Enstrophy inertial range dynamics in generalized two-dimensional turbulence

Iwayama, Takahiro

Watanabe, Takeshi

(Citation)
Physical Review Fluids,1(3):034403-034403

(Issue Date)
2016-07-26

(Resource Type)
journal article

(Version)
Version of Record

PHYSICAL REVIEW FLUIDS **1**, 034403 (2016)

Enstrophy inertial range dynamics in generalized two-dimensional turbulence

Takahiro Iwayama*
Department of Planetology, Graduate School of Science, Kobe University, Kobe 657-8501, Japan

Takeshi Watanabe†
Department of Physical Science and Engineering, Graduate School of Engineering, Nagoya Institute of Technology, Gokiso, Showa-ku, Nagoya 466-8555, Japan

(Received 12 January 2016; published 26 July 2016)

We show that the transition to a k^{-1} spectrum in the enstrophy inertial range of generalized two-dimensional turbulence can be derived analytically using the eddy damped quasinormal Markovianized (EDQNM) closure. The governing equation for the generalized two-dimensional fluid system includes a nonlinear term with a real parameter α. This parameter controls the relationship between the stream function and generalized vorticity and the nonlocality of the dynamics. An asymptotic analysis accounting for the overwhelming dominance of nonlocal triads allows the k^{-1} spectrum to be derived based upon a scaling analysis. We thereby provide a detailed analytical explanation for the scaling transition that occurs in the enstrophy inertial range at $\alpha = 2$ in terms of the spectral dynamics of the EDQNM closure, which extends and enhances the usual phenomenological explanations.

DOI: 10.1103/PhysRevFluids.1.034403

I. INTRODUCTION

As is well known, forced-dissipative turbulence governed by the two-dimensional (2D) Navier-Stokes (NS) equation has peculiar properties that are absent in three-dimensional NS turbulence. In forced-dissipative 2D NS turbulence, two inertial ranges coexist because there are two inviscid quadratic invariants, kinetic energy and enstrophy. If flow is forced in a narrow band of wave numbers around a wave number k_f, the energy and enstrophy are transferred to wave numbers smaller and larger than k_f, respectively. The associated inertial ranges are called the energy and enstrophy inertial ranges, respectively. According to Kraichnan-Leith-Batchelor (KLB) phenomenology [1–3], the enstrophy spectrum in the enstrophy inertial range should take the form

$$Q(k) = C'\eta^{2/3}k^{-1}, \tag{1}$$

where C' is a dimensionless constant and η is the enstrophy dissipation rate. The KLB phenomenology assumes local interactions among triad wave numbers. However, Kraichnan [4] pointed out that nonlocal triad interactions are important in the enstrophy inertial range. He then proposed the log-corrected form of (1),

$$Q(k) = C'\eta^{2/3}k^{-1}[\ln(k/k_\mathrm{b})]^{-1/3} \quad (k \gg k_\mathrm{b}), \tag{2}$$

where k_b is the bottom wave number of the enstrophy inertial range. The nonlocality of triad interactions is a central subject in 2D turbulence studies and has been extensively reported (see, e.g., [5,6]). The spectrum (2) has been observed in many numerical simulations forced near a low-wave-number cutoff, i.e., the enstrophy inertial range is resolved but the energy inertial range is not in those simulations (see, e.g., [7–9]). Note that recent high-resolution numerical simulations suggest that spectra observed in the energy and enstrophy inertial ranges significantly deviate from

*iwayama@kobe-u.ac.jp
†watanabe@nitech.ac.jp

the spectra predicted by the KLB phenomenology when both of the inertial ranges are resolved (see, e.g., [10,11]).

The present paper investigates a generalized 2D fluid system introduced by Pierrehumbert *et al.* [12]. This system was proposed as a tool for studying the effects of nonlocality of triad interactions on 2D NS turbulence in the enstrophy inertial range. The generalized 2D fluid system is governed by

$$\frac{\partial q}{\partial t} + J(\psi,q) = \mathcal{D} + \mathcal{F}, \tag{3a}$$

$$-(-\nabla^2)^{\alpha/2}\psi = q, \tag{3b}$$

where $\psi(\boldsymbol{x},t)$ is the stream function and $q(\boldsymbol{x},t)$ is a scalar advected by an incompressible velocity $\boldsymbol{v} = \boldsymbol{e}_z \times \nabla\psi$. Here $\boldsymbol{e}_z$ is a unit vector normal to the plane of motion, J is the 2D Jacobian defined by $J(a,b) \equiv \partial_x a \partial_y b - \partial_y a \partial_x b$, α is a real number, ∇^2 is the Laplacian in 2D space, and $\mathcal{D}$ and $\mathcal{F}$ are dissipation and forcing terms, respectively. When $\alpha = 2$, this system reduces to the 2D NS system. Moreover, for appropriate values of α, this system describes various geophysical 2D fluid systems [13–16]. Over the past 15 years, researchers have studied not only turbulent properties of the generalized 2D fluid system, but also linear stability of parallel shear flows in this system (see [17] and references cited therein). In the present paper we consider the original issue in this system, that is, the dynamics in the enstrophy inertial range in turbulence governed by (3) forced in a narrow band of wave numbers around k_f on an infinite plane, with appropriate decay conditions at infinity. In particular, we focus on $\alpha > 0$. In the following, we will refer to the turbulence governed by (3) as α turbulence.

Similar to the 2D NS system, (3) has two quadratic inviscid invariants, the generalized energy $\mathcal{E}_\alpha$ and the generalized enstrophy $\mathcal{Q}_\alpha$, defined as

$$\mathcal{E}_\alpha \equiv -\tfrac{1}{2}\overline{\psi q}, \tag{4}$$

$$\mathcal{Q}_\alpha \equiv \tfrac{1}{2}\overline{q^2}, \tag{5}$$

respectively. Here the overbar denotes a spatial average over the flow domain. Hereafter, we refer to $\mathcal{E}_\alpha$ and $\mathcal{Q}_\alpha$ as the energy and enstrophy, respectively. Furthermore, we call q vorticity for simplicity, although the advected quantity q is frequently referred to as the generalized vorticity.

As there are two inviscid invariants, the cascade phenomenon of both invariants can be inferred in α turbulence. Applying the KLB phenomenology to α turbulence [12], the enstrophy spectrum $Q_\alpha(k)$ in the enstrophy inertial range takes the form

$$Q_\alpha(k) \propto k^{-(7-2\alpha)/3}. \tag{6}$$

In direct numerical simulations of (3) [12,18], when the forcing wave number k_f is small and only the enstrophy inertial range is resolved, the prediction of KLB phenomenology (6) is verified in $0 < \alpha < 2$, but fails for $\alpha > 2$. The simulation results suggest the form

$$Q_\alpha \propto k^{-1} \tag{7}$$

for $\alpha > 2$. The violation of the KLB phenomenology is attributed to the locality assumption of the interactions in wave-number space [12,18]. That is, in the KLB phenomenology, the enstrophy flux $\Pi_\alpha^{\mathcal{Q}}(k)$ scales as

$$\Pi_\alpha^{\mathcal{Q}}(k) = \{k^{5-2\alpha}Q_\alpha(k)\}^{1/2} k Q_\alpha(k), \tag{8}$$

where the rate of strain acting on the scale k^{-1} is assumed to be dominated by eddy effects on scales similar to k^{-1}. Pierrehumbert *et al.* [12] explained the spectrum (7) by an analogy with a passive scaler spectrum. In contrast, Watanabe and Iwayama [19] modified the KLB phenomenology by introducing the nonlocality of the rate of strain. In their phenomenology, hereafter referred to as the revised KLB phenomenology, the enstrophy flux scales as

$$\Pi_\alpha^Q(k) = \left\{ \int_0^k k'^{4-2\alpha} Q_\alpha(k') dk' \right\}^{1/2} k Q_\alpha(k) \tag{9}$$

and (9) provides a unified form of the enstrophy spectrum in the enstrophy inertial range for all α. That is, their spectrum reduces to (6) for $\alpha < 2$, (2) for $\alpha = 2$, and (7) for $\alpha > 2$. In direct numerical simulations that resolve only the enstrophy inertial range, Watanabe and Iwayama [20] conducted a detailed analysis of enstrophy transfer due to triad interactions in the enstrophy inertial range for $\alpha = 1$, 2, and 3. They showed that in the enstrophy inertial range, enstrophy transfer by nonlocal triad interactions dominates the transfer dynamics even for $\alpha = 1$.

