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A reproducing kernel for nonsymmetric Macdonald polynomials

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Abstract. We present a new formula of Cauchy type for the nonsymmetric Macdonald polynomials which are joint eigenfunctions of q-Dunkl operators. This gives the explicit formula for a reproducing kernel on the polynomial ring of n variables.

§0: Introduction.

In this paper we propose a new formula of Cauchy type for the nonsymmetric Macdonald polynomials of type A_{n-1}. This can be regarded as an explicit formula for the reproducing kernel of a certain scalar product on the polynomial ring of n variables. A similar result for nonsymmetric Jack polynomials was recently given by Sahi [S].

The nonsymmetric Macdonald polynomials $E_\lambda(x|q,t)$, introduced by Macdonald [Ma1], are characterized as the joint eigenfunctions in the polynomial ring of n variables $x=(x_1,\dots,x_n)$, for the commuting family of q-Dunkl operators. (For the precise definition of $E_\lambda(x|q,t)$, see Section 1.) We define a meromorphic function $E(x;y|q,t)$ in $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ by

$$(0.1)\quad E(x;y|q,t)=\prod_{1\le j<i\le n}\frac{(qtx_iy_j;q)_\infty}{(qx_iy_j;q)_\infty}\prod_{1\le i\le n}\frac{(qtx_iy_i;q)_\infty}{(x_iy_i;q)_\infty}\prod_{1\le i<j\le n}\frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}.$$

The main result of this paper is the following.

Theorem A. *The function $E(x;y|q,t)$ has the following expansion in terms of nonsymmetric Macdonald polynomials:*

$$(0.2)\qquad E(x;y|q,t)=\sum_{\lambda\in\mathbb{N}^n}a_\lambda(q,t)E_\lambda(x|q,t)E_\lambda(y|q^{-1},t^{-1}).$$

For each composition $\lambda\in\mathbb{N}^n$, the coefficient $a_\lambda(q,t)$ is given by

$$(0.3)\qquad a_\lambda(q,t)=\prod_{s\in\lambda}\frac{1-q^{a(s)+1}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}},$$

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where, for each box $s \in \lambda$, $a(s)$ and $l(s)$ are the arm-length and the generalized leg-length of s in λ.

(See Theorems 2.1 and 2.2.)

Assuming that q and t are complex numbers with $0 < |q|, |t| < 1$, we now consider the meromorphic function $K(x;y|q,t) = E(x;y^{-1}|q,t)$ on the algebraic torus $(\mathbb{C}^*)^n \times (\mathbb{C}^*)^n$.

Theorem B. *For each composition $\lambda \in \mathbb{N}^n$, we have*

$$(0.4) \qquad \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{\mathbb{T}^n} K(x;y|q,t) E_\lambda(y|q,t) w(y|q,t) \frac{dy_1 \cdots dy_n}{y_1 \cdots y_n} = C_\lambda . E_\lambda(x|q,t)$$

for all $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ with $|x_j| < 1$ $(j = 1, \dots, n)$. Here $\mathbb{T}^n = \{y = (y_1, \dots, y_n) \in (\mathbb{C}^*)^n \; ; \; |y_j| = 1 \;\; (j = 1, \dots, n)\}$ is the n-dimensional torus with the standard orientation, and

$$(0.5) \qquad w(y|q,t) = \prod_{1 \le i < j \le n} \frac{(y_i/y_j;q)_\infty (q y_j/y_i;q)_\infty}{(t y_i/y_j;q)_\infty (q t y_j/y_i;q)_\infty}.$$

The constant C_λ is given by

$$(0.6) \qquad C_\lambda = C_{\lambda^+} = \left(\frac{(qt;q)_\infty}{(q;q)_\infty}\right)^n \prod_{i=1}^n \frac{(q^{\lambda_i^+ + 1} t^{n-i};q)_\infty}{(q^{\lambda_i^+ + 1} t^{n-i+1};q)_\infty},$$

where λ^+ is the partition obtained by reordering the parts of λ.

(See Theorem 3.2.)

After preliminaries on nonsymmetric Macdonald polynomials, we will state our main results in Section 2. In Section 2, we will prove that $E(x;y|q,t)$ has an expansion of the form (0.2), and reduce the determination of the coefficients $a_\lambda(q,t)$ to the case of partitions. In this paper we will present two ways of determining the coefficients $a_\lambda(q,t)$ for partitions λ. In Section 3, we determine these coefficients in an analytic way by asymptotic analysis of q-Selberg type integrals similarly as in [Mi2]. In this proof we will make use of the evaluation of Cherednik's scalar product for nonsymmetric Macdonald polynomials. Theorem B will also be formulated in Section 3. In Section 4, we give an algebraic proof of (0.3) by using the evaluation of the nonsymmetric Macdonald polynomials $E_\lambda(x;q,t)$ at $x = (t^{-n+1}, t^{-n+2}, \dots, 1)$ due to Cherednik [C2]. This second proof is an extension of the argument of Sahi [S] to the q-version.

§1: Nonsymmetric Macdonald polynomials.

We first recall the definition of *nonsymmetric Macdonald polynomials* of type A_{n-1} in the GL_n version. Although we follow the presentation by Macdonald [Ma2] in principle, we use a slightly different convention which is more convenient for our purpose.

Let $\mathbb{K}[x^{\pm 1}]$ be the ring of Laurent polynomials in n variables $x = (x_1, \ldots, x_n)$ with coefficients in the field $\mathbb{K} = \mathbb{Q}(q, t^{\frac{1}{2}})$. (Although we use this coefficient field for convenience, the use of $t^{\frac{1}{2}}$ is *not* essential; one could work within $\mathbb{Q}(q,t)$ by modifying the argument appropriately.) We denote by $\tau = (\tau_1, \ldots, \tau_n)$ the corresponding q-shift operators. For each $i = 1, \ldots, n$, the operator τ_i acts on $\mathbb{K}[x^{\pm 1}]$ as a $\mathbb{K}$-automorphism such that $\tau_i(x_j) = x_j q^{\delta_{ij}}$ $(j = 1, \ldots, n)$. The action of the symmetric group $W = \mathfrak{S}_n$ on $\mathbb{K}[x^{\pm 1}]$ will be fixed so that each $w \in W$ defines the $\mathbb{K}$-algebra automorphism such that $w.x_j = x_{w(j)}$ for $j = 1, \ldots, n$. The ring $\mathbb{K}[x^{\pm 1}]$ is identified with the group-ring $\mathbb{K}[P]$ of the integral weight lattice $P = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n$. As usual, we take the symmetric bilinear form $\langle\ ,\ \rangle$ on P such that $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$ $(1 \le i, j \le n)$. For each $\lambda \in P$, we use the notation of multi-indices

$$x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}, \quad \tau^\lambda = \tau_1^{\lambda_1} \cdots \tau_n^{\lambda_n}, \tag{1.1}$$

where $\lambda_j = \langle \lambda, \epsilon_j \rangle$ $(j = 1, \ldots, n)$. The action of the symmetric group $W = \mathfrak{S}_n$ of degree n on P will be fixed as $w.\epsilon_i = \epsilon_{w(i)}$, or equivalently as $(w.\lambda)_i = \lambda_{w^{-1}(i)}$ $(i = 1, \ldots, n)$ for each $w \in W$. Note that the commutation relations among the multiplication operators x^λ, the q-shift operators τ^μ and the permutations $w \in W$ are given as follows:

$$\tau^\mu x^\lambda = x^\lambda \tau^\mu q^{\langle \mu, \lambda \rangle}, \quad w x^\lambda = x^{w.\lambda} w, \quad w \tau^\mu = \tau^{w.\mu} w, \tag{1.2}$$

for $\lambda, \mu \in P$ and $w \in W$. We will use the standard notation of the set of positive roots

$$\Delta^+ = \{\epsilon_i - \epsilon_j \ ;\ 1 \le i < j \le n\}, \tag{1.3}$$

the simple roots $\alpha_i = \epsilon_i - \epsilon_{i+1}$ $(i = 1, \ldots, n-1)$ and the cone of dominant integral weights

$$P^+ = \{\lambda \in P \ ;\ \langle \alpha_i, \lambda \rangle \ge 0 \ \ (i = 1, \ldots, n-1)\} = \{\lambda \in P \ ;\ \lambda_1 \ge \cdots \ge \lambda_n\}. \tag{1.4}$$

We denote the set of *compositions* and the set of *partitions* with length $\le n$ by $L = \mathbb{N}\epsilon_1 \oplus \cdots \oplus \mathbb{N}\epsilon_n \subset P$ and by $L^+ = P^+ \cap L$, respectively, where $\mathbb{N} = \{0, 1, 2, \cdots\}$.

