18c) is positive but the value of it has not been determined yet. It is determined from the so-called Kolmogorov–Kraichnan constants in the energy and enstrophy cascading ranges. However, the Kolmogorov–Kraichnan constants for generalized 2D fluids except for the NS system have not been determined yet. Thus, μ is also undetermined. However, the concrete value of μ is unnecessary in the present study because the asymptotic wavenumber dependence of the eddy viscosity coefficient $\nu_{\mathrm{T}}(k|k_{\mathrm{c}})$ is independent of μ.

To the leading order in $\epsilon \equiv k/l$, the triad enstrophy transfer function $T_\alpha^Q(k,l,m)$ is given by Eq. (31) of IW14,

$$T_\alpha^Q(k,l,m) \simeq \frac{4z^2\sqrt{1-z^2}}{\pi}\theta_{kll} \times \Big[\alpha^2 l^{-(2\alpha+1)}\{Q_\alpha(l)\}^2 k^4 - \alpha l^{-\alpha}\left\{\frac{\partial\{lQ_\alpha(l)\}}{\partial l} - 2Q_\alpha(l)\right\} k^{3-\alpha}Q_\alpha(k)H(\alpha) + \alpha l^{-2\alpha}\left\{\frac{\partial\{lQ_\alpha(l)\}}{\partial l} - (\alpha+2)Q_\alpha(l)\right\} k^3 Q_\alpha(k)H(-\alpha)\Big], \tag{19}$$

where $H(x)$ is the Heaviside step function. The asymptotic form of $T_\alpha^{Q(>)}(k|k_c)$ in the case $k \ll k_c$, which is Eq. (32) of IW14, is

$$T_\alpha^{Q(>)}(k|k_c) \simeq \frac{\alpha^2}{4}\left[\int_{k_c}^{\infty} \theta_{kll}\, l^{-(2\alpha+1)}\{Q_\alpha(l)\}^2\,\mathrm{d}l\right] k^5 - 2\nu_T(k|k_c)k^{4-\alpha}Q_\alpha(k)H(\alpha) - 2\nu_T(k|k_c)k^4 Q_\alpha(k)H(-\alpha), \tag{20}$$

where

$$\nu_T(k|k_c) = \frac{\alpha}{8}\int_{k_c}^{\infty}\theta_{kll}\, l^{-\alpha}\left[\frac{\partial\{lQ_\alpha(l)\}}{\partial l} - 2Q_\alpha(l)\right]\mathrm{d}l, \quad (\alpha > 0), \tag{21}$$

$$\nu_T(k|k_c) = \frac{|\alpha|}{8}\int_{k_c}^{\infty}\theta_{kll}\, l^{-2\alpha}\left[\frac{\partial\{lQ_\alpha(l)\}}{\partial l} - (\alpha+2)Q_\alpha(l)\right]\mathrm{d}l, \quad (\alpha < 0). \tag{22}$$

IW14 devoted their discussion to the contribution of the first term of (19). In this paper, we focus our attention on the second or third terms of (19). Thus, we discuss the condition for neglecting the first term of (19) compared to the second or third terms of them. If we assume a self-similar spectrum

$$Q_\alpha(k) = Ck^{-n} \tag{23}$$

over the whole wavenumber range, where C is a positive constant, then the ratio of the first and second terms in (19) is $\{\alpha/(n+1)\}(k/l)^{\alpha+n+1}$. In contrast, the ratio of the first and third terms in (19) is $\{|\alpha|/(\alpha+n+1)\}(k/l)^{n+1}$. Thus, the first term is negligible compared with the second term when $n > -\alpha - 1$ for $\alpha > 0$. On the other hand, it is negligible compared with the third term when $n > -1$ for $\alpha < 0$. From this analysis, we conclude that, if the scale separation is sufficient, i.e., $k \ll k_c$, the first term of (19) or the first term of (20) is negligible for self-similar spectrum range (23) with $n > -1$.

The quantities μ_k in the relaxation time for the third-order moment θ_{klm} can be considered an increasing function of wavenumber for turbulent motions. Such condition is satisfied when self-similar spectrum (23) has the exponent $n < 5 - 2\alpha$. Then, we can use an approximation $\mu_k + \mu_l + \mu_m \simeq 2\mu_l$ for $k \ll l \simeq m$. Hence, θ_{kll} in eddy viscosity coefficients (21) and (22) are independent of k. Therefore, eddy viscosity coefficients (21) and (22) are constants independent of k.

We specifically evaluate the eddy viscosity coefficients, (21) and (22), in the case of self-similar spectrum (23) with $-1 < n < 5 - 2\alpha$. Furthermore, we assume an additional condition $-3 < n$ for $\alpha > 0$ and $2|\alpha| - 3 < n$ for $\alpha < 0$ to ensure that integrals (21) and (22) are finite. Then, we obtain

$$\nu_T(k|k_c) = -\frac{\alpha}{8\mu}\frac{n+1}{n+3}\left\{k_c^{-3}Q_\alpha(k_c)\right\}^{1/2}, \quad (\alpha > 0), \tag{24}$$

$$\nu_T(k|k_c) = -\frac{|\alpha|}{8\mu}\frac{n+\alpha+1}{n+2\alpha+3}\left\{k_c^{-(2\alpha+3)}Q_\alpha(k_c)\right\}^{1/2}, \quad (\alpha < 0). \tag{25}$$

The sign of eddy viscosity coefficients (21) and (22) is determined from the sign of the quantity in the angle bracket of the integrand because the relaxation time of the third-order moments θ_{kll} is a positive quantity. If we assume self-similar spectrum (23), (24) and (25) indicate that $\nu_T(k|k_c) > 0$ when $n < -1$ for $\alpha > 0$, while $\nu_T(k|k_c) > 0$ when $n < |\alpha| - 1$ for $\alpha < 0$.

We have formulated the eddy viscosity in terms of the enstrophy spectrum and the enstrophy transfer function because we would like to use the results of IW14. We can formulate the eddy viscosity in terms of the energy spectrum and the energy transfer function similar to Kraichnan.[2] However, the results are independent of which formulation is adopted. That is, the energy transfer function $T_\alpha^{\mathcal{E}(>)}(k|k_c) \equiv k^{-\alpha} T_\alpha^{\mathcal{Q}(>)}(k|k_c)$ is asymptotically proportional to $k^{4-\alpha}E_\alpha(k)$ for $\alpha > 0$ and $k^4 E_\alpha(k)$ for $\alpha < 0$, and the proportionality constant is independent of the local wavenumber k. Indeed, when $\alpha = 2$, (21) in terms of the energy spectrum is consistent with the eddy viscosity coefficient derived by Kraichnan[2] for the 2D NS system.