The present study investigates enstrophy transfer by triad interactions in detail using a closure approximation equation. Although the generalized 2D fluid system was proposed for studying nonlocality of triad interactions in the enstrophy inertial range, detailed analyses of these triad interactions have been conducted only by Watanabe and Iwayama [20]. Moreover, their analysis was limited to $\alpha = 1$, 2, and 3. Instead, we study enstrophy transfer due to triad interactions as a function of α. However, analyzing the triad interactions for continuously varying α via direct numerical simulations would consume enormous computational time, even on state-of-the-art computers. To avoid this problem, we adopt a closure approximation equation.

Study of enstrophy transfer in the enstrophy inertial range via a closure approximation equation aims at a further and main objective. As stated above, the power law of the enstrophy spectrum in the enstrophy inertial range for forced-dissipative α turbulence undergoes transition at $\alpha = 2$. This transition has been explained by the revised KLB phenomenology, but has yet to be analytically derived. In general, analytical statistical theories of turbulence predict the statistical quantities of turbulent flows such as the energy spectrum from the governing equations or by dynamical equations approximated by the governing equations. Noting this trend of turbulence study, we identify another need in the present paper, that is, to derive the power-law transition of the enstrophy spectrum in the enstrophy inertial range for forced-dissipative α turbulence from a closure approximation equation. Furthermore, this derivation would provide a dynamical basis for the revised KLB phenomenology (9).

We employ an eddy damped quasinormal Markovianized (EDQNM) closure approximation equation in this study. Among closure approximation equations, the EDQNM closure approximation [21], or a similar approximation equation, the so-called test field model [22], has been frequently used in analytical studies of 2D NS turbulence, because it is easy to treat compared to the other closure approximation equations [23]. As studied by Herring and McWilliams [24], closure equations cannot represent intermittency due to the existence of coherent vortices and fail predictions of statistical quantities in the case of decaying problems, because the coherent vortices dominate the dynamics in decaying problems. However, in forced-dissipative problems, predictions by closure equations are still valid when a forcing is strong and destroys the coherent vortices.

Recently, an EDQNM closure approximation equation for the generalized 2D fluid system was introduced by Burgess and Shepherd (BS) [25]. They investigated the validity of the KLB phenomenology in the energy inertial range in forced-dissipative α turbulence. Their analysis indicated that the energy flux is upscale for $\alpha < 5/2$ but downscale for $\alpha > 5/2$ in the energy inertial range with the KLB spectrum. Therefore, they concluded that the KLB phenomenology works well for $\alpha < 5/2$, but fails for $\alpha > 5/2$. Indeed, in the direct numerical simulations that resolve only the energy inertial range, the energy flux was upscale even for $\alpha > 5/2$ and the spectrum predicted from the KLB phenomenology were observed only for $\alpha < 5/2$. In contrast, the present authors previously examined the enstrophy spectrum in the infrared range ($k \to 0$) and the asymptotic form of the eddy viscosity in α turbulence by the EDQNM closure approximation equation [17,26]. We derived that the enstrophy spectrum in the infrared range should take the form $Q_\alpha(k) \propto k^5$ and the asymptotic wave-number dependence of the eddy viscosity is $k^{4-\alpha}$. Our EDQNM predictions were verified in direct numerical simulations. Therefore, the EDQNM closure approximation equation is a useful tool for studying α turbulence and is employed in the present study.

The remainder of this paper is organized as follows. In Sec. II we briefly introduce the EDQNM closure approximation equation for the generalized 2D fluid system. In Sec. III we investigate how the enstrophy fluxes in the enstrophy inertial range and the triad interactions responsible for those fluxes depend on the parameter α. From the results of Sec. III we derive the power-law transition of the enstrophy spectrum in the enstrophy inertial range through an asymptotic analysis of the EDQNM approximation equation in Sec. IV. In Sec. V we discuss the modeling of enstrophy flux under conserved energy and enstrophy.

II. EDQNM CLOSURE APPROXIMATION EQUATION

We consider 2D flows in an infinite plane with appropriate decay conditions at infinity. When we set the damping term $\mathcal{D}$ with the hyperviscosity of order p and the hypofriction of order s, the evolution equation of the enstrophy spectrum is given by

$$\left\{\frac{\partial}{\partial t} + 2\nu_p k^{2p} + 2\gamma_s\left(\frac{k_\gamma}{k}\right)^{2s}\right\} Q_\alpha(k) = -\frac{\partial \Pi_\alpha^Q(k)}{\partial k} + F(k), \tag{10}$$

where $F(k)$ is a forcing localized in a narrow wave-number range around k_f, k_γ is the friction wave number, and ν_p and γ_s are the hyperviscosity and hypofriction coefficients, respectively. The enstrophy flux $\Pi_\alpha^Q(k)$ is expressed in terms of the triad enstrophy transfer function $T_\alpha^Q(k,l,m)$ by

$$\Pi_\alpha^Q(k) = \frac{1}{2}\int_k^\infty dk' \int_0^\infty dl \int_0^\infty dm\, T_\alpha^Q(k',l,m), \tag{11}$$

where $T_\alpha^Q(k,l,m)$ satisfies the detailed conservation laws of energy and enstrophy

$$k^{-\alpha}T_\alpha^Q(k,l,m) + l^{-\alpha}T_\alpha^Q(l,m,k) + m^{-\alpha}T_\alpha^Q(m,k,l) = 0, \tag{12}$$

$$T_\alpha^Q(k,l,m) + T_\alpha^Q(l,m,k) + T_\alpha^Q(m,k,l) = 0 \tag{13}$$

for wave numbers forming the triangle $\boldsymbol{k} + \boldsymbol{l} + \boldsymbol{m} = 0$, here referred to as triads (k,l,m). Moreover, $T_\alpha^Q(k,l,m)$ is symmetric under permutations of l and m,

$$T_\alpha^Q(k,l,m) = T_\alpha^Q(k,m,l). \tag{14}$$

We concentrate our discussion on the enstrophy flux $\Pi_\alpha^Q(k)$, because we consider the enstrophy inertial range in which the damping and forcing terms are negligible. Because the detailed conservation law of enstrophy and the triad enstrophy transfer function are symmetric under l and m permutations, the enstrophy flux can be decomposed into two parts:

$$\Pi_\alpha^Q(k) = \Pi_\alpha^{Q(+)}(k) - \Pi_\alpha^{Q(-)}(k), \tag{15a}$$

where

$$\Pi_\alpha^{Q(+)}(k) = \int_k^\infty dk' \int_0^k dl \int_0^l dm\, T_\alpha^Q(k',l,m), \tag{15b}$$

$$\Pi_\alpha^{Q(-)}(k) = \int_0^k dk' \int_k^\infty dl \int_l^\infty dm\, T_\alpha^Q(k',l,m). \tag{15c}$$

Here $\Pi_\alpha^{Q(+)}(k)$ indicates the enstrophy flux into all wave numbers larger than k from interactions with l and m (both less than k) and $\Pi_\alpha^{Q(-)}(k)$ is the enstrophy flux into all wave numbers less than k from interactions with l and m (both larger than k).

Using an eddy damped quasinormal approximation [23,25], the triad enstrophy transfer function can be expressed in terms of the enstrophy spectrum as

$$T_\alpha^Q(k,l,m) = \frac{2k^{\alpha+2}}{\pi lm}\theta_{klm}\left\{2a_{klm}\frac{k}{(lm)^\alpha}Q_\alpha(l)Q_\alpha(m) - b_{klm}\frac{l}{(mk)^\alpha}Q_\alpha(m)Q_\alpha(k) - b_{kml}\frac{m}{(kl)^\alpha}Q_\alpha(k)Q_\alpha(l)\right\}, \tag{16}$$

where $T_\alpha^Q(k,l,m) = 0$ outside the domain in the (l,m) plane such that k, l, and m comprise the sides of the triangle $\boldsymbol{k}+\boldsymbol{l}+\boldsymbol{m} = 0$;

$$a_{klm} = \frac{b_{klm}+b_{kml}}{2}, \tag{17a}$$

$$b_{klm} = 2\frac{(l^\alpha - m^\alpha)(k^\alpha - m^\alpha)\sqrt{1-x^2}}{k^{\alpha+2}(lm)^{\alpha-2}}, \tag{17b}$$

$$b_{kml} = 2\frac{(m^\alpha - l^\alpha)(k^\alpha - l^\alpha)\sqrt{1-x^2}}{k^{\alpha+2}(lm)^{\alpha-2}}, \tag{17c}$$

are geometric coefficients; and x indicates the cosine of the interior angle of the triangle facing side k, i.e.,

$$x = \frac{\boldsymbol{l}\cdot\boldsymbol{m}}{lm}. \tag{18}$$

The quantity θ_{klm} is the relaxation time of the third-order moments associated with the triads (k,l,m). In Ref. [25], θ_{klm} was formulated as

$$\theta_{klm} = \frac{1}{\mu_{klm}}, \tag{19a}$$

$$\mu_{klm} = \mu_k + \mu_p + \mu_q, \tag{19b}$$

$$\mu_k = \mu[k^{5-2\alpha}Q_\alpha(k)]^{1/2}. \tag{19c}$$

In contrast, previous NS turbulence studies [4,27,28] and the revised KLB phenomenology [19] suggest that the quantity μ_k should be revised to

$$\mu_k = \lambda\left\{\int_{k_\mathrm{b}}^k m^{4-2\alpha}Q_\alpha(m)dm\right\}^{1/2}. \tag{20}$$

