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On a symmetric representation of Hermitian matrices and its applications to graph theory

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Abstract

We give an inequality on the inertia of Hermitian matrices with some symmetry and discuss algebraic conditions for equality. The basic results also have various applications in the theories of graph decompositions, graph embeddings, and block designs.

Key words: Fisher's inequality, Graham-Pollak theorem, Graph decomposition, Isometric embedding, Witsenhausen's inequality

AMS classifications: Primary 05C50, 05C70, 54E40, Secondary 05B05

1 Introduction

A classical problem in graph theory and discrete geometry is the isometric embedding of a finite connected graph G into a premetric space $\{0, 1, *\}^N$, called a (binary) squashed hypercube. In [9], Witsenhausen proved that if G is isometrically embedded in a premetric space $\{0, 1, *\}^N$, then the dimension N is bounded from below by the maximum of the number of positive and negative eigenvalues of the distance matrix of G. An *eigensharp graph*, a graph with equality in Witsenhausen bound, has been extensively studied by many researchers in graph theory, discrete geometry, and other related areas; for example, see [9,11,15,26] and the references therein. Recently, this result has been naturally generalized for q-ary squashed hypercubes in [23].

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The isometric embedding of graphs into squashed hypercubes is closely related to a certain graph decomposition problem. A famous theorem by Graham and Pollak [10] asserts that if the complete graph K_n of order n is the union of edge-disjoint complete bipartite subgraphs that are not necessarily isomorphic, then the number of bipartite subgraphs is no less than $n-1$. Liu and Schwenk [17] showed a q-analog of the Graham-Pollak theorem, namely, they proved that if the complete graph K_n is decomposed into edge-disjoint complete q-partite subgraphs, then the number of subgraphs is bounded from below by $(n-1)/(q-1)$. A full generalization of the Liu-Schwenk theorem was established by Gregory and vander Meulen [12] for general graphs.

On the other hand, as implied by van Lint and Wilson [16, p.433], the Graham-Pollak theorem brings to mind a well-known theorem by De Bruijn and Erdös [4], which asserts, in a quite different terminology in incidence geometry, that if the complete graph K_n is the union of edge-disjoint subgraphs each of which is isomorphic to a given complete subgraph, then the number of subgraphs is greater than or equal to n. The De Bruijn-Erdös theorem can be regarded as a special case of Fisher's inequality [7] in design of experiments.

The main aim of this paper is to provide a systematic treatment of the three topics mentioned above in an algebraic manner. The paper is organized as follows. In Section 2, we give two inequalities on the inertia of a Hermitian matrix with some "symmetry"; see Proposition 2.1 and Theorem 2.1. Our bounds are a kind of spectral inequalities and provide a q-analog of Witsenhausen's inequality and Graham-Pollak inequality. Proposition 2.1, which is a generalization of work in [13], also gives algebraic conditions for equality that are used for further arguments in Section 3. We enjoy two different proofs of the former inequality, to provide a new geometric insight of Witsenhausen's inequality and Graham-Pollak inequality. We also discuss how we can get a "symmetric" representation of a Hermitian matrix; for the details, see Subsection 2.3. In Section 3, Witsenhausen's inequality, Graham-Pollak inequality and Fisher's inequality are rephrased in a matrix form, respectively. We reprove them using the basic results of Section 2, and discuss their generalizations, together with some new related results. In Section 4 brief remarks will be made, in connection with a question by Graham and Lovász [8] on distance matrices of graphs as well as a problem posed by Colbourn and Rosa [6] on a set of Steiner triple systems with a certain "hierarchy structure".

2 Basic results

The famous spectral inequality due to A. Hirsch asserts that the modulus of an eigenvalue of an $n \times n$ $\mathbb{C}$-valued matrix A is less than n multiplied by the maximum of the entries of A. There are a number of similar spectral inequalities for positive definite matrices. In this section, we give a quite different type of spectral bounds together with two different proofs, and discuss algebraic conditions for equality. An improvement of this bound is also given.

2.1 Bounds and equality conditions

Let A be a Hermitian matrix of order n, that is, $A = A^*$, where A^* denotes the adjoint matrix of A. By $E_+(A), E_-(A), E_0(A)$, we denote the subspace of $\mathbb{C}^n$ spanned by the eigenvectors associated with the positive, negative, and zero eigenvalues of A, respectively. We denote the image and kernel of A by $\mathrm{Im}(A)$ and $\mathrm{Ker}(A)$, respectively. Clearly,

$$\mathrm{Im}(A) = E_+(A) \oplus E_-(A),$$

$$\mathrm{Im}(A) = \mathrm{Ker}(A)^\perp.$$

Here, for a subspace W of $\mathbb{C}^n$, $W^\perp$ denotes the orthogonal complement with respect to the usual Euclidean inner product. Let $n_+(A), n_-(A), n_0(A)$ be the dimension of the spaces $E_+(A), E_-(A)$, and $E_0(A)$, respectively.

The following is a slight generalization of Lemma 2.1 in [13]:

Proposition 2.1 Let n, q be positive integers with $q \geq 2$. Let A be a Hermitian matrix of order n. Let $X_1, \dots, X_q$ be $\mathbb{C}$-valued matrices of size $n \times N$ such that

$$A = \sum_{\substack{1 \leq i, i' \leq q \\ i \neq i'}} X_i X_{i'}^*. \tag{1}$$

Then the following inequality holds:

$$(q-1)N \geq \max\{n_+(A), n_-(A)\}. \tag{2}$$

In particular if $(q-1)N = n_-(A)$, then

$$\mathrm{Im}(A) = E_+(A) \oplus \bigoplus_{i \neq j} \mathrm{Im}(X_i) \qquad \text{for any } 1 \leq j \leq q, \tag{3}$$

$$\dim_{\mathbb{C}} \mathrm{Im}(X_j) = N \qquad \text{for any } 1 \leq j \leq q. \tag{4}$$

