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39. Askey-Wilson Polynomials and the Quantum Group $SU_q(2)$

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The *Askey-Wilson polynomials* are a 4-parameter family of q -orthogonal polynomials expressed by the basic hypergeometric series ${}_4\phi_3$. As special cases, it provides various types of q -Jacobi polynomials such as little, big and continuous q -Jacobi polynomials. In this note, we report that a (partially discrete) 4-parameter family of Askey-Wilson polynomials is realized as “doubly associated spherical functions” on the quantum group $SU_q(2)$.

In [2], Koornwinder realized a 2-parameter subfamily of Askey-Wilson polynomials as *zonal* spherical functions on $SU_q(2)$ in an *infinitesimal* sense. Generalizing his arguments to *non-zonal* cases, we obtain a 4-parameter family of Askey-Wilson polynomials that are connected to these polynomials as Jacobi polynomials are to Legendre polynomials in the $SU(2)$ case. From this interpretation, we also derive an addition formula for Koornwinder’s 2-parameter extension of the continuous q -Legendre polynomials. Details will be given elsewhere.

1. Throughout this note, we fix a real number q with $0 < q < 1$. The algebra of functions $A(G)$ on the quantum group $G = SU_q(2)$ is the C -algebra generated by x, u, v, y with fundamental relations

$$(1.1) \quad \begin{cases} qxu = ux, q xv = vx, qvy = yu, qvy = yv, \\ uv = vu, xy - q^{-1}uv = yx - qvu = 1, \end{cases}$$

and the $*$ -structure determined by $x^* = y$ and $v^* = -qu$. The quantized universal enveloping algebra $U_q(su(2))$ is the C -algebra generated by k, k^{-1}, e, f with relations

$$(1.2) \quad \begin{cases} kk^{-1} = k^{-1}k = 1, kek^{-1} = qe, kfk^{-1} = q^{-1}f, \\ ef - fe = (k^2 - k^{-2})/(q - q^{-1}), \end{cases}$$

and the $*$ -structure with $k^* = k$ and $e^* = f$. As for the Hopf algebra structure, we take the coproduct determined by

$$\Delta(k) = k \otimes k, \quad \Delta(e) = k^{-1} \otimes e + e \otimes k, \quad \Delta(f) = k^{-1} \otimes f + f \otimes k.$$

The algebra of functions $A(G)$ has a natural structure of two-sided $U_q(su(2))$ -module. For each $j \in (1/2)\mathbb{N}$, there exists a unique $2j+1$ dimensional irreducible representation of G of highest weight q^j with respect to $k \in U_q(su(2))$. By V_j we denote the corresponding right $A(G)$ -comodule with coaction $R: V_j \rightarrow V_j \otimes A(G)$. We fix a C -basis $(v_m^j)_{m \in I_j}$ for V_j , with $I_j = \{j, j-1, \dots, -j\}$, such that the differential representation takes the form

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$$(1.3) \quad \begin{cases} k. v_m^j = v_m^j q^m, \\ e. v_m^j = v_{m+1}^j ([j-m][j+1+m])^{1/2}, \\ f. v_m^j = v_{m-1}^j ([j+m][j+1-m])^{1/2}, \end{cases}$$

where $[m] = (q^m - q^{-m}) / (q - q^{-1})$. This representation is unitary with respect to the Hermitian form \langle, \rangle on V_j such that $\langle v_m^j, v_n^j \rangle = \delta_{mn}$ ($m, n \in I_j$) and the *-operation of $U_q(su(2))$. See also [3].

2. For each matrix

$$(2.1) \quad g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in GL(2; C),$$

we define the twisted primitive element $\theta(g) \in U_q(su(2))$ by

$$(2.2) \quad \theta(g) = -\alpha\beta q^{-1/2}e + (\alpha\delta + \beta\gamma)(k - k^{-1}) / (q - q^{-1}) + \gamma\delta q^{1/2}f.$$

When $q \rightarrow 1$, the element $\theta(g)$ corresponds to a generator of the Lie algebra of the subgroup $K(g) := gKg^{-1}$ of $SU(2)$, where K is the diagonal subgroup of $SU(2)$.