As an example of (23), we assume a self-similar enstrophy spectrum

$$Q_\alpha(k) = C|\epsilon|^{2/3} k^{-(7-4\alpha)/3} \tag{26}$$

over the whole wavenumber range. Here, C is the Kolmogorov–Kraichnan constant and ϵ is the energy dissipation rate. Spectrum (26) is derived by the KLB scaling in energy cascading range (7). Then, from the above discussion, the first term of (20) is negligible when $\alpha < 10$. Moreover, to ensure that μ_k is an increasing function of wavenumber, α must be less than 4. Furthermore, to ensure the convergence of integrals (13b) and (14b), it must be $-8 < \alpha < 4$. Then, the explicit forms of the asymptotic eddy viscosity coefficients are

$$\nu_T(k|k_c) = -\frac{\alpha}{16\mu}\frac{5-2\alpha}{4-\alpha}\left\{k_c^{-3}Q_\alpha(k_c)\right\}^{1/2}, \quad (\alpha > 0), \tag{27}$$

$$\nu_T(k|k_c) = -\frac{|\alpha|}{16\mu}\frac{10+|\alpha|}{8-|\alpha|}\left\{k_c^{-(2\alpha+3)}Q_\alpha(k_c)\right\}^{1/2}, \quad (\alpha < 0). \tag{28}$$

Therefore, the asymptotic values of the eddy viscosity coefficients are negative for $\alpha < 5/2$, positive for $\alpha > 5/2$, vanish for $\alpha = 5/2$. Note that the KLB spectra in the energy cascading range, (7) or (26), for $\alpha = 5/2$ and $\alpha = 10$ correspond to the enstrophy and energy equipartition spectra, respectively, as discussed by BS13.[18] Moreover, we note that (27) for $\alpha = 2$ is consistent with (4.7) of Kraichnan.[2]

IV. NUMERICAL CALCULATIONS

In this section, we examine the validity of the results obtained in Sec. III by numerical calculations of the EDQNM closure approximation equation and direct numerical simulations of (1).

A. Numerical calculations of the EDQNM equation

We evaluate the eddy viscosity coefficients defined by (13b) and (14b) by calculating (12c) with the triad enstrophy transfer function in EDQNM approximation (15). Here, we assume a self-similar enstrophy spectrum (26).

We numerically integrated (12c) by dividing the wavenumber into the interval $\Delta k = \Delta l = \Delta m = 5 \times 10^{-4}$ and setting the cutoff wavenumber $k_c = 1$ and the upper bound for the integration $k_{max} = 20$ instead of $k_{max} = \infty$. We also examined the dependence of our numerical results on the values of k_{max} and the wavenumber intervals, Δk, Δl, and Δm, by performing two additional calculations: one is setting $\Delta k = \Delta l = \Delta m = 2.5 \times 10^{-4}$ and the other $k_{max} = 40$, while the other parameters are unchanged. We checked that the subsequent results are independent of the values of the upper bound k_{max} and the wavenumber intervals, Δk, Δl, and Δm.

Fig. 1 shows the normalized eddy viscosity coefficients for some values of α. The eddy viscosity coefficients are evaluated in the interval $0.01k_c \le k \le k_c$ and are normalized by their absolute values at $k = 0.01k_c$, i.e., $|\nu_T(0.01k_c|k_c)|$. Fig. 1 indicates that the eddy viscosity coefficients are monotonically increasing or decreasing functions of wavenumber in general. (For $\alpha = -2$, the eddy viscosity coefficient decreases about 10% of the asymptotic value $\nu_T(0.01k_c|k_c)$ until $k \simeq 0.4k_c$ and rapidly increases for $k \gtrsim 0.4k_c$.) This shows that the eddy viscosity coefficients defined by (13b)

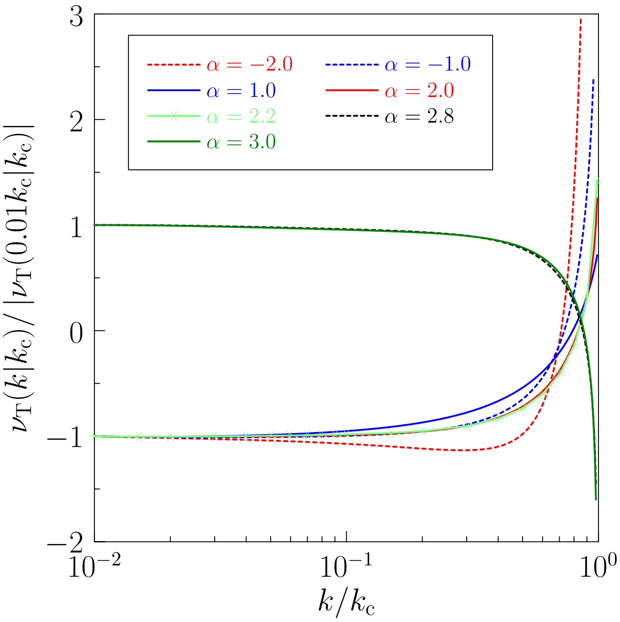

FIG. 1. Eddy viscosity coefficients (13b) and (14b) calculated by the EDQNM closure approximation equation. The abscissa indicates the ratio of the wavenumber of interest k to the cutoff wavenumber k_c, k/k_c. The ordinate is the normalized eddy viscosity, $\nu_T(k|k_c)/|\nu_T(0.01k_c|k_c)|$.

and (14b) are constants for the wavenumber range $k \ll k_c$. That is, even if the separation between the scale of interest and the subgrid-scale is sufficient, the subgrid-scale phenomena act as not normal diffusion processes but generally as anomalous diffusion processes on the resolved-scale.

For $\alpha = 5/2$, eddy viscosity coefficient (21) vanishes. This is because the self-similar spectrum (26) for $\alpha = 5/2$ is the enstrophy equipartition spectrum, and the triad enstrophy transfer function vanishes identically.[18] The sign of the eddy viscosity coefficients for $\alpha > 5/2$ is opposite that for $\alpha < 5/2$. This would arise from the downscale energy flux of KLB solution (26) of the EDQNM closure approximation equation, as pointed out by BS13.[18] If the energy flux is upward even for $\alpha > 5/2$, we expect that the eddy viscosity coefficient asymptotically possesses the negative value. This point will be discussed in Sec. IV D.