The proportionality coefficients μ in Eq. (19c) and λ in Eq. (20) are yet to be determined. However, the conclusion of the present study is independent of the values of μ and λ. To avoid infrared divergence, we adopt the lower bound k_b of the integral in Eq. (20), where $0 < k_\mathrm{b} \ll k$. Note that, in general, there are three contributions to μ_k, that is,

$$\mu_k = \lambda\left\{\int_{k_\mathrm{b}}^k m^{4-2\alpha}Q_\alpha(m)dm\right\}^{1/2} + 2\nu_p k^{2p} + 2\gamma_s(k_\gamma/k)^{2s}. \tag{21}$$

The first term in Eq. (21) arises from the nonlinearity of the governing equation and the second and third terms in Eq. (21) represent the contributions of hyperviscosity and hypofriction on μ_k, respectively [29]. However, the second and third terms would be negligible compared to the first term, because the present study considers the dynamics in the enstrophy inertial range where the nonlinearity dominates the friction and viscous effects. We thus use (20) instead of (21).

III. DETAILED ANALYSIS OF THE ENSTROPHY FLUX IN THE ENSTROPHY INERTIAL RANGE

Using the EDQNM closure approximation equation, we analyze the enstrophy fluxes in the enstrophy inertial range and examine the characteristics of the triads responsible for those fluxes. Our detailed analysis follows Kraichnan [4] and BS [25].

The triads (k',l,m) comprising $\Pi_\alpha^{Q(+)}(k)$ and $\Pi_\alpha^{Q(-)}(k)$ satisfy the relations $m \leqslant l \leqslant k \leqslant k'$ and $k' \leqslant k \leqslant l \leqslant m$, respectively. Following BS [25], we refer to the former and latter triads as type A and type B triads, respectively. Furthermore, we refer to $\Pi_\alpha^{Q(+)}(k)$ and $\Pi_\alpha^{Q(-)}(k)$ as the enstrophy fluxes from the type A and type B triads, respectively.

Normalizing the triads of the enstrophy fluxes and considering the detailed conservation laws of energy and enstrophy, the enstrophy fluxes from the type A and type B triads are rewritten as

$$\Pi_\alpha^{Q(+)}(k) = k^3 \int_0^1 dv \int_1^{1+v} dw \frac{1-v^\alpha}{v^\alpha - w^\alpha} w^\alpha \int_1^w du\, u^{-4} T_\alpha^Q(l,lv,lw), \tag{22a}$$

$$\Pi_\alpha^{Q(-)}(k) = k^3 \int_0^1 dv \int_1^{1+v} dw \frac{w^\alpha - 1}{v^\alpha - w^\alpha} v^\alpha \int_v^1 du\, u^{-4} T_\alpha^Q(l,lv,lw), \tag{22b}$$

where u^{-1} is the medium wave number normalized by the wave number of interest k, and v and w are the minimum and maximum wave numbers normalized by the medium wave number, respectively. Equation (22) is derived in Appendix A. Note that the form of (22) is independent of the closure approximations.

The following analysis uses (16), (19a), (19b), and (20). Furthermore, we assume the following power law for the enstrophy spectrum:

$$Q_\alpha(k) = C_\alpha k^{-n}. \tag{23}$$

That is, we assume the existence of the enstrophy inertial range with infinite width and ignore that of the energy inertial range. This assumption requires simplification of the following theoretical analysis. Moreover, as mentioned in Sec. I, the numerical simulations verifying the KLB scaling and the transition to the k^{-1} spectrum support the above assumption. Then the scaling of the triad enstrophy transfer function in the EDQNM approximation depends on the sign of $5 - 2\alpha - n$. When $5 - 2\alpha - n > 0$, the triad enstrophy transfer function in the EDQNM approximation is scaled as

$$T_\alpha^Q(l,lv,lw) = l^{(1-2\alpha-3n)/2} T_\alpha^Q(1,v,w) = \left(\frac{k}{u}\right)^{(1-2\alpha-3n)/2} T_\alpha^Q(1,v,w). \tag{24}$$

Inserting (24) into (22), we obtain

$$\Pi_\alpha^{Q(+)}(k) = k^{(7-2\alpha-3n)/2} \int_0^1 dv\, L_\alpha^{Q(+)}(v), \tag{25a}$$

$$L_\alpha^{Q(+)}(v) = \int_1^{1+v} dw\, W_\alpha^{Q(+)}(v,w,n) T_\alpha^Q(1,v,w), \tag{25b}$$

$$W_\alpha^{Q(+)}(v,w,n) = \frac{1-v^\alpha}{v^\alpha - w^\alpha} w^\alpha \int_1^w du\, u^{(3n+2\alpha-9)/2}, \tag{25c}$$

$$\Pi_\alpha^{Q(-)}(k) = k^{(7-2\alpha-3n)/2} \int_0^1 dv\, L_\alpha^{Q(-)}(v), \tag{25d}$$

$$L_\alpha^{Q(-)}(v) = \int_1^{1+v} dw\, W_\alpha^{Q(-)}(v,w,n) T_\alpha^Q(1,v,w), \tag{25e}$$

$$W_\alpha^{Q(-)}(v,w,n) = \frac{w^\alpha - 1}{v^\alpha - w^\alpha} v^\alpha \int_v^1 du\, u^{(3n+2\alpha-9)/2}, \tag{25f}$$

where $L_\alpha^{Q(\pm)}$ measure the localness of the enstrophy transfer and $W_\alpha^{Q(\pm)}$ are the weight functions. Note that in the above analysis, the enstrophy fluxes from the type A and type B triads have the same k dependence. As the enstrophy flux $\Pi_\alpha^Q(k)$ is independent of the wave number in the enstrophy inertial range, from (25) we obtain

$$n = \frac{7-2\alpha}{3}. \tag{26}$$

Equation (26) is exactly the KLB scaling (6). The condition $5-2\alpha-n>0$ with (26) reduces to $\alpha<2$. The weight functions are then given by

$$W_\alpha^{Q(+)}(v,w,(7-2\alpha)/3) = \frac{1-v^\alpha}{v^\alpha-w^\alpha}w^\alpha \ln w, \tag{27a}$$

$$W_\alpha^{Q(-)}(v,w,(7-2\alpha)/3) = \frac{1-w^\alpha}{v^\alpha-w^\alpha}v^\alpha \ln v. \tag{27b}$$

In contrast, when $5-2\alpha-n<0$, the scaling (24) is replaced by

$$T_\alpha^Q(l,lv,lw) = \left(\frac{k}{u}\right)^{3-2\alpha-2n} T_\alpha^Q(1,v,w). \tag{28}$$

Furthermore, (25) is replaced by

$$\Pi_\alpha^{Q(+)}(k) = k^{6-2\alpha-2n}\int_0^1 dv\, L_\alpha^{Q(+)}(v), \tag{29a}$$

$$W_\alpha^{Q(+)}(v,w,n) = \frac{1-v^\alpha}{v^\alpha-w^\alpha}w^\alpha\int_1^w du\, u^{2n+2\alpha-7}, \tag{29b}$$

$$\Pi_\alpha^{Q(-)}(k) = k^{6-2\alpha-2n}\int_0^1 dv\, L_\alpha^{Q(-)}(v), \tag{29c}$$

$$W_\alpha^{Q(-)}(v,w,n) = \frac{w^\alpha-1}{v^\alpha-w^\alpha}v^\alpha\int_v^1 du\, u^{2n+2\alpha-7}. \tag{29d}$$

Note that $L_\alpha^{Q(\pm)}$ are unchanged. By (29) the power-law exponent of the enstrophy spectrum in the enstrophy inertial range is

$$n = 3-\alpha. \tag{30}$$

Then the weight functions are equivalent to (27). Inserting (30) into the condition $5-2\alpha-n<0$, the condition reduces to $\alpha>2$. Thus, merely revising the form of θ_{klm} does not achieve the spectrum (7).

