A complete classification of threshold properties for one-dimensional discrete Schrödinger operators

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Abstract

We consider the one-dimensional discrete Schrödinger operator on $\mathbb{Z}$, and study the relation between the generalized eigenstates and the asymptotic expansion of the resolvent for the threshold 0. We decompose the generalized zero eigenspace into subspaces, some of which correspond to the bound states or the resonance states, only by their growth properties at infinity, and precisely describe the first few leading coefficients in the expansion using these subspaces. The generalized zero eigenspace we consider is the largest possible one, consisting of all solutions to the eigenequation. For the resolvent expansion we implement the expansion scheme of Jensen-Nenciu [Rev. Math. Phys. **13** (2001), 717–754, **16** (2004), 675–677] in its full generality.

Contents

1 Introduction

In this paper we investigate the threshold properties of the resolvent of a one-dimensional discrete Schrödinger operator with a large class of interactions. We do not discuss the general spectral properties of this class of operators, although we note that it is known that the absolutely continuous spectrum equals the interval $[0, 4]$ and the singular continuous spectrum is absent under some additional assumptions on the interaction, see Remark 1.2. The eigenvalues all have finite multiplicities and there are no accumulation points for the eigenvalues. For the thresholds the last result is a consequence of the results obtained here.

In Section 1.1 of this introduction we describe the setting and define the class of interactions. We then state the main results in a simplified form. The complete statements are rather complicated and require a substantial

amount of preparation. In Section 1.2 we outline the strategy of proof. In Section 1.3 we comment on the literature.

In Section 2 we prepare the results needed to obtain the asymptotic resolvent expansions. In Section 3 we describe an (iterated) inversion procedure and define some intermediate operators. Section 4 is devoted to a detailed analysis of the (generalized) eigenspaces associated with the thresholds, and an analysis of the intermediate operators, needed in the statement of the full results. Our full results on the asymptotic expansion of the resolvent are given in detail in Section 5. The Appendices contain results on the threshold 4 and a number of explicit examples of interactions.

1.1 Setting and overview of results

We introduce the setting of the paper, and give a quick and self-contained overview of the main results in a slightly weaker form. The refined versions of the results require long preparations. They are mostly gathered in Section 5.

Let H_0 be the positive one-dimensional discrete Laplacian, i.e., for any sequence $x\colon \mathbb{Z} \to \mathbb{C}$ we define $H_0 x\colon \mathbb{Z} \to \mathbb{C}$ by

$$(H_0 x)[n] = -(x[n+1] + x[n-1] - 2x[n]).$$

The restriction of H_0 to the Hilbert space

$$\mathcal{H} = \ell^2(\mathbb{Z}) = \left\{ x\colon \mathbb{Z} \to \mathbb{C};\ \|x\|^2 = \sum_{n \in \mathbb{Z}} |x[n]|^2 < \infty \right\}$$

defines a bounded self-adjoint operator. We call it the *free Schrödinger operator*, and denote it by the same notation H_0. The operator H_0 has an explicit spectral representation employing the Fourier series. As will be seen in Section 2, we have the spectrum:

$$\sigma(H_0) = \sigma_{\mathrm{ac}}(H_0) = [0, 4],$$

and the points $\lambda = 0, 4$ are the thresholds. It is not difficult to compute explicit asymptotic expansions of the *free resolvent* $R_0(z) = (H_0 - z)^{-1}$ around these thresholds.

The purpose of the present paper is to provide asymptotic expansions of $R(z) = (H - z)^{-1}$ for perturbed $H = H_0 + V$ around these thresholds, and investigate the relation between the coefficients and the generalized eigenspaces. As we will see in Appendix A, the expansion around the threshold $\lambda = 4$ is reduced to the one around $\lambda = 0$, whence in the present paper we shall consider only $\lambda = 0$ in detail.

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Let us first fix our class of perturbations. We introduce for $s \in \mathbb{R}$

$$\mathcal{L}^s = \ell^{1,s}(\mathbb{Z}) = \left\{ x \colon \mathbb{Z} \to \mathbb{C}; \ \|x\|_{1,s} = \sum_{n \in \mathbb{Z}} (1 + n^2)^{s/2} |x[n]| < \infty \right\},$$

$$(\mathcal{L}^s)^* = \ell^{\infty,-s}(\mathbb{Z}) = \left\{ x \colon \mathbb{Z} \to \mathbb{C}; \ \|x\|_{\infty,-s} = \sup_{n \in \mathbb{Z}} (1 + n^2)^{-s/2} |x[n]| < \infty \right\}.$$

The superscript s is dropped when $s = 1$: $\mathcal{L} = \mathcal{L}^1$, $\mathcal{L}^* = (\mathcal{L}^1)^*$. We denote the set of all bounded operators from a general Banach space $\mathcal{K}$ to another $\mathcal{K}'$ by $\mathcal{B}(\mathcal{K}, \mathcal{K}')$, and abbreviate $\mathcal{B}(\mathcal{K}) = \mathcal{B}(\mathcal{K}, \mathcal{K})$ and $\mathcal{B}^s = \mathcal{B}(\mathcal{L}^s, (\mathcal{L}^s)^*)$. We replace $\mathcal{B}$ by $\mathcal{C}$ when considering those for compact operators.

Assumption 1.1. Let $V \in \mathcal{B}(\mathcal{H})$ be self-adjoint, and assume that there exist a real number $\beta \geq 1$, a Hilbert space $\mathcal{K}$, an injective operator $v \in \mathcal{B}(\mathcal{K}, \mathcal{L}^\beta) \cap \mathcal{C}(\mathcal{K}, \mathcal{L})$ and a self-adjoint unitary operator $U \in \mathcal{B}(\mathcal{K})$ such that

$$V = vUv^* \in \mathcal{B}((\mathcal{L}^\beta)^*, \mathcal{L}^\beta) \cap \mathcal{C}(\mathcal{L}^*, \mathcal{L}).$$

Remark 1.2. Using the results in [1] one can show that under the above assumption with $\beta \geq 2$ the singular continuous spectrum of $H = H_0 + V$ is empty and $\sigma_{\mathrm{ac}}(H) = [0, 4]$. Eigenvalues of H have finite multiplicity and have no accumulation point. This last result is a consequence of the resolvent expansions obtained here. We expect that these results hold under the weaker assumption $\beta \geq 1$.

The possible range of $\beta \geq 1$ is subject to further restriction according to how high an order we require for the expansions, and hence we shall mention an appropriate range of β each time it is referred to. Though formulated abstractly, Assumption 1.1 includes plenty of examples and, in particular, allows non-local potentials. Here we only remark that for a multiplication operator V it is typical to choose

$$\mathcal{K} = \ell^2(\mathrm{supp}\, V), \quad v = \iota \circ \sqrt{|V|}, \quad U = \mathrm{sgn}\, V,$$

where ι denotes the extension of domain by zero, and refer the reader to Section B.1 for more general examples of potentials satisfying Assumption 1.1. We also note that our class of interactions is additive in the sense that if V_1 and V_2 satisfy Assumption 1.1 for some β, then $V_1 + V_2$ satisfies the assumption for this β, as can be seen by a straightforward direct sum construction.

Under Assumption 1.1 the perturbed Schrödinger operator $H = H_0 + V$ is bounded and self-adjoint on $\mathcal{H}$, and we can expand the perturbed resolvent

$$R(z) = (H - z)^{-1}$$

as follows. For $z \in \mathbb{C} \setminus [0, \infty)$ we take the determination of the square root with $\mathrm{Im}\, \sqrt{z} > 0$.

Theorem 1.3. *Suppose $\beta \geq 4$ in Assumption 1.1, and let $N \in [-2, \beta - 6]$ be any integer. Then as $z \to 0$ in $\mathbb{C} \setminus [0, \infty)$ the resolvent $R(z)$ has the asymptotic expansion in the uniform topology of $\mathcal{B}^{N+4}$:*

$$R(z) = \sum_{j=-2}^{N} z^{j/2} G_j + \mathcal{O}(z^{(N+1)/2}), \quad G_j \in \mathcal{B}^{j+3},$$

and the coefficients G_j can be computed explicitly.