Proof. Let us fix $1 \leq j \leq q$. For $x \in \bigcap_{i \neq j} \mathrm{Ker}(X_i^*)$ we have

$$x^* A x = \sum_{i \neq i'} (X_i^* x)^* (X_{i'}^* x) = 0$$

and thus

$$(E_-(A) \oplus E_0(A)) \cap \left(\bigcap_{i \neq j} \mathrm{Ker}(X_i^*) \right) \subseteq E_0(A).$$

It follows that

$$\sum_{i\neq j} \operatorname{Im}(X_i) + E_+(A) \supseteq \operatorname{Im}(A), \tag{5}$$

since

$$\begin{aligned}\sum_{i\neq j} \operatorname{Im}(X_i) + E_+(A) &= \sum_{i\neq j} \operatorname{Ker}(X_i^*)^\perp + (E_0(A) \oplus E_-(A))^\perp \\ &= \Big(\bigcap_{i\neq j} \operatorname{Ker}(X_i^*) \cap (E_0(A) \oplus E_-(A))\Big)^\perp \\ &\supseteq E_0(A)^\perp.\end{aligned}$$

Now, by (5) it follows that

$$\begin{aligned}(q-1)N + n_+(A) &\geq \sum_{i\neq j} \dim_{\mathbb{C}} \operatorname{Im}(X_i) + \dim_{\mathbb{C}} E_+(A) &(6)\\ &\geq \dim_{\mathbb{C}} \Big(\sum_{i\neq j} \operatorname{Im}(X_i) + E_+(A)\Big) &(7)\\ &\geq \dim_{\mathbb{C}} \operatorname{Im}(A) &(8)\\ &= n_-(A) + n_+(A),\end{aligned}$$

which completes the proof of (2).

Finally, equalities in (7) and (8) imply (3). Moreover equality in (6) means (4). □

Let us look at an explicit small example of (1) for $q = 3$, where we use the notation I_n, J_n to mean the all-one matrix and the identity matrix of order n, respectively.

Example 2.1 Let

$$A = J_3 - I_3,\ X_1 = e_1,\ X_2 = e_2,\ X_3 = e_3,$$

where e_1, e_2, e_3 are the standard basis vectors in $\mathbb{R}^3$. Then we have $A = \sum_{i\neq j} X_i X_j^*$. Clearly, $n_+(A) = 1$ and $n_-(A) = 2$, and so the bound (2) holds.

Remark 2.1 When $q = 2$ and A is taken to be the distance matrix of a graph, the bound in Proposition 2.1 was first proved by Witsenhausen in a famous paper by Graham and Pollak [9]; see Subsection 3.1 of this paper. Witsenhausen's inequality was generalized for real symmetric matrices by Orlin [20, Lemma 7.4]: Orlin's theorem states that for a real matrix C of order n and a symmetric matrix $A = C + C^T$, the rank of A is bounded from below by the maximum of $n_+(A)$ and $n_-(A)$. Proposition 2.1 can be viewed as a refinement of Orlin's theorem.

In the next subsection we improve the bound (2). But, before doing this, let us look at an alternative proof of (2). The bound (2) has been considered in various combinatorial as well as geometric situations, as will be explained in Section 3 later. It seems that the second proof below gives a new theoretical insight of the bound (2).

We begin with some basic terminologies and facts about Hermitian forms. Let V be an n-dimensional $\mathbb{C}$-vector space equipped with a Hermitian form $(\cdot,\cdot)_V$. By Sylvester's law of inertia, there exists a unique triple of non-negative integers a, b, c such that $a+b+c = n$ and there exists an ordered basis $v_1, \dots, v_n$ for which

$$(v_i, v_j)_V = \begin{cases} \delta_{ij} & \text{if } 1 \leq i \leq a, \\ -\delta_{ij} & \text{if } a+1 \leq i \leq a+b, \\ 0 & \text{if } a+b+1 \leq i \leq n, \end{cases} \tag{9}$$

where $\delta_{ij} = 0$ or 1 according when $i = j$ or not. The triple (a, b, c) is called the *inertia* of the form $(\cdot,\cdot)_V$. It is known that c is the dimension of the radical

$$\mathrm{rad}(V) := \{\, w \in V \mid (w, v)_V = 0 \text{ for any } v \in V \,\}$$

and the form $(\cdot,\cdot)_V$ induces a non-degenerate Hermitian form on the quotient $V/\mathrm{rad}(V)$. A *totally isotropic subspace* of V is a subspace on which the Hermitian form vanishes.

Lemma 2.1 Let V, W be finite-dimensional $\mathbb{C}$-vector spaces equipped with Hermitian form $(\cdot,\cdot)_V$ of inertia (a, b, c), and Hermitian form $(\cdot,\cdot)_W$ of inertia (a', b', c'), respectively. Suppose there exists a $\mathbb{C}$-linear map $f : V \to W$ preserving the forms $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_W$. Then, $a \leq a'$ and $b \leq b'$.

Proof. Let $n = \dim_{\mathbb{C}} V$ and take an ordered basis $v_1, \dots, v_n$ of V satisfying (9). Since f preserves the forms $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_W$, for $1 \leq i, j \leq a+b$, we have

$$(f(v_i), f(v_j))_W = \begin{cases} \delta_{ij} & \text{if } 1 \leq i \leq a; \\ -\delta_{ij} & \text{if } a+1 \leq i \leq a+b. \end{cases}$$

So $f(v_1), \dots, f(v_{a+b})$ are linearly independent in W, which can be extended to an ordered basis of W, and which completes the proof. $\square$

The next lemma plays a substantial role in the second proof of the bound (2), which provides a geometric interpretation of (2) in terms of totally isotropic subspaces.

Lemma 2.2 Let V be a finite-dimensional $\mathbb{C}$-vector space equipped with Hermitian form $(\cdot,\cdot)_V$ of inertia (a, b, c). Let U be a totally isotropic subspace of V. Then, $\dim_{\mathbb{C}} U \leq \min\{a, b\} + c$.