Although we have not derived the enstrophy spectrum (7) in the enstrophy inertial range for $\alpha>2$, we continue our discussion here and derive (7) in the next section. We first examine the α dependence of the enstrophy fluxes from the type A and type B triads in the enstrophy inertial range. In Fig. 1 the enstrophy fluxes from the type A and type B triads are normalized as $\Pi_\alpha^{Q(+)}(k)/\Pi_\alpha^Q(k)$ and $\Pi_\alpha^{Q(-)}(k)/\Pi_\alpha^Q(k)$ and plotted as functions of α. Note that the net enstrophy flux $\Pi_\alpha^Q(k)$ equals the enstrophy dissipation rate. The integrals in Eqs. (25) and (29) were resolved to $dv=dw=2\times10^{-4}$. We confirmed that the results are unchanged at higher resolution ($dv=dw=1\times10^{-4}$). According to Fig. 1, when α is small, $\Pi_\alpha^{Q(+)}(k)$ and $\Pi_\alpha^{Q(-)}(k)$ far exceed the net enstrophy flux and a small difference between them derives the net enstrophy flux downward. The upward enstrophy flux from the type B triads exceeds the net enstrophy flux when $\alpha\lesssim0.79$. As α increases, both upward and downward enstrophy fluxes decrease and type A fluxes dominate the net enstrophy flux up to and beyond $\alpha\simeq2$. The normalized fluxes $\Pi_\alpha^{Q(+)}(k)/\Pi_\alpha^Q(k)$ and $\Pi_\alpha^{Q(-)}(k)/\Pi_\alpha^Q(k)$ for various α are listed in Table I.

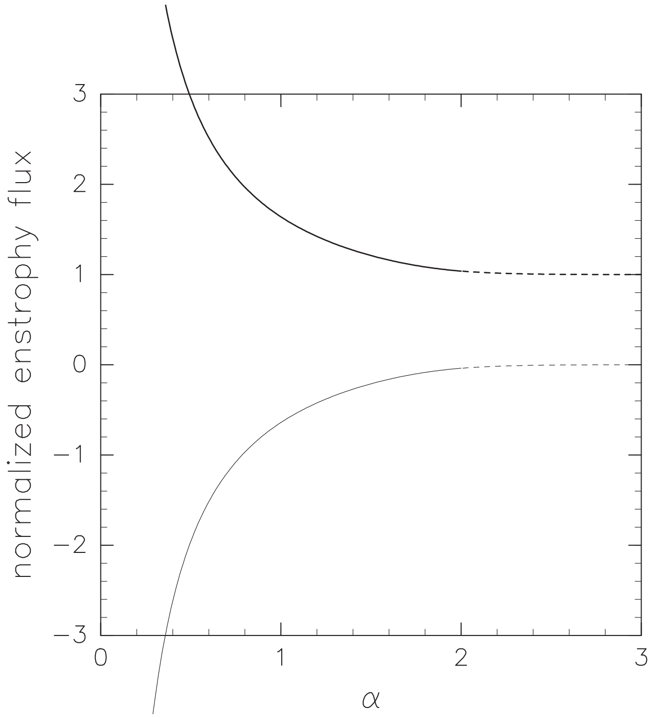

FIG. 1. Plot of α dependence of $\Pi_\alpha^{Q(+)}(k)$ and $\Pi_\alpha^{Q(-)}(k)$ normalized by the net enstrophy flux $\Pi_\alpha^{Q}(k)$ in the enstrophy inertial range. The bold and thin lines indicate $\Pi_\alpha^{Q(+)}(k)/\Pi_\alpha^{Q}(k)$ and $-\Pi_\alpha^{Q(-)}(k)/\Pi_\alpha^{Q}(k)$, respectively. Dotted lines indicate the normalized enstrophy fluxes for $\alpha > 2$, where our analysis fails to derive the spectrum (7).

We now compare the above results with those of Watanabe and Iwayama [20], who numerically simulated (3) for $\alpha = 1$, 2, and 3. They assumed large-scale drag and hyperviscosity and adopting forcing at small wave numbers. Their simulations adequately resolved the enstrophy inertial range but not the energy inertial range. They also introduced a coarse-grained triad enstrophy transfer function and its associated total enstrophy flux by dividing the wave-number space into circular logarithmic shells. As the coarse-grained triad enstrophy transfer function satisfies the detailed conservation laws, the associated total flux is decomposed into two parts, similar to (15a). In Fig. 6 of Ref. [20], $\Pi_\alpha^{Q(+)}(k) \simeq 1.4\eta_\alpha$ and $\Pi_\alpha^{Q(-)}(k) \simeq 0.4\eta_\alpha$ for $\alpha = 1$, although the plateau wave-number ranges for enstrophy fluxes from the type A and type B triads are narrow. In the enstrophy inertial range $\Pi_\alpha^{Q(+)}(k) \simeq \eta_\alpha$ for $\alpha = 2$ and 3. Thus, the enstrophy fluxes calculated by the EDQNM closure approximation equation agree reasonably well with the DNS results.

TABLE I. Normalized enstrophy fluxes $\Pi_\alpha^{Q(\pm)}(k)$ for various α.

α	$\Pi_\alpha^{Q(+)}(k)/\Pi_\alpha^{Q}(k)$	$\Pi_\alpha^{Q(-)}(k)/\Pi_\alpha^{Q}(k)$
0.5	2.96	1.96
1	1.64	0.64
2	1.05	0.05
3	1.00	0.00

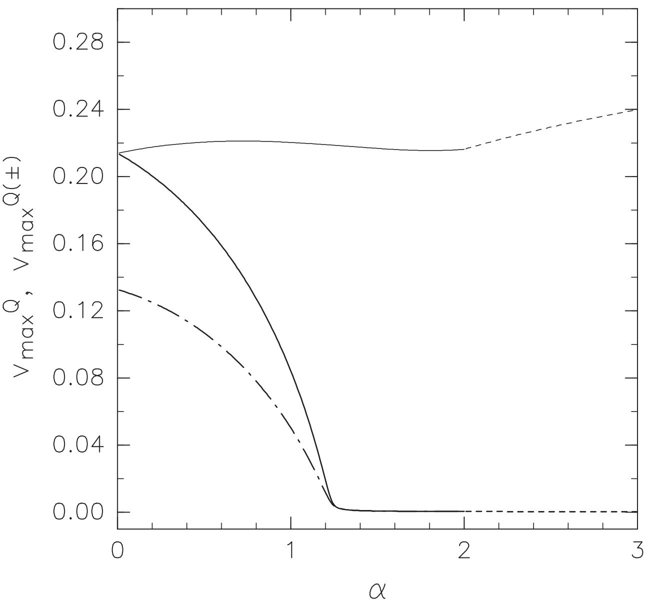

FIG. 2. Plot of α dependence of v at which $L_\alpha^{Q(+)}(v)$, $L_\alpha^{Q(-)}(v)$, and $L_\alpha^{Q}(v)$ are maximized. Bold and thin lines indicate $v_{\max}^{Q(+)}$ and $v_{\max}^{Q(-)}$, respectively. The dash-dotted line indicates $v_{\max}^{Q}$.

Next we examine the characteristics of the triads responsible for enstrophy transfer in the enstrophy inertial range. To this end, we examine the α dependence of the localness of the enstrophy transfer. The functions $L_\alpha^{Q(+)}(v)$, $L_\alpha^{Q(-)}(v)$, and $L_\alpha^{Q}(v) \equiv L_\alpha^{Q(+)}(v) - L_\alpha^{Q(-)}(v)$ are unimodal functions that increase as v decreases and vanish at $v \to 1$. Figure 2 plots v for which $L_\alpha^{Q(+)}(v)$, $L_\alpha^{Q(-)}(v)$, and $L_\alpha^{Q}(v)$ are maximized at each α. For $\alpha \simeq 0$, $L_\alpha^{Q(+)}$ and $L_\alpha^{Q(-)}$ are maximized at $v_{\max}^{Q(\pm)} \simeq 0.2$, while $L_\alpha^{Q}(v)$ is maximized at $v_{\max}^{Q} \simeq 0.13$. The peaks of $L_\alpha^{Q(+)}$ and L_α^{Q} shift to smaller v as α increases, approaching 0 at $\alpha \gtrsim 1.2$. Meanwhile the peak of $L_\alpha^{Q(-)}$ is insensitive to α. This indicates that the enstrophy transfers through all triads and the type A triads become nonlocal as α increases.