Remark 1.4. The lower bound of β and the operator classes of G_j can be refined if we know *a priori* the threshold type given in Definition 1.6 below. See Section 5 for these refinements. In particular, if the threshold is regular, it suffices to assume only $\beta \geq 2$. In the later sections we shall use the variable κ given by (1.7) below, and define the coefficients G_j with respect to the expansion in κ. Hence, they are different by factors. *We keep the convention of* Theorem 1.3 *only in this subsection.*

We shall investigate the first few coefficients in terms of the *(generalized) eigenfunctions*. We note that it immediately follows from the identity

$$HR(z) = R(z)H = 1 + zR(z) \tag{1.1}$$

that

$$
\begin{aligned}
HG_j &= G_j H = 0 \quad \text{for } j = -2, -1, \\
HG_0 &= G_0 H = 1 + G_{-2}, \\
HG_j &= G_j H = G_{j-2} \quad \text{for } j = 1, 2, \ldots.
\end{aligned}
\tag{1.2}
$$

Define the *(generalized) eigenspace* for the threshold $\lambda = 0$ and set its dimension as

$$\widetilde{\mathcal{E}} = \{\Psi \in (\mathcal{L}^\beta)^*; \ H\Psi = 0\}, \quad \widetilde{d} = \dim \widetilde{\mathcal{E}},$$

respectively. Then we can show that the eigenspace is finite-dimensional, and the eigenfunctions have special asymptotics at infinity. Define the sequences $\mathbf{1}, \boldsymbol{\sigma} \in (\mathcal{L}^0)^*$ and $\mathbf{n}, |\mathbf{n}| \in \mathcal{L}^*$ by

$$\mathbf{1}[n] = 1, \quad \boldsymbol{\sigma}[n] = \begin{cases} \pm 1 & \text{if } \pm n > 0, \\ 0 & \text{if } n = 0, \end{cases} \quad \mathbf{n}[n] = n, \quad |\mathbf{n}|[n] = |n|,$$

respectively.

Theorem 1.5. *Suppose $\beta \geq 1$ in Assumption 1.1. Then*

$$\widetilde{\mathcal{E}} \subset \mathbb{C}\mathbf{n} \oplus \mathbb{C}|\mathbf{n}| \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\boldsymbol{\sigma} \oplus \mathcal{L}^{\beta-2}, \quad \widetilde{d} < \infty.$$

In the classification of the singular part of the resolvent expansion the following subspaces are canonical. We set

$$\mathcal{E} = \widetilde{\mathcal{E}} \cap \left(\mathbb{C}\mathbf{1} \oplus \mathbb{C}\boldsymbol{\sigma} \oplus \mathcal{L}^{\beta-2} \right), \quad d = \dim \mathcal{E}; \tag{1.3}$$

$$E = \widetilde{\mathcal{E}} \cap \mathcal{L}^{\beta-2}, \quad d_0 = \dim E. \tag{1.4}$$

Obviously, we have $E \subset \mathcal{E} \subset \widetilde{\mathcal{E}}$ and $d_0 \leq d \leq \widetilde{d} \leq d_0 + 4$, which will be refined soon. Now let us introduce the notion of the *regular* and *exceptional properties* for the threshold (similar to the terminology in [3] for continuous Schrödinger operators in dimension three):

Definition 1.6. The threshold $\lambda = 0$ is said to be

1. a *regular point*, if $\mathcal{E} = E = \{0\}$;

2. an *exceptional point of the first kind*, if $\mathcal{E} \supsetneq E = \{0\}$;

3. an *exceptional point of the second kind*, if $\mathcal{E} = E \supsetneq \{0\}$;

4. an *exceptional point of the third kind*, if $\mathcal{E} \supsetneq E \supsetneq \{0\}$.

These properties *characterize and are characterized by* the expansion coefficients G_{-2} and G_{-1} of the singular part:

Theorem 1.7. *Suppose $\beta \geq 4$ in Assumption 1.1. Then $\widetilde{d} = d_0 + 2$ and there exist bases $\Psi_j \in E$, $j = 1, \ldots, d_0$, and $\Psi_j \in \mathcal{E}/E$, $j = d_0 + 1, \ldots, d$, such that*

$$G_{-2} = -\sum_{j=1}^{d_0} \langle \Psi_j, \cdot \rangle \Psi_j, \quad G_{-1} \equiv i \sum_{j=d_0+1}^{d} \langle \Psi_j, \cdot \rangle \Psi_j \mod \langle E, \cdot \rangle E, \tag{1.5}$$

where $\langle E, \cdot \rangle E \subset \mathcal{B}((\mathcal{L}^{\beta-2})^*, \mathcal{L}^{\beta-2})$ is the subspace spanned by the operators of the form $\langle \Psi, \cdot \rangle \Psi'$ with $\Psi, \Psi' \in E$. Furthermore, one can choose $\Psi_j \in E$, $j = 1, \ldots, d$, to be orthonormal with respect to the inner product of $\mathcal{H}$, and hence $-G_{-2}$ is the orthogonal projection from $\mathcal{H}$ onto E.

Remark 1.8. If the spaces E or $\mathcal{E}/E$ are trivial, we interpret the corresponding operators to the right of (1.5) as 0, respectively. A complete classification of dimensions of the eigenspaces is obtained in Section 4.2. The lower bound of β can also be refined, cf. Section 5.

Next, we consider G_0. We will see that the explicit formula for G_0 in fact belongs to $\mathcal{B}^3$ at worst and is well-defined for $\beta \geq 5$ if we do not know *a priori* the threshold type. We have already seen in (1.2) that G_0 is actually the *Green operator* for H. Here let us formulate it more precisely. Denote the identity operator by $1_{\mathcal{L}^s} = \mathrm{id}_{\mathcal{L}^s}$.

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Theorem 1.9. *Suppose $\beta \geq 5$ in Assumption 1.1. Then*

$$HG_0 = G_0 H = 1_{\mathcal{L}^3} + G_{-2}.$$

For the Green operator it would be natural to expect an expression of the form

$$
\begin{aligned}
G_0 &= G_0^0 - G_0^0 V G_0^0 + G_0^0 V G_0^0 V G_0^0 - \cdots \\
&= (1_{\mathcal{L}^*} + G_0^0 V)^{-1} G_0^0 \\
&= G_0^0 (1_{\mathcal{L}} + V G_0^0)^{-1},
\end{aligned}
$$

where G_0^0 is the coefficient appearing in the asymptotic expansion of $R_0(z) = (H_0 - z)^{-1}$ given in Section 2.1:

$$R_0(z) = \sum_{j=-1}^{N} z^{j/2} G_j^0 + \mathcal{O}(z^{(N+1)/2}), \tag{1.6}$$

with the same convention as in the statement of Theorem 1.3. However, the expression for G_0 is affected by the eigenspaces again, and we need to take care of eigenfunctions with certain *quasi-symmetric* asymptotics at infinity:

Definition 1.10. Define the *quasi-symmetric eigenspace* and set its dimension as

$$\widetilde{\mathcal{E}}_{\mathrm{qs}} = \widetilde{\mathcal{E}} \cap \left(\mathbb{C}|\mathbf{n}| \oplus \mathbb{C}\boldsymbol{\sigma} \oplus \mathcal{L}^{\beta-2} \right), \quad \widetilde{d}_{\mathrm{qs}} = \dim \widetilde{\mathcal{E}}_{\mathrm{qs}}.$$

It follows directly from the definition that $E \subset \widetilde{\mathcal{E}}_{\mathrm{qs}}$ and $d_0 \leq \widetilde{d}_{\mathrm{qs}} \leq d_0 + 2$.

Theorem 1.11. *Suppose $\beta \geq 5$ in Assumption 1.1. Then the following three conditions are equivalent:*

1. *The quasi-symmetric eigenspace is trivial, i.e., $\widetilde{\mathcal{E}}_{\mathrm{qs}} = \{0\}$;*

2. *The operator $1_{\mathcal{L}^*} + G_0^0 V$ is invertible in $\mathcal{B}(\mathcal{L}^*)$;*

3. *The operator $1_{\mathcal{L}} + V G_0^0$ is invertible in $\mathcal{B}(\mathcal{L})$.*

Furthermore, in the affirmative case, $d + 1 \leq \widetilde{d} \leq d + 2$, and there exist $\Psi_j \in \widetilde{\mathcal{E}}$, $j = d+1, \ldots, \widetilde{d}$ such that

$$
\begin{aligned}
G_0 &= (1_{\mathcal{L}^*} + G_0^0 V)^{-1} G_0^0 + \sum_{j=d+1}^{\widetilde{d}} \langle \Psi_j, \cdot \rangle \Psi_j \\
&= G_0^0 (1_{\mathcal{L}} + V G_0^0)^{-1} + \sum_{j=d+1}^{\widetilde{d}} \langle \Psi_j, \cdot \rangle \Psi_j.
\end{aligned}
$$

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Remark 1.12. We do not assert that $\{\Psi_j\}$ is a basis of $\widetilde{\mathcal{E}}/\mathcal{E}$, and in general it is not. Even when $\widetilde{\mathcal{E}}_{\mathrm{qs}}$ is nontrivial, there exist alternative modified expressions for G_0. See Section 5. The lower bound of β can also be refined.

We finally note that in the case where V is multiplicative, further dimensional relations hold true: $\widetilde{d} = 2$, $d \leq 1$ and $d_0 = 0$. In particular, the space E is always trivial for the multiplicative potentials, while the dimension $d_0 = \dim E$ can be an arbitrary finite number in general for non-local potentials. We shall see such examples in Appendix B.4.