Proof. Let U be a maximal totally isotropic subspace of V and $V_0 := \mathrm{rad}(V)$. Note that V_0 is included in U and U/V_0 is also a totally isotropic subspace of V/V_0 with respect to the induced non-degenerate form $(\cdot,\cdot)_{V/V_0}$ on V/V_0. It suffices to prove that $\dim_{\mathbb{C}}(U/V_0) \leq \min\{a,b\}$. Since $(\cdot,\cdot)_{V/V_0}$ has inertia $(a,b,0)$, we have a decomposition

$$V/V_0 = (V/V_0)_+ \oplus (V/V_0)_-$$

where $\dim_{\mathbb{C}}(V/V_0)_+ = a$ and $\dim_{\mathbb{C}}(V/V_0)_- = b$, and where $(\cdot,\cdot)_{V/V_0}$ is positive definite on $(V/V_0)_+$ and negative definite on $(V/V_0)_-$. By $\pi_+ : V/V_0 \to (V/V_0)_+$ and $\pi_- : V/V_0 \to (V/V_0)_-$ we denote the projection with respect to the decomposition. Now, for any $u \in U/V_0$, we see that $(u,u)_{V/V_0} = 0$. If $\pi_+(u) = \mathbf{0}$, then $u \in (V/V_0)_-$ and hence $u = \mathbf{0}$ since the form $(\cdot,\cdot)_{V/V_0}$ is negative definite on $(V/V_0)_-$. Therefore, $\pi_+|_{U/V_0}$ is injective and $\dim_{\mathbb{C}}(U/V_0) \leq a$. The same arguments show that $\pi_-|_{U/V_0}$ is injective and $\dim_{\mathbb{C}}(U/V_0) \leq b$. □

Proof II of the bound (2). We use the same notations $V, W, (\cdot,\cdot)_V, (\cdot,\cdot)_W$ as in the proof of Theorem 2.1. Let us denote the inertia of W by (a', b', c'), and define

$$U := \{(x, \mathbf{0}, \ldots, \mathbf{0}) \in W \mid x \in \mathbb{C}^N\}. \tag{10}$$

Then U is a totally isotropic subspace of $(\cdot,\cdot)_W$. It follows by Lemma 2.2 that

$$\begin{aligned}(q-1)N &= (a' + b' + c') - N \\ &\geq (a' + b' + c') - \min\{a', b'\} - c' = a' + b' - \min\{a', b'\} = \max\{a', b'\}.\end{aligned}$$

Therefore, the bound (2) follows by Lemma 2.1. □

2.2 Improving the bound (2)

There are many examples of matrices A which have a “symmetric” representation of the form (1), and for which $n_-(A) = (q-1)N$; for example, see [12,23] and Section 3 of this paper. Then, can we get the equality $n_+(A) = (q-1)N$ in the bound (2)? In the first proof of (2), the roles of $n_-(A)/(q-1)$ and $n_+(A)/(q-1)$ can be naturally interchanged and apparently $n_-(A)/(q-1)$ and $n_+(A)/(q-1)$ should be treated “symmetrically”. However, to our surprise, the equality never holds for $n_+(A)/(q-1)$ if $q \geq 3$:

Theorem 2.1 With the assumption of Proposition 2.1,

$$N \geq \max\{n_+(A), n_-(A)/(q-1)\}. \tag{11}$$

Proof. Let $V = \mathbb{C}^n$, and define a Hermitian form

$$(x,y)_V = x^* A y, \quad \text{for } x, y \in V.$$

Clearly, by the definition of A, we have

$$(x, y)_V = \sum_{i \neq i'} (X_i^* x)^* (X_{i'}^* y).$$

Let us consider a $\mathbb{C}$-linear map f from V to the q-th Cartesian product W of $\mathbb{C}^N$, that is, $f(x) := (X_1^* x, \ldots, X_q^* x)$. We also define

$$(\boldsymbol{z}, \boldsymbol{w})_W = \sum_{i \neq i'} \boldsymbol{z}_i^* \boldsymbol{w}_{i'}$$

for every $\boldsymbol{z} = (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_q), \boldsymbol{w} = (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_q) \in W$. Note that $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_W$ are forms on V and W, respectively, such that $(x, y)_V = (f(x), f(y))_W$ for every $x, y \in V$. Naturally identifying W with $\mathbb{C}^{qN}$, and note that

$$(\boldsymbol{z}, \boldsymbol{w})_W = \boldsymbol{z}^* ((J_q - I_q) \otimes I_N) \boldsymbol{w}.$$

Here $X \otimes Y$ denotes the Kronecker product of matrices X, Y. Therefore,

$$(a', b', c') = (N, (q-1)N, 0) \tag{12}$$

since $n_+(J_q - I_q) = 1$ and $n_-(J_q - I_q) = q - 1$. The result thus follows by Lemma 2.1. □

Theorem 2.1 generalizes a beautiful theorem by Gregory and vander Meulen [12, Theorem 4.1] for adjacency matrices of graphs related to a certain graph decomposition problem; see Subsection 3.2 for the details. A most important situation is the equality case for $q = 2$; in a graph-theoretic framework, many such situations have been reported. In this case, a Hermitian matrix with equality in (11) is called *eigensharp* [15].

2.3 Representing a Hermitian matrix by the form (1)

Let n, q be positive integers with $q \geq 2$, and A be a Hermitian matrix of order n. In this subsection we discuss how to find $\mathbb{C}$-valued matrices $X_1, \ldots, X_q$ satisfying the condition (1) in general.