The above results are summarized as follows. The net downward enstrophy flux in the enstrophy inertial range competes with the downward enstrophy flux from the type A triads and the upward flux from the type B triads. At smaller α, both enstrophy fluxes far exceed the net enstrophy flux. That is, very small difference between two enstrophy fluxes drives the net enstrophy flux downward. Both enstrophy fluxes decrease as α enlarges and the type B flux almost vanishes at $\alpha = 2$. When $\alpha = 2$, the net enstrophy flux comprises only the enstrophy flux from type A triads. Furthermore, as α increases, nonlocal triads with small ratios of minimum to medium wave number contribute significantly to the enstrophy transfer.

IV. DERIVATION OF THE ENSTROPHY SPECTRUM IN THE ENSTROPHY INERTIAL RANGE FOR $\alpha > 2$ BY THE EDQNM APPROXIMATION EQUATION

In the previous section we failed to derive the enstrophy spectrum (7) in the enstrophy inertial range for $\alpha > 2$. However, we inferred that for $\alpha > 2$, the net enstrophy flux in the enstrophy inertial range is contributed only by nonlocal type A triads. Therefore, in this section we estimate the enstrophy flux produced by these triads and hence derive the power-law exponent of the enstrophy spectrum in the enstrophy inertial range for $\alpha > 2$.

To calculate the contribution of nonlocal triads $m \ll l < k < k'$ to the enstrophy flux, we asymptotically expand the flux with respect to a small parameter $\epsilon \equiv m/k'$ (see Appendix B for details). When $\alpha > 0$, the enstrophy flux produced by the nonlocal triads $m \ll l < k < k'$ is given by

$$\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k) \simeq \frac{1}{4}\theta_{kk(ak)}\left[\underbrace{-k^3\frac{\partial k^{-1}Q_\alpha(k)}{\partial k}\int_{k_\mathrm{b}}^{ak} m^{4-2\alpha}Q_\alpha(m)dm}_{\equiv \mathrm{I}} + \underbrace{\alpha k^{-\alpha}\{Q_\alpha(k)\}^2\int_{k_\mathrm{b}}^{ak} m^{5-\alpha}dm}_{\equiv \mathrm{II}}\right]. \quad (31)$$

Here a is a small parameter such that $0 < k_\mathrm{b} \ll ak \ll k$. The nonlocal triads contribute to the enstrophy flux through two terms, I and II, in Eq. (31). Assuming the self-similar enstrophy spectrum (23) and $5 - 2\alpha - n \neq 0$, we now estimate these two terms. Note that the following estimation is independent of the form of θ_{klm}. The ratio of the terms I to II is given by

$$\frac{\mathrm{I}}{\mathrm{II}} = \begin{cases} \frac{(6-\alpha)(1+n)}{\alpha(5-2\alpha-n)}a^{-1-\alpha-n} & (5-2\alpha-n>0) \\ \frac{(6-\alpha)(1+n)}{\alpha(n+2\alpha-5)}a^{\alpha-6}\left(\frac{k_\mathrm{b}}{k}\right)^{5-2\alpha-n} & (5-2\alpha-n<0). \end{cases} \quad (32)$$

When the exponent of a in Eq. (32) is negative, the enstrophy flux $\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k)$ is contributed more by term I than by term II, because a is small. For a physically relevant system, we require $\alpha < 3$ [30]. Furthermore, now that we are considering a sufficient width of the enstrophy inertial range we may suppose that $k_\mathrm{b}/k = O(a^2)$ without loss of generality. Term I will then dominate over term II if, as expected, the turbulent spectrum is steeper than the enstrophy equipartition spectrum, i.e., $n > -1$. Therefore, we obtain

$$\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k) \simeq -\frac{1}{4}\theta_{kk0}k^3\frac{\partial k^{-1}Q_\alpha(k)}{\partial k}\int_{k_\mathrm{b}}^{ak} m^{4-2\alpha}Q_\alpha(m)dm. \quad (33)$$

Equation (33) is the α-turbulence version of the nonlocal enstrophy flux derived by Kraichnan [4] for 2D NS turbulence. As the integral in Eq. (33) has dimensions of $\mathcal{T}^{-2}$, it is interpreted as the square of the mean shear with wave numbers less than ak. Note that (33) vanishes in the enstrophy equipartition spectrum $Q_\alpha(k) \propto k^1$.

From (33) we derive the power-law exponent of the enstrophy spectrum in the enstrophy inertial range. Again, we assume the self-similarity of the enstrophy spectrum (23). The form of the nonlocal enstrophy flux depends on the sign of $5 - 2\alpha - n$.

When $5 - 2\alpha - n > 0$, the integral in Eq. (33) is evaluated as

$$\int_{k_\mathrm{b}}^{ak} m^{4-2\alpha}Q_\alpha(m)dm \simeq \frac{C_\alpha}{5-2\alpha-n}(ak)^{5-2\alpha-n}.$$

Thus, one obtains

$$\theta_{kk(ak)}\int_{k_\mathrm{b}}^{ak} m^{4-2\alpha}Q_\alpha(m)dm \simeq \frac{C_\alpha^{1/2}}{2\lambda\sqrt{5-2\alpha-n}}(a^2k)^{(5-2\alpha-n)/2},$$

$$\frac{\partial k^{-1}Q_\alpha(k)}{\partial k} = -(n+1)C_\alpha k^{-n-2}.$$

These results lead to

$$\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k) \simeq \frac{(n+1)C_\alpha^{3/2}}{8\lambda\sqrt{5-2\alpha-n}}a^{5-2\alpha-n}k^{(7-2\alpha-3n)/2}. \quad (34)$$

As $\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k)$ has the same k dependence with $\Pi_\alpha^{\mathcal{Q}(\pm)}(k)$ in Eq. (25), in the enstrophy inertial range, the exponent n in this case should be the KLB scaling (26). Moreover, the condition $5 - 2\alpha - n > 0$ yields $\alpha < 2$. As the exponent of the small parameter a in Eq. (34) is positive, the enstrophy flux generated by the nonlocal type A triads is negligible in this case. Note that this analysis is consistent with the behavior of $v_{\max}^{\mathcal{Q}(+)}$ as shown in Fig. 2. The small parameter a in Eq. (34) can be interpreted as a small v in Eq. (25a). When $v \ll 1$, (25a) reduces to

$$\Pi_\alpha^{\mathcal{Q}(+)}(k) = k^{(7-2\alpha-3n)/2} L_\alpha^{\mathcal{Q}(+)} v \tag{35}$$

and from (34) and (35) we obtain

$$L_\alpha^{\mathcal{Q}(+)}(k) \propto v^{4-2\alpha-n}. \tag{36}$$

Expressing n by (26), the exponent of v in Eq. (36) is positive for $\alpha < 5/4$ and negative for $\alpha > 5/4$. Therefore, $v_{\max}^{\mathcal{Q}(+)}$ collapses to zero when $\alpha \geqslant 5/4$ as shown in Fig. 2.

When $5 - 2\alpha - n < 0$, the integral in Eq. (33) becomes

$$\int_{k_\mathrm{b}}^{ak} m^{4-2\alpha} Q_\alpha(m)dm \simeq \frac{C_\alpha}{|5-2\alpha-n|} k_\mathrm{b}^{5-2\alpha-n}.$$

Thus, one obtains

$$\theta_{kk(ak)} \int_{k_\mathrm{b}}^{ak} m^{4-2\alpha} Q_\alpha(m)dm \simeq \frac{C_\alpha^{1/2}}{2\lambda\sqrt{|5-2\alpha-n|}} k_\mathrm{b}^{(5-2\alpha-n)/2}.$$

These results lead to

$$\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k) \simeq \frac{(n+1)C_\alpha^{3/2}}{8\lambda\sqrt{|5-2\alpha-n|}} k_\mathrm{b}^{(5-2\alpha-n)/2} k^{1-n}. \tag{37}$$

As the enstrophy flux is independent of wave number in the enstrophy inertial range, (37) yields

$$n = 1. \tag{38}$$

Moreover, inserting (38) into the condition $5 - 2\alpha - n < 0$, one obtains $\alpha > 2$. That is, we have identified the transition of the power-law exponent of the enstrophy spectrum in the enstrophy inertial range. Furthermore, as the exponent of k_b in Eq. (37) is negative, the enstrophy flux due to the nonlocal type A triads becomes significant when $\alpha > 2$.