The relations between the first few coefficients and the eigenspaces are mediated by certain operators, which we name P, m_0, q_0 and r_0, cf. Section 3.2. As mentioned at the beginning, if we formulate our statement directly in terms of these *intermediate operators*, we can have more refined statements than the ones above, which is one of our goals.

1.2 Strategy

We fix the determination of the square root with $\mathrm{Im}\sqrt{z} > 0$ for $z \in \mathbb{C}\backslash[0, \infty)$, as in the previous subsection, and introduce the new variable κ:

$$\kappa = -i\sqrt{z}; \quad z = -\kappa^2. \tag{1.7}$$

We use the two variables z and κ interchangeably, e.g., $R(z) = R(\kappa)$, without comment. Our first reduction is given in the following proposition, where the expansion of $R(\kappa)$ is reduced to that of $R_0(\kappa)$. Define the operator $M(\kappa) \in \mathcal{B}(\mathcal{K})$ for $\mathrm{Re}\,\kappa > 0$ by

$$M(\kappa) = U + v^* R_0(\kappa)v. \tag{1.8}$$

Due to the decay assumption on V there exists a $\kappa_0 > 0$ such that for $\mathrm{Re}\,\kappa \in (0, \kappa_0)$ we have that $z = -\kappa^2$ is in the resolvent set of H.

Proposition 1.13. *For any $0 < \mathrm{Re}\,\kappa < \kappa_0$ the operator $M(\kappa)$ is invertible in $\mathcal{B}(\mathcal{K})$, and*

$$R(\kappa) = R_0(\kappa) - R_0(\kappa)vM(\kappa)^{-1}v^* R_0(\kappa), \tag{1.9}$$

$$M(\kappa)^{-1} = U - Uv^* R(\kappa)vU. \tag{1.10}$$

Proof. By the definition (1.8) and Assumption 1.1 it is straightforward to see that

$$M(\kappa)\big[U - Uv^* R(\kappa)vU\big] = \big[U - Uv^* R(\kappa)vU\big]M(\kappa) = 1_{\mathcal{K}}.$$

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Hence (1.10) is verified. As for (1.9), we use the identities

$$R(\kappa) = R_0(\kappa) - R(\kappa)VR_0(\kappa) = R_0(\kappa) - R_0(\kappa)VR(\kappa)$$

to obtain

$$R(\kappa) = R_0(\kappa) - R_0(\kappa)v\big[U - Uv^*R(\kappa)vU\big]v^*R_0(\kappa).$$

Then it suffices to substitute (1.10) into the above equation. □

The expansion of $R_0(\kappa)$ is straightforward and will be given explicitly in Section 2.1. Then by (1.8) we can also expand $M(\kappa)$, cf. Section 2.2. Hence by (1.9) it suffices to have an expansion scheme that expands $M(\kappa)^{-1}$ in terms of the expansion coefficients of $M(\kappa)$. We will actually borrow the scheme from [4, 5], however, we emphasize that we will implement it under the most general assumption on perturbations. The first step of the scheme will be given in Section 3.1. The full expansion scheme consists of iterated applications of the inversion formula of Section 3.1, and the outline will be given in Section 3.2, proving that the iteration stops after a finite number of steps, as well as introducing the intermediate operators P, m_0, q_0 and r_0. We will investigate the relation between these intermediate operators and the eigenspaces in Section 4. An alternative proof of finiteness of iteration is also given here. The main refined results will be stated and proved in Section 5.

1.3 Comments on the literature

We are not aware of any previous complete treatments of the resolvent expansion problem, or the associated complete classification of the threshold (generalized) eigenspaces, for the discrete Schrödinger operator with the general class of interactions considered here.

For the case of discrete one-dimensional Schrödinger operators with multiplicative potentials there are results on the threshold expansions of the resolvent in the generic case in [8] and in the general case in [2]. The latter paper uses the Jost function for the threshold analysis.

The types of resolvent expansions considered here were first obtained for continuous Schrödinger operators in [9, 3, 7]. Further references concerning the continuous case can be found in [4].

There are many different applications of the asymptotic expansion of the resolvent of a Schrödinger operator around thresholds. They include the decay properties of the Schrödinger flow $e^{-itH}P_{\mathrm{ac}}$ and dispersive estimates for the flow. In the continuous case, see the survey [10]. This question has been studied for the discrete one-dimensional Schrödinger operators in the

regular case in [6, 8]. Other applications are the analysis of the scattering matrix at thresholds, and the mapping properties of the wave operators in L^p-spaces, for the latter see [2]. Thus the results in this paper can be used both to include the exceptional cases in these applications and to extend the results to our large class of interactions.

2 Expansions of $R_0(\kappa)$ and $M(\kappa)$

2.1 Expansion of $R_0(\kappa)$

For $x \in \mathcal{H}$ and $f \in L^2(\mathbb{T})$ we use the following conventions for the Fourier transform $\mathcal{F} \colon \mathcal{H} \to L^2(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

$$(\mathcal{F}x)(\theta) = (2\pi)^{-1/2} \sum_{n \in \mathbb{Z}} e^{-in\theta} x[n], \quad (\mathcal{F}^{-1}f)[n] = (2\pi)^{-1/2} \int_{\mathbb{T}} e^{in\theta} f(\theta) \, d\theta.$$

Then we have

$$\mathcal{F}(H_0 x)(\theta) = (2 - 2\cos\theta)(\mathcal{F}x)(\theta) = \left(4\sin^2 \tfrac{\theta}{2}\right)(\mathcal{F}x)(\theta),$$

which shows that the spectrum of H_0 is purely absolutely continuous and equals $[0, 4]$. It follows that, if we denote $R_0(z) = (H_0 - z)^{-1}$, then

$$\mathcal{F}(R_0(z)x)(\theta) = \frac{(\mathcal{F}x)(\theta)}{4\sin^2(\theta/2) - z}, \quad z \in \mathbb{C} \setminus [0, 4]. \tag{2.1}$$

For $z \in \mathbb{C} \setminus [0, 4]$ sufficiently close to 0 let us change the variable from z to ϕ through the correspondence

$$z = 4\sin^2 \tfrac{\phi}{2}, \quad \operatorname{Im}\phi > 0. \tag{2.2}$$

With the correspondence (2.2) the resolvent $R_0(z)$ is represented by convolution with the function

$$R_0(z; n) = \frac{ie^{i\phi|n|}}{2\sin\phi}. \tag{2.3}$$

We now combine (2.3), (2.2), and (1.7) to expand $R_0(\kappa) = R_0(z)$.

Proposition 2.1. *Let $N \geq -1$ be any integer. Then as $\kappa \to 0$ with $\operatorname{Re}\kappa > 0$, the resolvent $R_0(\kappa)$ has the expansion in $\mathcal{B}^{N+2}$:*

$$R_0(\kappa) = \sum_{j=-1}^{N} \kappa^j G_j^0 + \mathcal{O}(\kappa^{N+1}), \quad G_j^0 \in \mathcal{B}^{j+1}, \tag{2.4}$$

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and the coefficients G_j^0 are given explicitly as convolution operators with polynomials $G_j^0(n)$ of degree $j + 1$ in $|n|$. For instance,

$$G_{-1}^0(n) = \tfrac{1}{2}, \quad G_0^0(n) = -\tfrac{1}{2}|n|, \quad G_1^0(n) = \tfrac{1}{4}|n|^2 - \tfrac{1}{16},$$
$$G_2^0(n) = -\tfrac{1}{12}|n|^3 + \tfrac{1}{12}|n|, \quad G_3^0(n) = \tfrac{1}{48}|n|^4 - \tfrac{5}{96}|n|^2 + \tfrac{3}{256}. \tag{2.5}$$

Proof. The proof is straightforward and is omitted. □

2.2 Expansion of $M(\kappa)$

By (1.8) and Proposition 2.1 we have the expansion of $M(\kappa)$:

Proposition 2.2. *Suppose $\beta \geq 1$ in Assumption 1.1, and let $N \in [-1, \beta-2]$ be any integer. Then as $\kappa \to 0$ with $\operatorname{Re} \kappa > 0$, the operator $M(\kappa)$ has the expansion in $\mathcal{B}(\mathcal{K})$:*

$$M(\kappa) = \sum_{j=-1}^{N} \kappa^j M_j + \mathcal{O}(\kappa^{N+1}), \tag{2.6}$$

where the coefficients $M_j \in \mathcal{B}(\mathcal{K})$ are given by

$$M_0 = U + v^* G_0^0 v, \quad M_j = v^* G_j^0 v \ \text{for } j \neq 0. \tag{2.7}$$

Proof. The proof is straightforward by (1.8) and Proposition 2.1, and is omitted. □

3 Expansion scheme of Jensen-Nenciu

3.1 Inversion formula

In Section 2.2 we obtained the expansion of $M(\kappa)$. Based on this expansion we next want to expand $M(\kappa)^{-1}$. In this subsection we provide an inversion formula in a general setting, adapted from [4, Corollary 2.2]. We consider the following condition:

Assumption 3.1. Let $\mathcal{K}$ be a Hilbert space and $A(\kappa)$ a family of bounded operators on $\mathcal{K}$ with $\kappa \in D \subset \mathbb{C} \setminus \{0\}$. Suppose that

1. The set $D \subset \mathbb{C} \setminus \{0\}$ is invariant under complex conjugation and accumulates at $0 \in \mathbb{C}$.