We begin by the simple case that all X_i's are the same. By the well-known fact on matrix congruence (cf. [14, p.223]), there exists some invertible matrix P such that

$$PAP^* = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1, 0, \ldots, 0)$$

where the first $n_+(A)$ and the second $n_-(A)$ diagonal entries are 1 and -1, respectively. We consider the decomposition

$$PAP^* = \sum_{i=1}^{n_+(A)} E_i + \sum_{i=n_+(A)+1}^{n_+(A)+n_-(A)} (-E_i),$$

where each E_i is the matrix with the ith diagonal entry 1 and all others 0. With the standard basis vectors $e_1, ..., e_n$ of $\mathbb{R}^n$, it follows that for each $1 \le k \le n_+(A)$,

$$E_k = \sum_{\substack{1 \le i,j \le q, \\ i \ne j}} \left(\sqrt{\tfrac{1}{q(q-1)}} e_k \right) \left(\sqrt{\tfrac{1}{q(q-1)}} e_k \right)^*$$

where the subscripts k are independent of the choice of i and j, and similarly for $n_+(A) + 1 \le k \le n_+(A) + n_-(A)$. For each $1 \le i \le q$, let

$$X_i' := \sqrt{\tfrac{1}{q(q-1)}} (e_1, \ldots, e_{n_+(A)}, \sqrt{-1} e_{n_+(A)+1}, \ldots, \sqrt{-1} e_{n_+(A)+n_-(A)})$$

and $X_i := P^* X_i'$. Then we have the representation

$$A = \sum_{\substack{1 \le i,j \le q \\ i \ne j}} (P^* X_i')(P^* X_j')^* = \sum_{\substack{1 \le i,j \le q \\ i \ne j}} X_i X_j^*.$$

Next, what about X_i's that are not necessarily the same? In Theorem 2.2 below, we give an answer to the above questions only for the case where $n_-(A) \equiv 0 \pmod{q-1}$; similar results can also be proved for other cases, though they are omitted in detail.

For the arguments below, we use the following conventions:

$$\alpha = \min\{n_+(A), n_-(A)/(q-1)\}, \quad \beta = n_+(A) + n_-(A) - q\alpha,$$
$$\gamma = \begin{cases} 1 & \text{if } n_+(A) \ge n_-(A)/(q-1), \\ -1 & \text{otherwise.} \end{cases}$$

Theorem 2.2 Let A be a Hermitian matrix A of order n, and q be an integer with $q \ge 2$. Moreover, assume that $n_-(A) \equiv 0 \pmod{q-1}$. Then there exist $\mathbb{C}$-valued matrices $X_1, \ldots, X_q$ of size $n \times (\alpha + \beta)$, such that not all X_i's are the same and $A = \sum_{\substack{1 \le i,i' \le q, \\ i \ne i'}} X_i X_{i'}^*$.

Proof. First, take matrices $P, A_1, \ldots, A_{\alpha+\beta+1}$ such that

$$PAP^* = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_{\alpha+\beta+1} \end{pmatrix}, \tag{13}$$

where

$$A_1 = \cdots = A_\alpha = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}, \tag{14}$$

$$A_{\alpha+1} = \cdots = A_{\alpha+\beta} = (\gamma), \quad A_{\alpha+\beta+1} = 0_{n-q\alpha-\beta},$$

where the first α matrices are of order q, the next β matrices are of order 1, and $0_{n-q\alpha-\beta}$ is the zero matrix of order $n - q\alpha - \beta$; for our convenience, we do not consider A_1 if $n_-(A) = 0$ or $n_+(A) = 0$, and similarly for $A_{\alpha+1}$ and $A_{\alpha+\beta+1}$. We note, by Sylvester's law of inertia, that the diagonal matrix $\mathrm{diag}(1, -1, \ldots, -1)$ in (14) is conguent to the matrix $J_q - I_q$. So, there exists some invertible matrix Q such that $Q(J_q - I_q)Q^* = \mathrm{diag}(1, -1, \ldots, -1)$. Since

$$J_q - I_q = \sum_{\substack{1 \le i,j \le q, \\ i \ne j}} e_i e_j^*,$$

we have

$$\mathrm{diag}(1, -1, \ldots, -1) = \sum_{\substack{1 \le i,j \le q, \\ i \ne j}} (Qe_i)(Qe_j)^*. \tag{15}$$

Now, let us decompose the matrix PAP^* as follows:

$$PAP^* = \sum_{i=1}^{\alpha+\beta+1} B_i$$

where each B_i is a block diagonal matrix whose ith diagonal part is A_i and all other blocks are zero matrices. For each $1 \le k \le \alpha$, we consider a natural embedding of the vectors Qe_i (in (15)) into $\mathbb{C}^n$. Moreover, for each $1 \le k \le \beta$, $B_{k+\alpha}$ is represented by

$$B_{k+\alpha} = \sum_{\substack{1 \le i,j \le q, \\ i \ne j}} \left(\sqrt{\tfrac{\gamma}{q(q-1)}} e_{q\alpha+k}\right) \left(\sqrt{\tfrac{\gamma}{q(q-1)}} e_{q\alpha+k}\right)^*, \tag{16}$$

where the subscript $q\alpha+k$ is fixed and independent of the choice of i, j. In summary, we get $\alpha + \beta$ complex vectors, say $x_1^{(1)}, \ldots, x_1^{(\alpha+\beta)}, \ldots, x_q^{(1)}, \ldots, x_q^{(\alpha+\beta)}$, such that

$$PAP^* = \sum_{k=1}^{\alpha+\beta} \sum_{\substack{1 \le i,i' \le q, \\ i \ne i'}} x_i^{(k)} (x_{i'}^{(k)})^*,$$

which is moreover written by

$$\sum_{k=1}^{\alpha+\beta} \sum_{\substack{1\le i,i'\le q,\\ i\neq i'}} x_i^{(k)}(x_{i'}^{(k)})^* = \sum_{\substack{1\le i,i'\le q,\\ i\neq i'}} \sum_{k=1}^{\alpha+\beta} x_i^{(k)}(x_{i'}^{(k)})^* = \sum_{\substack{1\le i,i'\le q,\\ i\neq i'}} X_i'(X_{i'}')^*$$

where $X_i' := (x_i^{(1)}, \ldots, x_i^{(\alpha+\beta)})$. Writing $X_i := P^* X_i'$, we have

$$A = \sum_{\substack{1\le i,i'\le q,\\ i\neq i'}} X_i X_{i'}^*,$$

which completes the proof. □

As easily seen, the assumption "$n_-(A) \equiv 0 \pmod{q-1}$" can be relaxed by the argument similar to that used in the proof of Theorem 2.2; however, we omit the details to avoid tedious case-by-case arguments here.

