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Multiple elliptic hypergeometric series

— An approach from the Cauchy determinant —

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Dedicated to Professor Tom Koornwinder on his sixtieth birthday

Abstract

A multiple generalization of elliptic hypergeometric series is studied through the Cauchy determinant for the Weierstrass sigma function. In particular, a duality transformation for multiple hypergeometric series is proposed. As an application, two types of Bailey transformations for very well-poised multiple elliptic hypergeometric series are derived.

Mathematics Subject Classification: 33Cxx; 33C70, 33D15, 33D67

Introduction

In this paper we investigate a multiple generalization of elliptic hypergeometric series, and propose a *duality transformation* for multiple hypergeometric series. Our duality transformation is obtained from an identity arising from the Cauchy determinant formula for the Weierstrass sigma function, by means of specialization of a particular form. This formula for the multiple elliptic case, as well as the idea of proof, is a variant of the one previously studied by one of the authors [7, 8] for multiple basic hypergeometric series. As an application of the duality transformation, we prove two types of Bailey transformation formulas for very well-poised multiple elliptic hypergeometric series; one of them is the one proved recently by Rosengren [15], but the other seems to be new.

We remark that the Bailey transformation formula for very well-poised elliptic hypergeometric series ${}_{10}E_9$ was discovered by Frenkel-Turaev [5] in the context of elliptic 6-j symbols. It plays a crucial role in the theory of Spiridonov-Zhedanov [17] on biorthogonal rational functions on elliptic grids. Also, in the geometric approach to the elliptic Painlevé equation [9], this transformation formula is explained as the standard Cremona transformation applied to its hypergeometric solutions.

The Bailey transformation formula [2, 3] has been extended by the method of Bailey chains into various directions including multiple generalizations (see

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[1], [13] for instance). We expect that our approach, based on the Cauchy determinant and the duality transformation, would provide a unified and transparent way of looking at the symmetry structure of Bailey transformations for the hierarchy of elliptic, basic and ordinary hypergeometric series. Multiple elliptic hypergeometric series are also generalized by means of root systems [15]. In terms of root systems, our discussion is restricted to the cases of A type. It would be an intriguing problem to extend the method of Cauchy determinants to other root systems.

Notes. After completing this work, the authors were informed by H. Rosengren that he obtained independently the duality transformation formula (Theorem 2.1).

1 Starting from the Cauchy formula

Throughout this paper, by the symbol $[x]$ we denote a nonzero holomorphic odd function on $\mathbb{C}$ in the variable x, satisfying the Riemann relation:

$$\begin{aligned} &(1) \quad [-x] = -[x], \\ &(2) \quad [x+y]\,[x-y]\,[u+v]\,[u-v] \\ &\qquad = [x+u]\,[x-u]\,[y+v]\,[y-v] - [x+v]\,[x-v]\,[y+u]\,[y-u]. \end{aligned} \tag{1.1}$$

There are three classes of such functions; elliptic, trigonometric and rational. It is well known [19] that all $[x]$ satisfying the conditions above are obtained from the three functions $\sigma(x;\omega_1,\omega_2)$, $\sin(\pi x)$ and x by the transformation of the form $e^{ax^2+b}[cx]$ $(a,b,c \in \mathbb{C})$; $\sigma(x;\omega_1,\omega_2)$ stands for the Weierstrass sigma function with respect to the periods (ω_1,ω_2). The following determinant formula for $[x]$ plays a crucial role in our arguments.

Theorem 1.1 (Cauchy determinant formula) *For two sets of variables $z = (z_1,\ldots,z_M)$ and $w = (w_1,\ldots,w_M)$, we have*

$$\det\left(\frac{[\lambda+z_i+w_j]}{[\lambda]\,[z_i+w_j]}\right)_{i,j=1}^{M} = \frac{\left[\lambda+\sum_{i=1}^{M}(z_i+w_i)\right]\prod_{1\le i<j\le M}[z_i-z_j]\,[w_i-w_j]}{[\lambda]\prod_{1\le i,j\le M}[z_i+w_j]}, \tag{1.2}$$

where λ is a complex parameter.

This determinant formula can be regarded as the special case of Fay's trisecant formula [4], concerning the cases of elliptic curves and their degenerations. We remark that the elliptic case can already be found in Frobenius [6]. Also, from the rational case where $[x] = x$, by taking the limit $\lambda \to \infty$ we obtain the usual Cauchy determinant formula. Note that the case of $M = 2$ is precisely the Riemann relation; the general formula (1.2) can be proved by the induction on M, thanks to the formula

$$\det\left(a_{ij}\right)_{i,j=1}^{M} = a_{MM}^{2-M}\det\left(a_{ij}a_{MM} - a_{iM}a_{Mj}\right)_{i,j=1}^{M-1} \tag{1.3}$$

for any $M \times M$ matrix $\big(a_{ij}\big)_{i,j}^{M}$ with $a_{MM} \neq 0$. Regarding the Cauchy determinant (1.2) as a function in $z = (z_1, \ldots, z_M)$ and $w = (w_1, \ldots, w_M)$, we set

$$D(z|w) = \det\left(\frac{[\lambda + z_i + w_j]}{[\lambda]\,[z_i + w_j]}\right)_{i,j=1}^{M}. \tag{1.4}$$

Note that $D(z|w)$ is alternating under the permutation of variables z_i (resp. w_j), and symmetric with respect to exchanging z and w.

We now fix a generic constant $\delta \in \mathbb{C}$ such that $[k\delta] \neq 0$ for any nonzero $k \in \mathbb{Z}$; in the case of the sigma function $[x] = \sigma(x; \omega_1, \omega_2)$, this condition is equivalent to $\delta \notin \mathbb{Q}\omega_1 + \mathbb{Q}\omega_2$. We denote by T_{z_i} the shift operator in the variable z_i by δ:

$$T_{z_i} f(z_1, \ldots, z_M) = f(z_1, \ldots, z_i + \delta, \ldots, z_M). \tag{1.5}$$

Given a set of indices $K = \{i_1, \ldots, i_r\} \subset \{1, \ldots, M\}$, we also define

$$T_{z,K} = \prod_{i \in K} T_{z_i} = T_{z_{i_1}} \cdots T_{z_{i_r}}. \tag{1.6}$$

In this notation, the operator $E(T_z; u) = (1 + u\,T_{z_1}) \cdots (1 + u\,T_{z_M})$, containing a formal parameter u, is expanded into the sum

$$E(T_z; u) = (1 + u\,T_{z_1}) \cdots (1 + u\,T_{z_M}) = \sum_{K \subset \{1,\ldots,M\}} u^{|K|}\,T_{z,K} \tag{1.7}$$

over all subsets K of the index set $\{1, \ldots, M\}$. By applying the operator $E(T_z; u)$ to the Cauchy determinant $D(z|w)$ we obtain

$$\begin{aligned} E(T_z; u)\,D(z|w) &= \det\left((1 + u\,T_{z_i})\frac{[\lambda + z_i + w_j]}{[\lambda]\,[z_i + w_j]}\right)_{i,j=1}^{M} \\ &= \det\left(\frac{[\lambda + z_i + w_j]}{[\lambda]\,[z_i + w_j]} + u\,\frac{[\lambda + z_i + w_j + \delta]}{[\lambda]\,[z_i + w_j + \delta]}\right)_{i,j=1}^{M}. \end{aligned} \tag{1.8}$$

Since this formula is symmetric with respect to exchanging z and w, we have

$$E(T_z; u)\,D(z|w) = E(T_w; u)\,D(w|z). \tag{1.9}$$

As we will see below, this formula, combined with the expansion (1.7), gives rise to remarkable identities concerning the function $[x]$.

By using the notation

$$\Delta(z) = \prod_{1 \le i < j \le M} [z_i - z_j], \quad |z| = z_1 + \cdots + z_M, \tag{1.10}$$

let us write the Cauchy formula in the form

$$D(z|w) = \frac{[\,\lambda + |z| + |w|\,]\ \Delta(z)\,\Delta(w)}{[\lambda] \displaystyle\prod_{1 \le i,j \le M} [z_i + w_j]}. \tag{1.11}$$

By applying the shift operator $T_{z,K}$ to the difference product $\Delta(z)$, we obtain

$$\frac{T_{z,K}\Delta(z)}{\Delta(z)} = \prod_{i\in K;j\notin K} \frac{[z_i - z_j + \delta]}{[z_i - z_j]}, \tag{1.12}$$

and hence

$$\begin{aligned}\frac{T_{z,K}D(z|w)}{D(z|w)} = &\frac{[\,\lambda + |z| + |w| + |K|\delta\,]}{[\,\lambda + |z| + |w|\,]} \\ &\cdot \prod_{i\in K;\, j\notin K} \frac{[z_i - z_j + \delta]}{[z_i - z_j]} \prod_{i\in K;\, 1\le k\le M} \frac{[z_i + w_k]}{[z_i + w_k + \delta]}.\end{aligned} \tag{1.13}$$

This computation implies the following expansion of $E(T_z; u)D(z|w)$:

$$\begin{aligned}\frac{E(T_z;u)D(z|w)}{D(z|w)} &= \sum_{K\subset\{1,\ldots,M\}} u^{|K|} \frac{T_{z,K}D(z|w)}{D(z|w)} \\ &= \sum_{K\subset\{1,\ldots,M\}} u^{|K|} \frac{[\,\lambda + |z| + |w| + |K|\delta\,]}{[\,\lambda + |z| + |w|\,]} \\ &\quad\cdot \prod_{i\in K;\, j\notin K} \frac{[z_i - z_j + \delta]}{[z_i - z_j]} \prod_{i\in K;\, 1\le k\le M} \frac{[z_i + w_k]}{[z_i + w_k + \delta]}.\end{aligned} \tag{1.14}$$

In view of this formula, we denote the right-hand side by $F(z|w;u)$:

$$\begin{aligned}F(z|w;u) \;=\; &\sum_{K\subset\{1,\ldots,M\}} u^{|K|} \frac{[\,\lambda + |z| + |w| + |K|\delta\,]}{[\,\lambda + |z| + |w|\,]} \\ &\cdot \prod_{i\in K;\, j\notin K} \frac{[z_i - z_j + \delta]}{[z_i - z_j]} \prod_{i\in K;\, 1\le k\le M} \frac{[z_i + w_k]}{[z_i + w_k + \delta]}.\end{aligned} \tag{1.15}$$

From the symmetry $E(T_z; u)D(z; w) = E(T_w; u)D(w; z)$, we conclude that the function $F(z|w; u)$ defined as above is symmetric with respect to exchanging z and w.