In what follows, we will make use of the *Demazure-Lusztig operators* $T_1, \ldots, T_{n-1}$ defined by

$$T_i = t^{\frac{1}{2}} + t^{-\frac{1}{2}} \frac{1 - t x_i / x_{i+1}}{1 - x_i / x_{i+1}} (s_i - 1) \quad (i = 1, \ldots, n-1), \tag{1.5}$$

where $s_i = (i, i+1)$ stands for the reflection with respect to the simple root $\alpha_i = \epsilon_i - \epsilon_{i+1}$. Note that

$$(T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0 \quad (i = 1, \ldots, n-1), \tag{1.6}$$

and that the operators $T_1, \ldots, T_{n-1}$ satisfy the fundamental relations for the canonical generators of the Hecke algebra $H(\mathfrak{S}_n)$. Furthermore we define the *q-Dunkl operators* $Y_1, \ldots, Y_n$, due to Cherednik, by

$$Y_i = T_i T_{i+1} \cdots T_{n-1} \omega T_1^{-1} \cdots T_{i-1}^{-1} \quad (i = 1, \ldots, n), \tag{1.7}$$

where

$$\omega = s_{n-1}\cdots s_1\tau_1 = \ldots = \tau_n s_{n-1}\cdots s_1. \tag{1.8}$$

These operators $Y_1, \ldots, Y_n$ commute with each other and, for any *symmetric* Laurent polynomial $f(Y)$ of $Y = (Y_1, \ldots, Y_n)$, Macdonald's symmetric polynomials $P_\lambda(x) = P_\lambda(x|q,t)$ ($\lambda \in P^+$) [Ma2] satisfy the equation

$$f(Y)P_\lambda(x) = P_\lambda(x)f(q^\lambda t^\rho), \tag{1.9}$$

where $f(q^\lambda t^\rho)$ denotes the evaluation of f at the point $q^\lambda t^\rho = (q^{\lambda_1}t^{\rho_1}, \ldots, q^{\lambda_n}t^{\rho_n})$, and

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha = \frac{1}{2}\sum_{i=1}^{n}(n-2i+1)\epsilon_i. \tag{1.10}$$

One important fact is that the q-Dunkl operators have the triangularity with respect to a certain partial ordering of monomials. We define the partial ordering $\preceq$ in P as follows: For $\lambda, \mu \in P$,

$$\mu \preceq \lambda \quad\Leftrightarrow\quad \mu^+ < \lambda^+ \text{ or } (\mu^+ = \lambda^+,\ \ \mu \le \lambda), \tag{1.11}$$

where λ^+ stands for the unique dominant integral weight in the W-orbit $W.\lambda$ of λ and $\le$ is the dominance order ($\mu \le \lambda$ means that $\lambda - \mu$ is a linear combination of positive roots with coefficients in $\mathbb{N}$). Then it turns out that, for any $\lambda, \mu \in P$, one has

$$Y^\mu x^\lambda = x^\lambda q^{\langle\mu,\lambda\rangle}t^{\langle\mu,\rho(\lambda)\rangle} + (\text{lower order terms under } \preceq), \tag{1.12}$$

where

$$\rho(\lambda) = \frac{1}{2}\sum_{\alpha\in\Delta^+}\varepsilon(\langle\alpha,\lambda\rangle)\alpha, \tag{1.13}$$

where $\varepsilon(u) = 1$ if $u \ge 0$ and $\varepsilon(u) = -1$ if $u < 0$. Note that $\rho(\lambda)$ is precisely the element $w_\lambda.\rho$ in the W-orbit of ρ if one take the w_λ which has the minimal length among all $w \in W$ such that $\lambda = w.\lambda^+$.

Remark 1.1. Because of the definition of q-Dunkl operators mentioned above, the partial ordering $\preceq$ and the function $\varepsilon(u)$ is different from those in [Ma2]. Note that, under our definition of $\preceq$, the dominant weight λ^+ is maximal in the W-orbit $W.\lambda$.

By the triangularity of q-Dunkl operators mentioned above, one can show that, for each $\lambda \in P$, there exists a unique Laurent polynomial $E_\lambda(x) = E_\lambda(x|q,t)$ such that

$$E_\lambda(x) = x^\lambda + (\text{lower order terms under } \preceq), \tag{1.14}$$

and that

$$Y^\mu E_\lambda(x) = E_\lambda(x)q^{\langle\mu,\lambda\rangle}t^{\langle\mu,\rho(\lambda)\rangle} \tag{1.15}$$

for any $\mu \in P$. These Laurent polynomials $E_\lambda(x) = E_\lambda(x|q,t)$ are called the *nonsymmetric Macdonald polynomials* of type A_{n-1}. The first property (1.14) implies in particular that $E_\lambda(x)$ is homogeneous of degree $|\lambda| = \sum_{i=1}^n \lambda_i$, and is eventually a polynomial in x if $\lambda \in L$. Note also that the second property (1.15) is equivalent to saying that

$$f(Y)E_\lambda(x) = E_\lambda(x) f(q^\lambda t^{\rho(\lambda)}) \tag{1.16}$$

for *any* Laurent polynomial $f(Y)$ of the q-Dunkl operators. It is easy to see that the nonsymmetric Macdonald polynomials $E_\lambda(x)$ $(\lambda \in P)$ actually have coefficients in $\mathbb{Q}(q,t)$. We also remark that, as a function of t, each $E_\lambda(x) = E_\lambda(x|q,t)$ is regular at $t = q^k$ $(k = 0, 1, 2, \dots)$. These polynomials $E_\lambda(x)$ form a $\mathbb{K}$-basis of the ring $\mathbb{K}[x^{\pm 1}]$ of Laurent polynomials or of the ring $\mathbb{K}[x]$ of polynomials as follows:

$$\mathbb{K}[x^{\pm 1}] = \bigoplus_{\lambda \in P} \mathbb{K} E_\lambda(x), \quad \mathbb{K}[x] = \bigoplus_{\lambda \in L} \mathbb{K} E_\lambda(x). \tag{1.17}$$

It is known by [Ma1] that, for any dominant integral weight $\lambda \in P^+$, Macdonald's symmetric polynomial $P_\lambda(x)$ is expressed as a linear combination of nonsymmetric Macdonald polynomials $E_\mu(x)$ $(\mu \in W.\lambda)$. To be more explicit, one has

$$P_\lambda(x) = \sum_{\mu \in W.\lambda} a_{\lambda\mu} E_\mu(x) \quad \text{with} \quad a_{\lambda\mu} = \prod_{\substack{\alpha \in \Delta^+ \\ \langle \alpha, \mu \rangle < 0}} \frac{1 - q^{\langle \alpha, \mu \rangle} t^{\langle \alpha, \rho(\mu) \rangle - 1}}{1 - q^{\langle \alpha, \mu \rangle} t^{\langle \alpha, \rho(\mu) \rangle}}. \tag{1.18}$$

We also give a remark on the action of the Hecke algebra $H(\mathfrak{S}_n)$ on nonsymmetric Macdonald polynomials: For each $i = 1, \dots n-1$, one has

$$T_i E_\mu(x) = t^{\frac{1}{2}} E_\mu(x) \quad \text{if} \quad \langle \alpha_i, \mu \rangle = 0, \tag{1.19}$$

and

$$T_i E_\mu(x) = u_{i,\mu} E_\mu(x) + v_{i,\mu} E_{s_i\mu}(x) \quad \text{if} \quad \langle \alpha_i, \mu \rangle \neq 0. \tag{1.20}$$