Theorem 1. *Let g be a matrix of the form (2.1) and assume that*

$$\alpha\delta - q^{2k}\beta\gamma \neq 0 \quad \text{for all } k \in \mathbb{Z}.$$

For each $m \in (1/2)\mathbb{Z}$, set

$$(2.3) \quad \lambda_m(g) = (q^m\alpha\delta - q^{-m}\beta\gamma)(q^m - q^{-m}) / (q - q^{-1}).$$

Then the element $k\theta(g)$ is diagonalizable on each left $U_q(su(2))$ -module V_j ($j \in (1/2)\mathbb{N}$). Its eigenvalues are given by $\lambda_m(g)$ ($m = j, j-1, \dots, -j$).

We remark that Theorem 1 is also valid when q is a nonzero complex number as long as q is not a root of unity. It is essentially the same as Theorem 8.5 of Koornwinder [2].

Hereafter, we assume that the parameter of (2.1) satisfies the condition $\bar{\alpha} = \delta, \bar{\gamma} = -\beta$ so that $(k\theta(g))^* = k\theta(g)$. Then we see that there exists a family of orthogonal bases $(v_m^j(g))_{m \in I_j}$ for V_j , depending polynomially on $(\alpha, \beta, \gamma, \delta)$, such that

$$(2.4) \quad k\theta(g). v_m^j(g) = v_m^j(g) \lambda_m(g) \quad \text{for all } m \in I_j,$$

and

$$(2.5) \quad \langle v_m^j(g), v_n^j(g) \rangle = \delta_{mn} D_m^j(g) \quad \text{for } m, n \in I_j,$$

where

$$D_m^j(g) = \prod_{-j-m \leq k \leq j-m, k \neq -2m} (\alpha\delta - q^{2k}\beta\gamma).$$

We fix such a family of orthogonal bases $(v_m^j(g))_{m \in I_j}$ for V_j under a suitable normalization, although we do not give here its precise description. The connection coefficients between the bases $(v_m^j(g))_{m \in I_j}$ and $(v_m^i(g))_{m \in I_i}$ can be written explicitly by Stanton's q -Krawtchouk polynomials (see also [2]).

3. We now introduce the matrix elements of V_j relative to the two bases $(v_m^j(g_1))_m$ and $(v_m^j(g_2))_m$. Let (g_1, g_2) be a couple of elements in $GL(2; C)$ such that

$$(3.1) \quad g_i = \begin{bmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{bmatrix} \in GL(2; C); \quad \bar{\alpha}_i = \delta_i, \bar{\beta}_i = -\gamma_i \quad (i=1, 2).$$

We define the matrix element $\varphi_{mn}^j(g_1, g_2) \in A(G)$ ($m, n \in I_j$) of V_j by

$$(3.2) \quad \varphi_{mn}^j(g_1, g_2) := \langle v_m^j(g_1), R(v_n^j(g_2)) \rangle.$$

We also set $\psi_{mn}^j(g_1, g_2) := \varphi_{mn}^j(g_1, g_2) \cdot k$ by using the right action of $k \in U_q(\mathfrak{su}(2))$.

Proposition 2. a) *The element $\psi = \psi_{mn}^j(g_1, g_2)$ has the relative invariance*
 (3.3) $k\theta(g_2) \cdot \psi = \psi \lambda_n(g_2) \quad \text{and} \quad \psi \cdot \theta(g_1)k = \lambda_m(g_1)\psi.$

b) *The elements $\psi_{mn}^j(g_1, g_2)$ ($j \in (1/2)\mathbb{N}$, $m, n \in I_j$) form an orthogonal basis for $A(G)$ under the Hermitian form $\langle \cdot, \cdot \rangle_L$ defined by the Haar measure. The square length of $\psi_{mn}^j(g_1, g_2)$ is given by*

$$(3.4) \quad \langle \psi_{mn}^j(g_1, g_2), \psi_{mn}^j(g_1, g_2) \rangle_L = q^{2j} \frac{1 - q^2}{1 - q^{2(j+1)}} D_m^j(g_1) D_n^j(g_2).$$

c) *For any g , one has*

$$(3.5) \quad \Delta(\varphi_{mn}^j(g_1, g_2)) = \sum_k D_k^j(g)^{-1} \varphi_{mk}^j(g_1, g) \otimes \varphi_{kn}^j(g, g_2).$$

In view of the relative invariance (3.3), we say that the elements $\psi_{mn}^j(g_1, g_2)$ are doubly associated spherical functions on G .