When $\alpha = 2$, (33) reduces to the first term of (2.6) in Basdevant *et al.* [28]. To leading order, the enstrophy spectrum in the enstrophy inertial range is then given by (2). Indeed, if the enstrophy spectrum $Q_2(k)$ is given by (2),

$$\int_{k_\mathrm{b}}^{ak} Q_2(m)dm = \frac{3}{2} C' \eta^{2/3} [\ln(ak/k_\mathrm{b})]^{2/3}.$$

Furthermore, we obtain

$$\theta_{kk(ak)} \int_{k_\mathrm{b}}^{ak} Q_2(m)dm \simeq \sqrt{\frac{3}{8}} C'^{1/2} \eta^{1/3} [\ln(k/k_\mathrm{b})]^{1/3},$$
$$\frac{\partial k^{-1} Q_2(k)}{\partial k} = -C' \eta^{2/3} k^{-3} [\ln(k/k_\mathrm{b})]^{-1/3} \left[2 + \frac{1}{3}\{\ln(k/k_\mathrm{b})\}^{-1}\right]. \tag{39}$$

Therefore, to leading order, the enstrophy flux $\Pi_2^{\mathcal{Q}(+)}(k)$ is independent of k when the enstrophy spectrum is described by (2).

According to the above analysis (and that of Sec. III), to derive the power-law exponent of the enstrophy spectrum for $\alpha > 2$ in the enstrophy inertial range, we require not only the revision of the form of θ_{klm} but also nonlocal dominance of the triad interactions. The revised KLB phenomenology (9) merely modifies the characteristic time scale from $[k^{5-2\alpha} Q_k]^{-1/2}$ to the integral form. Here we showed that this modification embodies the nonlocal effect on θ_{klm} and the dominant nonlocality of triad interactions.

V. DISCUSSION

Basdevant *et al.* [28] proposed an expression for enstrophy flux from nonlocal triads that conserves both energy and enstrophy in NS turbulence. Here we discuss a generalization of their enstrophy flux to α turbulence. In Sec. IV we found that the nonlocal enstrophy flux $\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k)$ ceases as the nonlocal parameter $a \to 0$ when $\alpha < 2$, but increases when $\alpha \geqslant 2$. Therefore, we consider the case $\alpha \geqslant 2$ in the following. The enstrophy transfer function associated with the enstrophy flux $\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k)$ is given by

$$T_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k) = -\frac{\partial}{\partial k}\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k). \tag{40}$$

As the enstrophy conservation is expressed by $\int_0^\infty T_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k)dk = 0$, (40) indicates the enstrophy conservation provided that the enstrophy flux $\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k)$ vanishes at $k = 0$ and $k = \infty$. From the relation between energy and enstrophy, the energy transfer function associated with $\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k)$ is given by

$$T_{\alpha\mathrm{nl}}^{\mathcal{E}(+)}(k) = -k^{-\alpha}\frac{\partial}{\partial k}\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k) = -\frac{\partial}{\partial k}\left\{k^{-\alpha}\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k)\right\} - \alpha k^{-\alpha-1}\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k). \tag{41}$$

In order to satisfy the energy conservation, the second term in the last expression of (41) must be at least rewritten as the divergence of a function. Using (33), we have

$$\begin{aligned}\alpha k^{-\alpha-1}\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k) &= -\frac{1}{4}\alpha k^{2-\alpha}\theta_{kk(ak)}\frac{\partial k^{-1}Q_\alpha(k)}{\partial k}\int_{k_\mathrm{b}}^{ak} m^{4-2\alpha}Q_\alpha(m)dm \\ &= -\frac{1}{4}\alpha k^{2-\alpha}\frac{\partial}{\partial k}\left\{k^{-1}Q_\alpha(k)\theta_{kk(ak)}\int_{k_\mathrm{b}}^{ak} m^{4-2\alpha}Q_\alpha(m)dm\right\}.\end{aligned} \tag{42}$$

This modification is valid for $\alpha > 2$ because $\theta_{kk(ak)}\int_{k_\mathrm{b}}^{ak} m^{4-2\alpha}Q_\alpha(m)dm$ is independent of k for $\alpha > 2$ in the enstrophy inertial range. When $\alpha = 2$, $\theta_{kk(ak)}\int_{k_\mathrm{b}}^{ak} m^{4-2\alpha}Q_\alpha(m)dm$ is a logarithmic function of k, so the modification includes a logarithmic error. Because the last expression in Eq. (42) includes a factor $k^{2-\alpha}$, the energy transfer function associated with $\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k)$ cannot be expressed in an energy-conserving form, except when $\alpha = 2$. Therefore, the nonlocal enstrophy flux proposed by Basdevant *et al.* [28], which conserves both energy and enstrophy, cannot be generalized to $\alpha > 2$.

VI. SUMMARY

We have comprehensively investigated the spectral nonlocality of the forced-dissipative α turbulence by the EDQNM closure approximation equation proposed by BS [25]. Prior to the present study, the nonlocality of energy transfer by triad interactions in the energy inertial range in forced-dissipative α turbulence had been investigated by BS. In contrast, investigations of the nonlocality of enstrophy transfers by triad interactions in the enstrophy inertial range were lacking. To fill this gap, we employed the EDQNM approximation equation and investigated the enstrophy inertial range dynamics. For $\alpha \approx 0$, both upward and downward enstrophy fluxes far exceeded the net downward enstrophy flux. A small difference between them drove the net enstrophy flux downward. Both the upward and downward enstrophy fluxes decreased with increasing α. When $\alpha \gtrsim 2$ the downward enstrophy flux dominated the net enstrophy flux. Triads responsible for the downward enstrophy flux became nonlocal, while those responsible for the upward enstrophy flux were insensitive to α. Investigating the nature of the triad interactions responsible for the enstrophy transfer and asymptotic analysis of the EDQNM approximation equation, we theoretically derived the transition of the exponent of the enstrophy spectrum in the enstrophy inertial range. The transition occurred at $\alpha = 2$, consistent with previous numerical simulations and our revised KLB phenomenology [19]. As demonstrated in the present analysis, the revised KLB phenomenology modifies the characteristic time scale not only through the nonlocal effects on the decay of the third-order moments associated with the triads but also by the dominant nonlocality of the triad interactions.

As mentioned in Sec. I, there is the another transition of the enstrophy spectrum in forced-dissipative α turbulence. The power law of the enstrophy spectrum in the energy inertial range in forced-dissipative α turbulence undergoes transition at $\alpha = 5/2$ [25,31]. The power-law scaling of the enstrophy spectrum past the transition has not been theoretically derived yet; this is a subject for future study.

ACKNOWLEDGMENTS

We would like to thank the anonymous reviewers for their constructive suggestions on the manuscript. This study was supported by a Grant-in-aid for Scientific Research (C) No. 15K05288 from Japan Society for the Promotion of Science. The GFD-DENNOU library was used for drawing the figures.

APPENDIX A: DERIVATION OF (22)

To theoretically investigate the enstrophy flux, we first normalize type A and type B triads. The medium wave number l is scaled by the wave number of interest k. The minimum and maximum wave numbers are further scaled by the medium wave number as lv and lw, respectively. The normalization of type A and type B triads are given by (A1) and (A2), respectively:

$$l = \frac{k}{u}, \quad m = lv, \quad k' = lw, \tag{A1}$$

$$l = \frac{k}{u}, \quad k' = lv, \quad m = lw. \tag{A2}$$

The integrations (15b) and (15c) are rewritten as

$$\int_k^\infty dk' \int_0^k dl \int_0^l dm = \int_0^1 dv \int_1^{1+v} dw \int_1^w du \frac{k^3}{u^4} \tag{A3a}$$

and

$$\int_0^k dk' \int_k^\infty dl \int_l^\infty dm = \int_0^1 dv \int_1^{1+v} dw \int_v^1 du \frac{k^3}{u^4}, \tag{A3b}$$

respectively.