The coefficients $u_{i,\mu}$, $v_{i,\mu}$ are given by

$$u_{i,\mu} = \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - q^{-\langle \alpha_i, \mu \rangle} t^{-\langle \alpha_i, \rho(\mu) \rangle}}, \quad v_{i,\mu} = t^{\frac{1}{2}} \tag{1.21}$$

if $\langle \alpha_i, \mu \rangle < 0$, and

$$\begin{aligned} u_{i,\mu} &= \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - q^{-\langle \alpha_i, \mu \rangle} t^{-\langle \alpha_i, \rho(\mu) \rangle}}, \\ v_{i,\mu} &= t^{-\frac{1}{2}} \frac{(1 - q^{\langle \alpha_i, \mu \rangle} t^{\langle \alpha_i, \rho(\mu) \rangle + 1})(1 - q^{\langle \alpha_i, \mu \rangle} t^{\langle \alpha_i, \rho(\mu) \rangle - 1})}{(1 - q^{\langle \alpha_i, \mu \rangle} t^{\langle \alpha_i, \rho(\mu) \rangle})^2} \end{aligned} \tag{1.22}$$

if $\langle \alpha_i, \mu \rangle > 0$.

§2: Formula of Cauchy type.

It is well-known that the Macdonald polynomials $P_\lambda(x|q,t)$ ($\lambda \in L^+$) have the following formula of Cauchy type [Ma2]. Let now $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two sets of variables, and define the function $\Pi(x;y|q,t)$ by

$$\Pi(x;y|q,t) = \prod_{1\le i,j\le n} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}, \tag{2.1}$$

where $(x;q)_\infty = \prod_{i=0}^{\infty}(1-q^ix)$. The infinite products may be understood either in the sense of formal power series in appropriate variables, or in the analytic sense by assuming that q is a complex number with $0<|q|<1$. Then we have

$$\Pi(x;y|q,t) = \sum_{\lambda\in L^+} b_\lambda(q,t)P_\lambda(x|q,t)P_\lambda(y|q,t), \tag{2.2}$$

where the coefficients are given by

$$b_\lambda(q,t) = \prod_{s\in\lambda} \frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}} \quad (\lambda\in L^+) \tag{2.3}$$

in terms of the arm-length $a(s) = \lambda_i - j$ and the leg-length $l(s) = \lambda'_j - i$ of each box $s = (i,j)$ in the partition λ.

We now introduce the function $E(x;y|q,t)$ by setting

$$E(x;y|q,t) = \prod_{1\le j<i\le n} \frac{(qtx_iy_j;q)_\infty}{(qx_iy_j;q)_\infty} \prod_{1\le i\le n} \frac{(qtx_iy_i;q)_\infty}{(x_iy_i;q)_\infty} \prod_{1\le i<j\le n} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}. \tag{2.4}$$

Note that this function can be factored as follows:

$$E(x;y|q,t) = \Pi(x;y|q,t) \prod_{i=1}^{n} \frac{1}{1-tx_iy_i} \prod_{j<i} \frac{1-x_iy_j}{1-tx_iy_j}. \tag{2.5}$$

The ratio $E(x;y|q,t)\Pi(x;y|q,t)^{-1}$ is essentially one of the rational functions (before symmetrization) which are used in [Mi1] and [KN].

Theorem 2.1. *The function $E(x;y|q,t)$ has an expansion*

$$E(x;y|q,t) = \sum_{\lambda\in L} a_\lambda(q,t)E_\lambda(x|q,t)E_\lambda(y|q^{-1},t^{-1}) \quad (a_\lambda(q,t)\in\mathbb{Q}(q,t)) \tag{2.6}$$

summed over all compositions $\lambda \in L$.

In order to describe the coefficients $a_\lambda(q,t)$ ($\lambda\in L$) in the expansion (2.6), we use the notion of leg-length generalized to compositions, due to Knop and Sahi [KS]. For each box $s = (i,j)$ in a composition $\lambda \in L$, the *generalized leg-length* $l(s) = l_\lambda(s)$ is defined to be the sum

$$l(s) = l_{\mathrm{up}}(s) + l_{\mathrm{low}}(s) \tag{2.7}$$

of the upper and the lower leg-length, where

$$l_{\mathrm{low}}(s) = \#\{k>i\ ;\ j\le\lambda_k\le\lambda_i\}, \quad l_{\mathrm{up}}(s) = \#\{k<i\ ;\ j\le\lambda_k+1\le\lambda_i\}. \tag{2.8}$$

Note that this $l(s)$ is the same as the usual leg-length if λ is a partition.

Theorem 2.2. *For each composition $\lambda \in L$, the coefficient $a_\lambda(q,t)$ in expansion* (2.6) *is given by*

$$a_\lambda(q,t) = \prod_{s\in\lambda} \frac{1-q^{a(s)+1}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}} \quad (\lambda \in L), \tag{2.9}$$

where $l(s) = l_\lambda(s)$ $(s \in \lambda)$ stands for the generalized leg-length in λ.

In this section, we will give a proof of Theorem 2.1. Also, we will describe the ratio of $a_\lambda(q,t)$ and $a_\mu(q,t)$ when λ is a partition and μ is a composition in the orbit $W.\lambda$. Theorem 2.2 will be established in two ways in Sections 3 and 4, by determining $a_\lambda(q,t)$ for partitions $\lambda \in L^+$.

In what follows, we set $K(x;y|q,t) = E(x;y^{-1}|q,t)$, i.e.

$$K(x;y|q,t) = \prod_{1\le j<i\le n} \frac{(qtx_i/y_j;q)_\infty}{(qx_i/y_j;q)_\infty} \prod_{1\le i\le n} \frac{(qtx_i/y_i;q)_\infty}{(x_i/y_i;q)_\infty} \prod_{1\le i<j\le n} \frac{(tx_i/y_j;q)_\infty}{(x_i/y_j;q)_\infty}. \tag{2.10}$$

Proposition 2.3. *For each $i = 1,\dots,n$, one has*

$$Y_{i,x} K(x;y|q,t) = (Y^*_{i,y})^{-1} K(x;y|q,t), \tag{2.11}$$

where the suffix x or y indicates the variables on which the operator should act, and Y_i^* is the dual q-Dunkl operator (cf. [KN]) *defined by*

$$Y_i^* = T_i^{-1}\cdots T_{n-1}^{-1}\omega T_1\cdots T_{i-1}. \tag{2.12}$$

This proposition is a direct consequence of the following lemma.

Lemma 2.4.
(1) $T_{i,x}K(x;y|q,t) = T_{i,y}K(x;y|q,t) \quad (i = 1,\dots,n-1)$.
(2) $\omega_x K(x;y|q,t) = \omega_y^{-1} K(x;y|q,t)$.

Note that the function $K(x;y|q,t)$ can be factored as follows:

$$K(x;y|q,t) = \Pi(x;y^{-1}|q,t)\psi(x,y), \quad \psi(x,y) = \prod_{i=1}^{n}\left(\frac{1}{1-tx_i/y_i}\prod_{j<i}\frac{1-x_i/y_j}{1-tx_i/y_j}\right). \tag{2.13}$$

Since $\Pi(x;y^{-1}|q,t)$ is symmetric both in x and in y, the formula of Lemma 2.4.(1) is equivalent to

$$T_{i,x}\psi(x,y) = T_{i,y}\psi(x,y) \quad (i = 1,\dots,n-1). \tag{2.14}$$

For a fixed i, it reduces to the identity

$$T_{i,x}\psi_{i,i+1}(x,y) = T_{i,y}\psi_{i,i+1}(x,y) \tag{2.15}$$

for

$$\psi_{i,i+1}(x,y) = \frac{1-x_{i+1}/y_i}{(1-tx_i/y_i)(1-tx_{i+1}/y_i)(1-tx_{i+1}/y_{i+1})}, \tag{2.16}$$

which can be checked by a direct computation. (This computation is essentially contained in [Mi1], [MN]). The formula of Lemma 2.4.(2) can be proved directly by chasing the arguments of q-shift factorials under the action $\omega_y\omega_x$.