Theorem 1.2 *For two sets of variables $z = (z_1, \ldots, z_M)$ and $w = (w_1, \ldots, w_M)$, define the function $F(z|w; u)$ by* (1.15)*. Then $F(z|w; u)$ is symmetric with respect to z and w, namely, $F(z|w; u) = F(w|z; u)$.*

We remark that $F(z|w; u)$ is characterized by the identity

$$\begin{aligned}E(T_z;u)D(z|w) \;&= \det\left(\frac{[\lambda + z_i + w_j]}{[\lambda]\,[z_i + w_j]} + u\,\frac{[\lambda + z_i + w_j + \delta]}{[\lambda]\,[z_i + w_j + \delta]}\right)_{i,j=1}^{M} \\ &=\; F(z|w;u)\,\frac{[\lambda + |z| + |w|]\;\Delta(z)\,\Delta(w)}{[\lambda]\displaystyle\prod_{1\le i,j\le M}[z_i + w_j]}.\end{aligned} \tag{1.16}$$

Explicitly, Theorem 1.2 implies the identity

$$
\begin{aligned}
&\sum_{K\subset\{1,\ldots,M\}} u^{|K|}\,\frac{[\,\lambda+|z|+|w|+|K|\delta\,]}{[\,\lambda+|z|+|w|\,]} \\
&\qquad\cdot \prod_{i\in K;\, j\notin K}\frac{[z_i-z_j+\delta]}{[z_i-z_j]}\prod_{i\in K;\,1\le k\le M}\frac{[z_i+w_k]}{[z_i+w_k+\delta]} \\
&= \sum_{L\subset\{1,\ldots,M\}} u^{|L|}\,\frac{[\,\lambda+|z|+|w|+|L|\delta\,]}{[\,\lambda+|z|+|w|\,]} \\
&\qquad\cdot \prod_{k\in L;\, l\notin L}\frac{[w_k-w_l+\delta]}{[w_k-w_l]}\prod_{1\le i\le M;\,k\in L}\frac{[z_i+w_k]}{[z_i+w_k+\delta]}.
\end{aligned}
\tag{1.17}
$$

By taking the coefficients of u^d $(d=0,1,\ldots,M)$, we obtain

Theorem 1.3 *Given two sets of variables $z=(z_1,\ldots,z_M)$ and $w=(w_1,\ldots,w_M)$, the following identity holds for each $d=0,1,\ldots,M$:*

$$
\begin{aligned}
&\sum_{|K|=d}\prod_{i\in K;\, j\notin K}\frac{[z_i-z_j+\delta]}{[z_i-z_j]}\prod_{i\in K;\,1\le k\le M}\frac{[z_i+w_k]}{[z_i+w_k+\delta]} \\
&= \sum_{|L|=d}\prod_{k\in L;\, l\notin L}\frac{[w_k-w_l+\delta]}{[w_k-w_l]}\prod_{1\le i\le M;\,k\in L}\frac{[w_k+z_i]}{[w_k+z_i+\delta]},
\end{aligned}
\tag{1.18}
$$

where K and L run over all d-subsets of $\{1,\ldots,M\}$.

2 Passage to multiple hypergeometric series

From $F(z|w;u)$ introduced above, one can generate a class of multiple hypergeometric series, denoted by $\Phi_N^{m,n}$ below, by a particular specialization of the variables z and w. The symmetry property $F(z|w;u)=F(w|z;u)$ is then translated into a transformation formula between $\Phi_N^{m,n}$ and $\Phi_N^{n,m}$ (Theorem 2.2).

Taking m variables $x=(x_1,\ldots,x_m)$ and a multi-index $\alpha=(\alpha_1,\ldots,\alpha_m)\in\mathbb{N}^m$ such that $|\alpha|=\alpha_1+\cdots+\alpha_m=M$, we specialize the variables $z=(z_1,\ldots,z_M)$ as follows:

$$
\begin{aligned}
(z_1,\ldots,z_M) &= (-x_1,-x_1-\delta,\ldots,-x_1-(\alpha_1-1)\delta;-x_2,-x_2-\delta,\ldots \\
&\qquad \ldots;-x_m,-x_m-\delta,\ldots,-x_m-(\alpha_m-1)\delta).
\end{aligned}
\tag{2.1}
$$

Namely, we divide the index set $\{1,\ldots,M\}$ into m blocks of size $\alpha_1,\ldots,\alpha_m$ according to the multi-index α, and then specialize each block of z variables by the sequence $(-x_r,-x_r-\delta,\ldots,-x_r-(\alpha_r-1)\delta)$ for $r=1,\ldots,m$. Similarly, taking n variables $y=(y_1,\ldots,y_n)$ and a multi-index $\beta\in\mathbb{N}^n$ with $|\beta|=M$, we specialize the w variables as follows:

$$
\begin{aligned}
(w_1,\ldots,w_M) &= (-y_1,-y_1-\delta,\ldots,-y_1-(\beta_1-1)\delta;-y_2,-y_2-\delta,\ldots \\
&\qquad \ldots;-y_n,-y_n-\delta,\ldots,-y_n-(\beta_n-1)\delta).
\end{aligned}
\tag{2.2}
$$

In the following, we denote these specializations simply by $z \to -(x)_\alpha$, $w \to -(y)_\beta$, respectively. We first look at the factor

$$\prod_{i\in K;\, j\notin K} \frac{[z_i - z_j + \delta]}{[z_i - z_j]} \tag{2.3}$$

in the expansion (1.15) of $F(z|w;u)$. When we perform the specialization $z \to -(x)_\alpha$ as in (2.1), this factor becomes zero if $z_j = z_i + \delta$ for some (i,j) such that $i \in K$, $j \notin K$. Replace the index set $\{1,\ldots,M\}$ for the z variables by

$$\{(r,\mu) \mid 1 \le r \le m \;;\; 0 \le \mu < \alpha_r\} \tag{2.4}$$

according to the m blocks of x variables, so that $z_{(r,\mu)} = -x_r - \mu\delta$. Then by the specialization $z \to -(x)_\alpha$ of (2.1), the factor (2.3) becomes zero if there exists (r,μ) such that $(r,\mu) \notin K$, $(r,\mu+1) \in K$. This implies that, in the summation (1.15), nontrivial terms arise only from such K that all the elements of K are packed towards the left in each block. Such subsets K can be parametrized by the multi-indices $\gamma \in \mathbb{N}^m$ such that $\gamma \le \alpha$ (i.e., $\gamma_r \le \alpha_r$ for each $r = 1,\ldots,m$) as

$$K = \{(r,\mu) \mid 1 \le r \le m \;;\; 0 \le \mu < \gamma_r\}. \tag{2.5}$$

With this parametrization of K, we compute

$$\begin{aligned}
&\prod_{i\in K;\, j\notin K} \frac{[z_i - z_j + \delta]}{[z_i - z_j]}\Big|_{z\to-(x)_\alpha} \\
&= \prod_{1\le r,s\le m}\ \prod_{0\le\mu<\gamma_r}\ \prod_{\gamma_s\le\nu<\alpha_s} \frac{[-x_r + x_s + (-\mu+\nu+1)\delta]}{[-x_r + x_s + (-\mu+\nu)\delta]} \\
&= \prod_{1\le r,s\le m}\ \prod_{0\le\mu<\gamma_r} \frac{[-x_r + x_s + (-\mu+\alpha_s)\delta]}{[-x_r + x_s + (-\mu+\gamma_s)\delta]} \\
&= \prod_{1\le r,s\le m} \frac{[x_r - x_s - \alpha_s\delta]_{\gamma_r}}{[x_r - x_s - \gamma_s\delta]_{\gamma_r}},
\end{aligned} \tag{2.6}$$

where we have used the notation of shifted factorials

$$[x]_k = [x][x+\delta]\cdots[x+(k-1)\delta] \qquad (k = 0,1,2,\ldots). \tag{2.7}$$

By the specialization $z \to -(x)_\alpha$ and $w \to -(y)_\beta$, the remaining factor leads to

$$\begin{aligned}
&\prod_{i\in K;\, 1\le k\le M} \frac{[z_i + w_k]}{[z_i + w_k + \delta]}\Big|_{z\to-(x)_\alpha, w\to-(y)_\beta} \\
&= \prod_{r=1}^{m}\prod_{s=1}^{n}\ \prod_{0\le\mu<\gamma_r}\ \prod_{0\le\nu<\beta_s} \frac{[-x_r - y_s + (-\mu-\nu)\delta]}{[-x_r - y_s + (-\mu-\nu+1)\delta]} \\
&= \prod_{r=1}^{m}\prod_{s=1}^{n}\ \prod_{0\le\mu<\gamma_r} \frac{[-x_r - y_s + (-\mu-\beta_s+1)\delta]}{[-x_r - y_s + (-\mu+1)\delta]} \\
&= \prod_{r=1}^{m}\prod_{s=1}^{n} \frac{[x_r + y_s + (\beta_s-1)\delta]_{\gamma_r}}{[x_r + y_s - \delta]_{\gamma_r}}.
\end{aligned} \tag{2.8}$$

After all, we have

$$
\begin{aligned}
&F(z|w;u)\Big|_{z\to-(x)_\alpha,w\to-(y)_\beta}\\
&=\sum_{\gamma} u^{|\gamma|}\,\frac{[\lambda-\alpha x-\beta y-\binom{\alpha}{2}\delta-\binom{\beta}{2}\delta+|\gamma|\delta]}{[\lambda-\alpha x-\beta y-\binom{\alpha}{2}\delta-\binom{\beta}{2}\delta]}\\
&\quad\cdot\prod_{1\le r,s\le n}\frac{[x_r-x_s-\alpha_s\delta]_{\gamma_r}}{[x_r-x_s-\gamma_s\delta]_{\gamma_r}}\prod_{1\le r\le m;1\le s\le n}\frac{[x_r+y_s+(\beta_s-1)\delta]_{\gamma_r}}{[x_r+y_s-\delta]_{\gamma_r}},
\end{aligned}
\tag{2.9}
$$

summed over all $\gamma\in\mathbb{N}^m$ with $\gamma\le\alpha$. Here we have used the abbreviation

$$
\alpha x=\alpha_1x_1+\cdots+\alpha_mx_m,\qquad \binom{\alpha}{2}=\binom{\alpha_1}{2}+\cdots+\binom{\alpha_m}{2}
\tag{2.10}
$$

for $\alpha=(\alpha_1,\dots,\alpha_m)$. In the summation (2.9), notice that

$$
\prod_{1\le r,s\le m}\frac{[x_r-x_s-\alpha_s\delta]_{\gamma_r}}{[x_r-x_s-\gamma_s\delta]_{\gamma_r}}=\prod_{r=1}^{m}\frac{[-\alpha_r\delta]_{\gamma_r}}{[-\gamma_r\delta]_{\gamma_r}}\prod_{r\ne s}\frac{[x_r-x_s-\alpha_s\delta]_{\gamma_r}}{[x_r-x_s-\gamma_s\delta]_{\gamma_r}}.
\tag{2.11}
$$

Since this factor becomes zero for $\gamma\in\mathbb{N}^m$ such that $\gamma\not\le\alpha$, we can consider the summation (2.9) as being taken over all $\gamma\in\mathbb{N}^m$. For convenience, we rewrite this formula further by using the equality

$$
\begin{aligned}
&\prod_{i,j}\frac{1}{[x_i-x_j-\gamma_j\delta]_{\gamma_i}}\\
&=(-1)^{|\gamma|}\prod_{i<j}\frac{[x_i-x_j+(\gamma_i-\gamma_j)\delta]}{[x_i-x_j]}\prod_{i,j}\frac{1}{[x_i-x_j+\delta]_{\gamma_i}}\\
&=(-1)^{|\gamma|}\frac{\Delta(x+\gamma\delta)}{\Delta(x)}\prod_{i,j}\frac{1}{[x_i-x_j+\delta]_{\gamma_i}},
\end{aligned}
\tag{2.12}
$$

so that

$$
\begin{aligned}
&F(z|w;u)\Big|_{z\to-(x)_\alpha,w\to-(y)_\beta}\\
&=\sum_{\gamma\in\mathbb{N}^m}(-u)^{|\gamma|}\,\frac{[\lambda-\alpha x-\beta y-\binom{\alpha}{2}\delta-\binom{\beta}{2}\delta+|\gamma|\delta]}{[\lambda-\alpha x-\beta y-\binom{\alpha}{2}\delta-\binom{\beta}{2}\delta]}\,\frac{\Delta(x+\gamma\delta)}{\Delta(x)}\\
&\quad\cdot\prod_{1\le i,j\le m}\frac{[x_i-x_j-\alpha_j\delta]_{\gamma_i}}{[x_i-x_j+\delta]_{\gamma_i}}\prod_{1\le i\le m;\,1\le k\le n}\frac{[x_i+y_k+(\beta_k-1)\delta]_{\gamma_i}}{[x_i+y_k-\delta]_{\gamma_i}}.
\end{aligned}
\tag{2.13}
$$