2. For each $\kappa \in D$ the operator $A(\kappa)$ satisfies $A(\kappa)^* = A(\overline{\kappa})$ and has a bounded inverse $A(\kappa)^{-1} \in \mathcal{B}(\mathcal{K})$.

3. As $\kappa \to 0$ in D, the operator $A(\kappa)$ has an expansion in the uniform topology of the operators at $\mathcal{K}$:

$$A(\kappa) = A_0 + \kappa \widetilde{A}_1(\kappa); \quad \widetilde{A}_1(\kappa) = \mathcal{O}(1). \tag{3.1}$$

4. The spectrum of A_0 does not accumulate at $0 \in \mathbb{C}$ as a set.

If the leading operator A_0 is invertible in $\mathcal{B}(\mathcal{K})$, the Neumann series provides an inversion formula for the expansion of $A(\kappa)^{-1}$:

$$A(\kappa)^{-1} = \sum_{j=0}^{\infty} (-1)^j \kappa^j A_0^{-1} \big[\widetilde{A}_1(\kappa) A_0^{-1} \big]^j.$$

However, the expansion is not that simple when A_0 has no bounded inverse. In that case the leading operator A_0 alone cannot yield invertibility on the whole space, and the lower order remainder $\kappa \widetilde{A}_1(\kappa)$ has to compensate the missing invertibility. Note that by Assumption 3.1.2 we can still invert $A(\kappa)$ in $\mathcal{B}(\mathcal{K})$ for any $\kappa \in D$. But the inverse would have norm of order κ^{-1} or worse in a subspace where the remainder yields the invertibility and is involved in the leading part of expansion. This is how negative exponents may appear in the expansion of $A(\kappa)^{-1}$.

Before stating the formula rigorously we introduce some terminology. We define the *pseudo-inverse* $a^\dagger$ for a complex number $a \in \mathbb{C}$ by

$$a^\dagger = \begin{cases} 0 & \text{if } a = 0, \\ a^{-1} & \text{if } a \neq 0. \end{cases} \tag{3.2}$$

Let $\mathcal{K}' \subset \mathcal{K}$ be a closed subspace. We will always identify $\mathcal{B}(\mathcal{K}')$ with its embedding in $\mathcal{B}(\mathcal{K})$ in the standard way. For an operator $A \in \mathcal{B}(\mathcal{K}') \subset \mathcal{B}(\mathcal{K})$ we say that A is *invertible in* $\mathcal{B}(\mathcal{K}')$ if there exists an operator $A^\dagger \in \mathcal{B}(\mathcal{K}')$ such that $A^\dagger A = A A^\dagger = 1_{\mathcal{K}'}$, which we identify with the orthogonal projection onto $\mathcal{K}' \subset \mathcal{K}$ as noted. For a general self-adjoint operator A on $\mathcal{K}$ we abuse the notation $A^\dagger$ also to denote the operator defined by the usual operational calculus for the function (3.2). The operator $A^\dagger$ for a self-adjoint operator A belongs to $B(\mathcal{K})$ if and only if the spectrum of A does not accumulate at 0 as a set, and in such a case the above two $A^\dagger$ coincide. In either case we call $A^\dagger$ the *pseudo-inverse* of A. The reader should note that we always use the notation A^* for adjoint and the notation $A^\dagger$ for the pseudo-inverse.

Proposition 3.2. *Suppose Assumption 3.1. Let Q be the orthogonal projection onto* $\operatorname{Ker} A_0$*, and define the operator $a(\kappa) \in \mathcal{B}(Q\mathcal{K})$ by*

$$a(\kappa) = \tfrac{1}{\kappa} \big\{ 1_{Q\mathcal{K}} - Q(Q + A(\kappa))^{-1} Q \big\}$$
$$= \sum_{j=0}^{\infty} (-1)^j \kappa^j Q \widetilde{A}_1(\kappa) \big[(Q + A_0)^{-1} \widetilde{A}_1(\kappa) \big]^j Q. \tag{3.3}$$

Then $a(\kappa)$ is bounded in $\mathcal{B}(Q\mathcal{K})$ as $\kappa \to 0$ in D. Moreover, for each $\kappa \in D$ sufficiently close to 0 the operator $a(\kappa)$ is invertible in $\mathcal{B}(Q\mathcal{K})$, and

$$A(\kappa)^{-1} = (Q + A(\kappa))^{-1} + \tfrac{1}{\kappa}(Q + A(\kappa))^{-1}a(\kappa)^{\dagger}(Q + A(\kappa))^{-1}. \qquad (3.4)$$

Before the proof let us describe the inversion procedure briefly. We may assume that the leading operator A_0 has a nontrivial kernel. Then by (3.4) and the boundedness of $a(\kappa)$ we can see that negative powers of κ, at least κ^{-1}, show up in the expansion of $A(\kappa)^{-1}$. Assume that we have a higher-order expansion of $A(\kappa)$ than (3.1), i.e., $\widetilde{A}_1(\kappa) = A_1 + \kappa\widetilde{A}_2(\kappa)$, $\widetilde{A}_2(\kappa) = \mathcal{O}(1)$. Then the expansion of $a(\kappa)$ follows from (3.3):

$$a(\kappa) = a_0 + \kappa\widetilde{a}_1(\kappa); \quad a_0 = QA_1Q, \quad \widetilde{a}_1(\kappa) = \mathcal{O}(1).$$

If the leading operator a_0 is invertible in $\mathcal{B}(Q\mathcal{K})$, then substitution of the Neumann series for $a(\kappa)^{\dagger}$ into (3.4) yields the expansion of $A(\kappa)^{-1}$. Otherwise, by applying Proposition 3.2 for $a(\kappa)$ again we obtain the expansion of $a(\kappa)^{\dagger}$, and find that $A(\kappa)^{-1}$ has at least a κ^{-2} singularity in its expansion. We repeat this argument. The iteration procedure stops when we encounter a leading operator with trivial kernel. The asymptotic expansion of $A(\kappa)$ is then completed, and the number of the iterations gives the worst order of the negative powers of κ.

We note that the sequence of the kernels of the leading operators is monotone non-increasing, however, the inversion procedure may not stop after a finite number of steps. But in our application this procedure always stops after a few steps, due to the self-adjointness of H. We can also prove this finiteness in a direct way. We will see this in Sections 3.2 and 4.2, respectively.

Proof of Proposition 3.2. We first note that for each $\kappa \in D$ close to 0 the operator $Q + A(\kappa)$ has a bounded inverse, and hence $a(\kappa)$ is well-defined. Then the assertion follows if we can verify

$$(Q + QA(\kappa)^{-1}Q)a(\kappa) = a(\kappa)(Q + QA(\kappa)^{-1}Q) = \tfrac{1}{\kappa}Q,$$

and

$$1_{\mathcal{K}} = \Big[(Q + A(\kappa))^{-1} + \tfrac{1}{\kappa}(Q + A(\kappa))^{-1}a(\kappa)^{\dagger}(Q + A(\kappa))^{-1}\Big]A(\kappa)$$
$$= A(\kappa)\Big[(Q + A(\kappa))^{-1} + \tfrac{1}{\kappa}(Q + A(\kappa))^{-1}a(\kappa)^{\dagger}(Q + A(\kappa))^{-1}\Big].$$

But these are straightforward and we omit the computations. $\qquad \square$

3.2 Finiteness of inversion iteration

In this subsection we prove that the inversion iteration, when applied to $M(\kappa)$, stops after a finite number of steps when $\beta \geq 4$. We also introduce the intermediate operators P, m_0, q_0 and r_0, outlining the procedure of the expansion for the resolvent $R(\kappa)$. The detailed computations will be given in Section 5.