Proof of Theorem 2.1. We begin with expanding the function $E(x;y|q,t)$ in the form

$$E(x;y|q,t)=\sum_{\lambda\in L}E_\lambda(x|q,t)F_\lambda(y|q,t)\quad (F_\lambda(y|q,t)\in\mathbb{Q}(q,t)[y]). \tag{2.17}$$

We will show that each $F_\lambda(x|q,t)$ is a constant multiple of $E_\lambda(y|q^{-1},t^{-1})$. Note that Proposition 2.3 implies

$$Y_x^{-\mu}K(x;y|q,t)=(Y_y^*)^\mu K(x;y|q,t) \tag{2.18}$$

for any $\mu\in P$. Since

$$K(x;y|q,t)=\sum_{\lambda\in L}E_\lambda(x|q,t)F_\lambda(y^{-1}|q,t), \tag{2.19}$$

we have

$$Y_y^{*\mu}F_\lambda(y^{-1}|q,t)=F_\lambda(y^{-1}|q,t)q^{-\langle\mu,\lambda\rangle}t^{-\langle\mu,\rho(\lambda)\rangle}. \tag{2.20}$$

As is shown in [KN], for each $i=1,\dots,n$ the q-Dunkl operator Y_i and its dual Y_i^* are interchanged by the involution ι on $\mathbb{K}[y]$ such that $\iota(y_j)=y_j^{-1}$ $(j=1,\dots,n)$, $\iota(q)=q^{-1}$, $\iota(t^{\frac12})=t^{-\frac12}$. Hence we have

$$Y_y^{\mu}F_\lambda(y|q^{-1},t^{-1})=F_\lambda(y|q^{-1},t^{-1})q^{\langle\mu,\lambda\rangle}t^{\langle\mu,\rho(\lambda)\rangle} \tag{2.21}$$

for all $\mu\in P$. This implies that $F_\lambda(y|q^{-1},t^{-1})$ is a constant multiple of $E_\lambda(y|q,t)$. Namely, $F_\lambda(y|q,t)$ is a constant multiple of $E_\lambda(y|q^{-1},t^{-1})$. $\square$

Before determining the coefficients $a_\lambda(q,t)$, we will describe the relation between $a_\lambda(q,t)$ and $a_\mu(q,t)$ when λ is dominant and μ is in the orbit $W.\lambda$.

Lemma 2.5. *If $\lambda\in L^+$, $\mu\in L$ and $\mu\in W.\lambda$, then one has*

$$a_\mu(q,t)=a_\lambda(q,t)\prod_{\substack{\alpha\in\Delta^+\\ \langle\alpha,\mu\rangle<0}}\frac{(1-q^{-\langle\alpha,\mu\rangle}t^{-\langle\alpha,\rho(\mu)\rangle+1})(1-q^{-\langle\alpha,\mu\rangle}t^{-\langle\alpha,\rho(\mu)\rangle-1})}{(1-q^{-\langle\alpha,\mu\rangle}t^{-\langle\alpha,\rho(\mu)\rangle})^2} \tag{2.22}$$

Proof. By Lemma 2.4.(1), we have

$$\sum_{\mu\in L}a_\mu T_{i,x}E_\mu(x)\iota(E_\mu(y))=\sum_{\mu\in L}a_\mu E_\mu(x)T_{i,y}\iota(E_\mu(y)), \tag{2.23}$$

for each $i=1,\dots,n-1$, where $a_\mu=a_\mu(q,t)$. As we remarked at the end of Section 2, for each $\mu\in L$, we have $T_{i,x}E_\mu(x)=t^{\frac12}E_\mu(x)$ if $\langle\alpha_i,\mu\rangle=0$, and

$$T_{i,x}E_\mu(x)=u_{i,\mu}E_\mu(x)+v_{i,\mu}E_{s_i\mu}(x). \tag{2.24}$$

when $\langle \alpha_i, \mu \rangle \neq 0$. Since $T_{i,y}\iota(E_\mu(y)) = \iota(T_{i,y}^{-1}E_\mu(y))$, and $T_{i,y}^{-1} = T_{i,y} - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$, we can determine the action of $T_{i,y}$ on $\iota(E_\mu(y))$ as follows: $T_{i,\mu}\iota(E_\mu(y)) = t^{\frac{1}{2}}\iota(E_\mu(y))$ if $\langle \alpha_i, \mu \rangle = 0$, and

$$T_{i,y}\iota(E_\mu(y)) = (\iota(u_{i,\mu}) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}))\iota(E_\mu(y)) + \iota(v_{i,\mu})\iota(E_{s_i\mu}(y)) \tag{2.25}$$

if $\langle \alpha_i, \mu \rangle \neq 0$. By substituting these formulas into (2.23), we obtain the recurrence formulas

$$a_{s_i\mu}v_{s_i\mu} = a_\mu \iota(v_\mu) \tag{2.26}$$

for $\mu \in L$ with $\langle \alpha_i, \mu \rangle \neq 0$. Hence, by (2.21) and (2.22), we have

$$a_\mu = a_{s_i\mu}\frac{(1 - q^{-\langle \alpha_i, \mu\rangle}t^{-\langle \alpha_i, \rho(\mu)\rangle + 1})(1 - q^{-\langle \alpha_i, \mu\rangle}t^{-\langle \alpha_i, \rho(\mu)\rangle - 1})}{(1 - q^{-\langle \alpha_i, \mu\rangle}t^{-\langle \alpha_i, \rho(\mu)\rangle})^2} \tag{2.27}$$

for $\mu \in L$ with $\langle \alpha_i, \mu \rangle < 0$. Assume now that $\lambda \in L^+$ is a partition and $\mu \in W.\lambda$. Then one can find a sequence of simple roots $\alpha_{j_1}, \dots, \alpha_{j_p}$ such that $\mu = s_{j_1} \cdots s_{j_p}(\lambda)$ and that

$$\langle \alpha_{j_1}, \mu \rangle < 0, \langle \alpha_{j_2}, s_{j_1}(\mu) \rangle < 0, \dots, \langle \alpha_{j_p}, s_{j_{p-1}} \cdots s_{j_1}(\mu) \rangle < 0. \tag{2.28}$$

Note also that

$$\{\alpha \in \Delta^+ \ ; \ \langle \alpha, \mu \rangle < 0\} = \{\alpha_{j_1}, s_{j_1}(\alpha_{j_2}), \dots, s_{j_1} \cdots s_{j_{p-1}}(\alpha_{j_p})\}. \tag{2.29}$$

Applying formula (2.27) to $\mu^{(0)} = \mu, \mu^{(1)} = s_{j_1}(\mu), \cdots, \mu^{(p)} = s_{j_p} \cdots s_{j_1}(\mu) = \lambda$ repeatedly, we obtain formula (2.22) since $\langle \alpha_{j_r}, \mu^{(r-1)} \rangle = \langle s_{j_1} \cdots s_{j_{r-1}}(\alpha_{j_r}), \mu \rangle$ and $\langle \alpha_{j_r}, \rho(\mu^{(r-1)}) \rangle = \langle s_{j_1} \cdots s_{j_{r-1}}(\alpha_{j_r}), \rho(\mu) \rangle$ for $r = 1, \dots, p$. $\square$

Lemma 2.5 can be rewritten in the combinatorial language. Imitating Sahi's notation [S], we set

$$d_\mu(q,t) = \prod_{s \in \mu}(1 - q^{a(s)+1}t^{l(s)+1}), \quad d'_\mu(q,t) = \prod_{s \in \mu}(1 - q^{a(s)+1}t^{l(s)}) \tag{2.30}$$

for each $\mu \in L$. In this notation, Theorem 2.2 is equivalent to the equality $a_\mu(q,t) = d_\mu(q,t)/d'_\mu(q,t)$.