We now specialize the equality $F(z|w;u)=F(w|z;u)$ by $z\to-(x)_\alpha,w\to-(y)_\beta$, and take the coefficients of u^d $(d=0,1,\dots,M)$. Then, for any multi-indices $\alpha\in\mathbb{N}^m$, $\beta\in\mathbb{N}^n$ such that $|\alpha|=|\beta|$, we obtain the identity

$$
\begin{aligned}
&\sum_{\mu\in\mathbb{N}^m;|\mu|=d}\frac{\Delta(x+\mu\delta)}{\Delta(x)}\prod_{i,j}\frac{[x_i-x_j-\alpha_j\delta]_{\mu_i}}{[x_i-x_j+\delta]_{\mu_i}}\prod_{i,k}\frac{[x_i+y_k+(\beta_k-1)\delta]_{\mu_i}}{[x_i+y_k-\delta]_{\mu_i}}\\
&=\sum_{\nu\in\mathbb{N}^n;|\nu|=d}\frac{\Delta(y+\nu\delta)}{\Delta(y)}\prod_{k,l}\frac{[y_k-y_l-\beta_l\delta]_{\nu_k}}{[y_k-y_l+\delta]_{\nu_k}}\prod_{k,i}\frac{[y_k+x_i+(\alpha_i-1)\delta]_{\nu_k}}{[y_k+x_i-\delta]_{\nu_k}}
\end{aligned}
\tag{2.14}
$$

for each $d = 0, 1, \ldots, M$, where $i, j \in \{1, \ldots, m\}$ and $k, l \in \{1, \ldots, n\}$. Replacing y_k by $y_k + \delta$ ($k = 1, \ldots, n$) in this identity, we obtain

Theorem 2.1 *Take two sets of variables* $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_n)$ *and two multi-indices* $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ *such that* $|\alpha| = |\beta|$, *i.e.,*

$$\alpha_1 + \cdots + \alpha_m = \beta_1 + \cdots + \beta_n. \tag{2.15}$$

Then the following identity holds for each $d = 0, 1, 2, \ldots$:

$$\begin{aligned} &\sum_{\mu \in \mathbb{N}^m;\, |\mu| = d} \frac{\Delta(x + \mu\delta)}{\Delta(x)} \prod_{i,j} \frac{[x_i - x_j - \alpha_j\delta]_{\mu_i}}{[x_i - x_j + \delta]_{\mu_i}} \prod_{i,k} \frac{[x_i + y_k + \beta_k\delta]_{\mu_i}}{[x_i + y_k]_{\mu_i}} \\ &= \sum_{\nu \in \mathbb{N}^n;\, |\nu| = d} \frac{\Delta(y + \nu\delta)}{\Delta(y)} \prod_{k,l} \frac{[y_k - y_l - \beta_l\delta]_{\nu_k}}{[y_k - y_l + \delta]_{\nu_k}} \prod_{k,i} \frac{[y_k + x_i + \alpha_i\delta]_{\nu_k}}{[y_k + x_i]_{\nu_k}}, \end{aligned} \tag{2.16}$$

where $i, j \in \{1, \ldots, m\}$ *and* $k, l \in \{1, \ldots, n\}$.

The identity (2.16) can be thought of as a transformation formula for certain multiple hypergeometric series. In order to clarify the idea, we introduce the multiple hypergeometric series $\Phi_N^{m,n}$ with respect to $[x]$ as follows. Given four vectors of variables

$$(a_1, \ldots, a_m), (x_1, \ldots, x_m) \in \mathbb{C}^m, \quad \text{and} \quad (b_1, \ldots, b_n), (c_1, \ldots, c_n) \in \mathbb{C}^n, \tag{2.17}$$

we define

$$\begin{aligned} &\Phi_N^{m,n}\left(\begin{matrix} a_1, \ldots, a_m \\ x_1, \ldots, x_m \end{matrix} \middle| \begin{matrix} b_1, \ldots, b_n \\ c_1, \ldots, c_n \end{matrix} \right) \\ &= \sum_{\mu \in \mathbb{N}^m,\, |\mu| = N} \frac{\Delta(x + \mu\delta)}{\Delta(x)} \\ &\quad \cdot \prod_{1 \le i,j \le m} \frac{[x_i - x_j + a_j]_{\mu_i}}{[x_i - x_j + \delta]_{\mu_i}} \prod_{1 \le i \le m; 1 \le k \le n} \frac{[x_i + b_k]_{\mu_i}}{[x_i + c_k]_{\mu_i}} \end{aligned} \tag{2.18}$$

for each $N = 0, 1, 2, \ldots$. Here the summation is taken over the finite set of $\mu \in \mathbb{N}^m$ with $|\mu| = N$. Note that $\Phi_N^{m,n}$ defined as above is invariant with respect to the simultaneous permutations of $(a_1, \ldots, a_m)$ and $(x_1, \ldots, x_m)$, and symmetric in the variables $(b_1, \ldots, b_n)$, and in $(c_1, \ldots, c_n)$, respectively. We also remark that our parametrization of $\Phi_N^{m,n}$ is redundant in the sense that

$$\begin{aligned} &\Phi_N^{m,n}\left(\begin{matrix} a_1, & \ldots, & a_m \\ x_1 + t, & \ldots, & x_m + t \end{matrix} \middle| \begin{matrix} b_1, \ldots, b_n \\ c_1, \ldots, c_n \end{matrix} \right) \\ &= \Phi_N^{m,n}\left(\begin{matrix} a_1, \ldots, a_m \\ x_1, \ldots, x_m \end{matrix} \middle| \begin{matrix} b_1 + t, \ldots, b_n + t \\ c_1 + t, \ldots, c_n + t \end{matrix} \right). \end{aligned} \tag{2.19}$$

We keep this redundancy for the sake of symmetry.

In this notation of $\Phi_N^{m,n}$, Theorem 2.1 can be understood as the identity

$$
\begin{aligned}
&\Phi_N^{m,n}\left(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix}\,\middle|\,\begin{matrix} y_1-b_1,\ldots,y_n-b_n \\ y_1,\quad \ldots,\quad y_n \end{matrix}\right) \\
&= \Phi_N^{n,m}\left(\begin{matrix} b_1,\ldots,b_n \\ y_1,\ldots,y_n \end{matrix}\,\middle|\,\begin{matrix} x_1-a_1,\ldots,x_m-a_m \\ x_1,\quad \ldots,\quad x_m \end{matrix}\right)
\end{aligned}
\tag{2.20}
$$

for the special values

$$
a_i = -\alpha_i\delta \quad (i=1,\ldots,m) \quad \text{and} \quad b_k = -\beta_k\delta \quad (k=1,\ldots,n) \tag{2.21}
$$

for all $\alpha \in \mathbb{N}^m$, $\beta \in \mathbb{N}^n$ such that $|\alpha| = |\beta|$. Theorem 2.1 is then generalized to the following transformation formula between $\Phi_N^{m,n}$ and $\Phi_N^{n,m}$.

Theorem 2.2 (Duality transformation) *Suppose that* $(a_1,\ldots,a_m) \in \mathbb{C}^m$ *and* $(b_1,\ldots,b_n) \in \mathbb{C}^n$ *satisfy the balancing condition*

$$
a_1 + \cdots + a_m = b_1 + \cdots + b_n. \tag{2.22}
$$

Then, for two sets of variables $(x_1,\ldots,x_m)$ *and* $(y_1,\ldots,y_n)$*, the following transformation formula holds between the multiple hypergeometric series* $\Phi_N^{m,n}$ *and* $\Phi_N^{n,m}$*:*

$$
\begin{aligned}
&\Phi_N^{m,n}\left(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix}\,\middle|\,\begin{matrix} y_1-b_1,\ldots,y_n-b_n \\ y_1,\quad \ldots,\quad y_n \end{matrix}\right) \\
&= \Phi_N^{n,m}\left(\begin{matrix} b_1,\ldots,b_n \\ y_1,\ldots,y_n \end{matrix}\,\middle|\,\begin{matrix} x_1-a_1,\ldots,x_m-a_m \\ x_1,\quad \ldots,\quad x_m \end{matrix}\right)
\end{aligned}
\tag{2.23}
$$

for each $N = 0,1,2,\ldots$.

We remark that Theorem 2.2 is equivalent to the transformation formula

$$
\begin{aligned}
&\Phi_N^{m,n}\left(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix}\,\middle|\,\begin{matrix} b_1,\ldots,b_n \\ c_1,\ldots,c_n \end{matrix}\right) \\
&= \Phi_N^{n,m}\left(\begin{matrix} c_1-b_1,\ldots,c_n-b_n \\ c_1,\quad \ldots,\quad c_n \end{matrix}\,\middle|\,\begin{matrix} x_1-a_1,\ldots,x_m-a_m \\ x_1,\quad \ldots,\quad x_m \end{matrix}\right)
\end{aligned}
\tag{2.24}
$$

for $N = 0,1,2,\ldots$, under the balancing condition $a_1+\cdots+a_m+b_1+\cdots+b_n = c_1+\cdots+c_n$, or to

$$
\begin{aligned}
&\Phi_N^{m,n}\left(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix}\,\middle|\,\begin{matrix} b_1+y_1,\ldots,b_n+y_n \\ c+y_1,\ \ldots,\ c+y_n \end{matrix}\right) \\
&= \Phi_N^{n,m}\left(\begin{matrix} c-b_1,\ldots,c-b_n \\ y_1,\quad \ldots,\quad y_n \end{matrix}\,\middle|\,\begin{matrix} c-a_1+x_1,\ldots,c-a_m+x_m \\ c+x_1,\quad \ldots,\quad c+x_m \end{matrix}\right)
\end{aligned}
\tag{2.25}
$$

under the balancing condition $a_1+\cdots+a_m+b_1+\cdots+b_n = nc$. Note that this transformation formula makes sense only in the setting of multiple hypergeometric series; it becomes tautological when $(m,n) = (1,1)$. In what follows, we refer to the transformation formula (2.23) as the *duality transformation* between $\Phi_N^{m,n}$ and $\Phi_N^{n,m}$. As we will see in the next section, this duality transformation formula implies various transformation formulas and summation formulas for very well-poised multiple elliptic hypergeometric series.

For the proof of Theorem 2.2, we first remark that equality (2.23) is reduced to the three typical cases, $[x] = x, \sin(\pi x)$ and $\sigma(x;\omega_1,\omega_2)$. In fact, if we replace $[x]$ by $e^{ax^2+b}[x]$, we obtain the same exponential factor from the both sides of (2.23). Assuming that $[x] = x, \sin(\pi x)$ or $\sigma(x;\omega_1,\omega_2)$, we investigate the dependence of

$$\text{LHS} = \sum_{|\mu|=N} \frac{\Delta(x+\mu\delta)}{\Delta(x)} \prod_{i,j} \frac{[x_i - x_j + a_j]_{\mu_i}}{[x_i - x_j + \delta]_{\mu_i}} \prod_{i,k} \frac{[x_i + y_k - b_k]_{\mu_i}}{[x_i + y_k]_{\mu_i}} \tag{2.26}$$

and

$$\text{RHS} = \sum_{|\nu|=N} \frac{\Delta(y+\nu\delta)}{\Delta(y)} \prod_{k,l} \frac{[y_k - y_l + b_l]_{\nu_k}}{[y_k - y_l + \delta]_{\nu_k}} \prod_{k,i} \frac{[y_k + x_i - a_i]_{\nu_k}}{[y_k + x_i]_{\nu_k}} \tag{2.27}$$

on the variables a_i and b_k. In view of $a_1 + \cdots + a_m = b_1 + \cdots + b_n$, we consider the $(m+n-1)$-dimensional affine space $\mathbb{C}^{m+n-1}$ with coordinates $(a_1,\ldots,a_m,b_1,\ldots,b_{n-1})$, regarding b_n as a linear function on this affine space. Let $F = F(a_1,\ldots,a_m,b_1,\ldots,b_{n-1})$ be the difference of the two sides regarded as a function in the variables $(a_1,\ldots,a_m,b_1,\ldots,b_{n-1})$. In the case where $[x] = x$, F is a polynomial in a_i, b_k. Also, in the case where $[x] = \sin(\pi x)$, it is a Laurent polynomial in $e^{\pi\sqrt{-1}a_i}, e^{\pi\sqrt{-1}b_k}$. We already know by Theorem 2.1 that F has zeros at $a_i = -\alpha_i\delta$ $(i = 1,\ldots,m)$, $b_k = -\beta_k\delta$ $(k = 1,\ldots,n-1)$ for all $(\alpha_1,\ldots,\alpha_m,\beta_1,\ldots,\beta_{n-1}) \in \mathbb{N}^{m+n-1}$. Under the assumption $[l\delta] \neq 0$ $(l \in \mathbb{Z}, l \neq 0)$, we conclude by induction on the number of variables that F is identically zero. Namely, the both sides coincide identically as holomorphic functions on $\mathbb{C}^{m+n-1}$ (and also as meromorphic functions in all the variables).