Let $P \in \mathcal{B}(\mathcal{K})$ be the orthogonal projection onto the subspace spanned by $v^*\mathbf{1} \in \mathcal{K}$. We can write it as

$$P = \tfrac{\gamma}{2}\langle \Phi_1, \cdot \rangle \Phi_1; \quad \gamma = 2\|\Phi_1\|^{\dagger 2}, \ \Phi_1 = v^*\mathbf{1}. \tag{3.5}$$

In order to ensure the finiteness of the iteration procedure it suffices to assume $\beta \geq 4$, at worst. This lower bound of β can be improved if we come across invertible P, m_0 or q_0 before r_0, but for the moment we let $\beta \geq 4$. Under this assumption we can write (2.6) as

$$M(\kappa) = \tfrac{\gamma^\dagger}{\kappa}P + M_0 + \kappa M_1 + \kappa^2 M_2 + \mathcal{O}(\kappa^3). \tag{3.6}$$

Note that, as a consequence of (1.10) and the self-adjointness of H, we already know that $M(\kappa)^{-1}$ satisfies, at worst,

$$M(\kappa)^{-1} = \mathcal{O}(\kappa^{-2}). \tag{3.7}$$

If P is invertible in $\mathcal{B}(\mathcal{K})$, i.e., $\Phi_1 \neq 0$ and $\mathcal{K} = \mathbb{C}\Phi_1$, we can use the Neumann series to compute the inverse $M(\kappa)^{-1}$. Then the inversion procedure for $M(\kappa)^{-1}$ stops, and $R(\kappa)$ can be expanded by using Proposition 1.13. Note that in this case we in fact need only $\beta \geq 1$. Hence we may assume that P is not invertible. Apply Proposition 3.2 for $\kappa M(\kappa)$, and then we obtain

$$M(\kappa)^{-1} = \kappa(Q + \kappa M(\kappa))^{-1} + (Q + \kappa M(\kappa))^{-1}m(\kappa)^\dagger(Q + \kappa M(\kappa))^{-1}, \tag{3.8}$$

where $Q = 1_\mathcal{K} - P$ and $m(\kappa) \in \mathcal{B}(Q\mathcal{K})$ has an expansion of the form:

$$m(\kappa) = m_0 + \kappa m_1 + \kappa^2 m_2 + \mathcal{O}(\kappa^3), \quad m_j \in \mathcal{B}(Q\mathcal{K}). \tag{3.9}$$

The explicit expressions for m_j are listed in Lemma 3.3. Now it is reduced to expand $m(\kappa)^\dagger$ in $\mathcal{B}(\mathcal{K})$. Note that from (3.8) and (3.7) it follows

$$m(\kappa)^\dagger = QM(\kappa)^{-1}Q - \kappa Q = \mathcal{O}(\kappa^{-2}). \tag{3.10}$$

Similarly to the above, if m_0 is invertible in $\mathcal{B}(Q\mathcal{K})$, then $m(\kappa)^\dagger$ can be computed by the Neumann series, in fact with $\beta \geq 2$. Hence we may assume m_0 is not invertible in $\mathcal{B}(Q\mathcal{K})$, and then by Proposition 3.2

$$m(\kappa)^\dagger = (S + m(\kappa))^\dagger + \tfrac{1}{\kappa}(S + m(\kappa))^\dagger q(\kappa)^\dagger(S + m(\kappa))^\dagger, \tag{3.11}$$

where $S \in \mathcal{B}(\mathcal{K})$ is the orthogonal projection onto $Q\mathcal{K} \cap \operatorname{Ker} m_0$ and $q(\kappa) \in \mathcal{B}(S\mathcal{K})$ has an expansion of the form:

$$q(\kappa) = q_0 + \kappa q_1 + \mathcal{O}(\kappa^2), \quad q_j \in \mathcal{B}(S\mathcal{K}). \tag{3.12}$$

Next, we expand $q(\kappa)^\dagger$ in $\mathcal{B}(\mathcal{K})$. Similarly to the above by (3.11) and (3.10) we have

$$q(\kappa)^\dagger = \kappa(Sm(\kappa)^\dagger S - S) = \mathcal{O}(\kappa^{-1}). \tag{3.13}$$

Assuming q_0 is not invertible in $\mathcal{B}(S\mathcal{K})$, we have by Proposition 3.2

$$q(\kappa)^\dagger = (T + q(\kappa))^\dagger + \tfrac{1}{\kappa}(T + q(\kappa))^\dagger r(\kappa)^\dagger (T + q(\kappa))^\dagger, \tag{3.14}$$

where $T \in \mathcal{B}(\mathcal{K})$ is the orthogonal projection onto $S\mathcal{K} \cap \operatorname{Ker} q_0$ and $r(\kappa) \in \mathcal{B}(T\mathcal{K})$ has an expansion of the form:

$$r(\kappa) = r_0 + \mathcal{O}(\kappa), \quad r_0 \in \mathcal{B}(T\mathcal{K}).$$

By (3.14) and (3.13) we have

$$r(\kappa)^\dagger = \kappa(Tq(\kappa)^\dagger T - T) = \mathcal{O}(1).$$

This implies that r_0 has to be invertible in $\mathcal{B}(T\mathcal{K})$, and now the iteration stops. Note that we can also show the finiteness by directly computing r_0, cf. Corollary 4.15.

Finally for later use we collect some of coefficients of $m(\kappa)$, $q(\kappa)$ and $r(\kappa)$.

Lemma 3.3. *One has the explicit formulas: For $\beta \geq 1$*

$$m_0 = QM_0Q;$$

For $\beta \geq 2$

$$m_1 = QM_1Q - QM_0(Q + \gamma P)M_0Q,$$
$$q_0 = Sm_1S;$$

For $\beta \geq 3$

$$m_2 = QM_2Q - QM_0(Q + \gamma P)M_1Q - QM_1(Q + \gamma P)M_0Q$$
$$\quad + QM_0(Q + \gamma P)M_0(Q + \gamma P)M_0Q,$$
$$q_1 = Sm_2S - Sm_1(S + m_0^\dagger)m_1S,$$
$$r_0 = Tq_1T.$$

Proof. These coefficients are computed from the formula (3.3). $\qquad \square$

Remark 3.4. These formulas will be used in Section 5. The above lower bounds for β are the least ones needed for the definitions to make sense. In the actual expansions where these coefficients appear we need to increase the lower bound by 1 in order to dominate the error terms.

4 Intermediate operators

In Section 3.2 we saw that the singular part of the expansion of $R(\kappa)$ depends heavily on the operators P, m_0, q_0 and r_0. In this section, as a preliminary step before the detailed computations of the expansion, we investigate these operators P, m_0, q_0, r_0 and, in particular, their kernels, cf. Propositions 4.11–4.14.

4.1 Characterization of eigenspaces

Lemma 4.1. *For any $x \in \mathcal{L}^s$, $s \geq 1$, the sequence $G_0^0 x \in \mathcal{L}^*$ has the representations:*

$$(G_0^0 x)[n] = -\tfrac{n}{2}\langle \mathbf{1}, x \rangle + \tfrac{1}{2}\langle \mathbf{n}, x \rangle - \sum_{k \geq n}(k-n)x[k] \qquad (4.1)$$

$$= \tfrac{n}{2}\langle \mathbf{1}, x \rangle - \tfrac{1}{2}\langle \mathbf{n}, x \rangle - \sum_{k \leq n}(n-k)x[k]. \qquad (4.2)$$

Moreover, $G_0^0 x \in (\mathcal{L}^0)^*$ if and only if $\langle \mathbf{1}, x \rangle = 0$, and $G_0^0 x \in \mathcal{L}^{s-2}$ if and only if $\langle \mathbf{1}, x \rangle = \langle \mathbf{n}, x \rangle = 0$.

Proof. The identities (4.1) and (4.2) follow immediately from the formula

$$(G_0^0 x)[n] = -\tfrac{1}{2}\sum_{k \geq n}(k-n)x[k] - \tfrac{1}{2}\sum_{k \leq n}(n-k)x[k].$$

As for the last assertions, we note that

$$\sum_{n \geq 0}(1+n^2)^{(s-2)/2}\left|\sum_{k \geq n}(k-n)x[k]\right| \leq C\|x\|_{1,s} < \infty.$$

This implies that the last summation of (4.1) belongs to $\ell^{1,s-2}(\mathbb{Z}_+)$, and similarly for that of (4.2). Since we have $\boldsymbol{\sigma}, |\mathbf{n}| \notin \mathcal{L}^{s-2}$ for $s \geq 1$, we are done. $\qquad \square$

Lemma 4.2. *The compositions $H_0 G_0^0$ and $G_0^0 H_0$ are the identity on $\mathcal{L}$:*

$$H_0 G_0^0 = G_0^0 H_0 = 1_{\mathcal{L}}. \qquad (4.3)$$

Moreover, the composition $G_0^0 H_0$ is well-defined also on the extended space $\mathbb{C}\mathbf{n} \oplus \mathbb{C}|\mathbf{n}| \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\boldsymbol{\sigma} \oplus \mathcal{L}$, and coincides with the projection Π given as follows:

$$G_0^0 H_0 = \Pi \colon \mathbb{C}\mathbf{n} \oplus \mathbb{C}|\mathbf{n}| \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\boldsymbol{\sigma} \oplus \mathcal{L} \to \mathbb{C}|\mathbf{n}| \oplus \mathbb{C}\boldsymbol{\sigma} \oplus \mathcal{L}. \qquad (4.4)$$

Remark 4.3. Lemmas 4.1 and 4.2 in particular imply that for $s \geq 1$

$$\mathbb{C}|\mathbf{n}| \oplus \mathbb{C}\boldsymbol{\sigma} \oplus \mathcal{L}^s \subset G_0^0(\mathcal{L}^s) \subset \mathbb{C}|\mathbf{n}| \oplus \mathbb{C}\boldsymbol{\sigma} \oplus \mathcal{L}^{s-2}. \tag{4.5}$$

Thus, the sequences belonging to the image $G_0^0(\mathcal{L})$ have quasi-symmetric asymptotics at infinity.