Now, let $N_{q,A}$ be the minimum number of columns of X_i's satisfying the condition (1). By the argument done at the beginning of this subsection, we have

$$N_{q,A} \le n_+(A) + n_-(A), \tag{17}$$

with equality if all X_i's are the same. However, in some other cases, the bound (17) can be further improved as follows:.

Corollary 2.1 With the assumption of Theorem 2.2,

$$N_{q,A} \le n_+(A) + n_-(A) - \min\{(q-1)n_+(A), n_-(A)\}. \tag{18}$$

Moreover, the equality holds if $n_-(A) \equiv 0 \pmod{q-1}$ and $(q-1)n_+(A) \ge n_-(A)$, or $q = 2$.

Proof. The bound (18) directly follows by Theorem 2.2. The equality case can be proved by Theorem 2.1 and Theorem 2.2. □

Remark 2.2 In particular when $q = 2$, by Theorem 2.1 and Corollary 2.1, we have

$$\begin{aligned}\max\{n_+(A), n_-(A)\} \le N_{2,A} &\le n_+(A) + n_-(A) - \min\{n_+(A), n_-(A)\}\\ &= \max\{n_+(A), n_-(A)\},\end{aligned}$$

that is, $N_{2,A} = \max\{n_+(A), n_-(A)\}$. This is known as *Hermitian rank* of A [13].

At this point, we do not know whether the bound (18) is best if $n_-(A) \equiv 0$ (mod $q-1$) and $(q-1)n_+(A) < n_-(A)$, which is left for a future work. In the next section, we discuss how to construct 'integer-valued' matrices $X_1, \ldots, X_q$ with the property (1), through various problems in graph theory and related areas.

3 Applications

In this section we discuss the implications of Proposition 2.1 and Theorem 2.1 in the theories of graph decompositions, isometric embeddings of graphs, and block designs. We shall also provide various constructions of a set of integer-valued matrices satisfying the property (1).

3.1 Isometric embeddings

Throughout this subsection we are only concerned with a finite simple connected graph G. Let V be the set of vertices of G. Let d_G be the usual graph metric for G.

Let q, N be positive integers such that $q \geq 2$. Let $*$ be a symbol and

$$\mathcal{A}_{q,N} = ([q] \cup \{*\})^N,$$

where $[q] = \{0, 1, \cdots, q-1\}$. The set $\mathcal{A}_{q,N}$ is often called a *squashed (q-ary) hypercube*. We define a map $d_{\tilde{H}} : \mathcal{A}_{q,N} \times \mathcal{A}_{q,N} \to \mathbb{N} \cup \{0\}$ by

$$d_{\tilde{H}}(\mathbf{a}, \mathbf{b}) = |\{i \mid 1 \leq i \leq N,\ a_i \neq b_i,\ * \notin \{a_i, b_i\}\}|$$

for every $\mathbf{a} = (a_1, \cdots, a_N),\ \mathbf{b} = (b_1, \cdots, b_N) \in \mathcal{A}_{q,N}$. Note that the symbol $*$ does not contribute to the values of $d_{\tilde{H}}$. The pair $(\mathcal{A}_{q,N}, d_{\tilde{H}})$ is a premetric space. Namely, $d_{\tilde{H}}$ satisfies only two conditions $d_{\tilde{H}}(\mathbf{x}, \mathbf{y}) > 0$ and $d_{\tilde{H}}(\mathbf{x}, \mathbf{x}) = 0$ for $\mathbf{x}, \mathbf{y} \in \mathcal{A}_{q,N}$. The restriction of $d_{\tilde{H}}$ to the Cartesian power (hypercube) $[q]^N$ is the so-called Hamming distance d_H.

A classical problem in graph theory and discrete geometry asks when there exists a mapping $\sigma : V \to \mathcal{A}_{q,N}$ such that

$$d_G(u, v) = d_{\tilde{H}}(\sigma(u), \sigma(v)) \tag{19}$$

for every $u, v \in V$. This embedding is called an *addressing of G into $\mathcal{A}_{q,N}$*. A binary addressing was originally considered by Graham and Pollak [9] in connection with early work on routing algorithms for packet switching in data networks. An addressing is an isometric embedding from the metric space (V, d_G) to the premetric space $(\mathcal{A}_{q,N}, d_{\tilde{H}})$; in particular if $\sigma(V) \subset [q]^N$, then σ

is an isometric embedding from the graph metric space (V, d_G) to the Hamming metric space $([q]^N, d_H)$, which has been traditionally considered in graph theory, discrete geometry, and other related areas; for example, see [11,25].

A main subject in the study of graph addressings is to determine the minimum number $N = N_q(G)$ for which a graph G has an addressing into $\mathcal{A}_{q,N}$. The value $N_q(G)$ is called the *squashed-cube dimension of* G [24].