In the case where $[x]$ is elliptic, Theorem 2.2 is proved by using the following lemma.

Lemma 2.3 *Let $F(z_1,\ldots,z_N)$ be a holomorphic function on $\mathbb{C}^N$. Let $\omega_1,\omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$, and suppose that $F(z_1,\ldots,z_N)$ is quasi-periodic in each variable z_i $(i = 1,\ldots,N)$ with respect to the lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ in the following sense:*

$$F(z_1,\ldots,z_i+\omega_k,\ldots,z_N) = e^{f_{i,k}(z_i)} F(z_1,\ldots,z_N) \tag{2.28}$$

for some linear functions $f_{ik}(u) = a_{ik}u + b_{ik}$ $(i = 1,\ldots,N; k = 1,2)$. Let $\delta \in \mathbb{C}$ be a constant such that $\delta \notin \mathbb{Q}\omega_1 + \mathbb{Q}\omega_2$, and suppose that

$$F(\nu_1\delta,\ldots,\nu_N\delta) = 0 \tag{2.29}$$

for all $\nu = (\nu_1,\ldots,\nu_N) \in \mathbb{N}^N$. Then $F(z_1,\ldots,z_N)$ is identically zero on $\mathbb{C}^N$.

This lemma is reduced by the induction on N to the fact in the one variable case that any quasi-periodic holomorphic function may have only a finite number of zeros in the period-parallelogram.

Let us regard the both sides of (2.23) as holomorphic functions in the variables $(a_1,\ldots,a_m,b_1,\ldots,b_{n-1}) \in \mathbb{C}^{m+n-1}$. Then one can check directly that they are quasi-periodic with the same multiplicative factors, with respect to each of the coordinate functions of $\mathbb{C}^{m+n-1}$. We know by Theorem 2.1 that the difference of the two sides has zeros at $a_i = -\alpha_i\delta$ $(i = 1, \ldots, m)$, $b_j = -\beta_k\delta$ $(k = 1, \ldots, n-1)$ for all $(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_{n-1}) \in \mathbb{N}^{m+n-1}$. This is impossible unless the both sides coincide identically as holomorphic functions by Lemma 2.3. This completes the proof of Theorem 2.2.

Remark 2.4 In the trigonometric case, our multiple hypergeometric series $\Phi_N^{m,n}$ gives rise to the multiple q-hypergeometric series

$$\begin{aligned} &\phi_N^{m,n}\left(\begin{matrix} a_1, \ldots, a_m \\ x_1, \ldots, x_m \end{matrix}\,\middle|\, \begin{matrix} b_1, \ldots, b_n \\ c_1, \ldots, c_n \end{matrix}\right) \\ &= \sum_{\mu \in \mathbb{N}^m; |\mu|=N} \prod_{i<j} \frac{q^{\mu_i} x_i - q^{\mu_j} x_j}{x_i - x_j} \prod_{i,j} \frac{(a_j x_i/x_j; q)_{\mu_i}}{(q x_i/x_j; q)_{\mu_i}} \prod_{i,k} \frac{(b_k x_i; q)_{\mu_i}}{(c_k x_i; q)_{\mu_i}}, \end{aligned} \tag{2.30}$$

where $i, j \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, n\}$. Setting $x = e^{2\pi\sqrt{-1}\xi}$, let us consider the case where

$$[\xi] = e^{\pi\sqrt{-1}\xi} - e^{-\pi\sqrt{-1}\xi} = x^{\frac{1}{2}} - x^{-\frac{1}{2}}. \tag{2.31}$$

Then the shifted factorials $[\xi]_k$ $(k = 0, 1, 2, \ldots)$ are rewritten into

$$[\xi]_k = (-x^{-\frac{1}{2}})^k q^{-\frac{1}{2}\binom{k}{2}} (x; q)_k, \quad (x; q)_k = \prod_{i=0}^{k-1} (1 - xq^i) \tag{2.32}$$

with base $q = e^{2\pi\sqrt{-1}\delta}$. In this setting, $\Phi_N^{m,n}$ and $\phi_N^{m,n}$ are related by the formula

$$\begin{aligned} &\Phi_N^{m,n}\left(\begin{matrix} \alpha_1, \ldots, \alpha_m \\ \xi_1, \ldots, \xi_m \end{matrix}\,\middle|\, \begin{matrix} \beta_1, \ldots, \beta_n \\ \gamma_1, \ldots, \gamma_n \end{matrix}\right) \\ &= \left(\frac{q c_1 \cdots c_n}{a_1 \cdots a_m b_1 \cdots b_n}\right)^{\frac{N}{2}} \phi_N^{m,n}\left(\begin{matrix} a_1, \ldots, a_m \\ x_1, \ldots, x_m \end{matrix}\,\middle|\, \begin{matrix} b_1, \ldots, b_n \\ c_1, \ldots, c_n \end{matrix}\right), \end{aligned} \tag{2.33}$$

where

$$\begin{aligned} a_i &= e^{2\pi\sqrt{-1}\alpha_i}, \quad x_i = e^{2\pi\sqrt{-1}\xi_i} \quad (i = 1, \ldots, m), \\ b_k &= e^{2\pi\sqrt{-1}\beta_k}, \quad c_k = e^{2\pi\sqrt{-1}\gamma_k} \quad (k = 1, \ldots, n). \end{aligned} \tag{2.34}$$

Hence Theorem 2.2 implies the transformation formula

$$\begin{aligned} &\phi_N^{m,n}\left(\begin{matrix} a_1, \ldots, a_m \\ x_1, \ldots, x_m \end{matrix}\,\middle|\, \begin{matrix} y_1/b_1, \ldots, y_n/b_n \\ y_1, \ \ldots, \ y_n \end{matrix}\right) \\ &= \phi_N^{n,m}\left(\begin{matrix} b_1, \ldots, b_n \\ y_1, \ldots, y_n \end{matrix}\,\middle|\, \begin{matrix} x_1/a_1, \ldots, x_m/a_m \\ x_1, \ \ldots, \ x_m \end{matrix}\right) \end{aligned} \tag{2.35}$$

for each $N = 0, 1, 2, \ldots$, under the balancing condition $a_1 \cdots a_m = b_1 \cdots b_n$. This formula can also be written in the form

$$\begin{aligned} &\phi_N^{m,n}\left(\begin{matrix} a_1, \ldots, a_m \\ x_1, \ldots, x_m \end{matrix}\,\middle|\, \begin{matrix} b_1 y_1, \ldots, b_n y_n \\ c y_1, \ \ldots, \ c y_n \end{matrix}\right) \\ &= \phi_N^{n,m}\left(\begin{matrix} c/b_1, \ldots, c/b_n \\ y_1, \ \ldots, \ y_n \end{matrix}\,\middle|\, \begin{matrix} c x_1/a_1, \ldots, c x_m/a_m \\ c x_1, \ \ldots, \ c x_m \end{matrix}\right) \end{aligned} \tag{2.36}$$

for $N = 0, 1, 2, \ldots$ under the balancing condition $a_1 \cdots a_m b_1 \cdots b_n = c^n$.

As is already known in [7], the duality transformation formula (2.35) for multiple basic hypergeometric series can be further generalized to non-balanced cases. Introducing a variable u, consider the infinite series

$$\phi^{m,n}\left(\begin{matrix}a_1,\ldots,a_m\\x_1,\ldots,x_m\end{matrix}\,\middle|\,\begin{matrix}b_1,\ldots,b_n\\c_1,\ldots,c_n\end{matrix};u\right)=\sum_{N=0}^{\infty}u^N\phi_N^{m,n}\left(\begin{matrix}a_1,\ldots,a_m\\x_1,\ldots,x_m\end{matrix}\,\middle|\,\begin{matrix}b_1,\ldots,b_n\\c_1,\ldots,c_n\end{matrix}\right). \tag{2.37}$$

Then the following *Euler transformation formula* holds for general parameters a_i, b_k and c ([7], Theorem 1.1):

$$\begin{aligned}&\phi^{m,n}\left(\begin{matrix}a_1,\ldots,a_m\\x_1,\ldots,x_m\end{matrix}\,\middle|\,\begin{matrix}b_1y_1,\ldots,b_ny_n\\cy_1,\ \ldots,\ cy_n\end{matrix};u\right)\\&=\frac{(a_1\cdots a_mb_1\cdots b_nu/c^n;q)_\infty}{(u;q)_\infty}\\&\quad\phi^{n,m}\left(\begin{matrix}c/b_1,\ldots,c/b_n\\y_1,\ \ldots,\ y_n\end{matrix}\,\middle|\,\begin{matrix}cx_1/a_1,\ldots,cx_m/a_m\\cx_1,\ \ldots,\ cx_m\end{matrix};a_1\cdots a_mb_1\cdots b_nu/c^n\right).\end{aligned} \tag{2.38}$$

It would be an interesting problem to find an elliptic extension of this Euler transformation formula.

3 Very well-poised multiple series $E^{m,n}$

In this section we show that our duality transformation between $\Phi_N^{m,n}$ and $\Phi_N^{n,m}$ (Theorem 2.2) implies various transformation and summation formulas for *very well-poised* multiple elliptic hypergeometric series.

We first remark that, when $m=1$, $\Phi_N^{m,n}$ reduces to a single term:

$$\Phi_N^{1,n}\left(\begin{matrix}a\\x\end{matrix}\,\middle|\,\begin{matrix}b_1,\ldots,b_n\\c_1,\ldots,c_n\end{matrix}\right)=\frac{[a]_N}{[\delta]_N}\,\frac{[x+b_1]_N\cdots[x+b_n]_N}{[x+c_1]_N\cdots[x+c_n]_N}. \tag{3.1}$$

In general, $\Phi_N^{1+m,n}$ is rewritten into a terminating m-dimensional sum which can be regarded as a *very well-poised* terminating multiple hypergeometric series:

$$\begin{aligned}&\Phi_N^{1+m,n}\left(\begin{matrix}a_0,a_1,\ldots,a_m\\x_0,x_1,\ldots,x_m\end{matrix}\,\middle|\,\begin{matrix}b_1,\ldots,b_n\\c_1,\ldots,c_n\end{matrix}\right)\\&=\frac{[a_0]_N}{[\delta]_N}\prod_{i=1}^{m}\frac{[x_0-x_i+a_i]_N}{[x_0-x_i]_N}\prod_{k=1}^{n}\frac{[x_0+b_k]_N}{[x_0+c_k]_N}\\&\quad\cdot\sum_{\mu\in\mathbb{N}^m}\frac{\Delta(x_1+\mu_1\delta,\ldots,x_m+\mu_m\delta)}{\Delta(x_1,\ldots,x_m)}\prod_{i=1}^{m}\frac{[x_i-x_0-N\delta+(|\mu|+\mu_i)\delta]}{[x_i-x_0-N\delta]}\\&\qquad\cdot\frac{[-N\delta]_{|\mu|}}{[(1-N)\delta-a_0]_{|\mu|}}\prod_{i=1}^{m}\frac{[x_i-x_0+a_0]_{\mu_i}}{[x_i-x_0+\delta]_{\mu_i}}\\&\qquad\cdot\prod_{j=1}^{m}\left(\frac{[-N\delta-x_0+x_j]_{|\mu|}}{[(1-N)\delta-x_0+x_j-a_j]_{|\mu|}}\prod_{i=1}^{m}\frac{[x_i-x_j+a_j]_{\mu_i}}{[x_i-x_j+\delta]_{\mu_i}}\right)\\&\qquad\cdot\prod_{k=1}^{n}\left(\frac{[(1-N)\delta-x_0-c_k]_{|\mu|}}{[(1-N)\delta-x_0-b_k]_{|\mu|}}\prod_{i=1}^{m}\frac{[x_i+b_k]_{\mu_i}}{[x_i+c_k]_{\mu_i}}\right).\end{aligned} \tag{3.2}$$

This expression is obtained simply by applying the formula

$$[a]_{N-r} = \frac{[a]_N}{[a+(N-r)\delta]\cdots[a+(N-1)\delta]} = \frac{(-1)^r[a]_N}{[(1-N)\delta-a]_r} \tag{3.3}$$

to every appearance of the shifted factorial of the form $[a]_{\mu_0}$, $\mu_0 = N-(\mu_1+\cdots+\mu_n)$. Notice that by definition (2.18) this series is symmetric with respect to the simultaneous permutations of the variables $(a_0, a_1, \ldots, a_m)$ and $(x_0, x_1, \ldots, x_m)$, although it is not apparent in this expression.