As the wavenumber k is increased to the cutoff wavenumber k_c, the eddy viscosity coefficient exhibits a cusp. The eddy viscosity coefficient at the cutoff wavenumber reverses the sign with its asymptotic value. As the absolute value of α increases, the cusp is enhanced.

We have to confirm the conclusion obtained above by direct numerical simulations of (1). It is well known that closure equations in the quasi-normal approximation filter out coherent structures in turbulent fields. It is also well known that coherent structures or coherent vortices are important ingredients of 2D turbulence, and they play significant roles in the dynamics of 2D turbulence. They are responsible for the deviation of the spectrum from the KLB scaling. Thus, we are interested in the effect of the existence of coherent structures on the wavenumber dependence of the asymptotic eddy viscosity. Furthermore, the EDQNM closure approximation equation fails the direction of the energy flux and the slope of the energy spectrum in the energy cascading range for $\alpha \geq 5/2$.[18] Therefore, the wavenumber dependence of the eddy viscosity and the sign of the asymptotic eddy viscosity coefficient for $\alpha \geq 5/2$ must be investigated by direct numerical simulations.

B. Overview of direct numerical simulations

Next, we report direct numerical simulations of forced and dissipative turbulence governed by (1). First, we outline the numerical method, the forcing function $\mathcal{F}$ used, the dissipation term $\mathcal{D}$ adopted, and the resolution of the calculations.

We consider the system that is confined within a square domain of the side length L with periodic boundary conditions. Furthermore, we set the damping term $\mathcal{D}$ of (1a) as a hyper-viscosity term of order p. Then, the Fourier space version of governing equation (1) is

$$\left(\frac{\partial}{\partial t} + \nu_p k^{2p}\right)\hat{q}(\boldsymbol{k}) = \sum_{\boldsymbol{l}}\sum_{\boldsymbol{m}} \delta_{\boldsymbol{k}+\boldsymbol{l}+\boldsymbol{m},\boldsymbol{0}} \frac{\boldsymbol{e}_z\cdot(\boldsymbol{l}\times\boldsymbol{m})\,(l^\alpha - m^\alpha)}{2l^\alpha m^\alpha}\,\hat{q}^*(\boldsymbol{l})\,\hat{q}^*(\boldsymbol{m}) + \hat{\mathcal{F}}(\boldsymbol{k}), \tag{29}$$

where $\boldsymbol{k} = 2\pi\boldsymbol{n}/L$ is the wavenumber vector, $\hat{q}(\boldsymbol{k})$ and $\hat{\mathcal{F}}(\boldsymbol{k})$ are the Fourier coefficients of q and $\mathcal{F}$ with the wavenumber vector $\boldsymbol{k}$, respectively, and the summation is taken over the integer vector $\boldsymbol{n} = (n_x, n_y)$. The asterisk denotes the complex conjugate. For brevity, the time argument is omitted. A pseudo-spectral method was used in the double-precision arithmetic at a resolution of N^2, which is the number of grid points in the computational domain $[0,L]\times[0,L]$. The truncation wavenumber k_{T} is chosen according to the two-third dealiasing technique. We set $N = 2048$ and $L = 2\pi$. Then, the truncation wavenumber is $k_{\mathrm{T}} = 682$.

Time integration was performed by the third-order Adams-Bashforth scheme. The dissipation term was calculated implicitly using an integral factor. The next time step of $\hat{q}(\boldsymbol{k},t)$ was then calculated according to

$$\hat{q}(\boldsymbol{k},t+\Delta t) = e^{-\nu_p k^{2p}\Delta t}\hat{q}(\boldsymbol{k},t) + \frac{1}{12}\Delta t\, e^{-\nu_p k^{2p}\Delta t} \times\left\{23\hat{\mathcal{N}}(\boldsymbol{k},t) - 16 e^{-\nu_p k^{2p}\Delta t}\hat{\mathcal{N}}(\boldsymbol{k},t-\Delta t) + 5e^{-2\nu_p k^{2p}\Delta t}\hat{\mathcal{N}}(\boldsymbol{k},t-2\Delta t)\right\},$$

where $\hat{\mathcal{N}}$ stands for the right hand side of (29) and Δt is the time step of integration. The initial condition is a state of no flow.

The forcing term $\hat{\mathcal{F}}(\boldsymbol{k},t)$ was determined by a random Markovian formula,[24,25]

$$\hat{\mathcal{F}}(\boldsymbol{k},t+\Delta t) = A(1-R^2)^{1/2}e^{i\theta} + R\hat{\mathcal{F}}(\boldsymbol{k},t), \tag{30}$$

where A is an amplitude that has a non-zero value only in the band $|k - k_{\mathrm{f}}| \le 2$. Here, k_{f} is the forcing wavenumber. R is a function depending on both the time step Δt and the correlation time τ. It is zero for white noise and unity for the infinite correlation time. The relation among Δt, τ, and R is given by

$$R = \frac{1 - \frac{1}{2}\frac{\Delta t}{\tau}}{1 + \frac{1}{2}\frac{\Delta t}{\tau}}. \tag{31}$$

Alternatively, solving the above equation for τ leads to

$$\tau = \frac{1}{2}\frac{1+R}{1-R}\Delta t. \tag{32}$$

θ is a random uniform number between 0 and 2π. We set $A = \sqrt{2}$, $\tau = 0.95\Delta t$, and $k_{\mathrm{f}} = 280$. Then, $R = 0.31$ from (31). Because the correlation time of the forcing is shorter than the time step of the integration, the forcing employed here can be considered as δ correlated forcing in time. In fact, the spectral properties of results of the present simulations are consistent with those of δ correlated forcing simulations by Burgess *et al.*[26] Numerical simulations were performed up to $t = t_{\mathrm{fin}}$ and their results were saved at intervals of 0.2. A time average of the interval between t_{ini} and t_{fin} was taken for each run to improve the statistical convergence of the results.

We define the dissipation wavenumber k_{d} as

$$k_{\mathrm{d}} \equiv \left(\frac{\eta_{\mathrm{d}}}{\nu_p^3}\right)^{1/\{2(\alpha+3p-2)\}}, \tag{33}$$

where η_{d} is the enstrophy dissipation rate. Furthermore, we define a non-dimensional time $T \equiv \sqrt{2Q_2}t$ using the ordinary enstrophy Q_2.

The other conditions of the simulations are tabulated in Table I.