Lemma 2.6. *If $\lambda \in L^+$, $\mu \in L$ and $\mu \in W.\lambda$, then formula* (2.22) *has an alternative expression*

$$a_\mu(q,t) = a_\lambda(q,t)\frac{d'_\lambda(q,t)d_\mu(q,t)}{d_\lambda(q,t)d'_\mu(q,t)}. \tag{2.31}$$

Proof. For each $\mu \in L$, set $a'_\mu = a'_\mu(q,t) = d_\mu(q,t)/d'_\mu(q,t)$. For the proof of formula (2.31), it is enough to show

$$a_\mu(q,t) = a_{s_i\mu}(q,t)\frac{a'_\mu(q,t)}{a'_{s_i\mu}(q,t)} \tag{2.32}$$

assuming that $\langle\alpha_i,\mu\rangle<0$ ($i=1,\dots,n-1$); one can use (2.32) repeatedly to prove (2.31) in view of the expression $\mu=s_{j_1}\cdots s_{j_p}(\lambda)$ as in the proof of Lemma 2.5. When $\langle\alpha_i,\mu\rangle<0$, it is easy to see that the only difference between $a'_\mu(q,t)$ and $a'_{s_i\mu}(q,t)$ arises at the box $s=(i+1,\mu_i+1)\in\mu$ (or at $s'=(i,\mu_i+1)\in s_i\mu$). In fact we have

$$a'_\mu=a'_{s_i\mu}\frac{(1-q^{\mu_{i+1}-\mu_i}t^{l_\mu(s)+1})(1-q^{\mu_{i+1}-\mu_i}t^{l_\mu(s)-1})}{(1-q^{\mu_{i+1}-\mu_i}t^{l_\mu(s)})^2}. \tag{2.33}$$

On the other hand, one can directly check that

$$-\langle\alpha_i,\mu\rangle=\mu_{i+1}-\mu_i,\quad -\langle\alpha_i,\rho(\mu)\rangle=l_\mu(s) \tag{2.34}$$

by the definition (1.13) of $\rho(\mu)$. Hence we have (2.32) by comparing (2.27) and (2.33). □

§3: First proof of Theorem 2.2.

In this section, we calculate the coefficients $a_\lambda=a_\lambda(q,t)$ for partitions $\lambda\in L^+$ by means of asymptotic analysis of a q-Selberg type integral. Such an argument has been employed in [Mi2] to evaluate the scalar product for Macdonald's symmetric polynomials.

We now assume that q and t are complex numbers with $0<|q|,|t|<1$, and recall Cherednik's scalar product [C1]. For $f=f(x|q,t), g=g(x|q,t)\in\mathbb{K}[x]=\mathbb{K}[x_1,\dots,x_n]$, we define

$$(f,g)=\left(\frac{1}{2\pi\sqrt{-1}}\right)^n\int_{\mathbb{T}^n}f(x|q,t)g(x^{-1}|q^{-1},t^{-1})w(x|q,t)\frac{dx_1\cdots dx_n}{x_1\cdots x_n}\,, \tag{3.1}$$

where

$$w(x|q,t)=\prod_{1\le i<j\le n}\frac{(x_i/x_j;q)_\infty(qx_j/x_i;q)_\infty}{(tx_i/x_j;q)_\infty(qtx_j/x_i;q)_\infty} \tag{3.2}$$

and $\mathbb{T}^n=\{(x_1,\dots,x_n)\in\mathbb{C}^n;|x_i|=1\ (1\le i\le n)\}$ with the standard orientation. Note that

$$w(x|q,q^k)=\prod_{1\le i<j\le n}(x_i/x_j;q)_k(qx_j/x_i;q)_k \tag{3.3}$$

if $t=q^k$ ($k=0,1,2,\dots$), where $(a;q)_k=(a;q)_\infty/(at;q)_\infty=\prod_{0\le i\le k-1}(1-aq^i)$. The nonsymmetric Macdonald polynomials $E_\lambda(x;q,t)$ ($\lambda\in L$) form an orthogonal basis of $\mathbb{K}[x]$ with respect to this scalar product:

$$(E_\lambda,E_\mu)=0\quad\text{if}\quad\lambda\ne\mu\,. \tag{3.4}$$

It is known furthermore that, if $t=q^k$ ($k\in\mathbb{N}$) and $\lambda\in L^+$ is a partition, then

$$(E_\lambda,E_\lambda)=\prod_{1\le i<j\le n}\frac{(q^{\lambda_i-\lambda_j+1+k(j-i)};q)_k}{(q^{\lambda_i-\lambda_j+1+k(j-i-1)};q)_k}. \tag{3.5}$$

(See [Ma1], [C1].) For general values of t with $|t|<1$, one has

$$(E_\lambda,E_\lambda)=\prod_{1\le i<j\le n}\frac{(q^{\lambda_i-\lambda_j+1}t^{j-i};q)_\infty^2}{(q^{\lambda_i-\lambda_j+1}t^{j-i+1};q)_\infty(q^{\lambda_i-\lambda_j+1}t^{j-i-1};q)_\infty} \tag{3.6}$$

for any partition $\lambda\in L^+$. Note that, as functions in t, the both sides of (3.6) are meromorphic in $\{|t|<1\}$. Since this formula is valid at the points $t=q^k$ ($k\in\mathbb{N}$) accumulating at the origin, one can conclude that the left hand side of (3.6) is eventually holomorphic near $t=0$, and that (3.6) is valid as an equality of analytic functions. It is also known that, if $\mu\in W.\lambda$ is a composition in the W-orbit of a partition λ, then we have

$$\frac{(E_\mu,E_\mu)}{(E_\lambda,E_\lambda)}=\prod_{\substack{\alpha\in\Delta^+\\ \langle\alpha,\mu\rangle<0}}\frac{(1-q^{-\langle\alpha,\mu\rangle}t^{-\langle\alpha,\rho(\mu)\rangle})^2}{(1-q^{-\langle\alpha,\mu\rangle}t^{-\langle\alpha,\rho(\mu)\rangle+1})(1-q^{-\langle\alpha,\mu\rangle}t^{-\langle\alpha,\rho(\mu)\rangle-1})}. \tag{3.7}$$

We now consider $x=(x_1,\ldots,x_n)$ as variables inside the polydisc $\{|x_j|<1\,;\ j=1,\ldots,n\}\subset\mathbb{C}^n$. Note that the series expansion

$$K(x;y|q,t)=\sum_{\mu\in L}a_\mu(q,t)E_\mu(x|q,t)E_\mu(y^{-1}|q^{-1},t^{-1})\,, \tag{3.8}$$

in Theorem 2.1 is then uniformly convergent on $\mathbb{T}^n$ in y. Hence by the orthogonality relations (3.4) we have

$$\begin{aligned}&\left(\frac{1}{2\pi\sqrt{-1}}\right)^n\int_{\mathbb{T}^n}K(x;y|q,t)E_\lambda(y|q,t)w(y|q,t)\frac{dy_1\cdots dy_n}{y_1\cdots y_n}\\ &=\sum_{\mu\in L}a_\mu(q,t)E_\mu(x|q,t)(E_\lambda,E_\mu)\\ &=a_\lambda(q,t)E_\lambda(x|q,t)(E_\lambda,E_\lambda)\,.\end{aligned} \tag{3.9}$$

We study the asymptotic behavior of the left hand side of (3.9) in x in the region

$$1>|x_1|\gg|x_2|\gg\cdots\gg|x_n|\,. \tag{3.10}$$

For the moment, we assume that $t=q^k$ ($k=0,1,2,\ldots$) and that λ is a partition.

Proposition 3.1. *If $t=q^k$ ($k\in\mathbb{N}$) and $\lambda\in L^+$, then one has*

(3.11)
$$\begin{aligned}&\left(\frac{1}{2\pi\sqrt{-1}}\right)^n\int_{\mathbb{T}^n}K(x;y|q,q^k)E_\lambda(y|q,q^k)w(y|q,q^k)\frac{dy_1\cdots dy_n}{y_1\cdots y_n}\\ &\sim\quad x^\lambda\ \prod_{i=1}^n\frac{(q^{\lambda_i+(n-i)k+1};q)_k}{(q;q)_k},\end{aligned}$$

as $\max\{|x_2/x_1|,|x_3/x_2|,\ldots,|x_n/x_{n-1}|\}$ *tends to* 0.