In view of the formula (3.2), we introduce the following notation for *very well-poised* multiple hypergeometric series $E^{m,n}$:

$$\begin{aligned}
&E^{m,n}\left(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix}\,\middle|\, s; u_1,\ldots,u_n; v_1,\ldots,v_n\right) \\
&= \sum_{\mu\in\mathbb{N}^m} \frac{\Delta(x_1+\mu_1\delta,\ldots,x_m+\mu_m\delta)}{\Delta(x_1,\ldots,x_m)} \prod_{i=1}^m \frac{[x_i+s+(|\mu|+\mu_i)\delta]}{[x_i+s]} \\
&\quad\cdot \prod_{j=1}^m \left(\frac{[s+x_j]_{|\mu|}}{[\delta+s+x_j-a_j]_{|\mu|}} \prod_{i=1}^m \frac{[x_i-x_j+a_j]_{\mu_i}}{[x_i-x_j+\delta]_{\mu_i}} \right) \\
&\quad\cdot \prod_{k=1}^n \left(\frac{[v_k]_{|\mu|}}{[\delta+s-u_k]_{|\mu|}} \prod_{i=1}^m \frac{[x_i+u_k]_{\mu_i}}{[x_i+\delta+s-v_k]_{\mu_i}} \right).
\end{aligned} \tag{3.4}$$

This series contains "well-poised" combinations of factors

$$\frac{[x_1+u]_{\mu_1}\cdots[x_m+u]_{\mu_m}}{[\delta+s-u]_{|\mu|}}, \quad \frac{[v]_{|\mu|}}{[\delta+s+x_1-v]_{\mu_1}\cdots[\delta+s+x_m-v]_{\mu_m}} \tag{3.5}$$

for $u = a_j-x_j, u_k$ and $v = s+x_j, v_k$. In order to make this series (3.4) terminate, we always work in such a situation where one of the following conditions is satisfied: either

$$\begin{array}{lll}
\text{(A)} & v_k = -N\delta & \text{for some } k=1,\ldots,n \text{ and } N=0,1,2,\ldots; \text{ or} \\
\text{(B)} & a_i = -\alpha_i\delta \quad (i=1,\ldots,m) & \text{for some } \alpha=(\alpha_1,\ldots,\alpha_m)\in\mathbb{N}^m.
\end{array} \tag{3.6}$$

Note that our parametrization of $E^{m,n}$ has the following redundancy:

$$\begin{aligned}
&E^{m,n}\left(\begin{matrix} a_1,\ldots,a_m \\ x_1+t,\ldots,x_m+t \end{matrix}\,\middle|\, s; u_1,\ldots,u_n; v_1,\ldots,v_n\right) \\
&= E^{m,n}\left(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix}\,\middle|\, s+t; u_1+t,\ldots,u_n+t; v_1,\ldots,v_n\right).
\end{aligned} \tag{3.7}$$

When $m=1$, $E^{1,n}$ reduces to the very well-poised elliptic hypergeometric series ${}_{2n+4}E_{2n+3}$:

$$\begin{aligned}
&E^{1,n}\left(\begin{matrix} a \\ x \end{matrix}\,\middle|\, s; u_1,\ldots,u_n; v_1,\ldots,v_n\right) \\
&= {}_{2n+4}E_{2n+3}\Big(x+s; a, x+u_1,\ldots,x+u_n, v_1,\ldots,v_n\Big),
\end{aligned} \tag{3.8}$$

where

$$ {}_{r+1}E_r(s;u_1,\ldots,u_{r-2}) = \sum_{k=0}^{\infty} \frac{[s+2k\delta]}{[s]} \frac{[s]_k}{[\delta]_k} \prod_{i=1}^{r-2} \frac{[u_i]_k}{[\delta+s-u_i]_k}. \tag{3.9} $$

In the particular case where $m=1$ and $x=0$,

$$ \begin{aligned} &E^{1,n}\Big(\begin{matrix} a \\ 0 \end{matrix} \Big| s;u_1,\ldots,u_n;v_1,\ldots,v_n \Big) \\ &= {}_{2n+4}E_{2n+3}\Big(s;a,u_1,\ldots,u_n,v_1,\ldots,v_n \Big). \end{aligned} \tag{3.10} $$

is completely symmetric in the $(2n+1)$ variables $(a,u_1,\ldots,u_n,v_1,\ldots,v_n)$.

It would be worthwhile to note here that $E^{m,n}$ carries a remarkable property concerning periodicity(cf. [16]). We now use the abbreviated notation

$$ E^{m,n}\Big(\begin{matrix} a \\ x \end{matrix} \Big| s;u;v \Big) = E^{m,n}\Big(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix} \Big| s;u_1,\ldots,u_n;v_1,\ldots,v_n \Big) \tag{3.11} $$

for $a=(a_1,\ldots,a_m)$, $x=(x_1,\ldots,x_m)$, $u=(u_1,\ldots,u_n)$, $v=(v_1,\ldots,v_n)$.

Lemma 3.1 *Suppose that the function $[x]$ is quasi-periodic with respect to a period $\omega \in \mathbb{C}$ in the sense that*

$$ [x+\omega] \;=\; e^{\xi x+\eta}\,[x] \qquad (x\in\mathbb{C}) \tag{3.12} $$

for some constants $\xi,\eta\in\mathbb{C}$. Then we have

$$ E^{m,n}\Big(\begin{matrix} a+l\,\omega \\ x \end{matrix} \Big| s;u+p\,\omega;v+q\,\omega \Big) = E^{m,n}\Big(\begin{matrix} a \\ x \end{matrix} \Big| s;u;v \Big) \tag{3.13} $$

for any $l\in\mathbb{Z}^m$, $p,q\in\mathbb{Z}^n$ such that $|l|+|p|+|q|=0$.

Relation between $\Phi_N^{m,n}$ and $E^{m,n}$

Proposition 3.2 *The two types of multiple hypergeometric series $\Phi_N^{m,n}$ and $E^{m,n}$ are related as follows*:

$$ \begin{aligned} &\Phi_N^{1+m,n}\Big(\begin{matrix} a_0,a_1,\ldots,a_m \\ x_0,x_1,\ldots,x_m \end{matrix} \Big| \begin{matrix} b_1,\ldots,b_n \\ c_1,\ldots,c_n \end{matrix} \Big) \\ &= \frac{[a_0]_N}{[\delta]_N} \prod_{i=1}^{m} \frac{[x_0-x_i+a_i]_N}{[x_0-x_i]_N} \prod_{k=1}^{n} \frac{[x_0+b_k]_N}{[x_0+c_k]_N} \\ &\quad \cdot\; E^{m,n+1}\Big(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix} \Big| -N\delta-x_0\,;a_0-x_0,b_1,\ldots,b_n\,; \\ &\qquad -N\delta,(1-N)\delta-x_0-c_1,\ldots,(1-N)\delta-x_0-c_n \Big). \end{aligned} \tag{3.14} $$

Conversely,

$$ \begin{aligned} &E^{m,n+1}\Big(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix} \Big| s;u_0,u_1,\ldots,u_n;-N\delta,v_1,\ldots,v_n \Big) \\ &= \frac{[-N\delta]_N}{[\delta+s+u_0]_N} \prod_{i=1}^{m} \frac{[\delta+s+x_i]_N}{[\delta+s+x_i-a_i]_N} \prod_{k=1}^{n} \frac{[v_k]_N}{[\delta+s-u_k]_N} \\ &\quad \cdot\; \Phi_N^{1+m,n}\Big(\begin{matrix} -N\delta-s+u_0,a_1,\ldots,a_m \\ -N\delta-s,\quad x_1,\ldots,x_m \end{matrix} \Big| \begin{matrix} u_1,\quad\ldots,\quad u_n \\ \delta+s-v_1,\ldots,\delta+s-v_n \end{matrix} \Big). \end{aligned} \tag{3.15} $$

This proposition is an elliptic version of [11], Lemma 1.22(see also [10]). Note in particular that $\Phi_N^{2,n}$ reduces to a very well-poised terminating ${}_{2n+6}E_{2n+5}$:

$$
\begin{aligned}
&\Phi_N^{2,n}\left(\begin{matrix} a_0, a_1 \\ x_0, x_1 \end{matrix}\,\middle|\,\begin{matrix} b_1 \ldots, b_n \\ c_1, \ldots, c_n \end{matrix}\right) \\
&= \frac{[a_0]_N[x_0 - x_1 + a_1]_N}{[\delta]_N[x_0 - x_1]_N} \frac{[x_0 + b_1]_N \cdots [x_0 + b_n]_N}{[x_0 + c_1]_N \cdots [x_0 + c_n]_N} \\
&\quad \cdot {}_{2n+6}E_{2n+5}\big(x_1 - x_0 - N\delta\ ;\ x_1 - x_0 + a_0, a_1, x_1 + b_1, \ldots, x_1 + b_n, \\
&\qquad\qquad -N\delta, (1-N)\delta - x_0 - c_1, \ldots, (1-N)\delta - x_0 - c_n\big).
\end{aligned}
\tag{3.16}
$$

Formula (3.14) is also valid for $m = 0$ if we set $E^{0,n+1} = 1$.