4. For each $m, n \in \frac{1}{2}\mathbb{Z}$, we set

$$e_{mn}(g_1, g_2) := \psi_{mn}^j(g_1, g_2) \quad \text{with } j = \max\{|m|, |n|\}.$$

This element is a basic relative invariant in the sense that it appears with smallest j among all relative invariants ψ satisfying (3.3). These $e_{mn}(g_1, g_2)$ are expressed as products of linear combinations of the generators x, u, v, y for $A(G)$.

The general matrix elements $\psi_{mn}^j(g_1, g_2)$ are expressed by the Askey-Wilson polynomials [1]:

$$p_n(x; a, b, c, d | q) = a^{-n} (ab, ac, ad; q)_n {}_4\phi_3 \left(\begin{matrix} q^{-n}, abcdq^{n-1}, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q \right),$$

where $x = (z + z^{-1})/2$. To describe the matrix elements, we introduce the following 2-parameter extension of the continuous q -Jacobi polynomials:

$$(4.1) \quad p_n^{(\alpha, \beta)}(x; s, t; q) := p_n \left(x; \frac{t}{s} q^{1/2}, \frac{s}{t} q^{a+1/2}, -\frac{1}{st} q^{1/2}, -stq^{\beta+1/2} | q \right),$$

where s and t are continuous parameters. If $(\alpha, \beta) = (0, 0)$, then formula (4.1) gives Koornwinder's 2-parameter extension of the continuous q -Legendre polynomials in [2]. If $(s, t) = (1, 1)$, (4.1) is Rahman's parametrization of continuous q -Jacobi polynomials.

For a couple (g_1, g_2) of (3.1), we define the zonal element $X = X(g_1, g_2)$ by

$$(4.2) \quad 2|\alpha_1 \gamma_1 \alpha_2 \gamma_2| X = \frac{1}{q + q^{-1}} (\psi_{00}^1(g_1, g_2) - (\alpha_1 \delta_1 + \beta_1 \gamma_1)(\alpha_2 \delta_2 + \beta_2 \gamma_2)),$$

assuming that $\alpha_i \neq 0$, $\gamma_i \neq 0$ ($i = 1, 2$). Note that $X = X(g_1, g_2)$ satisfies

$$k\theta(g_2) \cdot X = 0, \quad X \cdot \theta(g_1)k = 0, \quad X^* = X.$$

Theorem 3. *The doubly associated spherical functions $\psi_{mn}^j(g_1, g_2)$ are represented by the Askey-Wilson polynomials (4.1) in X .*

Case I. $m + n \geq 0$, $m \leq n$:

$$q^{-k(k+\mu+2\nu)} C_{\mu\nu k} |\alpha_1 \gamma_1 \alpha_2 \gamma_2|^k p_k^{(\mu, \nu)}(X; |\alpha_2/\gamma_2|, |\alpha_1/\gamma_1|; q^2) e_{mn}(g_1, g_2),$$

Case II. $m + n \geq 0$, $m \geq n$:

$$q^{-k(k+\mu+2\nu)} C_{\mu\nu k} |\alpha_1 \gamma_1 \alpha_2 \gamma_2|^k p_k^{(\mu, \nu)}(X; |\alpha_1/\gamma_1|, |\alpha_2/\gamma_2|; q^2) e_{mn}(g_1, g_2),$$

Case III. $m+n \leq 0$, $m \geq n$:

$$q^{-k(k+\mu)} C_{\mu\nu k} |\alpha_1 \gamma_1 \alpha_2 \gamma_2|^k p_k^{(\mu, \nu)}(X; |\gamma_2/\alpha_2|, |\gamma_1/\alpha_1|; q^2) e_{mn}(g_1, g_2),$$