Proposition will be proved later in this section.

We apply Proposition 3.1 to compare the coefficients of x^λ in equality (3.9). Since we know by (3.9) that the integral has the asymptotic behavior $x^\lambda a_\lambda(q,t)(E_\lambda,E_\lambda)+\cdots$, we obtain

$$a_\lambda(q,q^k)\,(E_\lambda,E_\lambda)=\prod_{i=1}^{n}\frac{(q^{\lambda_i+(n-i)k+1};q)_k}{(q;q)_k} \tag{3.12}$$

for any partition $\lambda\in L^+$, provided that $t=q^k$ ($k\in\mathbb{N}$). This is equivalent to

$$\begin{aligned} a_\lambda(q,q^k)&=\prod_{1\le i\le j\le n}\frac{(q^{\lambda_i-\lambda_{j+1}+1+k(j-i)};q)_k}{(q^{\lambda_i-\lambda_j+1+k(j-i)};q)_k}\\ &=\prod_{i=1}^{n}\prod_{j=i}^{n}\frac{(q^{\lambda_i-\lambda_j+1+k(j-i+1)};q)_{\lambda_j-\lambda_{j+1}}}{(q^{\lambda_i-\lambda_j+1+k(j-i)};q)_{\lambda_j-\lambda_{j+1}}}\end{aligned} \tag{3.13}$$

by formula (3.5), where we set $\lambda_{n+1}=0$. By analytic continuation as before, we get

$$a_\lambda(q,t)=\prod_{i=1}^{n}\prod_{j=i}^{n}\frac{(q^{\lambda_i-\lambda_j+1}\,t^{j-i+1};q)_{\lambda_j-\lambda_{j+1}}}{(q^{\lambda_i-\lambda_j+1}\,t^{j-i};q)_{\lambda_j-\lambda_{j+1}}}, \tag{3.14}$$

since $a_\lambda(q,t)$ is a rational function in t. Formula (3.14) implies that $a_0=1$ and that

$$a_{\lambda+(1^m)}(q,t)=a_\lambda(q,t)\prod_{i=1}^{m}\frac{1-q^{\lambda_i+1}t^{m-i+1}}{1-q^{\lambda_i+1}t^{m-i}} \tag{3.15}$$

for any $\lambda\in L^+$ with $l(\lambda)\le m$. In fact, the difference between $a_\lambda(q,t)$ and $a_{\lambda+(1)^m}(q,t)$ appears only in the factors in (3.14) with $1\le i\le m$ and $j=m$. From (3.15) it follows immediately that

$$a_\lambda(q,t)=\prod_{s\in\lambda}\frac{1-q^{a(s)+1}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}}=\frac{d_\lambda(q,t)}{d'_\lambda(q,t)} \tag{3.16}$$

for all $\lambda\in L^+$. Hence by Lemma 2.6, the same formula (3.16) holds for all compositions $\lambda\in L$ if $l(s)$ is understood as the generalized leg-length. This completes the proof of Theorem 2.2.

Note that formula (3.12) extends to the equality

$$a_\lambda(q,t)\,(E_\lambda,E_\lambda)=\prod_{i=1}^{n}\frac{(q^{\lambda_i+1}t^{n-i};q)_\infty(qt;q)_\infty}{(q^{\lambda_i+1}t^{n-i+1};q)_\infty(q;q)_\infty}\quad(\lambda\in L^+) \tag{3.17}$$

of analytic functions in t. On the other hand, by comparing formula (2.22) of Lemma 2.5 and (3.7), we see that

$$a_\mu(q,t)\,(E_\mu,E_\mu)=a_\lambda(q,t)\,(E_\lambda,E_\lambda) \tag{3.18}$$

for all compositions $\mu\in W.\lambda$. Summarizing these remarks, we obtain the following theorem.

Theorem 3.2. *For each composition $\lambda \in \mathbb{N}^n$, we have*

$$(3.19)\quad \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{\mathbb{T}^n} K(x;y|q,t)E_\lambda(y|q,t)w(y|q,t)\frac{dy_1\cdots dy_n}{y_1\cdots y_n} = C_\lambda . E_\lambda(x|q,t)$$

for $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ with $|x_j| < 1$ $(j = 1, \ldots, n)$. Here C_λ is constant on each W-orbit, and is given by

$$(3.20)\qquad C_\lambda = \left(\frac{(qt;q)_\infty}{(q;q)_\infty}\right)^n \prod_{i=1}^{n} \frac{(q^{\lambda_i+1}t^{n-i};q)_\infty}{(q^{\lambda_i+1}t^{n-i+1};q)_\infty}$$

when $\lambda \in L^+$ is a partition.

In this sense, our function $K(x;y|q,t)$ is a reproducing kernel for nonsymmetric Macdonald polynomials.

In the rest of this section, we will prove Proposition 3.1. From now on we assume that $t = q^k$ for a fixed $k \in \mathbb{N}$, and omit $(q,t) = (q,q^k)$ in the notation unless it might lead to confusion.

In proving Proposition, we may assume that $0 < |x_j| < 1$ for $j = 1\ldots, n$ and that all x_j's are mutually distinct. Recall that

$$(3.21)\qquad \begin{aligned} K(x;y) &= \prod_{j<i} \frac{1}{(qx_i/y_j;q)_k} \prod_{i} \frac{1}{(x_i/y_i;q)_{k+1}} \prod_{i<j} \frac{1}{(x_i/y_j;q)_k} \\ &= \prod_{i,j} \frac{1}{(q^{\theta(i>j)}x_i/y_j;q)_{k+\delta_{ij}}}, \end{aligned}$$

where $\theta(i > j) = 1$ if $i > j$, and $\theta(i > j) = 0$ if $i \le j$, and

$$(3.22)\qquad w(y) = \prod_{i<j} (y_i/y_j;q)_k (qy_j/y_i)_k.$$

Note first that, as a function of y_j $(1 \le j \le n)$, the integrand

$$(3.23)\qquad K(x;y)E_\lambda(y)w(y)(y_1\cdots y_n)^{-1}$$

of (3.11) is regular at $y_j = 0$ and has poles only at $y_j = x_s q^l$ $(1 \le s \le n, l \in \mathbb{N})$. The range of l can be specified as follows:

$$(3.24)\qquad \begin{array}{llll} (1) & 0 \le l < k & \text{if} & 1 \le s < j, \\ (2) & 0 \le l \le k & \text{if} & s = j, \qquad \text{and} \\ (3) & 0 < l \le k & \text{if} & j < s \le n. \end{array}$$

The integral (3.11) will be computed by picking up successively the residues at $y_j = x_s q^l$ $(1 \le j, s \le n)$ with l satisfying (3.24). To make clear this inductive process, we propose a lemma.

For any pair (I, J) of subsets of $\{1, \ldots, n\}$ with $|I| = |J|$, we extend the notation of (3.21), (3.23) as follows:

$$(3.25)\quad K(x_I;y_J) = \prod_{\substack{i\in I\\ j\in J}} \frac{1}{(q^{\theta(i>j)}x_i/y_j;q)_{k+\delta_{ij}}}, \quad w(y_J) = \prod_{\substack{i,j\in J\\ i<j}} (y_i/y_j;q)_k (qy_j/y_i)_k,$$

where $x_I = (x_i)_{i\in I}$ and $y_J = (y_j)_{j\in J}$.

Lemma 3.3. *Fix two indices $j \in J$, $s \in I$ and $l \in \mathbb{N}$ satisfying (3.24), and set $J' = J\backslash\{j\}$ and $I' = I\backslash\{s\}$. Then the residue*

$$f(x_I; y_{J'}) = \mathrm{Res}_{y_j = x_s q^l}(K(x_I; y_J) w(y_J) \frac{dy_j}{y_j}) \tag{3.26}$$

at $y_j = x_s q^l$ has an expression

$$f(x_I; y_{J'}) = g(x_I; y_{J'}) K(x_{I'}; y_{J'}) w(y_{J'}), \tag{3.27}$$

where $g(x_I; y_{J'})$ is a polynomial in $y_{J'}$ with coefficients in $\mathbb{K}(x_I)$.