Duality transformation between $E^{m,n+2}$ and $E^{n,m+2}$

The duality transformation (2.23) between $\Phi_N^{1+m,1+n}$ and $\Phi_N^{1+n,1+m}$ is now translated into a transformation formula between $E^{m,n+2}$ and $E^{n,m+2}$:

$$
\begin{aligned}
&\frac{[\delta + s - c_1]_N[\delta + s - c_2]_N}{[d_1]_N[d_2]_N} \prod_{i=1}^{m} \frac{[\delta + s + x_i - a_i]_N}{[\delta + s + x_i]_N} \prod_{k=1}^{n} \frac{[\delta + s - u_k]_N}{[v_k]_N} \\
&\quad \cdot E^{m,n+2}\left(\begin{matrix} a_1, \ldots, a_m \\ x_1, \ldots, x_m \end{matrix}\,\middle|\, s; c_1, c_2, u_1, \ldots, u_n; d_1, d_2, v_1, \ldots, v_n\right) \\
&= \frac{[\delta + t + c_1]_N[\delta + t + c_2]_N}{[d_1]_N[d_2]_N} \prod_{k=1}^{n} \frac{[\delta + t + y_k - b_k]_N}{[\delta + t + y_k]_N} \prod_{i=1}^{m} \frac{[\delta + t - z_i]_N}{[w_i]_N} \\
&\quad \cdot E^{n,m+2}\left(\begin{matrix} b_1, \ldots, b_n \\ y_1, \ldots, y_n \end{matrix}\,\middle|\, t; -c_1, -c_2, z_1, \ldots, z_m; d_1, d_2, w_1, \ldots, w_m\right)
\end{aligned}
\tag{3.17}
$$

under the balancing condition

$$
\sum_{i=1}^{m} a_i + \sum_{k=1}^{2} (c_k + d_k) + \sum_{k=1}^{n} (u_k + v_k) = (n+1)\delta + (n+2)s,
\tag{3.18}
$$

together with the termination condition $d_2 = -N\delta$. Here the variables b_k, y_k, t, z_i, w_i are specified by

$$
\begin{aligned}
&t = d_1 + d_2 - s - \delta, \\
&b_k = \delta + s - u_k - v_k, \quad y_k = \delta + s - v_k \qquad (k = 1, \ldots, n), \\
&z_i = x_i - a_i, \qquad\qquad\quad\ w_i = d_1 + d_2 - s - x_i \quad (i = 1, \ldots, m).
\end{aligned}
\tag{3.19}
$$

Equivalently,

$$
\begin{aligned}
&E^{m,n+2}\left(\begin{matrix} a_1, \ldots, a_m \\ x_1, \ldots, x_m \end{matrix}\,\middle|\, s; c_1, c_2, u_1, \ldots, u_n; d_1, d_2, v_1, \ldots, v_n\right). \\
&= \frac{[\delta + s - c_1 - d_1]_N[\delta + s - c_2 - d_1]_N}{[\delta + s - c_1]_N[\delta + s - c_2]_N} \\
&\cdot \prod_{i=1}^{m} \frac{[\delta + s + x_i]_N[\delta + s + x_i - a_i - d_1]_N}{[\delta + s + x_i - a_i]_N[\delta + s + x_i - d_1]_N} \prod_{k=1}^{n} \frac{[v_k]_N[\delta + s - u_k - d_1]_N}{[\delta + s - u_k]_N[v_k - d_1]_N} \\
&\cdot E^{n,m+2}\left(\begin{matrix} b_1, \ldots, b_n \\ y_1, \ldots, y_n \end{matrix}\,\middle|\, t; -c_1, -c_2, z_1, \ldots, z_m; d_1, d_2, w_1, \ldots, w_m\right).
\end{aligned}
\tag{3.20}
$$

This transformation formula for the basic case was previously given in [7], and called the Bailey type transformation.

Dougall/Jackson summation formula for $E^{m,2}$

When $n=0$, formula (3.20) gives rise to the following summation formula ([15], Corollary 5.2):

$$
\begin{aligned}
&E^{m,2}\left(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix}\,\middle|\, s; c_1,c_2; d_1,d_2\right) \\
&= \prod_{k=1}^{2} \frac{[\delta+s-c_k-d_1]_N}{[\delta+s-c_k]_N} \prod_{i=1}^{m} \frac{[\delta+s+x_i]_N[\delta+s+x_i-a_i-d_1]_N}{[\delta+s+x_i-a_i]_N[\delta+s+x_i-d_1]_N}, \\
&(a_1+\cdots+a_m+c_1+c_2+d_1+d_2=\delta+2s, \quad d_2=-N\delta).
\end{aligned} \tag{3.21}
$$

As the special case where $m=1$ and $x_1=0$, this contains the Frenkel-Turaev summation formula ([5], Theorem 5.5.2) for balanced ${}_8E_7$:

$$
\begin{aligned}
&{}_8E_7\big(s; a,b,c,d,e\big) \\
&= \frac{[\delta+s]_N[\delta+s-b-c]_N[\delta+s-b-d]_N[\delta+s-c-d]_N}{[\delta+s-b]_N[\delta+s-c]_N[\delta+s-d]_N[\delta+s-b-c-d]_N}, \\
&(a+b+c+d+e=\delta+2s, \quad e=-N\delta).
\end{aligned} \tag{3.22}
$$

Transformation from $E^{m,3}$ to ${}_{2m+8}E_{2m+7}$

When $n=1$, the series $E^{1,m+2}$ in the right-hand side of (3.20) gives a very well-poised elliptic hypergeometric series ${}_{2m+8}E_{2m+7}$. Hence, our duality transformation provides a formula for rewriting balanced multiple elliptic hypergeometric series $E^{m,3}$ in terms of ${}_{2m+8}E_{2m+7}$:

$$
\begin{aligned}
&E^{m,3}\left(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix}\,\middle|\, s; c_0,c_1,c_2; d_0,d_1,d_2\right). \\
&= \frac{[d_0]_N[\delta+s-c_0-d_1]_N[\delta+s-c_1-d_1]_N[\delta+s-c_2-d_1]_N}{[d_0-d_1]_N[\delta+s-c_0]_N[\delta+s-c_1]_N[\delta+s-c_2]_N} \\
&\cdot \prod_{i=1}^{m} \frac{[\delta+s+x_i]_N[\delta+s+x_i-a_i-d_1]_N}{[\delta+s+x_i-a_i]_N[\delta+s+x_i-d_1]_N} \\
&\cdot {}_{2m+8}E_{2m+7}\big(t; d_1,d_2,e_0,e_1,e_2,u_1,\ldots,u_m,v_1,\ldots,v_m\big)
\end{aligned} \tag{3.23}
$$

under the condition

$$
\sum_{i=1}^{m} a_i + \sum_{k=0}^{2} (c_k+d_k) = 2\delta+3s, \quad d_2=-N\delta, \tag{3.24}
$$

where

$$
\begin{aligned}
&t=d_1+d_2-d_0, & &e_k=\delta+s-d_0-c_k \quad (k=0,1,2), \\
&u_i=\delta+s-d_0+x_i-a_i, & &v_i=d_1+d_2-s-x_i \quad (i=1,\ldots,m).
\end{aligned} \tag{3.25}
$$

This formula for the case where $m=1$ and $x_1=0$ recovers the following transformation formula for balanced ${}_{10}E_9$:

$$
\begin{aligned}
&{}_{10}E_9\big(s;c_0,c_1,c_2,c_3,d_0,d_1,d_2\big)\\
&= \frac{[d_0]_N[\delta+s]_N}{[d_0-d_1]_N[\delta+s-d_1]_N}\prod_{k=0}^{3}\frac{[\delta+s-c_k-d_1]_N}{[\delta+s-c_k]_N}\\
&\cdot\ {}_{10}E_9\big(\widetilde{s};\widetilde{c}_0,\widetilde{c}_1,\widetilde{c}_2,\widetilde{c}_3,\widetilde{d}_0,d_1,d_2\big)\\
&(c_0+c_1+c_2+c_3+d_0+d_1+d_2=2\delta+3s,\quad d_2=-N\delta),
\end{aligned}
\tag{3.26}
$$

where

$$
\begin{aligned}
&\widetilde{s}=d_1+d_2-d_0, &&\widetilde{d}_0=d_1+d_2-s,\\
&\widetilde{c}_k=\delta+s-d_0-c_k &&(k=0,1,2,3).
\end{aligned}
\tag{3.27}
$$

Remark 3.3 In the trigonometric case where $[\xi]=x^{\frac{1}{2}}-x^{-\frac{1}{2}}$, $x=e^{2\pi\sqrt{-1}\xi}$, the very well-poised multiple series $E^{m,n}$ gives

$$
\begin{aligned}
&W^{m,n}\bigg(\begin{matrix}a_1,\ldots,a_m\\ x_1,\ldots,x_m\end{matrix}\,\bigg|\,s;u_1,\ldots,u_n;v_1,\ldots,v_n\bigg)\\
&=\sum_{\mu\in\mathbb{N}^m} z^{|\mu|}\prod_{1\le i<j\le m}\frac{q^{\mu_i}x_i-q^{\mu_j}x_j}{x_i-x_j}\prod_{i=1}^{m}\frac{1-q^{|\mu|+\mu_i}sx_i}{1-sx_i}\\
&\quad\cdot\prod_{j=1}^{m}\frac{(sx_j;q)_{|\mu|}}{(qsx_j/a_j;q)_{|\mu|}}\left(\prod_{i=1}^{m}\frac{(a_jx_i/x_j;q)_{\mu_i}}{(qx_i/x_j;q)_{\mu_i}}\right)\\
&\quad\cdot\prod_{k=1}^{n}\frac{(v_k;q)_{|\mu|}}{(qs/u_k;q)_{|\mu|}}\left(\prod_{i=1}^{m}\frac{(x_iu_k;q)_{\mu_i}}{(qsx_i/v_k;q)_{\mu_i}}\right),
\end{aligned}
\tag{3.28}
$$

where $z=q^ns^n/a_1\cdots a_mu_1\cdots u_nv_1\cdots v_n$ (hence $z=q$ in the balanced case). In fact we have

$$
\begin{aligned}
&E^{m,n}\bigg(\begin{matrix}\alpha_1,\ldots,\alpha_m\\ \xi_1,\ldots,\xi_m\end{matrix}\,\bigg|\,\sigma;\kappa_1,\ldots,\kappa_n;\lambda_1,\ldots,\lambda_n\bigg)\\
&=W^{m,n}\bigg(\begin{matrix}a_1,\ldots,a_m\\ x_1,\ldots,x_m\end{matrix}\,\bigg|\,s;u_1,\ldots,u_n;v_1,\ldots,v_n\bigg)
\end{aligned}
\tag{3.29}
$$

under the obvious correspondence of variables $a_i=e^{2\pi\sqrt{-1}\alpha_i},\ldots$.

4 Multiple Bailey transformations for $E^{m,3}$

In this section, we derive two types of multiple generalizations of the elliptic Bailey transformation formulas due to Frenkel-Turaev [5]; one of them is the same as the one recently proved by Rosengren [15], and the other seems to be new.

Recall that our duality transformation (3.17) transforms the balanced multiple series $E^{m,n+1}$'s into $E^{n,m+1}$. As we have seen in (3.23), in the particular case of $n=2$ it transforms the balanced $E^{m,3}$ into $E^{1,m+2}\propto {}_{2m+8}E_{2m+7}$; the corresponding ${}_{2m+8}E_{2m+7}(s;u_1,\ldots,u_{2m+5})$ are completely symmetric with respect to the $2m+5$ parameters $u_1,\ldots,u_{2m+5}$. We can use this symmetry of

${}_{2m+8}E_{2m+7}$ to produce nontrivial transformation formulas for $E^{m,3}$ through the diagram

$$\begin{array}{ccc} E^{m,3} & \overset{\text{Bailey}}{\longrightarrow} & E^{m,3} \\ \downarrow & & \uparrow \\ {}_{2m+8}E_{2m+7} & \overset{\text{Symmetry}}{\longrightarrow} & {}_{2m+8}E_{2m+7}, \end{array} \tag{4.1}$$

where the vertical arrows represent the duality transformation (3.23). In this way, we obtain two types of Bailey transformation formulas for multiple elliptic hypergeometric series $E^{m,3}$. (A similar argument is used in [8] for studying multiple Sears' transformation formulas of type A in the basic case.)