Proof. For example by Lemma 4.1 we can easily verify $H_0 G_0^0 x = G_0^0 H_0 = x$ for $x \in \mathcal{L}$, and this implies (4.3). We can also compute

$$H_0\mathbf{n} = H_0\mathbf{1} = 0, \quad G_0^0 H_0|\mathbf{n}| = |\mathbf{n}|, \quad G_0^0 H_0\boldsymbol{\sigma} = \boldsymbol{\sigma},$$

and thus the second assertion (4.4) follows. □

We set

$$
\begin{aligned}
\Psi_1^0 &= \mathbf{1}, \quad \Psi_2^0 = \mathbf{n} - \langle v^*\mathbf{1}, v^*\mathbf{n}\rangle \|v^*\mathbf{1}\|^{\dagger 2}\mathbf{1}, \\
\Phi_1 &= v^*\Psi_1^0, \quad \Phi_1^* = \|\Phi_1\|^{\dagger 2}\Phi_1, \\
\Phi_2 &= v^*\Psi_2^0 = Qv^*\mathbf{n}, \quad \Phi_2^* = \|\Phi_2\|^{\dagger 2}\Phi_2,
\end{aligned}
\tag{4.6}
$$

where Φ_1 is the same as that in (3.5), and define $\widetilde{P}, \widetilde{Q} \in \mathcal{B}(\mathcal{K})$ by

$$\widetilde{P} = \langle\Phi_1^*, \cdot\rangle\Phi_1 + \langle\Phi_2^*, \cdot\rangle\Phi_2, \quad \widetilde{Q} = 1_\mathcal{K} - \widetilde{P}.$$

Obviously, $\widetilde{P}$ is the orthogonal projection onto the subspace $\mathbb{C}\Phi_1 \oplus \mathbb{C}\Phi_2 \subset \mathcal{K}$. We define the operators $w \in \mathcal{B}((\mathcal{L}^\beta)^*, \mathcal{K})$ and $z \in \mathcal{B}(\mathcal{K}, \mathcal{L}^*)$ by

$$w = Uv^*, \quad z = \langle M_0\Phi_1^*, \cdot\rangle\Psi_1^0 + \langle M_0\Phi_2^*, \cdot\rangle\Psi_2^0 - G_0^0 v. \tag{4.7}$$

Proposition 4.4. *Suppose $\beta \geq 1$ in Assumption 1.1. Then*

$$w \circ z|_{\operatorname{Ker}\widetilde{Q}M_0} = 1_{\operatorname{Ker}\widetilde{Q}M_0}, \tag{4.8}$$

and

$$
\begin{aligned}
z^{-1}(\widetilde{\mathcal{E}}) &= \operatorname{Ker}\widetilde{Q}M_0, & \operatorname{Ker}w|_{\widetilde{\mathcal{E}}} &= \operatorname{Ker}v^*|_{\mathbb{C}\Psi_1^0 \oplus \mathbb{C}\Psi_2^0}, & (4.9) \\
z^{-1}(\mathcal{E}) &= \operatorname{Ker}P \cap \operatorname{Ker}QM_0, & \operatorname{Ker}w|_{\mathcal{E}} &= \operatorname{Ker}v^*|_{\mathbb{C}\Psi_1^0}, & (4.10) \\
z^{-1}(E) &= \operatorname{Ker}\widetilde{P} \cap \operatorname{Ker}M_0, & \operatorname{Ker}w|_E &= \{0\}, & (4.11) \\
z^{-1}(\widetilde{\mathcal{E}}_{\mathrm{qs}}) &= \operatorname{Ker}M_0, & \operatorname{Ker}w|_{\widetilde{\mathcal{E}}_{\mathrm{qs}}} &= \{0\}. & (4.12)
\end{aligned}
$$

Remark 4.5. We have not yet verified the asymptotics in Theorem 1.5, however, definitions (1.3), (1.4) and Definition 1.10 themselves make sense in any case. The above $\mathcal{E}$, E and $\widetilde{\mathcal{E}}_{\mathrm{qs}}$ are understood in this way.

Proof. Step 1. We can compute for $\Phi \in \operatorname{Ker} \widetilde{Q} M_0$, using $v^* G_0^0 v = M_0 - U$,

$$\begin{aligned}
w \circ z\Phi &= \langle M_0 \Phi_1^*, \Phi \rangle U \Phi_1 + \langle M_0 \Phi_2^*, \Phi \rangle U \Phi_2 + U(U - M_0)\Phi \\
&= U\widetilde{P} M_0 \Phi + \Phi - U M_0 \Phi \\
&= \Phi.
\end{aligned}$$

This implies (4.8).

Step 2. Next, we prove (4.9). For $\Phi \in \mathcal{K}$ by using (4.3) and $v^* G_0^0 v = M_0 - U$,

$$\begin{aligned}
Hz\Phi &= \langle M_0 \Phi_1^*, \Phi \rangle v U \Phi_1 + \langle M_0 \Phi_2^*, \Phi \rangle v U \Phi_2 - v\Phi - vU v^* G_0^0 v \Phi \\
&= v U \widetilde{P} M_0 \Phi - v U M_0 \Phi \\
&= -vU\widetilde{Q} M_0 \Phi.
\end{aligned}$$

Hence, by the injectivity of v it follows that $z\Phi \in \widetilde{\mathcal{E}}$ if and only if $\Phi \in \operatorname{Ker} \widetilde{Q} M_0$, which implies the first identity of (4.9). As for the second identity, we note that for any $\Psi \in (\mathcal{L}^\beta)^*$ we have $\Psi \in \operatorname{Ker} w|_{\widetilde{\mathcal{E}}}$ if and only if

$$H_0 \Psi = 0, \quad v^* \Psi = 0.$$

Since $H_0 \Psi = 0$ gives a difference equation of order 2, we can rephrase it as $\Psi \in \mathbb{C}\Psi_1^0 \oplus \mathbb{C}\Psi_2^0$. Thus we obtain the second identity of (4.9).

Step 3. Let us prove (4.10). Let $\Phi \in \mathcal{K}$. By Lemma 4.1 we have two expressions for $z\Phi$:

$$\begin{aligned}
z\Phi[n] &= \left[\langle \Phi_2^*, M_0 \Phi \rangle + \tfrac{1}{2} \langle \Phi_1, \Phi \rangle \right] \Psi_2^0[n] + \left[\langle \Phi_1^*, M_0 \Phi \rangle - \tfrac{1}{2} \langle \Phi_2, \Phi \rangle \right] \Psi_1^0[n] \\
&\quad + \sum_{k \geq n} (k - n)(v\Phi)[k] \\
&= \left[\langle \Phi_2^*, M_0 \Phi \rangle - \tfrac{1}{2} \langle \Phi_1, \Phi \rangle \right] \Psi_2^0[n] + \left[\langle \Phi_1^*, M_0 \Phi \rangle + \tfrac{1}{2} \langle \Phi_2, \Phi \rangle \right] \Psi_1^0[n] \\
&\quad + \sum_{k \leq n} (n - k)(v\Phi)[k].
\end{aligned}$$

(4.13)

As in the proof of Lemma 4.1, the two summations in (4.13) belong to $\ell^{1,\beta-2}(\mathbb{Z}_\pm)$, respectively. This fact combined with the first identity of (4.9) implies that $z\Phi \in \mathcal{E}$ if and only if

$$\Phi \in \operatorname{Ker} \widetilde{Q} M_0, \quad \langle \Phi_2^*, M_0 \Phi \rangle = \langle \Phi_1, \Phi \rangle = 0.$$

Hence the first identity of (4.10) is obtained. As for the second one we can proceed as in Step 2, and it is almost obvious.