Proof. The only factors to be checked are

$$\frac{(q^{-l} y_\mu / x_s; q)_k (q^{l+1} x_s / y_\mu; q)_k}{(q^{\theta(s>\mu)} x_s / y_\mu; q)_{k+\delta_{s\mu}}} \tag{3.28}$$

for $\mu \in J'$ with $\mu < j$, and

$$\frac{(q^{l} x_s / y_\mu; q)_k (q^{1-l} y_\mu / x_s; q)_k}{(q^{\theta(s>\mu)} x_s / y_\mu; q)_{k+\delta_{s\mu}}} \tag{3.29}$$

for $\mu \in J'$ with $\mu > j$. As a function of y_μ, the numerators of (3.28) and (3.29) have zeros at $y_\mu = x_s q^m$ for $m \in \{l-k+1, l-k+2, \cdots, l+k\}$ and for $m \in \{l-k, l-k+1, \cdots, l+k-1\}$, respectively. From this, it turns out that the rational functions (3.28) and (3.29) are in fact regular at $y_\mu = x_s q^m$ for all $m \in \mathbb{N}$, provided that l satisfies the condition of (3.24). $\square$

Let us now integrate $K(x; y) E_\lambda(y) w(y)/y_1$ with respect to the variable y_1. Then we have

$$\begin{aligned} &\frac{1}{2\pi\sqrt{-1}} \int_{|y_1|=1} K(x; y) E_\lambda(y) w(y) \frac{dy_1}{y_1} \\ &= \sum_{s=1}^{n} \sum_{l=0}^{k} \mathrm{Res}_{y_1 = x_s q^l}(K(x; y) E_\lambda(y) w(y) \frac{dy_1}{y_1}) \end{aligned} \tag{3.30}$$

since the integrand is regular at $y_1 = 0$. If we regard each summand of the right-hand side as a function of y_2, it has a zero at $y_2 = 0$ and has poles only at $y_2 = x_r q^m$ for $r \neq s$, $0 \leq m \leq k$ by Lemma 3.3. Hence we have

$$\begin{aligned} &\left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_{|y_1|=|y_2|=1} K(x; y) E_\lambda(y) w(y) \frac{dy_1 dy_2}{y_1 y_2} \\ &= \sum_{\substack{1 \leq s, r \leq n \\ s \neq r}} \sum_{0 \leq l, m \leq k} \mathrm{Res}_{\substack{y_1 = x_s q^l \\ y_2 = x_r q^m}} (K(x; y) E_\lambda(y) w(y) \frac{dy_1 dy_2}{y_1 y_2}). \end{aligned} \tag{3.31}$$

Applying Lemma 3.3 repeatedly from $j = 1$ to n, we obtain the equality

$$\begin{aligned} &\left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{\mathbb{T}^n} K(x; y) E_\lambda(y) w(y) \frac{dy}{y} \\ &= \sum_{\sigma \in \mathfrak{S}_n} \sum_{0 \leq l_1, \cdots, l_n \leq k} \mathrm{Res}_{y = (x_{\sigma(1)} q^{l_1}, \ldots, x_{\sigma(n)} q^{l_n})} \left(K(x; y) E_\lambda(y) w(y) \frac{dy}{y}\right), \end{aligned} \tag{3.32}$$

where we used the abbreviation $dy/y = dy_1 \cdots dy_n / y_1 \cdots y_n$.

We now investigate the asymptotic behavior of this function in the region (3.10). Note that we have

$$\begin{aligned} &\mathrm{Res}_{y=(x_{\sigma(1)}q^{l_1},\dots,x_{\sigma(n)}q^{l_n})}\left(K(x;y)E_\lambda(y)w(y)\frac{dy}{y}\right) \\ &= \mathrm{Res}_{y=(q^{l_1},\dots,q^{l_n})}\left(K(x;x_\sigma y)E_\lambda(x_\sigma y)w(x_\sigma y)\frac{dy}{y}\right), \end{aligned} \tag{3.33}$$

where $x_\sigma y$ stands for $(x_{\sigma(1)}y_1, \dots, x_{\sigma(n)}y_n)$. The function $K(x; x_\sigma y)w(x_\sigma y)$ can be written in the following form:

(3.34)

$$\prod_{i,j} \frac{1}{(q^{\theta(i\geq j)} x_i / x_{\sigma(j)} y_j; q)_k} \prod_{i\neq j} (q^{\theta(i>j)} x_{\sigma(i)} y_i / x_{\sigma(j)} y_j; q)_k \times \prod_{j=1}^{n} \frac{1}{1 - x_j / x_{\sigma(j)} y_j}.$$

The product of the first two factors altogether is bounded in the limit (3.10). If $\sigma \in \mathfrak{S}_n$ is *not* the identity element, one can take a suffix i such that $i < \sigma(i)$. As an effect of the factor $1/(1 - x_i/x_{\sigma(i)}y_i)$ in the third factors of (3.34), we then have

$$|K(x; x_\sigma y)w(x_\sigma y)| = O\left(\left|\frac{x_{\sigma(i)}}{x_i}\right|\right). \tag{3.35}$$

Since λ is a partition, $x^{-\lambda}E_\lambda(x_\sigma y)$ is also bounded in the limit (3.10). Hence we have

$$x^{-\lambda}\mathrm{Res}_{y=(q^{l_1},\dots,q^{l_n})}\left(K(x;x_\sigma y)E_\lambda(x_\sigma y)w(x_\sigma y)\frac{dy}{y}\right) = O\left(\left|\frac{x_{\sigma(i)}}{x_i}\right|\right), \tag{3.36}$$

provided that σ is not the identity element. If σ is the identity element, the function

$$\mathrm{Res}_{y=(q^{l_1},\dots,q^{l_n})}\left(K(x;xy)E_\lambda(xy)w(xy)\frac{dy}{y}\right) \tag{3.37}$$

tends to

(3.38)

$$\begin{aligned} &\mathrm{Res}_{y=(q^{l_1},\dots,q^{l_n})}\left(\prod_{i=1}^{n} \frac{y_i^{(n-i)k}}{(1/y_i;q)_{k+1}} \left\{(x_1y_1)^{\lambda_1}\cdots(x_ny_n)^{\lambda_n} + \cdots\right\} \frac{dy}{y}\right) \\ &= x^\lambda \ \mathrm{Res}_{y=(q^{l_1},\dots,q^{l_n})}\left(\prod_{i=1}^{n} \frac{y_i^{(n-i)k+\lambda_i}}{(1/y_i;q)_{k+1}} \frac{dy_1\cdots dy_n}{y_1\cdots y_n}\right) + \text{lower terms} \\ &= x^\lambda \ \prod_{i=1}^{n}\left\{\frac{(q^{-k};q)_{l_i}}{(q;q)_k(q;q)_{l_i}}(q^{\lambda_i+(n-i+1)k+1})^{l_i}\right\} + \text{lower terms}. \end{aligned}$$

Combining (3.32) with (3.36) and (3.38), we obtain

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{\mathbb{T}^n} K(x;y)E_\lambda(y)w(y)\frac{dy}{y} = C_\lambda x^\lambda + \text{lower terms} \tag{3.39}$$

in the region (3.10), with the leading coefficient

$$C_\lambda = \sum_{0\le l_1,\dots,l_n\le k} \prod_{i=1}^n \left\{ \frac{(q^{-k};q)_{l_i}}{(q;q)_k (q;q)_{l_i}} (q^{\lambda_i+(n-i+1)k+1})^{l_i} \right\}. \tag{3.40}$$

By using the q-binomial theorem

$$\sum_{l\ge 0} \frac{(a;q)_l}{(q;q)_l} z^l = \frac{(az;q)_\infty}{(z;q)_\infty} \quad (|z|<1)\,, \tag{3.41}$$

the constant C_λ is determined as

$$C_\lambda = \prod_{i=1}^n \frac{(q^{\lambda_i+(n-i)k+1};q)_k}{(q;q)_k}. \tag{3.42}$$

This completes the proof of Proposition 3.1.