Let us write again the duality transformation formula between $E^{m,3}$ and ${}_{2m+8}E_{2m+7}$: Under the conditions

$$\sum_{i=1}^{m} a_i + \sum_{k=0}^{2}(c_k + d_k) = 2\delta + 3s, \quad d_2 = -N\delta, \tag{4.2}$$

we have

$$\begin{aligned} &\frac{[d_0-d_1]_N[\delta+s-c_0]_N[\delta+s-c_1]_N[\delta+s-c_2]_N}{[d_0]_N[\delta+s-c_0-d_1]_N[\delta+s-c_1-d_1]_N[\delta+s-c_2-d_1]_N} \\ &\cdot \prod_{i=1}^{m}\frac{[\delta+s+x_i-a_i]_N[\delta+s+x_i-d_1]_N}{[\delta+s+x_i]_N[\delta+s+x_i-a_i-d_1]_N} \\ &\cdot E^{m,3}\left(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix}\,\middle|\, s; c_0,c_1,c_2; d_0,d_1,d_2\right) \\ &= {}_{2m+8}E_{2m+7}\big(t; d_1, d_2, e_0, e_1, e_2, u_1,\ldots,u_m, v_1,\ldots,v_m\big). \end{aligned} \tag{4.3}$$

Here the variables t, e_k $(k=0,1,2)$ and u_i, v_i $(i=1,\ldots,m)$ are defined by

$$\begin{aligned} &t = d_1+d_2-d_0, & &e_k = \delta+s-d_0-c_k \quad (k=0,1,2), \\ &u_i = \delta+s-d_0+x_i-a_i, & &v_i = d_1+d_2-s-x_i \quad (i=1,\ldots,m). \end{aligned} \tag{4.4}$$

Note that (4.2) implies

$$d_1+d_2+e_0+e_1+e_2+\sum_{i=1}^{n}(u_i+v_i) = (m+1)\delta+(m+2)t. \tag{4.5}$$

On the set of variables

$$(a_1,\ldots,a_m,x_1,\ldots,x_m,s,c_0,c_1,c_2,d_0,d_1,d_2), \tag{4.6}$$

we first consider the following change of variables:

$$\text{(I)} \quad \begin{aligned} &\widetilde{s} = \delta+2s-c_2-d_0-d_1, \quad \widetilde{c}_2 = \delta+s-d_0-d_1, \\ &\widetilde{d}_0 = \delta+s-c_2-d_1, \qquad \widetilde{d}_1 = \delta+s-c_2-d_0, \\ &\widetilde{c}_0 = c_0, \quad \widetilde{c}_1 = c_1, \quad \widetilde{d}_2 = d_2, \\ &\widetilde{a}_i = a_i, \quad \widetilde{x}_i = x_i \quad (i=1,\ldots,m). \end{aligned} \tag{4.7}$$

For a given function $\varphi = \varphi(a_1, a_2, \ldots)$, we will denote by $\widetilde{\varphi} = \varphi(\widetilde{a}_1, \widetilde{a}_2, \ldots)$ the function obtained by replacing the variables $a_1, a_2, \ldots$ by $\widetilde{a}_1, \widetilde{a}_2, \ldots$. Then the variables for ${}_{2m+8}E_{2m+7}$ are transformed into

$$\begin{array}{lll} \widetilde{t}=t, & \widetilde{d}_1=e_2, & \widetilde{d}_2=d_2, \\ \widetilde{e}_0=e_0, & \widetilde{e}_1=e_1, & \widetilde{e}_2=d_1, \\ \widetilde{u}_i=u_i, & \widetilde{v}_i=v_i & (i=1,\ldots,m), \end{array} \tag{4.8}$$

which is the transposition of d_1 and e_2. Hence the right-hand side of (4.3) is invariant under this change of variables. In terms of $E^{m,3}$ on the left-hand side, this invariance implies the following multiple generalization of the Bailey transformation due to Rosengren ([15], Corollary 8.2).

Theorem 4.1 (Multiple Bailey transformation I) *Under the balancing condition*

$$\sum_{i=1}^{m} a_i + \sum_{k=0}^{2}(c_k+d_k) = 2\delta+3s, \tag{4.9}$$

the following identify holds for $d_2=-N\delta$ $(N=0,1,2,\ldots,)$:

$$\begin{aligned} &E^{m,3}\left(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix}\,\middle|\, \widetilde{s};c_0,c_1,\widetilde{c}_2;\widetilde{d}_0,\widetilde{d}_1,d_2\right) \\ &=\frac{[\delta+s-c_0]_N[\delta+s-c_1]_N}{[\delta+\widetilde{s}-c_0]_N[\delta+\widetilde{s}-c_1]_N}\prod_{i=1}^{m}\frac{[\delta+s+x_i-a_i]_N[\delta+\widetilde{s}+x_i]_N}{[\delta+s+x_i]_N[\delta+\widetilde{s}+x_i-a_i]_N} \\ &\cdot E^{m,3}\left(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix}\,\middle|\, s;c_0,c_1,c_2;d_0,d_1,d_2\right), \end{aligned} \tag{4.10}$$

where

$$\begin{array}{ll} \widetilde{s}=\delta+2s-c_2-d_0-d_1, & \widetilde{c}_2=\delta+s-d_0-d_1, \\ \widetilde{d}_0=\delta+s-c_2-d_1, & \widetilde{d}_1=\delta+s-c_2-d_0. \end{array} \tag{4.11}$$

Formula (4.10) can be written alternatively in the form

$$\begin{aligned} &E^{m,3}\left(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix}\,\middle|\, \widetilde{s};c_0,c_1,\widetilde{c}_2;\widetilde{d}_0,\widetilde{d}_1,d_2\right) \\ &=\frac{[\delta+s-c_0]_N[\delta+s-c_1]_N}{[\delta+s-c_0-|a|]_N[\delta+s-c_1-|a|]_N} \\ &\quad\cdot\prod_{i=1}^{m}\frac{[\delta+s+x_i-a_i]_N[\delta+s-x_i-c_0-c_1-|a|]_N}{[\delta+s+x_i]_N[\delta+s-x_i+a_i-c_0-c_1-|a|]_N} \\ &\quad\cdot E^{m,3}\left(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix}\,\middle|\, s;c_0,c_1,c_2;d_0,d_1,d_2\right), \end{aligned} \tag{4.12}$$

where $|a|=a_1+\cdots+a_m$.

Next we consider the following change of variables:

$$\begin{array}{lll} \text{(II)} & \widetilde{s}=\delta+2s-c_0-c_1-c_2, & \widetilde{c}_0=\delta+s-c_1-c_2, \\ & \widetilde{c}_1=\delta+s-c_0-c_2, & \widetilde{c}_2=\delta+s-c_0-c_1, \\ & \widetilde{d}_k=d_k \quad (k=0,1,2), & \\ & \widetilde{a}_i=a_i, \quad \widetilde{x}_i=a_i-x_i-|a| & (i=1,\ldots,m), \end{array} \tag{4.13}$$

where $|a|=a_1+\cdots+a_m$. Then the variables for ${}_{2m+8}E_{2m+7}$ are transformed into

$$
\begin{array}{lll}
\widetilde{t}=t, & \widetilde{d}_1=d_1, & \widetilde{d}_2=d_2,\\
\widetilde{e}_k=e_k & (k=0,1,2), & \\
\widetilde{u}_i=v_i, & \widetilde{v}_i=u_i & (i=1,\ldots,m)
\end{array}
\tag{4.14}
$$

under the balancing condition (4.2), which is the simultaneous transposition of u_i and v_i $(i=1,\ldots,m)$. Since the right-hand side of (4.3) is invariant under this transformation of variables, we obtain another transformation formula for $E^{m,3}$.

Theorem 4.2 (Multiple Bailey transformation II) *Under the balancing condition*

$$
\sum_{i=1}^{m} a_i+\sum_{k=0}^{2}(c_k+d_k)=2\delta+3s,
\tag{4.15}
$$

the following identity holds for $d_2=-N\delta$ $(N=0,1,2,\ldots)$:

$$
\begin{aligned}
&E^{m,3}\left(\begin{matrix}a_1,\ldots,a_m\\ \widetilde{x}_1,\ldots,\widetilde{x}_m\end{matrix}\,\middle|\,\widetilde{s};\widetilde{c}_0,\widetilde{c}_1,\widetilde{c}_2;d_0,d_1,d_2\right)\\
&=\prod_{i=1}^{m}\frac{[\delta+s+x_i-a_i]_N[\delta+s+x_i-d_1]_N}{[\delta+s+x_i]_N[\delta+s+x_i-a_i-d_1]_N}\\
&\cdot\prod_{i=1}^{m}\frac{[\delta+\widetilde{s}+\widetilde{x}_i]_N[\delta+\widetilde{s}+\widetilde{x}_i-a_i-d_1]_N}{[\delta+\widetilde{s}+\widetilde{x}_i-a_i]_N[\delta+\widetilde{s}+\widetilde{x}_i-d_1]_N}\\
&\cdot E^{m,3}\left(\begin{matrix}a_1,\ldots,a_m\\ x_1,\ldots,x_m\end{matrix}\,\middle|\,s;c_0,c_1,c_2;d_0,d_1,d_2\right),
\end{aligned}
\tag{4.16}
$$

where

$$
\begin{array}{ll}
\widetilde{s}=\delta+2s-c_0-c_1-c_2, & \widetilde{c}_0=\delta+s-c_1-c_2,\\
\widetilde{c}_1=\delta+s-c_0-c_2, & \widetilde{c}_2=\delta+s-c_0-c_1,\\
\widetilde{x}_i=a_i-x_i-|a| & (i=1,\ldots,m).
\end{array}
\tag{4.17}
$$

The second multiple Bailey transformation (4.16) can also be written in the form

$$
\begin{aligned}
&E^{m,3}\left(\begin{matrix}a_1,\ldots,a_m\\ \widetilde{x}_1,\ldots,\widetilde{x}_m\end{matrix}\,\middle|\,\widetilde{s};\widetilde{c}_0,\widetilde{c}_1,\widetilde{c}_2;d_0,d_1,d_2\right)\\
&=\prod_{i=1}^{m}\frac{[\delta+s+x_i-d_0]_N[\delta+s+x_i-d_1]_N}{[\delta+s+x_i]_N[\delta+s+x_i-d_0-d_1]_N}\\
&\cdot\prod_{i=1}^{m}\frac{[\delta+s+x_i-a_i]_N[\delta+s+x_i-a_i-d_0-d_1]_N}{[\delta+s+x_i-a_i-d_0]_N[\delta+s+x_i-a_i-d_1]_N}\\
&\cdot E^{m,3}\left(\begin{matrix}a_1,\ldots,a_m\\ x_1,\ldots,x_m\end{matrix}\,\middle|\,s;c_0,c_1,c_2;d_0,d_1,d_2\right).
\end{aligned}
\tag{4.18}
$$

We remark that, when $m=1$ (and $x_1=0$), both (4.10) and (4.16) reduce to the elliptic Bailey transformation formula due to Frenkel-Turaev ([5], Theorem 5.5.1):

$$
\begin{aligned}
&{}_{10}E_9\big(\widetilde{s};\widetilde{c}_0,\widetilde{c}_1,\widetilde{c}_2,d_0,d_1,d_2,d_3\big)\\
&=\frac{[\delta+s-d_0]_N[\delta+s-d_1]_N[\delta+s-d_2]_N[\delta+s-d_0-d_1-d_2]_N}{[\delta+s]_N[\delta+s-d_0-d_1]_N[\delta+s-d_0-d_2]_N[\delta+s-d_1-d_2]_N}\\
&\cdot\ {}_{10}E_9\big(s;c_0,c_1,c_2,d_0,d_1,d_2,d_3\big),\\
&\widetilde{s}=\delta+2s-c_0-c_1-c_2,\quad \widetilde{c}_0=\delta+s-c_1-c_2,\\
&\widetilde{c}_1=\delta+s-c_0-c_2,\quad \widetilde{c}_2=\delta+s-c_0-c_1,
\end{aligned}
\tag{4.19}
$$

where $c_0+c_1+c_2+d_0+d_1+d_2+d_3=2\delta+3s,\quad d_3=-N\delta$.