Case IV. $m+n \leq 0$, $m \leq n$:

$$q^{-k(k+\mu)} C_{\mu\nu k} |\alpha_1 \gamma_1 \alpha_2 \gamma_2|^k p_k^{(\mu, \nu)}(X; |\gamma_1/\alpha_1|, |\gamma_2/\alpha_2|; q^2) e_{mn}(g_1, g_2).$$

Here $\mu = |m-n|$, $\nu = |m+n|$, $k = \min\{j+m, j-m, j+n, j-n\}$ and $C_{\mu\nu k}$ stands for

$$C_{\mu\nu k} = \left(\frac{q^{2(\mu+\nu+1)}; q^2_k}{(q^2, q^{2(\mu+1)}, q^{2(\nu+1)}; q^2)_k} \right)^{1/2}.$$

Theorem 3 is a generalization of Theorem 8.3 of Koornwinder [2] to non-zonal cases. The expressions in Theorem 3 make sense even when some of the $\alpha_1, \gamma_1, \alpha_2, \gamma_2$ are zero. We also remark that the orthogonality in Proposition 2 is interpreted as the orthogonality relation for the Askey-Wilson polynomials.

By the above interpretation, we obtain an addition formula for $p_n^{(0,0)}(x; s, t; q)$. In fact, property (3.5) is translated into an addition formula for them.

Theorem 4. *The polynomials $p_n^{(0,0)}(x; s, t; q)$ ($n \in N$) have the following addition formula involving an extra parameter u :*

$$\begin{aligned} (4.3) \quad & q^{-n/2}(q; q)_n p_n^{(0,0)}(x(zw); s, t; q) \\ &= \frac{1}{(-u^2q, -u^{-2}q; q)_n} p_n^{(0,0)}(x(z); u, s; q) p_n^{(0,0)}(x(w); u, t; q) \\ &+ \sum_{k=1}^n \frac{(q; q)_{n+k} (1+u^2q^{2k}) z^{-k} w^{-k} \left(\frac{u}{s} z, -usz, \frac{u}{t} w, -utw; q \right)_k}{(q; q)_{n-k} (1+u^2)(-u^2q; q)_{n+k} (-u^{-2}q; q)_{n-k}} \\ &\quad \times p_{n-k}^{(k,k)}(x(z); u, s; q) p_{n-k}^{(k,k)}(x(w); u, t; q) \\ &+ \sum_{k=1}^n \frac{(q; q)_{n+k} (1+u^{-2}q^{2k}) z^{-k} w^{-k} \left(\frac{s}{u} z, -\frac{1}{us} z, \frac{t}{u} w, -\frac{1}{ut} w; q \right)_k}{(q; q)_{n-k} (1+u^{-2})(-u^2q; q)_{n-k} (-u^{-2}q; q)_{n+k}} \\ &\quad \times p_{n-k}^{(k,k)}\left(x(z); \frac{1}{u}, \frac{1}{s}; q\right) p_{n-k}^{(k,k)}\left(x(w); \frac{1}{u}, \frac{1}{t}; q\right), \end{aligned}$$

where z and w are independent variables and $x(z) = (q^{-1/2}z + q^{1/2}z^{-1})/2$.

We remark that Rahman and Verma [4] have obtained an addition formula for Rogers' q -ultraspherical polynomials $p_n^{(a,a)}(x; 1, 1; q)$ by analytic methods. Their work suggests that Theorem 4 may be extended to an addition formula containing one more parameter.

References

- [1] R. Askey and J. Wilson: Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials. *Memoirs Amer. Math. Soc.*, **54**, no. 319 (1985).
- [2] T. H. Koornwinder: Orthogonal polynomials in connections with quantum groups. *Orthogonal Polynomials, Theory and Practice* (ed. P. Nevai). NATO ASI Series, Kluwer Academic Publishers, pp. 257-292 (1990).
- [3] M. Noumi and K. Mimachi: Quantum 2-spheres and big q -Jacobi polynomials. *Commun. Math. Phys.*, **128**, 521-531 (1990).
- [4] M. Rahman and A. Verma: Product and addition formulas for the continuous q -ultraspherical polynomials. *SIAM J. Math. Anal.*, **17**, 1461-1474 (1986).