TABLE I. Parameters for direct numerical simulations. ν_4 is the hyper-viscosity coefficient of order $p = 4$, $k_\mathrm{d} \equiv (\eta_\mathrm{d}/\nu_p^3)^{1/(6p-4+2\alpha)}$ is the dissipation wavenumber, η_d is the enstrophy dissipation rate, and ε_in is the energy input rate. t_ini and t_fin are the initial and final times for the time average, respectively. T_ave is the non-dimensional time corresponding to the interval $t_\mathrm{fin} - t_\mathrm{ini}$.

α	ν_4	Δt	k_d	η_d	ε_in	t_ini	t_fin	T_ave
1.0	5.0×10^{-21}	5.0×10^{-5}	553	0.650 97	2.3796×10^{-3}	25.0	35.2	1582.75
1.5	5.0×10^{-22}	2.5×10^{-4}	642	3.295 8	7.1035×10^{-4}	50.0	73.8	977.97
2.0	8.0×10^{-23}	1.0×10^{-3}	657	13.124	1.6943×10^{-4}	100	130	289.131
2.2	4.0×10^{-23}	1.0×10^{-3}	644	13.249	5.5065×10^{-5}	70	124	225.23
2.5	7.5×10^{-24}	1.0×10^{-3}	656	6.666 9	5.0848×10^{-6}	250.0	280.8	30.823
2.8	1.0×10^{-24}	1.0×10^{-3}	615	13.369	1.8734×10^{-6}	500	561	26.382
3.0	1.0×10^{-24}	1.0×10^{-3}	661	13.422	6.0897×10^{-7}	700.0	734.2	7.829

C. Fundamental results of direct numerical simulations

First, we examine the evolution of some elementary quantities: the energy, the enstrophy, and the energy and enstrophy dissipation rates of the results of direct numerical simulations (figures not shown). Because there is no linear drag or hypo-viscosity at large scales, the energy monotonically increased with time for all values of α studied. In contrast, the enstrophy rapidly increased with time in the early stage of evolution and then rapidly decreased for all the values of α studied. After that the evolution of enstrophy depended on α. Although the enstrophy increased again for $\alpha \leq 2.2$, the rate of enstrophy increase decreased as α increased. For $\alpha \geq 5/2$, the enstrophy gradually decreased, and the decreasing rate of enstrophy became faster as α increased. The energy and enstrophy dissipation rates remained constant after transient evolution. The initial time t_ini of the averaging time is much later than the time at which the enstrophy dissipation rate takes the maximum. In this sense, turbulence at high wavenumbers would be well developed after t_ini. Furthermore, in the interval $t_\mathrm{fin} - t_\mathrm{ini}$, the energy and enstrophy dissipation rates were considered as constant values, while the energy monotonically increased with time. Although the enstrophy for $\alpha = 1$ increased about 30% in this interval, those except for $\alpha = 1$ could be considered as constants there. Thus, turbulence at high wavenumbers, which would be cut off in the evaluation of the eddy viscosity, can be considered to be in a statistical steady state.

Figs. 2 and 3 show the energy and enstrophy fluxes normalized by their input rates, respectively. Here, the energy and enstrophy fluxes are determined by $\Pi_\alpha^{\mathcal{E}}(k) \equiv -\int_k^\infty k'^{-\alpha} T_\alpha^Q(k')\,\mathrm{d}k'$ and $\Pi_\alpha^Q(k) \equiv -\int_k^\infty T_\alpha^Q(k')\,\mathrm{d}k'$, respectively. A wavenumber range where the energy flux remains constant in lower wavenumbers less than the forcing wavenumber k_f is formed for all the values of α under study. This range is interpreted as the energy cascading range. All systems, even those with $\alpha \geq 5/2$, exhibit upward energy flux in wavenumbers less than k_f, as pointed out by BS13.[18] For $\alpha = 1$, half of the inputted energy is transferred upscale from k_f. The inputted energy is transferred upscale more and more as α increases. The simulations for $\alpha \geq 1.5$ resolve to some extent the enstrophy cascading range because the ratio of k_f to the dissipation wavenumber is $k_\mathrm{f}/k_\mathrm{d} \simeq 0.3$. In fact, as is evident from Fig. 3, there is a wavenumber range where the enstrophy flux is constant with wavenumbers. The wavenumber range where the enstrophy flux is constant expands to larger wavenumbers as α increases, although the dissipation wavenumbers are almost the same for $\alpha > 1.5$. In contrast to the energy flux, almost all the inputted enstrophy are transferred to larger wavenumbers than k_f, although it is slightly transferred to smaller wavenumbers for $\alpha = 1$.

Fig. 4 shows the enstrophy spectra at $t = t_\mathrm{fin}$ compensated by KLB scaling (26). These are consistent with the energy spectra compensated by KLB scaling (7). Low-resolution simulations forced in the dissipation range by BS13 indicate that the energy spectral slope in the energy cascading range is consistent with the KLB prediction $-(7-\alpha)/3$ for $\alpha < 5/2$, but it is steeper than the KLB prediction for $\alpha \geq 5/2$. In contrast, as evident from Fig. 4, the energy spectral slope of the present simulations is steeper than the KLB prediction for all the values of α under study. Thus, the energy spectrum for $\alpha < 5/2$ deviates from the KLB scaling when the energy and enstrophy cascading ranges coexist. For forced-dissipative 2D NS turbulence, it is well known that the energy

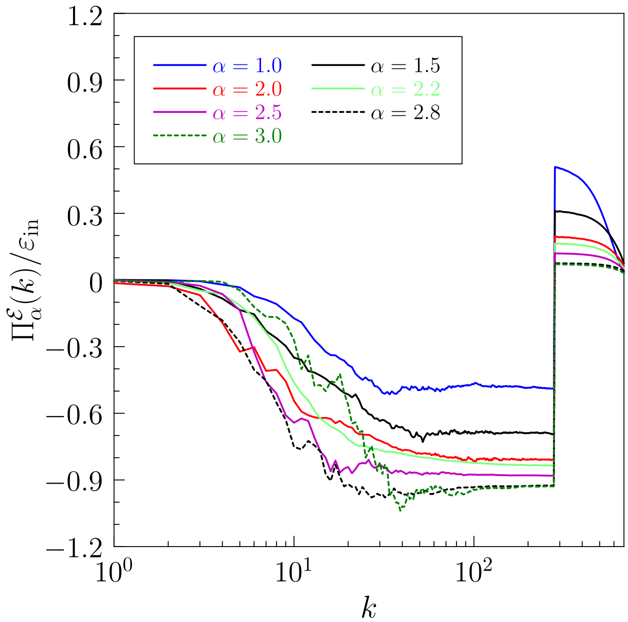

FIG. 2. Energy fluxes normalized by the energy input rates $\varepsilon_{\mathrm{in}}$.