Until now, we have discussed terminating multiple hypergeometric series

$$
E^{m,n}\left(\begin{matrix}a_1,\dots,a_m\\ x_1,\dots,x_m\end{matrix}\,\middle|\, s;u_1,\dots,u_n;v_1,\dots,v_n\right)
\tag{4.20}
$$

such that

$$
\text{(A)}\qquad v_k=-N\delta\quad\text{for some}\quad k=1,\dots,n\ \ \text{and}\ \ N=0,1,2,\dots.
\tag{4.21}
$$

As we remarked before, we can also consider terminating series under the condition

$$
\text{(B)}\qquad a_i=-\alpha_i\delta\ \ (i=1,\dots,m)\ \ \text{for some}\ \ \alpha=(\alpha_1,\dots,\alpha_m)\in\mathbb{N}^m.
\tag{4.22}
$$

There is a standard procedure for deriving identities for terminating series of type (B) from those of type (A), and *vice versa*. In the following, we derive two types of multiple Bailey transformation formulas for terminating series of type (B). In the first multiple Bailey transformation (4.12), besides the balancing condition

$$
\sum_{i=1}^{m}a_i+\sum_{k=0}^{2}(c_k+d_k)=2\delta+3s
\tag{4.23}
$$

we assume that both the two termination conditions

$$
a_i=-\alpha_i\delta\quad(i=1,\dots,m)\quad\text{and}\quad d_2=-N\delta
\tag{4.24}
$$

are satisfied. Then by using the identity

$$
\frac{[x+k\delta]_l}{[x]_l}=\frac{[x+l\delta]_k}{[x]_k}\qquad(k,l\in\mathbb{N}),
\tag{4.25}
$$

we can rewrite (4.12) into

$$
\begin{aligned}
&E^{m,3}\left(\begin{matrix}a_1,\dots,a_m\\ x_1,\dots,x_m\end{matrix}\,\middle|\,\widetilde{s};c_0,c_1,\widetilde{c}_2;\widetilde{d}_0,\widetilde{d}_1,d_2\right)\\
&=\frac{[\delta+s-c_0]_{|\alpha|}[\delta+s-c_1]_{|\alpha|}}{[\delta+s-c_0-d_2]_{|\alpha|}[\delta+s-c_1-d_2]_{|\alpha|}}\\
&\quad\cdot\prod_{i=1}^{m}\frac{[\delta+s+x_i-d_2]_{\alpha_i}[\delta+s-x_i+a_i-|a|-c_0-c_1-d_2]_{\alpha_i}}{[\delta+s+x_i]_{\alpha_i}[\delta+s-x_i+a_i-|a|-c_0-c_1]_{\alpha_i}}\\
&\quad\cdot E^{m,3}\left(\begin{matrix}a_1,\dots,a_m\\ x_1,\dots,x_m\end{matrix}\,\middle|\, s;c_0,c_1,c_2;d_0,d_1,d_2\right),
\end{aligned}
\tag{4.26}
$$

where $|a| = a_1 + \cdots + a_m$. This formula is valid for the values $d_2 = -N\delta$ ($N = 0, 1, 2, \ldots$). We now regard each of the two-sides of this formula as a function of d_2; we regard c_0 as a linear function of d_2 in the form $c_0 = \lambda - d_2$, keeping the other variables a_i, x_i ($i = 1, \ldots, m$) and s, c_1, c_2, d_0, d_1 constant. In the elliptic case, it can be checked that the both sides of (4.26) are periodic in the variable d_2 in this sense (see Lemma 3.1). Hence we conclude that (4.26) holds identically as a meromorphic function of d_2, only on the condition (B). Similarly, we can rewrite (4.18) into

$$\begin{aligned}
&E^{m,3}\left(\begin{matrix} a_1, \ldots, a_m \\ \widetilde{x}_1, \ldots, \widetilde{x}_m \end{matrix} \,\middle|\, \widetilde{s}; \widetilde{c}_0, \widetilde{c}_1, \widetilde{c}_2; d_0, d_1, d_2\right) \\
&= \prod_{i=1}^{m} \frac{[\delta + s + x_i - d_0]_{\alpha_i} [\delta + s + x_i - d_1]_{\alpha_i}}{[\delta + s + x_i]_{\alpha_i} [\delta + s + x_i - d_0 - d_1]_{\alpha_i}} \\
&\cdot \prod_{i=1}^{m} \frac{[\delta + s + x_i - d_2]_{\alpha_i} [\delta + s + x_i - d_0 - d_1 - d_2]_{\alpha_i}}{[\delta + s + x_i - d_0 - d_2]_{\alpha_i} [\delta + s + x_i - d_1 - d_2]_{\alpha_i}} \\
&\cdot E^{m,3}\left(\begin{matrix} a_1, \ldots, a_m \\ x_1, \ldots, x_m \end{matrix} \,\middle|\, s; c_0, c_1, c_2; d_0, d_1, d_2\right).
\end{aligned} \tag{4.27}$$

By a similar reasoning for the variables d_2 and d_0, we see that this formula is valid under the balancing condition together with the termination condition (B). After all, we also obtain two types of multiple Bailey transformations under the condition (B).

Theorem 4.3 *Suppose the parameters a_i ($i = 1, \ldots, m$) and c_k, d_k ($k = 0, 1, 2$) satisfy the balancing condition*

$$\sum_{i=1}^{m} a_i + \sum_{k=0}^{2} (c_k + d_k) = 2\delta + 3s \tag{4.28}$$

and the termination condition

$$a_i = -\alpha_i \delta \quad (i = 1, \ldots, m) \quad \text{for some} \quad \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m. \tag{4.29}$$

Then we have the following two types of multiple Bailey transformations:

$$\begin{aligned}
\text{(I)} \quad &E^{m,3}\left(\begin{matrix} a_1, \ldots, a_m \\ x_1, \ldots, x_m \end{matrix} \,\middle|\, \widetilde{s}; c_0, c_1, \widetilde{c}_2; \widetilde{d}_0, \widetilde{d}_1, d_2\right) \\
&= \frac{[\delta + s - c_0]_{|\alpha|} [\delta + s - c_1]_{|\alpha|}}{[\delta + s - c_0 - d_2]_{|\alpha|} [\delta + s - c_1 - d_2]_{|\alpha|}} \\
&\cdot \prod_{i=1}^{m} \frac{[\delta + s + x_i - d_2]_{\alpha_i} [\delta + s - x_i + a_i - |a| - c_0 - c_1 - d_2]_{\alpha_i}}{[\delta + s + x_i]_{\alpha_i} [\delta + s - x_i + a_i - |a| - c_0 - c_1]_{\alpha_i}} \\
&\cdot E^{m,3}\left(\begin{matrix} a_1, \ldots, a_m \\ x_1, \ldots, x_m \end{matrix} \,\middle|\, s; c_0, c_1, c_2; d_0, d_1, d_2\right),
\end{aligned} \tag{4.30}$$

where

$$\begin{aligned}
&\widetilde{s} = \delta + 2s - c_2 - d_0 - d_1, \quad &\widetilde{c}_2 = \delta + s - d_0 - d_1, \\
&\widetilde{d}_0 = \delta + s - c_2 - d_1, \quad &\widetilde{d}_1 = \delta + s - c_2 - d_0,
\end{aligned} \tag{4.31}$$

and

$$\text{(II)}\qquad E^{m,3}\left(\begin{matrix} a_1,\ldots,a_m \\ \widetilde{x}_1,\ldots,\widetilde{x}_m \end{matrix}\,\middle|\, \widetilde{s};\widetilde{c}_0,\widetilde{c}_1,\widetilde{c}_2;d_0,d_1,d_2\right)$$

$$\begin{aligned} &= \prod_{i=1}^{m} \frac{[\delta+s+x_i-d_0]_{\alpha_i}[\delta+s+x_i-d_1]_{\alpha_i}}{[\delta+s+x_i]_{\alpha_i}[\delta+s+x_i-d_0-d_1]_{\alpha_i}} \\ &\cdot \prod_{i=1}^{m} \frac{[\delta+s+x_i-d_2]_{\alpha_i}[\delta+s+x_i-d_0-d_1-d_2]_{\alpha_i}}{[\delta+s+x_i-d_0-d_2]_{\alpha_i}[\delta+s+x_i-d_1-d_2]_{\alpha_i}} \\ &\cdot\ E^{m,3}\left(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix}\,\middle|\, s;c_0,c_1,c_2;d_0,d_1,d_2\right), \end{aligned} \tag{4.32}$$

where

$$\begin{aligned} &\widetilde{s}=\delta+2s-c_0-c_1-c_2, && \widetilde{c}_0=\delta+s-c_1-c_2, \\ &\widetilde{c}_1=\delta+s-c_0-c_2, && \widetilde{c}_2=\delta+s-c_0-c_1, \\ &\widetilde{x}_i=a_i-x_i-|a| && (i=1,\ldots,m). \end{aligned} \tag{4.33}$$

(Formula (4.30) is proved in [15], Corollary 8.1.)

Remark 4.4 The second Bailey transformation formulas (4.18) and (4.32) appear to be new even in the case of multiple basic and ordinary hypergeometric series. In the basic case, under the balancing condition

$$a_1\cdots a_m c_0c_1c_2d_0d_1d_2 = q^2s^3, \tag{4.34}$$

the change of parameters

$$\begin{aligned} &\widetilde{s}=qs^2/c_0c_1c_2, && \widetilde{c}_0=qs/c_1c_2, \\ &\widetilde{c}_1=qs/c_0c_2, && \widetilde{c}_2=qs/c_0c_1, \\ &\widetilde{x}_i=a_i/a_1\cdots a_m x_i && (i=1,\ldots,m) \end{aligned} \tag{4.35}$$

implies the following two transformation formulas for $W^{m,3}$ of Remark 3.3:

$$\begin{aligned} &W^{m,3}\left(\begin{matrix} a_1,\ldots,a_m \\ \widetilde{x}_1,\ldots,\widetilde{x}_m \end{matrix}\,\middle|\, \widetilde{s};\widetilde{c}_0,\widetilde{c}_1,\widetilde{c}_2;d_0,d_1,d_2\right) \\ &= \prod_{i=1}^{m} \frac{(qsx_i/d_0;q)_N(qsx_i/d_1;q)_N}{(qsx_i;q)_N(qsx_i/d_0d_1;q)_N} \\ &\cdot \prod_{i=1}^{m} \frac{(qsx_i/a_i;q)_N(qsx_i/a_id_0d_1;q)_N}{(qsx_i/a_id_0;q)_N(qsx_i/a_id_1;q)_N} \\ &\cdot\ W^{m,3}\left(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix}\,\middle|\, s;c_0,c_1,c_2;d_0,d_1,d_2\right), \end{aligned} \tag{4.36}$$

for $d_2=q^{-N}$, $N=0,1,2\ldots$; and

$$\begin{aligned} &W^{m,3}\left(\begin{matrix} a_1,\ldots,a_m \\ \widetilde{x}_1,\ldots,\widetilde{x}_m \end{matrix}\,\middle|\, \widetilde{s};\widetilde{c}_0,\widetilde{c}_1,\widetilde{c}_2;d_0,d_1,d_2\right) \\ &= \prod_{i=1}^{m} \frac{(qsx_i/d_0;q)_{\alpha_i}(qsx_i/d_1;q)_{\alpha_i}}{(qsx_i;q)_{\alpha_i}(qsx_i/d_0d_1;q)_{\alpha_i}} \\ &\cdot \prod_{i=1}^{m} \frac{(qsx_i/d_2;q)_{\alpha_i}(qsx_i/d_0d_1d_2;q)_{\alpha_i}}{(qsx_i/d_0d_2;q)_{\alpha_i}(qsx_i/d_1d_2;q)_{\alpha_i}} \\ &\cdot\ W^{m,3}\left(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix}\,\middle|\, s;c_0,c_1,c_2;d_0,d_1,d_2\right), \end{aligned} \tag{4.37}$$

for $a_i = q^{-\alpha_i}$, $\alpha_i \in \mathbb{N}$ $(i = 1, \ldots, m)$.

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