Next we rewrite the triad enstrophy transfer functions in Eqs. (15b) and (15c). From the detailed conservations of energy (12) and enstrophy (13), the triad enstrophy transfer function is expressed as

$$T_\alpha^Q(k',l,m) = \frac{m^\alpha - l^\alpha}{k'^\alpha - m^\alpha} \frac{k'^\alpha}{l^\alpha} T_\alpha^Q(l,m,k'). \tag{A4}$$

Inserting the scaled wave numbers (A1) and (A2) in Eq. (A4), the triad enstrophy transfer functions for type A and type B triads are expressed as

$$T_\alpha^Q(k',l,m) = \frac{1 - v^\alpha}{v^\alpha - w^\alpha} w^\alpha T_\alpha^Q(l,lv,lw) \tag{A5a}$$

and

$$T_\alpha^Q(k',l,m) = \frac{w^\alpha - 1}{v^\alpha - w^\alpha} v^\alpha T_\alpha^Q(l,lv,lw), \tag{A5b}$$

respectively. Using (A3) and (A5), (15b) and (15c) are rewritten as (22).

APPENDIX B: DERIVATION OF (31)

First, we recognize the order of the integration in $\Pi_\alpha^{\mathcal{Q}(+)}(k)$ as follows:

$$\int_k^\infty dk' \int_0^k dl \int_0^l dm = \int_k^\infty dk' \int_0^k dm \int_m^k dl = \int_0^k dm \int_k^\infty dk' \int_m^k dl.$$

We explicitly evaluate the nonlocal part of the enstrophy flux $\Pi_\alpha^{\mathcal{Q}(+)}(k)$ from triads satisfying $m \ll l < k < k'$, $\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k)$. Introducing a small parameter a, $\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k)$ is expressed as

$$\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k) = \int_0^{ak} dm \int_k^{k+m} dk' \int_{k'-m}^k dl\, T_\alpha^{\mathcal{Q}}(k',l,m). \tag{B1}$$

The intervals of the integrals with respect to k' and l are constrained by $\boldsymbol{k}' + \boldsymbol{l} + \boldsymbol{m} = 0$.

Next we transform the variables in Eq. (B1) from l to $y \equiv \boldsymbol{k} \cdot \boldsymbol{m}/km$, where y is the cosine of the interior angle of the triangle $\boldsymbol{k}' + \boldsymbol{l} + \boldsymbol{m} = 0$ facing side l and y is determined from the cosine formula

$$l^2 = k'^2 + m^2 - 2k'my. \tag{B2}$$

Thus, we obtain

$$dl = -\frac{k'm}{l} dy, \tag{B3}$$

Solving (B2) with respect to y, we get

$$y = \frac{m^2 + k'^2 - l^2}{2k'm}. \tag{B4}$$

Equation (B4) expresses y as a function of l. Noting that

$$y = \frac{m^2 + k'^2 - k^2}{2k'm} \equiv \xi \tag{B5}$$

when $l = k$ and $y = 1$ when $l = k' - m$, (B1) is rewritten as

$$\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k) = \int_0^{ak} dm \int_k^{k+m} dk' \int_\xi^1 dy \frac{k'm}{l} T_\alpha^{\mathcal{Q}}(k',l,m). \tag{B6}$$

Furthermore, we transform the integration variables from k' to ξ. Equation (B5) expresses ξ as a function of k'; thus we obtain

$$k'md\xi = (k' - m\xi)dk'. \tag{B7}$$

Note that $\xi = m/(2k)$ when $k' = k$ and $\xi = 1$ when $k' = k + m$. As m is of order of ak and a is a small parameter, the integration with respect to ξ is performed to the leading order in a over the interval $[0,1]$. Moreover, noting that $\xi = O(1)$ and $m = O(ak)$, (B7) is approximated by

$$md\xi \simeq dk'. \tag{B8}$$

Therefore, (B6) is rewritten as

$$\begin{aligned}\Pi_{\alpha\mathrm{nl}}^{\mathcal{Q}(+)}(k) &\simeq \int_0^{ak} dm \int_0^1 d\xi \int_\xi^1 dy \frac{k'm^2}{l} T_\alpha^{\mathcal{Q}}(k',l,m) \\ &= \int_0^{ak} dm \int_0^1 dy \int_0^y d\xi \frac{k'm^2}{l} T_\alpha^{\mathcal{Q}}(k',l,m).\end{aligned} \tag{B9}$$

Next we expand the integrand in Eq. (B9) with respect to the small parameter $\epsilon = m/k'$. The details of the expansion are given in Appendix C. To the leading order in ϵ, the triad enstrophy transfer function $T^{Q}_{\alpha}(k',l,m)$ is expressed as

$$T^{Q}_{\alpha}(k',l,m) \simeq \frac{4k'y\sqrt{1-y^2}}{\pi m^{\alpha-1}}\theta_{k'k'm}\left[\left\{-\frac{k'^2}{m^{\alpha-1}}\frac{\partial k'^{-1}Q_\alpha(k')}{\partial k'}Q_\alpha(m)+\frac{\alpha m^2}{k'^{\alpha+1}}\{Q_\alpha(k')\}^2\right\}H(\alpha)\right.$$
$$\left.+\frac{m}{k'^{\alpha}}\left\{-\alpha Q_\alpha(k')+k'^2\frac{\partial k'^{-1}Q_\alpha(k')}{\partial k'}\right\}Q_\alpha(m)H(-\alpha)\right], \quad \text{(B10)}$$

where H denotes Heaviside's step function. Note that to the leading order in ϵ, $k'm^2/l$ in the integrand of (B9) is approximated by m^2.

Finally, inserting (B10) into (B9), we integrate (B9) with respect to ξ and y. When integrating with respect to ξ, the lower bound with respect to k' in Eq. (B6) was replaced by 0 in Eq. (B9). Note that the integral with respect to m covers a small interval of width $O(ak)$. Therefore, we can replace the independent variable k' with k and place it outside the integral with respect to ξ. Moreover, we use

$$\int_0^1 dy \int_0^y d\xi\, y\sqrt{1-y^2} = \int_0^1 dy\, y^2\sqrt{1-y^2} = \frac{\pi}{16}. \quad \text{(B11)}$$

Then we obtain the enstrophy flux produced by the nonlocal type A triads $m \ll l < k < k'$,

$$\Pi^{Q(+)}_{\alpha\mathrm{nl}}(k)$$
$$\simeq \frac{1}{4}\theta_{kk(ak)}\left[-k^3\frac{\partial k^{-1}Q_\alpha(k)}{\partial k}\int_{k_\mathrm{b}}^{ak} m^{4-2\alpha}Q_\alpha(m)dm + \alpha k^{-\alpha}\{Q_\alpha(k)\}^2\int_{k_\mathrm{b}}^{ak} m^{5-\alpha}dm\right]H(\alpha)$$
$$+\frac{1}{4}\theta_{kk(ak)}\left\{|\alpha|Q_\alpha(k)+k^2\frac{\partial k^{-1}Q_\alpha(k)}{\partial k}\right\}\left\{\int_{k_\mathrm{b}}^{ak} m^{4+|\alpha|}Q_\alpha(m)dm\right\}k^{|\alpha|+1}H(-\alpha). \quad \text{(B12)}$$

Here a is a small parameter such that $0 < k_\mathrm{b} \ll ak \ll k$. Further, $\Pi^{Q(+)}_{\alpha\mathrm{nl}}(k)$ is dominated by the first term on the right-hand side of (B12) for $\alpha > 0$ and by the second term for $\alpha < 0$.

APPENDIX C: DERIVATION OF (B10)

Inserting (17) into (16) and using the sine formula

$$\frac{\sqrt{1-x^2}}{k} = \frac{\sqrt{1-y^2}}{l}, \quad \text{(C1)}$$

the triad enstrophy transfer function (16) becomes

$$T^{Q}_{\alpha}(k',l,m) = \frac{4k'\sqrt{1-y^2}}{\pi l^\alpha m^{\alpha-1}}\theta_{k'lm}(l^\alpha - m^\alpha)$$
$$\times\left[\left\{\frac{l^\alpha-m^\alpha}{(lm)^\alpha}k'Q_\alpha(l)-\frac{k'^\alpha-m^\alpha}{(k'm)^\alpha}lQ_\alpha(k')\right\}Q_\alpha(m)+\frac{k'^\alpha-l^\alpha}{k^\alpha l^\alpha}mQ_\alpha(l)Q_\alpha(k')\right]. \quad \text{(C2)}$$