spectrum in the energy cascading range steepens compared with the KLB scaling $k^{-5/3}$.[27] Table II gives the energy spectral slopes for $\alpha \geq 2.5$ of BS13,[18] those of the present study, and the corresponding enstrophy spectral slopes of the present study. Fig. 5 shows the compensated enstrophy spectra for $\alpha \geq 2.5$. The energy spectral slopes in the energy cascading range for $\alpha > 5/2$ of the present study are close to the results of low-resolution simulations by BS13.[18] Therefore, the energy spectral slope in the energy cascading range for $\alpha \geq 5/2$ seems to be independent of the existence of the enstrophy cascading range. We have checked that the domain size of the flow does not affect the enstrophy spectrum. IW14 showed that the infrared enstrophy spectrum evolved from a narrow

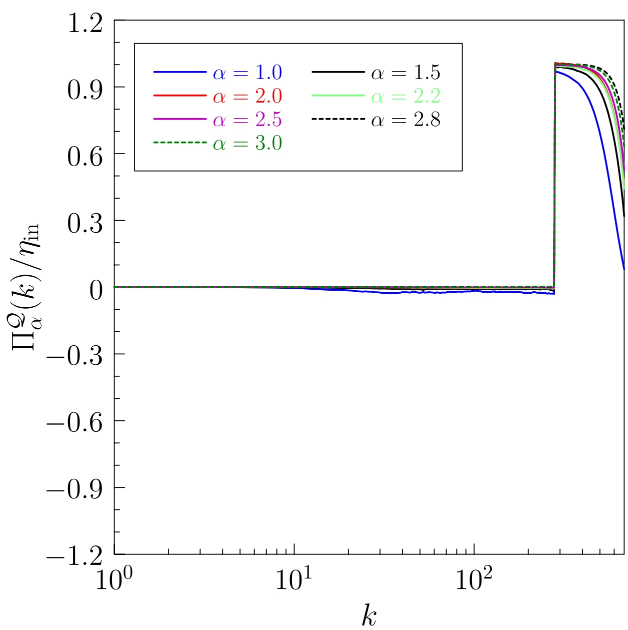

FIG. 3. Enstrophy fluxes normalized by the enstrophy input rates η_{in}.

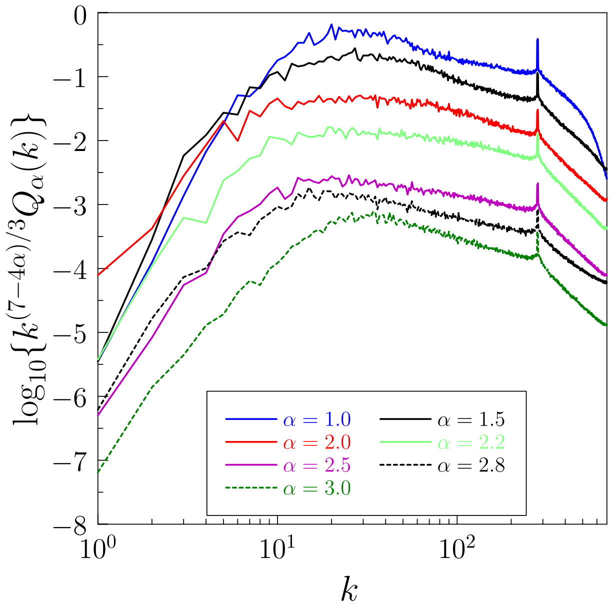

FIG. 4. Enstrophy spectra at $t = t_{\text{fin}}$ compensated by KLB scaling (7).

initial spectrum obeys $Q_\alpha(k \to 0) \propto k^5$ from asymptotic analysis of the EDQNM approximation equation. Indeed, all of the enstrophy spectral slopes at lower wavenumbers than the peak of the spectrum are approximately 5 at the end of the simulation time t_{fin}.

Note in passing that the energy cascading range spectra do not follow a single clear power law scaling for $\alpha < 5/2$, while those follow a single power law for $\alpha \geq 5/2$. This is consistent with the study by Burgess *et al.*[26] They performed high resolution simulations of forced α-turbulence for $\alpha = 1, 2, 3$ and obtained the deviation of the energy spectrum in the energy cascading range from the KLB scaling.

D. Eddy viscosity obtained from direct numerical simulations

We now address our main topic: the wavenumber dependence of the eddy viscosity for α-turbulence. $T_\alpha^{Q(>)}(k|k_c)$ in eddy viscosity coefficient (13b) was evaluated by subtracting $T_\alpha^{Q(<)}(k|k_c)$ from $T_\alpha^Q(k)$. We calculated eddy viscosity coefficient (13b) for three cases of the cutoff wavenumber $k_c = 220, 300$, and 600. These wavenumbers correspond to the higher bound of the energy cascading range, the lower bound of the enstrophy cascading range, and the dissipation range. As discussed in Sec. III, EDQNM analysis (24) indicates that the asymptotic eddy viscosity coefficient is negative when the enstrophy spectral slope is smaller than unity. Additionally, Figs. 4 and 5 indicate that the enstrophy spectral slopes for $k > k_c$ are smaller than unity for all the values of α under study except for the forcing wavenumbers. Therefore, we expect that the asymptotic eddy viscosity coefficients calculated from the direct simulations are negative for all the values of α under study.

TABLE II. Energy and enstrophy spectral slopes for $\alpha \geq 2.5$.