Theorem 3.1 Let G be a simple connected graph with distance matrix D. Then the following inequality holds:

$$N_q(G) \geq \max\{n_+(D), n_-(D)/(q-1)\}. \tag{20}$$

Proof. An addressing f of a graph G into $\mathcal{A}_{q,N}$ can be represented by a $|V| \times N$ array, say $Y := (y_{ij})$, with each $f(v)$ appearing in a row. For each $1 \leq \ell \leq q$, we define a zero-one matrix $X_\ell := (x_{ij}^{(\ell)})$ as follows:

$$x_{ij}^{(\ell)} = 1 \text{ if and only if } y_{ij} = \ell.$$

Then the distance matrix D of G is represented by

$$D = \sum_{i \neq j} X_i X_j^*. \tag{21}$$

The theorem thus follows by Theorem 2.1. □

Corollary 3.1 (Witsenhausen bound [9]) Let G be a simple connected graph with distance matrix D. Then

$$N_2(G) \geq \max\{n_+(D), n_-(D)\}.$$

Remark 3.1 Theorem 3.1 also improves a result by Ishii et al. [23] that for a simple connected graph G, $N_q(G) \geq \max\{n_+(D)/(q-1), n_-(D)/(q-1)\}$. In fact, Theorem 2.1 is a rewording of Theorem 4.1 of [12]. This information was not recognized by the authors of [23].

The following result is trivial in the binary case but not in higher-order cases; see [23] for other gaps between the binary and higher-order cases.

Theorem 3.2 With the notation D, Y of Theorem 3.1, we moreover assume that $N_q(G) = n_-(D)/(q-1)$. Then each element in $[q]$ appears in each column of the array Y at least once.

Proof. The theorem follows from (4) and (21). □

The following observation was made through private communication with Peter Winkler, which was communicated to us initially by Ronald Graham.

A graph is of *negative type* if it can be embedded in Euclidean space in such a way that the graph distance is the square of the Euclidean distance [26]. If two graphs G and H are of negative type, with $g \in V(G)$ represented by $(g_1, \ldots, g_m)$ in $\mathbb{R}^m$ and $h \in V(H)$ by $(h_1, \ldots, h_n) \in \mathbb{R}^n$, we can express $(g, h) \in V(G \times H)$ by the concatenation $(g_1, \ldots, g_m, h_1, \ldots, h_n)$ in $\mathbb{R}^{m+n}$. Since

$$\begin{aligned} d_{\mathbb{R}^{m+n}}((g,h),(g',h')) &= \sum (g_i - g_i')^2 + \sum (h_j - h_j')^2 \\ &= d_{\mathbb{R}^m}(g,g') + d_{\mathbb{R}^n}(h,h') \\ &= d_G(g,g') + d_G(h,h') \\ &= d_G((g,h),(g',h')), \end{aligned}$$

where $d_{\mathbb{R}^l}$ means the usual Euclidean distance in $\mathbb{R}^l$, the product graph $G \times H$ is also of negative type. Now, note that the complete graph K_q is of negative type since its vertices can be embedded as the corners of a regular simplex in $\mathbb{R}^{q-1}$ and so all Euclidean and all graph distances are $1 = 1^2$, which implies that the N-th Cartesian product K_q^N is of negative type. Let us recall the following theorem in geometry [21]:

Theorem 3.3 (Schoenberg's theorem). Let M be a finite metric space with points $v_0, \ldots, v_m$ and distances $d_{ij} := d(v_i, v_j)$. Then M is isometrically embedded in $\mathbb{R}^m$ if and only if the form

$$\sum_{i,j=0}^{m} d_{ij}^2 x_i x_j$$

is negative semi-definite on the hyperplane $x_0 + x_1 + \cdots + x_m = 0$.

By Schoenberg's characterization theorem, the distance matrix of K_q^N has exactly one positive eigenvalue, and similarly for isometrically embedded subgraphs of the q-ary hypercube.

Summarizing the arguments above and using Theorem 3.1, we get the following result.

Theorem 3.4 Let G be a finite connected graph with distance matrix D. If G is isometrically embedded in K_q^N, then $N \geq n_-(D)/(q-1)$.

3.2 Graph decompositions

A particularly interesting case of Theorem 3.1 is when a graph G is a complete graph. In this case, the complete graph K_n is decomposed into N edge-disjoint complete q-partite subgraphs if and only if K_n has a q-ary addressing of length N (cf. [10]). So, Theorem 3.1 is equivalent to a beautiful theorem by Liu and Schwenk [17], since K_n has one positive and $n-1$ negative eigenvalues.

Theorem 3.5 (Liu and Schwenk). If K_n is decomposed into N edge-disjoint complete q-partite subgraphs, then $N \geq (n-1)/(q-1)$.

We remark that our proof of Theorem 3.5 is based on Proposition 2.1 and so different from the previously known proofs by Gregory and vander Meulen [12] and Liu and Schwenk [17].

The binary case of Liu-Schwenk theorem is the famous result by Graham and Pollak published in 1973 [10].

Corollary 3.2 (Graham and Pollak). If K_n is decomposed into N edge-disjoint complete bipartite subgraphs, then $N \geq n-1$.

Now, by translating Theorem 3.2 with graph-decomposition terminology, we get the following:

Theorem 3.6 With the notation of Liu-Schwenk theorem, we moreover assume $N = (n-1)/(q-1)$ and denote N q-partite subgraphs by $K^{(1)}, \dots, K^{(N)}$. Then each $K^{(i)}$ must be proper [2].

The bound presented in the next theorem was first proved by Gregory and Vander Meulen [12, Theorem 4.1]. Below we shall prove it using Theorem 2.1, to make the paper self-contained.

Theorem 3.7 Let G be a finite graph with adjacency matrix A. Assume that G is decomposed into N edge-disjoint complete q-partite subgraphs $K^{(1)}, \dots, K^{(N)}$. Then the following hold:

(i) $N \geq \max\{n_+(A), n_-(A)/(q-1)\}$.
(ii) When $N = n_-(A)/(q-1)$, we moreover assume that p is a prime number and there exist p independent vertices Q with the same neighbors in G. In this case, if $v \in Q$ belongs to a partite set $K_j^{(i)}$ in a q-partite subgraph $K^{(i)}$, then $Q \subseteq K_j^{(i)}$.