Step 4. The assertions (4.11) and (4.12) can be shown as in Step 3, using in particular (4.13). We omit the details. $\qquad\square$

The identity (4.8) combined with the first of (4.9) implies that the restrictions

$$z|_{\operatorname{Ker} \widetilde{Q}M_0}\colon \operatorname{Ker}\widetilde{Q}M_0 \to \widetilde{\mathcal{E}}, \quad w|_{\widetilde{\mathcal{E}}}\colon \widetilde{\mathcal{E}} \to \operatorname{Ker}\widetilde{Q}M_0$$

are injective and surjective, respectively. Hence, we have a very important corollary:

Corollary 4.6. *The eigenspaces are identified as follows:*

$$\widetilde{\mathcal{E}} = z(\operatorname{Ker}\widetilde{Q}M_0) \oplus \operatorname{Ker} v^*|_{\mathbb{C}\Psi_1^0 \oplus \mathbb{C}\Psi_2^0}, \tag{4.14}$$

$$\mathcal{E} = z(\operatorname{Ker} P \cap \operatorname{Ker} QM_0) \oplus \operatorname{Ker} v^*|_{\mathbb{C}\Psi_1^0}, \tag{4.15}$$

$$E = z(\operatorname{Ker}\widetilde{P} \cap \operatorname{Ker} M_0), \tag{4.16}$$

$$\widetilde{\mathcal{E}}_{\mathrm{qs}} = z(\operatorname{Ker} M_0). \tag{4.17}$$

In particular, the eigenfunctions have the asymptotics:

$$\widetilde{\mathcal{E}} \subset \mathbb{C}\mathbf{n} \oplus \mathbb{C}|\mathbf{n}| \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\boldsymbol{\sigma} \oplus \mathcal{L}^{\beta-2}. \tag{4.18}$$

Proof. The isomorphisms (4.14)–(4.17) are direct consequences of Proposition 4.4. The asymptotics (4.18) follows from isomorphism (4.14), the definition (4.7) of z and (4.5). $\square$

We have yet another characterization for $\widetilde{\mathcal{E}}_{\mathrm{qs}}$ and also the finite dimensionality of eigenspaces:

Proposition 4.7. *Suppose $\beta \geq 1$ in Assumption 1.1. Then for the compact operators $G_0^0 V \in \mathcal{C}(\mathcal{L}^*)$, $VG_0^0 \in \mathcal{C}(\mathcal{L})$ and $v^* G_0^0 v \in \mathcal{C}(\mathcal{K})$ there exists a well-defined circular sequence of isomorphisms between the finite dimensional vector spaces:*

$$\cdots \overset{-G_0^0}{\to} \operatorname{Ker}(1_{\mathcal{L}^*} + G_0^0 V) \overset{Uv^*}{\to} \operatorname{Ker} M_0 \overset{v}{\to} \operatorname{Ker}(1_{\mathcal{L}} + VG_0^0) \\ \overset{-G_0^0}{\to} \operatorname{Ker}(1_{\mathcal{L}^*} + G_0^0 V) \overset{Uv^*}{\to} \cdots . \tag{4.19}$$

In particular, $\widetilde{d} < \infty$, and the quasi-symmetric eigenspace is characterized by

$$\widetilde{\mathcal{E}}_{\mathrm{qs}} = -G_0^0 v \operatorname{Ker} M_0 = \operatorname{Ker}(1_{\mathcal{L}^*} + G_0^0 V) = -G_0^0 \operatorname{Ker}(1_{\mathcal{L}} + VG_0^0). \tag{4.20}$$

Proof. The compactness of the operators $G_0^0 V$, VG_0^0 and M_0 is an immediate consequence of Assumption 1.1, and hence the kernels in the sequence (4.19) are of finite dimensions. It is not difficult to show that the sequence (4.19) is

actually well-defined and circular, and, furthermore, that any compositions of three adjacent operators are the identities. This implies that all operators in (4.19) are bijective. Then noting $z = -G_0^0 v$ on $\mathrm{Ker}\, M_0$, we obtain the characterization (4.20) by Corollary 4.6 and (4.19), and also $\widetilde{d}_{\mathrm{qs}} < \infty$. Finally, $\widetilde{d} \leq d_0 + 4 \leq \widetilde{d}_{\mathrm{qs}} + 4 < \infty$. Hence we are done. $\qquad\square$

We finally add a remark on the dimensions of eigenspaces for multiplicative potentials. With it for multiplicative potentials we can skip a part of the following subsection.

Proposition 4.8. *Suppose that $\beta \geq 1$ in Assumption 1.1, and that V is multiplicative. Then*

$$\widetilde{d} \leq 2, \quad d \leq 1, \quad d_0 = 0.$$

Proof. For a multiplicative potential the equation $H\Psi = 0$ is a difference equation of order 2, and hence it is clear that $\widetilde{d} \leq 2$.

For the remaining assertions it suffices to show that the subspace

$$\widetilde{\mathcal{E}}^- = \widetilde{\mathcal{E}} \cap \left(\mathbb{C}(\mathbf{n} - |\mathbf{n}|) \oplus \mathbb{C}(\mathbf{1} - \boldsymbol{\sigma}) \oplus \mathcal{L}^{\beta-2} \right)$$

is actually trivial. We let $\Psi \in \widetilde{\mathcal{E}}^-$ and write down the relation $\Psi = zw\Psi$ explicitly. Taking into account the asymptotics of Ψ as $n \to \infty$, we have to have

$$\Psi[n] = \sum_{k \geq n} (k - n) V[k] \Psi[k],$$

cf. (4.13). Note that by $V \in \mathcal{L}^\beta$ we can choose large $n_0 \geq 0$ such that

$$\sum_{n \geq n_0} n |V[n]| \leq \tfrac{1}{2}.$$

Then we obtain for $n \geq n_0$

$$|\Psi[n]| \leq \tfrac{1}{2} \sup_{k \geq n_0} |\Psi[k]|,$$

which implies $\Psi[n] = 0$ for $n \geq n_0$. Since the equation $H\Psi = 0$ is a difference equation of order 2, the above initial condition yields $\Psi = 0$. Hence $\widetilde{\mathcal{E}}^- = \{0\}$, and we are done. $\qquad\square$

4.2 Eigenspaces and intermediate operators

In this subsection we provide a complete classification of the eigenspaces. We can find explicit bases for them modulo E.

We shall investigate structures of the eigenspaces through the *block decomposition argument* for M_0. Let us decompose the action of M_0 into the blocks corresponding to the spatial decomposition:

$$\mathcal{K} = \mathbb{C}\Phi_1 \oplus S^\perp Q \mathcal{K} \oplus S\mathcal{K}, \quad S^\perp = 1_{\mathcal{K}} - S. \tag{4.21}$$

Lemma 4.9. *Suppose $\beta \geq 1$ in Assumption 1.1. Then the identities*

$$M_0^* = M_0, \quad m_0^\dagger M_0 Q = Q M_0 m_0^\dagger = S^\perp Q, \quad S M_0 Q = Q M_0 S = 0 \tag{4.22}$$

hold. In addition, $M_0^\dagger M_0 = M_0 M_0^\dagger = S^\perp$ hold if and only if $\Phi_1 = 0$ or

$$\langle \Phi_1, (M_0 - M_0 m_0^\dagger M_0)\Phi_1 \rangle \neq 0. \tag{4.23}$$

Remark 4.10. Stated differently, Lemma 4.9 says that according to the decomposition (4.21) M_0 has the following 3×3 block matrix representation

$$M_0 = \begin{pmatrix} ? & ? & ? \\ ? & * & 0 \\ ? & 0 & 0 \end{pmatrix}, \tag{4.24}$$

where $*$ and ? mean invertible and undetermined components, respectively. In case (4.21) has trivial subspaces as components we understand the corresponding columns and rows in (4.24) as eliminated. When $\Phi_1 \neq 0$, the upper left 2×2 submatrix of (4.24) is invertible if and only if (4.23) holds.

Proof. The identities in (4.22) are obvious from the definitions. Next, we note that the identities $M_0^\dagger M_0 = M_0 M_0^\dagger = S^\perp$ are equivalent to the invertibility of $S^\perp M_0 S^\perp$ in $\mathcal{B}(S^\perp \mathcal{K})$. If $\Phi_1 = 0$, then it is obvious by the definition of S that $S^\perp M_0 S^\perp$ is invertible in $\mathcal{B}(S^\perp \mathcal{K})$. Hence we may consider only the case $\Phi_1 \neq 0$. Then since the operator $S^\perp M_0 S^\perp$ is expressed by the upper left 2×2 submatrix of (4.24), the assertion follows as an easy consequence of the Gaussian elimination for matrices, cf. the Schur complement. We omit the details. $\qquad\square$

Let us introduce a set of important vectors in addition to (4.6):

$$\Psi_3 = z\Phi_3, \quad \Phi_3 = 2S M_0 \Phi_1^*, \quad \Phi_3^* = \|\Phi_3\|^{\dagger 2} \Phi_3$$

$$\Psi_4 = z\Phi_4, \quad \Phi_4 = S\Phi_2 - \langle \Phi_3^*, \Phi_2 \rangle \Phi_3, \quad \Phi_4^* = \|\Phi_4\|^{\dagger 2} \Phi_4,$$

$$\Psi_5 = z\Phi_5, \quad \Phi_5 = (1_{\mathcal{K}} - m_0^\dagger M_0)\Phi_1^*,$$
$$\Psi_6 = z\Phi_6, \quad \Phi_6 = m_0^\dagger \Phi_2 + 2\langle \Phi_5 - 2\Delta\Phi_3^*, \Phi_2 \rangle \Phi_3^* + 2\langle \Phi_3^*, \Phi_2 \rangle \Phi_5,$$

where

$$\Delta = \langle \Phi_1^*, M_0\Phi_5 \rangle = \|\Phi_1\|^{\dagger 4}\langle \Phi_1, (M_0 - M_0 m_0^\dagger M_0)\Phi_1 \rangle.$$