α	2.5	2.8	3.0
$d\ln E_\alpha(k)/d\ln k$ by BS13[18]	−1.7	−1.86	−2.0
$d\ln E_\alpha(k)/d\ln k$	−1.85	−1.9	−2.05
$d\ln Q_\alpha(k)/d\ln k$	0.65	0.9	0.95

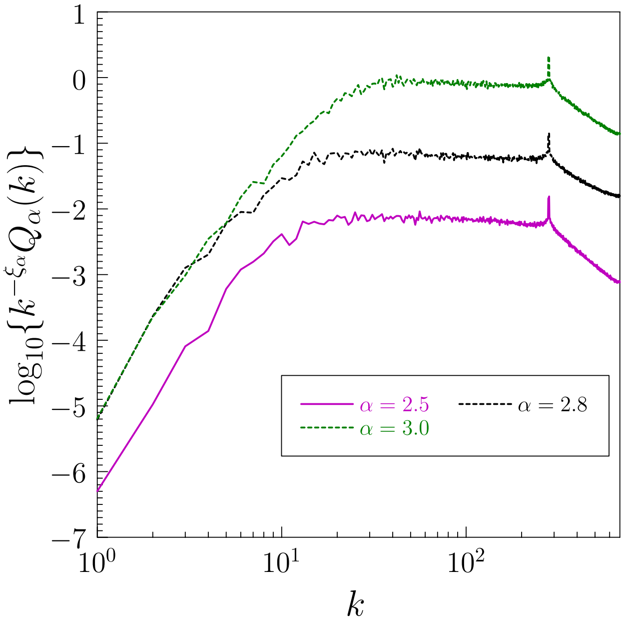

FIG. 5. Enstrophy spectra at $t = t_{\text{fin}}$ compensated by the self-similar spectra k^{ξ_α}. The exponents ξ_α are 0.65 for $\alpha = 2.5$, 0.9 for $\alpha = 2.8$, and 0.95 for $\alpha = 3$. The spectra for $\alpha = 2.8$ and $\alpha = 3$ are shifted one and two decades upward, respectively.

First, we show the results in the case $k_c = 220$. Fig. 6 shows eddy viscosity coefficients (13b) calculated from the direct numerical simulations. The eddy viscosity coefficients are normalized by their absolute values at $k = 0.02k_c$, i.e., $|\nu_T(0.02k_c|k_c)|$. Fig. 6 indicates that the eddy viscosity coefficients are increasing functions of wavenumber in general. Even for $\alpha \geq 5/2$, they are increasing functions of wavenumber. For small wavenumber ranges $k \ll k_c$, the eddy viscosity coefficients

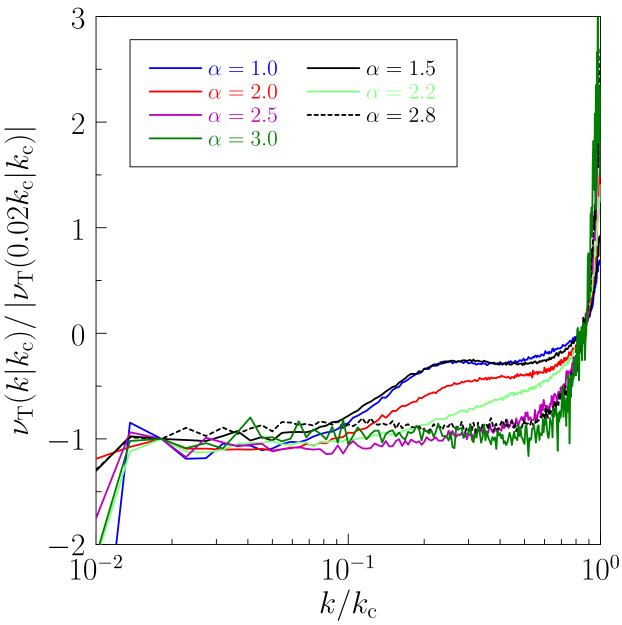

FIG. 6. Eddy viscosity coefficients (13b) obtained from direct numerical simulations for $k_c = 220$. The abscissa is the ratio of the wavenumber of interest k and the cutoff wavenumber k_c, k/k_c. The ordinate is the normalized eddy viscosity coefficient $\nu_T(k|k_c)/|\nu_T(0.02k_c|k_c)|$.

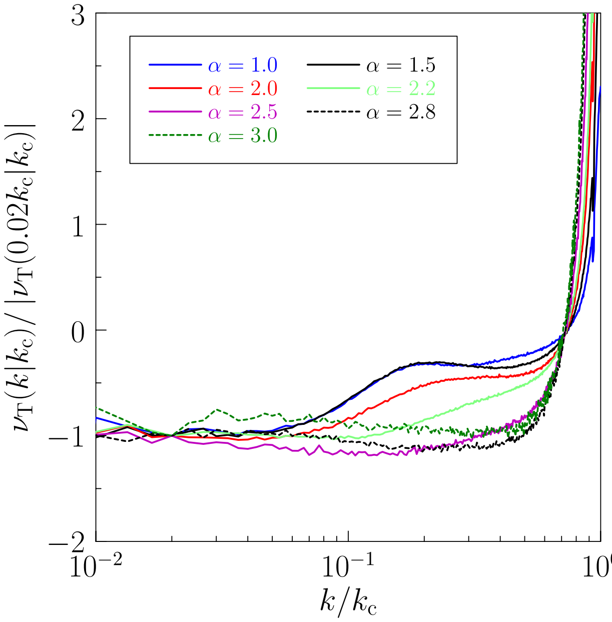

FIG. 7. Same with Fig. 6 but for $k_c = 300$.

are asymptotically negative constant as expected from the EDQNM analysis and the form of the enstrophy spectrum. Therefore, when the separation between the wavenumber of interest and the cutoff wavenumber is enough, the effect of subgrid-scale phenomena on the resolved-scale dynamics is proportional to $k^{4-\alpha}Q_\alpha(k)$ for $\alpha > 0$. For $k_c - k \ll k_c$, the eddy viscosity coefficients rise sharply to a cusp at $k = k_c$. The sign of the eddy viscosity coefficients for $k_c - k \ll k_c$ is opposite their asymptotic values. The cusp at $k = k_c$ is sharper as α increases. These features are consistent with the analysis of the EDQNM closure approximation equation. The transition wavenumber of the sign of the eddy viscosity coefficient is insensitive to the value of α. Figs. 7 and 8 show the

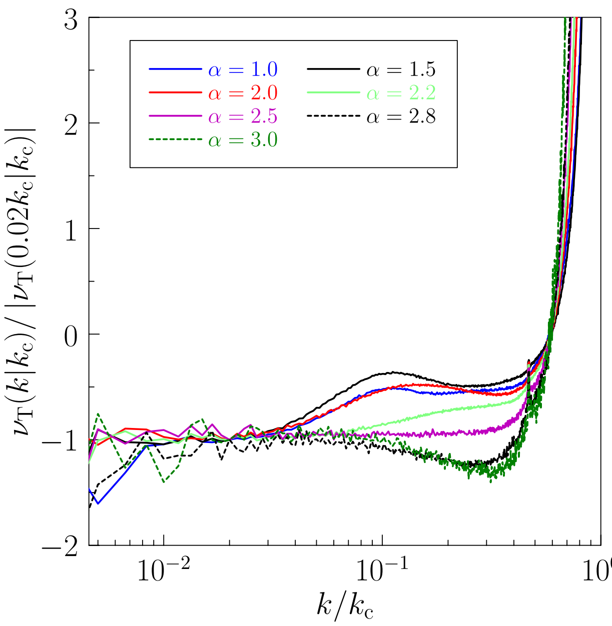

FIG. 8. Same with Fig. 6 but for $k_c = 600$.