In the proof below we denote $K^{(i)} := K(V_1^{(i)}, \dots, V_q^{(i)})$, where $1 \leq i \leq N$ and each $V_j^{(i)}$ is the j-th partite set in the i-th subgraph $K^{(i)}$. Let $x_j^{(i)}$ be the characteristic column vector of $V_j^{(i)}$ in $K^{(i)}$. Namely, for each $1 \leq k \leq |V(G)|$,

[2] *Proper* means that for any $p \leq q-1$, a complete q-partite graph cannot be viewed as a p-partite graph.

the k-th coordinate of $x_j^{(i)}$ takes value 1 if the k-th vertex (say v_k) belongs to $V_j^{(i)}$, and 0 otherwise.

Proof of (i). First, note that the adjacency matrix of $K^{(i)}$ is expressed as

$$\sum_{\substack{1\le j,j'\le q\\ j\ne j'}} x_j^{(i)} x_{j'}^{(i)*}.$$

Hence it follows that

$$A = \sum_{i=1}^{N} \sum_{j\ne j'} x_j^{(i)} x_{j'}^{(i)*} = \sum_{j\ne j'} \sum_{i=1}^{N} x_j^{(i)} x_{j'}^{(i)*} = \sum_{j\ne j'} X_j X_{j'}^*, \tag{22}$$

where $X_j := (x_j^{(1)}, \dots, x_j^{(N)})$; see also the proof of Theorem 2.2. Theorem 2.1 thus implies the assertion (i).

Proof of (ii). Let us fix $1 \le j \le q$ by (3) in Proposition 2.1 we have

$$\bigoplus_{i\ne j} \mathrm{Im}(X_i) \subseteq \mathrm{Ker}(A)^{\perp} \tag{23}$$

in particular, for each $i \ne j$ every column of $\mathrm{Im}(X_i)$ is included in $\mathrm{Ker}(A)^{\perp}$. We now denote the vertices of G by $v_1, \dots, v_n$; without loss of generality we may assume that $v_1, \dots, v_p$ form an independent set, each having the same neighbors in G. Let us take a (column) vector $z := (z_1, \dots, z_n)^* \in \mathbb{C}^n$ such that

$$z_\ell = \begin{cases} \zeta^{\ell-1} & \text{if } 1 \le \ell \le p; \\ 0 & \text{if } p+1 \le \ell \le n, \end{cases}$$

where $\zeta = \exp(2\pi\sqrt{-1}/p)$. Since $v_1, \dots, v_p$ are independent and have the same neighbors, we have $Az = 0$. So, by (23),

$$z \in \mathrm{Ker}(A) \subset \left(\bigoplus_{i\ne j} \mathrm{Im}(X_i) \right)^{\perp},$$

meaning, every column of each X_i, where $i \ne j$, is orthogonal to z. We now consider the cyclotomic polynomial

$$\Phi_p(x) = \prod_{k=1}^{p-1} (x - \zeta^k) = \sum_{i=0}^{p-1} x^i,$$

which is the minimal polynomial of ζ over $\mathbb{Q}$. Since any column of X_i is a zero-one vector, the theorem follows. $\square$

Theorem 3.7 is a natural q-ary extension of a result of Boyer and Shader [3, Theorem 3].

In [16, p.433], van Lint and Wilson said that

"This (Corollary 3.2 of this paper) brings to mind the De Bruijn-Erdös theorem, Theorem 19.1, which asserts, in a quite different terminology, and which was given a quite different proof, that if K_n is the union of N edge-disjoint complete subgraphs, then $N \geq n$."

In this section we reprove Fisher's inequality [7], a design-theoretic generalization of the De Bruijn-Erdös theorem in incidence geometry, by using the basic results of Section 2.

Let v, b, r, k, λ be positive integers such that $v \geq k$. A *BIB design with parameters* (v, b, r, k, λ) is a zero-one matrix of size $v \times b$, say $M := (m_{ij})$, which satisfies the following conditions:

$$\sum_{i=1}^{v} m_{ij} = k \quad \text{for any } 1 \leq j \leq b, \tag{24}$$

$$\sum_{j=1}^{b} m_{ij} = r \quad \text{for any } 1 \leq i \leq v, \tag{25}$$

$$\sum_{j=1}^{b} m_{i'j} m_{ij} = \lambda \quad \text{for any distinct } 1 \leq i, i' \leq v. \tag{26}$$

We say M is a BIBD(v, b, r, k, λ). Here parameters r, k, λ are called *replication number*, *block size*, and *coincidence number*, respectively [3].

By a standard counting argument we can verify (cf. [22]) that a BIBD(v, b, r, k, λ) exists only if

$$vr = bk \quad \text{and} \quad \lambda(v-1) = r(k-1). \tag{27}$$

Clearly we have $r > \lambda$ if $v > k$; a design with $v > k$ is said to be *nontrivial*. For the arguments below, note that the conditions (25) and (26) are summarized in a matrix form as follows:

$$MM^* = (r - \lambda) I_v + \lambda J_v. \tag{28}$$

Theorem 3.8 (Fisher's inequality). For any nontrivial BIB design of size $v \times b$, we have $b \geq v$.

Proof. Let X_1 be a BIBD(v, b, r, k, λ) and X_2 be its complement with parameters $(v, b, b - r, v - k, b - 2r + \lambda)$. Note that

$$X_1 X_2^* + X_2 X_1^* = 2(r - \lambda)(J_v - I_v). \tag{29}$$

[3] These terminologies originally come from design of experiments in statistics.