§4: Second proof of Theorem 2.2.

In this section, we will give a proof of Theorem 2.2 based on the principal specialization of nonsymmetric Macdonald polynomials, along the line similar to that in Sahi [S].

We begin with a lemma.

Lemma 4.1. *The function $\prod_{i=1}^n (ux_i;q)_\infty/(x_i;q)_\infty$ has an expansion*

$$\prod_{i=1}^n \frac{(ux_i;q)_\infty}{(x_i;q)_\infty} = \sum_{\lambda\in L^+} P_\lambda(x|q,t) f_\lambda(u|q,t), \tag{4.1}$$

in terms of Macdonald polynomials. The coefficients are given by

$$f_\lambda(u|q,t) = t^{n(\lambda)} \prod_{s\in\lambda} \frac{1-q^{a'(s)}t^{-l'(s)}u}{1-q^{a(s)+1}t^{l(s)}} \tag{4.2}$$

for each partition λ. Here, for each box $s=(i,j)$ in λ, $a'(s)=j-1$ and $l'(s)=i-1$ stand for the coarm-length and the coleg-length of s in λ, and $n(\lambda)=\sum_{s\in\lambda} l(s) = \sum_{s\in\lambda} l'(s)$.

Proof. Let m be an integer with $m\ge n$ and take the variables $y=(y_1,\dots,y_m)$. Then we have

$$\prod_{i=1}^n \prod_{j=1}^m \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty} = \sum_{\lambda\in L^+} b_\lambda(q,t) P_\lambda(x|q,t) P_\lambda(y|q,t). \tag{4.3}$$

By the evaluation at $y=(1,t,\dots,t^{m-1})$, we get

$$\prod_{i=1}^n \frac{(t^m x_i;q)_\infty}{(x_i;q)_\infty} = \sum_{\lambda\in L^+} b_\lambda(q,t) P_\lambda(x|q,t) P_\lambda(1,t,\dots,t^{m-1}|q,t). \tag{4.4}$$

Hence we have

$$f_\lambda(t^m|q,t) = b_\lambda(q,t) P_\lambda(1,t,\ldots,t^{m-1}|q,t). \tag{4.5}$$

It is known by [Ma2] that

$$P_\lambda(1,t,\ldots,t^{m-1}|q,t) = t^{n(\lambda)} \prod_{s\in\lambda} \frac{1-q^{a'(s)}t^{m-l'(s)}}{1-q^{a(s)}t^{l(s)+1}}. \tag{4.6}$$

Combining this with the formula for $b_\lambda(q,t)$, we obtain

$$f_\lambda(t^m|q,t) = t^{n(\lambda)} \prod_{s\in\lambda} \frac{1-q^{a'(s)}t^{m-l'(s)}}{1-q^{a(s)+1}t^{l(s)}}. \tag{4.7}$$

Since $f_\lambda(u|q,t)$ is a polynomial in u, and $m \geq n$ is arbitrary, we obtain the desired formula. $\square$

We will prove Theorem 2.2 by evaluating of the function $E(x;y|q,t)$ at the point $y = t^\delta = (t^{n-1}, t^{n-2}, \ldots, 1)$. From the definition of $E(x;y|q,t)$, we easily see that

$$E(x;t^\delta|q,t) = \prod_{i=1}^{n} \frac{(qt^n x_i;q)_\infty}{(x_i;q)_\infty}. \tag{4.8}$$

By Lemma 4.1, we can expand this into the sum over all $P_\lambda(x|q,t)$:

$$E(x;t^\delta|q,t) = \sum_{\lambda\in L^+} P_\lambda(x|q,t) f_\lambda(qt^n|q,t). \tag{4.9}$$

For each partition $\lambda \in L^+$, Macdonald's symmetric polynomial $P_\lambda(x|q,t)$ can be written as a linear combination of nonsymmetric ones $E_\mu(x|q,t)$, summed over all compositions μ in the W-orbit $W.\lambda$ (see [Ma1]) :

$$P_\lambda(x|q,t) = \sum_{\mu\in W.\lambda} a_{\lambda\mu}(q,t) E_\mu(x|q,t) \quad (a_{\lambda\mu}(q,t) \in \mathbb{Q}(q,t)) \tag{4.10}$$

with $a_{\lambda\lambda}(q,t) = 1$. Hence we have

$$E(x;t^\delta|q,t) = \sum_{\mu\in L} f_{\mu^+}(qt^n|q,t) a_{\mu^+\mu}(q,t) E_\mu(x|q,t). \tag{4.11}$$

On the other hand, by Theorem 2.1.(1), $E(x;y|q,t)$ has the expansion

$$E(x;y|q,t) = \sum_{\mu\in L} a_\mu(q,t) E_\mu(x|q,t) E_\mu(y|q^{-1},t^{-1}). \tag{4.12}$$

Hence we have

$$E(x;t^\delta|q,t) = \sum_{\mu\in L} a_\mu(q,t) E_\mu(t^\delta|q^{-1},t^{-1}) E_\mu(x|q,t). \tag{4.13}$$

Comparing the two expansions (4.11) and (4.13) of $E(x;t^\delta|q,t)$, we obtain

$$f_{\mu^+}(qt^n|q,t)a_{\mu^+\,\mu}(q,t)=a_\mu(q,t)E_\mu(t^\delta|q^{-1},t^{-1}). \tag{4.14}$$

In particular, if λ is a partition, then we have

$$f_\lambda(qt^n|q,t)=a_\lambda(q,t)E_\lambda(t^\delta|q^{-1},t^{-1}). \tag{4.15}$$

Evaluation of nonsymmetric Macdonald polynomials at $t^{-\delta}$ is already carried out by Cherednik [C2]. If $\lambda\in L^+$ is a partition, the value $E_\lambda(t^{-\delta}|q,t)$ can be rewritten as follows:

$$E_\lambda(t^{-\delta}|q,t)=t^{-(n-1)|\lambda|+n(\lambda)}\prod_{s\in\lambda}\frac{1-q^{a'(s)+1}t^{n-l'(s)}}{1-q^{a(s)+1}t^{l(s)+1}}. \tag{4.16}$$

From this formula, we have

$$E_\lambda(t^\delta|q^{-1},t^{-1})=t^{n(\lambda)}\prod_{s\in\lambda}\frac{1-q^{a'(s)+1}t^{n-l'(s)}}{1-q^{a(s)+1}t^{l(s)+1}}. \tag{4.17}$$

Substituting (4.2) and (4.17) into (4.15), we have

$$a_\lambda(q,t)=\frac{f_\lambda(qt^n|q,t)}{E_\lambda(t^\delta|q^{-1},t^{-1})}=\prod_{s\in\lambda}\frac{1-q^{a(s)+1}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}} \tag{4.18}$$

for all partition $\lambda\in L^+$. Namely we have $a_\lambda(q,t)=d_\lambda(q,t)/d'_\lambda(q,t)$ for all $\lambda\in L^+$ with the notation of Section 2. Hence by Lemma 2.5 we have

$$a_\mu(q,t)=\frac{d_\mu(q,t)}{d'_\mu(q,t)}=\prod_{s\in\mu}\frac{1-q^{a(s)+1}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}} \tag{4.19}$$

for all composition $\mu\in L$. This completes the proof of Theorem 2.2.

Remark 4.2. In Section 3, we determined the coefficients $a_\lambda(q,t)$ by combining asymptotic analysis of q-Selberg type integrals and the formulas for scalar products (E_λ,E_λ). Since we have derived the formulas for $a_\lambda(q,t)$ along a different route in this section, we can also use the argument of Section 3 conversely to determine the scalar products (E_λ,E_λ) (cf. [Mi2]).

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