eddy viscosity coefficients for $k_c = 300$ and 600, respectively. As evident from these figures, the basic characteristics of the eddy viscosity coefficients shown above, i.e., an asymptotically negative constant for $k \ll k_c$ and sharp rise for $k_c - k \ll k_c$, are independent of the values of the cutoff wavenumbers k_c. This indicates that the wavenumber dependence of asymptotic eddy viscosity, i.e., $k^{4-\alpha}Q_\alpha(k)$, is universal in the sense that it is independent of the characteristics of subgrid-scale motions.

One realizes the bulges of the eddy viscosity coefficient in the intermediate k/k_c from Figs. 6–8. Because the bulges move toward lower wavenumbers as k_c increases, we expect that the origin of the bulges would be properties of resolved scale motions rather than those of subgrid-scale motions. Furthermore, the bulges exist only for $\alpha < 2.5$ and the enstrophy spectra for $\alpha < 2.5$ do not follow a single power law scaling in the energy cascading range as shown in Fig. 4. As suggested by Burgess *et al.*,[26] the non-single power law scaling of the spectra would be the existence of coherent vortices. Therefore, we expect that the existence of coherent vortices and the resulting non-single power law scaling of the enstrophy spectrum would be the origin of the bulges. We would verify the above consideration if we will perform numerical simulations suppressing the formation of coherent vortices and calculate the eddy viscosity from the results. However, this is beyond the scope of the present work, because we are interested in the asymptotic form of the eddy viscosity in this study.

We have focused our attention on the dependence of the eddy viscosity on the wavenumber of interest and the sign of the asymptotic eddy viscosity coefficients. Next, we examine the cutoff wavenumber dependence of the magnitude of the asymptotic eddy viscosity coefficient. Fig. 9 shows the dependence of the magnitude of the asymptotic eddy viscosity coefficient on the cutoff wavenumber. For the asymptotic eddy viscosity coefficient, we concentrate our attention on the eddy viscosity coefficient at $k = 4$. Fig. 9 shows that the magnitude of the asymptotic eddy viscosity coefficient decreases as k_c increases. The magnitude of the asymptotic eddy viscosity coefficient for $k_c = 600$ is approximately one order of magnitude smaller than that for $k_c = 220$. This implies that the eddy viscosity is less important for the lower wavenumber dynamics as the cutoff wavenumber increases.

As shown in Fig. 9, the eddy viscosity coefficient decreases significantly when k_c increases into the dissipation range. We shall interpret this feature using the formula of the eddy viscosity

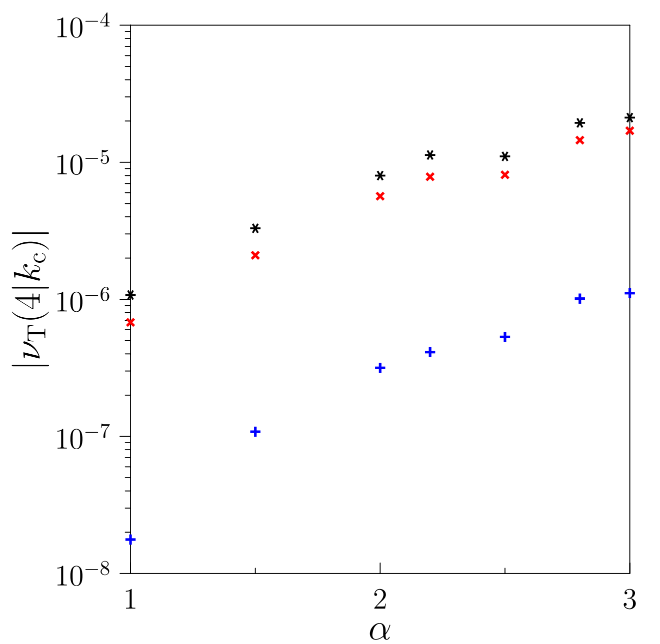

FIG. 9. Dependence of the magnitude of the asymptotic eddy viscosity coefficient on the cutoff wavenumber k_c. Asterisks denote the magnitude of the eddy viscosity coefficients at $k = 4$ for $k_c = 220$, crosses for $k_c = 300$ and pluses for $k_c = 600$.

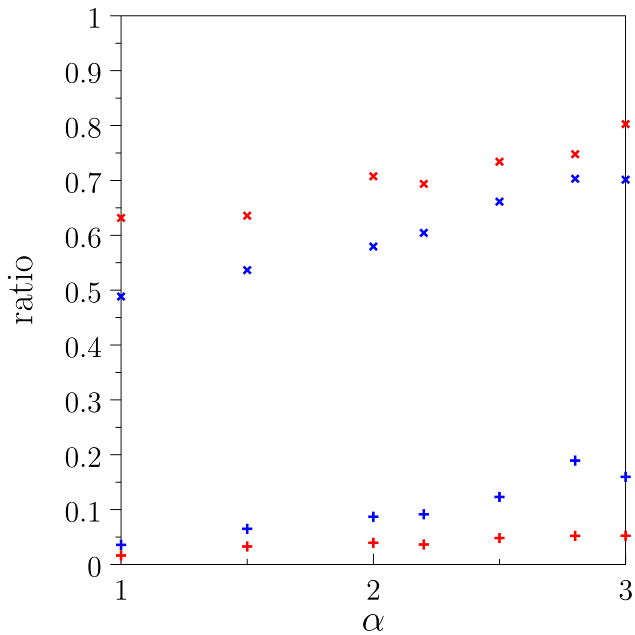

FIG. 10. Red marks denote the relative eddy viscosity $\nu_T(4|k_c)/\nu_T(4|220)$. Blue marks denote the ratio $\mathcal{R}(k_c)/\mathcal{R}(220)$, where $\mathcal{R}$ is defined by (34). Crosses indicate for $k_c = 300$, and pluses for $k_c = 600$.