We denote this matrix by A. Since $r \neq \lambda$, we have $n_+(A) = 1, n_-(A) = v - 1$; in particular, the all-one vector $\mathbf{1}$ is an eigenvector associated with the positive eigenvalue $2(r - \lambda)(v - 1)$. By (11), $b \geq v - 1$. Now, the equality never holds. In fact, note that $\mathbf{1} \in E_+(A) \cap \mathrm{Im}(X_1)$, since A has a constant row sum and similarly for X_1. But this is impossible by (3). $\square$

Remark 3.2 The original proof by Fisher [7] was fairly complicated and so Bose [2] gave a simple proof using arguments on the rank of $X_1 X_1^*$. Our proof is also based on linear algebraic arguments and the complement relation of designs. Bose's proof is simpler than ours, but our proof theoretically supports the comment by van Lint and Wilson. For example, see [22] and the references therein for other proof techniques of Fisher's inequality. de Caen and Gregory [5] proved a generalization of Fisher's inequality in terms of clique partitions of multigraphs; see [5] for the proof and some related observations.

The notion of BIB design can be extended to a design with various block sizes, called a *pairwise balanced (PB) design*, which is a $v \times b$ zero-one matrix with the condition (26). A PBD is *regular* if the condition (25) is satisfied. Since the complement of a regular PBD is also regular, the same argument as above shows the following theorem.

Theorem 3.9 (cf. [22, Theorem 1.34]). For any nontrivial regular PBD of size $v \times b$, we have $b \geq v$.

The ideas used in this subsection may also be applied to other types of block designs. Specifically, it would be interesting if we can prove the Fisher-type lower bound for combinatorial t-designs. A regular PBD can be naturally regarded as a cubature formula or a Euclidean design [19, Proposition 3.1]. It is also interesting to give an alternative new proof of a Fisher-type bound for cubature and Euclidean designs.

4 Further remarks and future works

Graham and Lovász [8] remarked that it is not known whether there exists a connected graph with $n_-(D) < n_+(D)$. Recently, Azarija [1] showed that the Paley graphs of order at least 13 are such examples. By a computer search, we have checked that all graphs of order at most 10 satisfy $n_-(D) \geq n_+(D)$. This result may possibly imply there is a bias in favor of the negative eigenvalues in general.

The equality case $n_-(D) = n_+(D)$ can be first observed for a graph with 10 vertices (see Figure 1) except for the trivial graph with two vertices and only one edge. The distance matrix of this graph has inertia $(n_+(D), n_-(D), n_0(D)) = (5, 5, 0)$, which is moreover, to our surprise, a unique graph of order at most 10 with “distance-inertia” $(a, a, 0)$, except for the trivial graph with 2 vertices. This graph is also eigensharp, or equivalently, there exist two zero-one matrices of size 10×5 with the property (1). This graph, which was found by Yamada [27], seems to be very attractive, but the author could not find whether it has a special name. Similar observations hold for some other graphs. For example, the binary hypercube K_2^3 is eigensharp and also the unique graph with distance-inertia $(1, 3, 4)$. Similarly, the unique graph with distance-inertia $(1, 3, 3)$ is the subgraph of K_2^3 induced by deletion of one vertex and its incident edges, often called the gear graph. In general, it would be interesting to characterize a graph from its distance-inertia or “distance-spectrum”. As far as the author knows, only a few publications have been devoted to general theories on this topic.

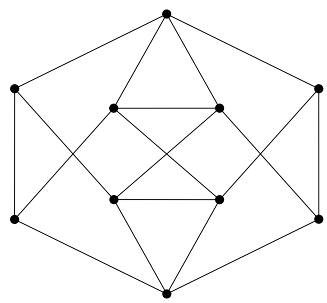

Fig. 1. Unique graph with 10 vertices and distance-inertia $(5, 5, 0)$

Now, in Section 3 we have observed some phenomena where various problems in combinatorics and geometry can be phrased in terms of $(0, 1)$-matrices with the property (1). We shall give one more combinatorial problem involving a set of matrices with the property (1).

A BIBD$(v, b, r, 3, 1)$ is called a *Steiner triple system (STS) of order* v. By (27) we have

$$r = (v - 1)/2 \quad \text{and} \quad b = v(v - 1)/6.$$

It is well known (cf. [6,22]) that an STS of order v exists if and only if $v \equiv 1, 3 \pmod 6$.

The following problem was posed by Colbourn and Rosa [6].

Problem 4.1 (Colbourn and Rosa). Let $p := \lfloor (v - 1)/3 \rfloor$. Find a set of p distinct STS of order v, say $X_0, \ldots, X_{p-1}$, such that

$$X_i + X_j \text{ is a BIBD}(v, v(v - 1)/6, v - 1, 6, 5) \text{ for any distinct } 0 \le i, j \le p - 1. \tag{30}$$

Colbourn and Rosa [6, pp.404-406] called such STSs a *compatibly minimal nesting* if $v \equiv 1 \pmod 6$, and a *compatible minimal partition* if $v \equiv 3 \pmod 6$.

If there exist such STSs, then for any q-subset Q of $[p]$

$$\sum_{\substack{i,j \in Q \\ i \neq j}} X_i X_j^* = cJ_v + dI_v$$

for some c, d. In fact, by (28) and (30) it follows that

$$\begin{aligned}
\sum_{i \neq j} X_i X_j^* &= \Big(\sum_{i \in Q} X_i\Big)\Big(\sum_{i \in Q} X_i\Big)^* - \sum_{i \in Q} X_i X_i^* \\
&= \sum_{\substack{i,j \in Q \\ i<j}} (X_i + X_j)(X_i + X_j)^* - (q-1) \sum_{i \in Q} X_i X_i^* \\
&= \sum_{\substack{i,j \in Q \\ i<j}} ((v-6)I_v + 5J_v) - (q-1) \sum_{i \in Q} \left(\left(\frac{v-3}{2} \right) I_v + J_v \right).
\end{aligned}$$

Thus, we can naturally obtain a set of matrices with the property (1) by a set of STS with the above nested structure. To find compatibly minimal nestings or compatible minimal partitions is a challenging problem in combinatorial design theory. However, to our best knowledge, no such examples have been found for a non-prime-power v and in particular the smallest open case is when $v = 15$. We also refer the reader to [18] for a generalization of the Colbourn-Rosa problem and their recent related results.

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