To expand (C2) with respect to $\epsilon = m/k'$, we perform some preliminary calculations. From the cosine formula (B2), one obtains the following formulas:

$$l^\alpha = k'^\alpha\left[1-\alpha y\epsilon+\frac{\alpha}{2}\{1+(\alpha-2)y^2\}\epsilon^2+O(\epsilon^3)\right], \quad \text{(C3)}$$

$$l^\alpha - m^\alpha = k'^\alpha\left[1-\epsilon^\alpha-\alpha y\epsilon+\frac{\alpha}{2}\{1+(\alpha-2)y^2\}\epsilon^2+O(\epsilon^3)\right], \quad \text{(C4)}$$

$$k'^\alpha - m^\alpha = k'^\alpha(1-\epsilon^\alpha), \quad \text{(C5)}$$

$$k'^{\alpha} - l^{\alpha} = k'^{\alpha}\left[\alpha y\epsilon - \frac{\alpha}{2}\{1 + (\alpha - 2)y^2\}\epsilon^2 + O(\epsilon^3)\right], \tag{C6}$$

$$(l^{\alpha} - m^{\alpha})^2 = k'^{2\alpha}[1 - 2\epsilon^{\alpha} + \epsilon^{2\alpha} + 2\alpha y\epsilon^{1+\alpha} - 2\alpha y\epsilon + \alpha\{1 + 2(\alpha - 1)y^2\}\epsilon^2 + O(\epsilon^3)]. \tag{C7}$$

Moreover, the enstrophy spectrum is expanded as

$$\begin{aligned} Q_{\alpha}(l) &= Q_{\alpha}\left(k'\left\{1 - \epsilon y + \frac{1}{2}(1 - y^2)\epsilon^2 + O(\epsilon^3)\right\}\right) \\ &= Q_{\alpha}(k') - k'y\epsilon\frac{\partial Q_{\alpha}(k')}{\partial k'} + \frac{1}{2}\left\{(1 - y^2)k'\frac{\partial Q_{\alpha}(k')}{\partial k'} + y^2k'^2\frac{\partial^2 Q_{\alpha}(k')}{\partial k'^2}\right\}\epsilon^2 + O(\epsilon^3). \end{aligned} \tag{C8}$$

Using the above expressions, the quantities in the curly brackets in Eq. (C2) are written to the leading order as

$$\begin{aligned} &\frac{l^{\alpha} - m^{\alpha}}{(lm)^{\alpha}}k'Q_{\alpha}(l) - \frac{k'^{\alpha} - m^{\alpha}}{(k'm)^{\alpha}}lQ_{\alpha}(k') \\ &\quad \simeq \frac{k'}{m^{\alpha}}y\left[\epsilon^{\alpha+1}\left\{-\alpha Q_{\alpha}(k') + k'^2\frac{\partial k'^{-1}Q_{\alpha}(k')}{\partial k'}\right\} - \epsilon k'^2\frac{\partial k'^{-1}Q_{\alpha}(k')}{\partial k'}\right]. \end{aligned} \tag{C9}$$

To the leading order, the last term in the square brackets in Eq. (C2) is

$$\frac{k'^{\alpha} - l^{\alpha}}{(k'l)^{\alpha}}mQ_{\alpha}(k')Q_{\alpha}(l) \simeq \frac{\alpha m y\epsilon}{k'^{\alpha}}\{Q_{\alpha}(k')\}^2. \tag{C10}$$

When $\alpha < 0$, the first and second terms in the square brackets of (C9) dominate, while the third term dominates when $\alpha > 0$. Inserting (C9) and (C10) into (C2), one obtains (B10).

---

[1] R. H. Kraichnan, Inertial ranges in two-dimensional turbulence, Phys. Fluids **10**, 1417 (1967).
[2] C. E. Leith, Diffusion approximation for two-dimensional turbulence, Phys. Fluids **11**, 671 (1968).
[3] G. K. Batchelor, Computation of the energy spectrum in homogeneous two-dimensional turbulence, Phys. Fluids **12**, II-233 (1969).
[4] R. H. Kraichnan, Inertial-range transfer in two- and three-dimensional turbulence, J. Fluid Mech. **47**, 525 (1971).
[5] K. Ohkitani, Nonlocality in a forced two-dimensional turbulence, Phys. Fluids A **2**, 1529 (1990).
[6] Y. Kaneda and T. Ishihara, Nonuniversal k^{-3} energy spectrum in stationary two-dimensional homogeneous turbulence, Phys. Fluids **13**, 1431 (2001).
[7] V. Borue, Inverse energy cascade in stationary two-dimensional homogeneous turbulence, Phys. Rev. Lett. **72**, 1475 (1994).
[8] T. Gotoh, Energy spectrum in the inertial and dissipation ranges of two-dimensional steady turbulence, Phys. Rev. E **57**, 2984 (1998).
[9] T. Ishihara and Y. Kaneda, Energy spectrum in the enstrophy transfer range of two-dimensional forced turbulence, Phys. Fluids **13**, 544 (2001).
[10] R. K. Scott, Nonrobustness of the two-dimensional turbulent inverse cascade, Phys. Rev. E **75**, 046301 (2007).
[11] J. Fontane, D. G. Dritschel, and R. K. Scott, Vortical control of forced two-dimensional turbulence, Phys. Fluids **25**, 015101 (2013).
[12] R. T. Pierrehumbert, I. M. Held, and K. L. Swanson, Spectra of local and nonlocal two-dimensional turbulence, Chaos Solitons Fractals **4**, 1111 (1994).

[13] I. M. Held, R. T. Pierrehumbert, S. T. Garner, and K. L. Swanson, Surface quasi-geostrophic dynamics, J. Fluid Mech. **282**, 1 (1995).
[14] W. Blumen, Uniform potential vorticity flow: Part I. Theory of wave interactions and two-dimensional turbulence, J. Atmos. Sci. **35**, 774 (1978).
[15] T. Iwayama, T. G. Shepherd, and T. Watanabe, An 'ideal' form of decaying two-dimensional turbulence, J. Fluid Mech. **456**, 183 (2002).
[16] K. S. Smith, G. Boccaletti, C. C. Henning, I. Marinov, C. Y. Tam, I. M. Held, and G. K. Vallis, Turbulent diffusion in the geostrophic inverse cascade, J. Fluid Mech. **469**, 13 (2002).
[17] T. Iwayama, S. Murakami, and T. Watanabe, Anomalous eddy viscosity for two-dimensional turbulence, Phys. Fluids **27**, 045104 (2015).
[18] N. Schorghofer, Energy spectra of steady two-dimensional turbulent flows, Phys. Rev. E **61**, 6572 (2000).
[19] T. Watanabe and T. Iwayama, Unified scaling theory for local and non-local transfers in generalized two-dimensional turbulence, J. Phys. Soc. Jpn. **73**, 3319 (2004).
[20] T. Watanabe and T. Iwayama, Interacting scales and triad enstrophy transfers in generalized two-dimensional turbulence, Phys. Rev. E **76**, 046303 (2007).
[21] S. A. Orszag, Analytical theories of turbulence, J. Fluid Mech. **41**, 363 (1970).
[22] R. H. Kraichnan, An almost-Markovian Galilean-invariant turbulence model, J. Fluid Mech. **47**, 513 (1971).
[23] M. Lesieur, *Turbulence in Fluids*, 4th ed. (Springer, Berlin, 2008).
[24] J. Herring and J. C. McWilliams, Comparison of direct numerical simulation of two-dimensional turbulence with two-point closure: the effects of intermittency, J. Fluid Mech. **153**, 229 (1985).
[25] B. H. Burgess and T. G. Shepherd, Spectral nonlocality, absolute equilibria, and Kraichnan-Leith-Batchelor phenomenology in two-dimensional turbulent energy cascades, J. Fluid Mech. **725**, 332 (2013).
[26] T. Iwayama and T. Watanabe, Universal spectrum in the infrared range of two-dimensional turbulent flows, Phys. Fluids **26**, 025105 (2014).
[27] A. Pouquet, M. Lesieur, J. C. André, and C. Basdevant, Evolution of high Reynolds number two-dimensional turbulence, J. Fluid Mech. **72**, 305 (1975).
[28] C. Basdevant, M. Lesieur, and R. Sadourny, Subgrid-scale modeling of enstrophy transfer in two-dimensional turbulence, J. Atmos. Sci. **35**, 1028 (1978).
[29] W. J. Bos and J.-P. Bertoglio, Large-scale bottleneck effect in two-dimensional turbulence, J. Turbul. **10**, N30 (2009).
[30] T. Iwayama and T. Watanabe, Green's function for a generalized two-dimensional fluid, Phys. Rev. E **82**, 036307 (2010).
[31] B. H. Burgess, R. K. Scott, and T. G. Shepherd, Kraichnan-Leith-Batchelor similarity theory and two-dimensional inverse cascades, J. Fluid Mech. **767**, 467 (2015).