coefficient given by the EDQNM calculation, (24). Equation (24) indicates that the cutoff wavenumber dependence of the eddy viscosity coefficient comes from two factors, $(n + 1)/(n + 3)$ and

$$\mathcal{R}(k_c) \equiv \left\{Q_\alpha(k_c)/k_c^3\right\}^{1/2}, \tag{34}$$

which can be interpreted as $\ell_c^2 q_c$, where ℓ_c and q_c are characteristic scales of length and generalized vorticity at the cutoff wavenumber, respectively. Because the scaling exponent n of the enstrophy spectrum in higher wavenumber range would be much larger than unity, the factor $(n + 1)/(n + 3)$ would be order of unity there. Thus, the decrease of the eddy viscosity coefficient as k_c increases would originate from the smallness of $\mathcal{R}$. In order to verify this consideration, we compare $\mathcal{R}(k_c)/\mathcal{R}(220)$ with $\nu_T(4|k_c)/\nu_T(4|220)$ for $k_c = 300$ and 600 using the results of numerical simulations in Fig. 10. Fig. 10 shows that the quantities $\mathcal{R}(k_c)/\mathcal{R}(220)$ are compatible with the ratio of eddy viscosity coefficients. Thus, we conclude that the decrease of the eddy viscosity as k_c increases originates from the decreases of the characteristic scales of length and generalized vorticity.

V. DISCUSSION

Kraichnan[2] proposed a model of the negative eddy viscosity coefficient for 2D NS turbulence. In his model, a small-scale vorticity blob is passively advected by a uniform straining field. His model does not rely on any turbulent models. Moreover, the generalized 2D fluid system is similarly governed by an advection equation. In this sense, Kraichnan's model for the negative eddy viscosity coefficient would be applicable to the generalized 2D fluid system.

In contrast, we propose an alternative explanation for the negative eddy viscosity coefficient using Fjørtoft's theorem[18,23] and a spreading hypothesis for the spectrum[28] in this section. As discussed by BS13,[18] Fjørtoft's theorem is also satisfied, even for the generalized 2D fluid system. That is, if the energy and the enstrophy are transferred among a triad with the constraint of the energy and enstrophy conservations, the energy and the enstrophy are simultaneously transferred from the middle wavenumber to the larger and smaller wavenumbers or transferred to the middle wavenumber from the larger and smaller wavenumbers. That is, if we suppose that $k < l < m$, the sign of $T_\alpha^Q(k,l,m)$ is opposite to that of $T_\alpha^Q(l,m,k)$. We cannot *a priori* determine which direction of transfer is realized. Here, we assume a spreading hypothesis for the spectrum, that is, localized energy and enstrophy distributions broaden, which is frequently assumed in the literature.[28,29] Then, energy and enstrophy transfers from the middle wavenumber to the larger and smaller wavenumbers would be realized, i.e., $T_\alpha^Q(l,m,k) < 0$. Therefore, Fjørtoft's theorem and the spreading hypothesis lead to $T_\alpha^Q(k,l,m) > 0$. Note that this statement holds for the triad energy transfer function $T_\alpha^{\mathcal{E}}(k,l,m) \equiv k^{-\alpha} T_\alpha^Q(k,l,m)$, i.e., $T_\alpha^{\mathcal{E}}(k,l,m) > 0$. Then, all the integrands of (12c) for $k \ll k_c$ are positive. Hence, $T_\alpha^{Q(>)}(k)$ is positive and the eddy viscosity coefficients, (13b) and (14b), are negative. Thus, Fjørtoft's theorem and the spreading hypothesis for the spectrum lead to energy and enstrophy transfers from wavenumbers larger than k_c to the wavenumber of interest k when $k \ll k_c$. This up-scale transfer is the origin of the negative eddy viscosity coefficient for $k \ll k_c$. Note also that, as shown by BS13,[18] the triad transfer functions $T_\alpha^{\mathcal{E}}(l,m,k)$ of the EDQNM solutions for $\alpha > 5/2$ are positive. This is contradictory to the spreading hypothesis and is the origin of the asymptotic eddy viscosity coefficient having the positive value shown in Sec. IV A.

We further note that the up-scale transfers of enstrophy and energy in the above explanation for the negative eddy viscosity are different from the so-called upscale energy cascade. The so-called upscale energy cascade is realized by the all triad interactions, i.e., by the transfer function $T_\alpha^{\mathcal{E}}(k)$. However, the up-scale transfers of enstrophy and energy responsible for the negative eddy viscosity originate from the transfers of them among particular triad interactions indicated by $T_\alpha^{Q(>)}(k|k_c)$ and $T_\alpha^{\mathcal{E}(>)}(k|k_c)$.

VI. SUMMARY

We have discussed the eddy viscosity for the generalized 2D fluid system. We formulated the eddy viscosity in wavenumber space. The starting point of our discussion was the enstrophy spectrum equation. When the enstrophy spectrum equation is truncated at a wavenumber k_c, the effect of the truncation on the evolution of the enstrophy spectrum $Q_\alpha(k)$ with wavenumbers $k < k_c$ was investigated by asymptotic analysis and numerical calculations of the EDQNM closure approximation equation for the enstrophy spectrum and direct numerical simulations of the governing equation for the generalized 2D fluid system. It has been presumed that subgrid-scale phenomena asymptotically act as normal diffusion processes on resolved-scale phenomena, when the scale of interest and the subgrid scale are well separated. The asymptotic analysis of the enstrophy transfer function due to interactions in which the wavenumbers $k > k_c$ intervene shows that the transfer function is asymptotically proportional to $k^{4-\alpha} Q_\alpha(k)$ for $\alpha > 0$ while $k^4 Q_\alpha(k)$ for $\alpha < 0$, and the proportionality constant, i.e., the eddy viscosity coefficient, is negative for $k \ll k_c$. That is, even if the separation between the scale of interest and that of truncation is great enough, the effect of the truncation cannot be modeled by a normal diffusion $k^2 Q_\alpha(k)$. Rather, it is generally modeled by an anomalous diffusion. The modeling by the normal diffusion is realized only for $\alpha = 2$, i.e., the NS system. The asymptotic analysis of the EDQNM equation was verified by numerical calculations of the EDQNM equation and direct numerical simulations. The wavenumber dependence of asymptotic eddy viscosity derived in this paper is universal in the sense that it is independent of the characteristics of subgrid-scale motions. The mechanism of the negative eddy viscosity coefficient was explained in terms of the energy and enstrophy transfers based on Fjørtoft's theorem and a spreading hypothesis for the spectrum.

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