We will see that combinations of these vectors span the eigenspaces. The behavior of these vectors under M_0 is essential in our argument because it determines the block matrix components of M_0. We can compute it here:

$$\begin{align} M_0\Phi_3 &= \tfrac{1}{2}\|\Phi_3\|^2\Phi_1, \quad M_0\Phi_4 = 0, \\ M_0\Phi_5 &= \Delta\Phi_1 + \tfrac{1}{2}\Phi_3, \quad M_0\Phi_6 = \Phi_2 - \Phi_4. \end{align} \tag{4.25}$$

In particular, $M_0\Phi_5 \in \mathbb{C}\Phi_1$ and $M_0\Phi_6 \in \mathbb{C}\Phi_2$ if and only if $\Phi_3 = 0$ and $\Phi_4 = 0$, respectively. For later use in Section 5 we also note that

$$\begin{align} (1_{\mathcal{L}^*} + G_0^0 V)\Psi_3 &= \tfrac{1}{2}\|\Phi_3\|^2\Psi_1^0, \\ (1_{\mathcal{L}^*} + G_0^0 V)\Psi_5 &= \Delta\Psi_1^0 + \tfrac{1}{2}\langle \Phi_2^*, \Phi_3 \rangle\Psi_2^0 - \tfrac{1}{2}G_0^0 vU\widetilde{Q}\Phi_3, \\ (1_{\mathcal{L}^*} + G_0^0 V)\Psi_6 &= \Psi_2^0 - \langle \Phi_2^*, \Phi_4 \rangle\Psi_2^0 + G_0^0 vU\widetilde{Q}\Phi_4. \end{align} \tag{4.26}$$

In particular, $(1_{\mathcal{L}^*} + G_0^0 V)\Psi_5 = \Delta\Psi_1^0$ and $(1_{\mathcal{L}^*} + G_0^0 V)\Psi_6 = \Psi_2^0$ if $\Phi_3 = 0$ and $\Phi_4 = 0$, respectively. Now we can state the main propositions of this subsection. Their proofs will be given later.

Proposition 4.11. *Suppose $\beta \geq 1$ in Assumption 1.1. Then P is invertible in $\mathcal{B}(\mathcal{K})$ if and only if $\mathcal{K} = \mathbb{C}\Phi_1$. Furthermore, if P is invertible in $\mathcal{B}(\mathcal{K})$, bases of the eigenspaces are given as in Table 2 according to the Cases defined in Table 1, where the entries with parentheses are automatically determined by the assumption and those without.*

	Φ_1	Φ_2	Φ_3	Φ_4	Δ
Case i.	$= 0$	$(= 0)$	$(= 0)$	$(= 0)$	$(= 0)$
Case ii.	$\neq 0$	$(= 0)$	$(= 0)$	$(= 0)$	$= 0$
Case iii.	$\neq 0$	$(= 0)$	$(= 0)$	$(= 0)$	$\neq 0$

Table 1: Definitions of Cases (Proposition 4.11).

Proposition 4.12. *Suppose that $\beta \geq 2$ in Assumption 1.1, and that P is not invertible in $\mathcal{B}(\mathcal{K})$. Then*

$$Q\mathcal{K} \cap \operatorname{Ker} m_0 = \operatorname{Ker} P \cap \operatorname{Ker} QM_0 \cong \mathcal{E}/(\operatorname{Ker} v^*|_{\mathbb{C}\Psi_1^0}). \tag{4.27}$$

	$\widetilde{\mathcal{E}}$	$\mathcal{E}$	E	$\widetilde{\mathcal{E}}_{\mathrm{qs}}$	type
Case i.	$\{\Psi_1^0, \Psi_2^0\}$	$\{\Psi_1^0\}$	$\emptyset$	$\emptyset$	exceptional I
Case ii.	$\{\Psi_5, \Psi_2^0\}$	$\emptyset$	$\emptyset$	$\{\Psi_5\}$	regular
Case iii.	$\{\Psi_5, \Psi_2^0\}$	$\emptyset$	$\emptyset$	$\emptyset$	regular

Table 2: Bases of the resulting eigenspaces (Proposition 4.11).

Furthermore, if m_0 is invertible in $\mathcal{B}(Q\mathcal{K})$, bases of the eigenspaces are given as in Table 4 according to the Cases defined in Table 3, where the entries with parentheses are automatically determined by the assumption and those without.

	Φ_1	Φ_2	Φ_3	Φ_4	Δ
Case i.	$= 0$	$= 0$	$(= 0)$	$(= 0)$	$(= 0)$
Case ii.	$\neq 0$	$= 0$	$(= 0)$	$(= 0)$	$= 0$
Case iii.	$\neq 0$	$= 0$	$(= 0)$	$(= 0)$	$\neq 0$
Case iv.	$= 0$	$\neq 0$	$(= 0)$	$(= 0)$	$(= 0)$
Case v.	$\neq 0$	$\neq 0$	$(= 0)$	$(= 0)$	$= 0$
Case vi.	$\neq 0$	$\neq 0$	$(= 0)$	$(= 0)$	$\neq 0$

Table 3: Definitions of Cases (Proposition 4.12).

	$\widetilde{\mathcal{E}}$	$\mathcal{E}$	E	$\widetilde{\mathcal{E}}_{\mathrm{qs}}$	type
Case i.	$\{\Psi_1^0, \Psi_2^0\}$	$\{\Psi_1^0\}$	$\emptyset$	$\emptyset$	exceptional I
Case ii.	$\{\Psi_5, \Psi_2^0\}$	$\emptyset$	$\emptyset$	$\{\Psi_5\}$	regular
Case iii.	$\{\Psi_5, \Psi_2^0\}$	$\emptyset$	$\emptyset$	$\emptyset$	regular
Case iv.	$\{\Psi_1^0, \Psi_6\}$	$\{\Psi_1^0\}$	$\emptyset$	$\emptyset$	exceptional I
Case v.	$\{\Psi_5, \Psi_6\}$	$\emptyset$	$\emptyset$	$\{\Psi_5\}$	regular
Case vi.	$\{\Psi_5, \Psi_6\}$	$\emptyset$	$\emptyset$	$\emptyset$	regular

Table 4: Bases of the resulting eigenspaces (Proposition 4.12).

Proposition 4.13. *Suppose that $\beta \geq 3$ in Assumption 1.1, and that P and m_0 are not invertible in $\mathcal{B}(\mathcal{K})$ and $\mathcal{B}(Q\mathcal{K})$, respectively. Then*

$$q_0 = -\tfrac{1}{2}\big[\langle S\Phi_2, \cdot \rangle S\Phi_2 + \langle \Phi_3, \cdot \rangle \Phi_3\big], \tag{4.28}$$

$$S\mathcal{K} \cap \operatorname{Ker} q_0 = \operatorname{Ker} \widetilde{P} \cap \operatorname{Ker} M_0 \cong E. \tag{4.29}$$

Furthermore, if q_0 is invertible in $\mathcal{B}(S\mathcal{K})$, bases of the eigenspaces are given as in Table 6 according to the Cases defined in Table 5, where the entries with parentheses are automatically determined by the assumption and those without.

	Φ_1	Φ_2	Φ_3	Φ_4	Δ
Case vii.	$= 0$	$(\neq 0)$	$(= 0)$	$(\neq 0)$	$(= 0)$
Case viii.	$(\neq 0)$	$= 0$	$\neq 0$	$= 0$	arbitrary
Case ix.	$\neq 0$	$(\neq 0)$	$= 0$	$(\neq 0)$	$= 0$
Case x.	$\neq 0$	$(\neq 0)$	$= 0$	$(\neq 0)$	$\neq 0$
Case xi.	$(\neq 0)$	$\neq 0$	$\neq 0$	$= 0$	arbitrary
Case xii.	$(\neq 0)$	$(\neq 0)$	$\neq 0$	$\neq 0$	arbitrary

Table 5: Definitions of Cases (Proposition 4.13).

	$\widetilde{\mathcal{E}}$	$\mathcal{E}$	E	$\widetilde{\mathcal{E}}_{\mathrm{qs}}$	type
Case vii.	$\{\Psi_1^0, \Psi_4\}$	$\{\Psi_1^0, \Psi_4\}$	$\emptyset$	$\{\Psi_4\}$	exceptional I
Case viii.	$\{\Psi_3, \Psi_2^0\}$	$\{\Psi_3\}$	$\emptyset$	$\emptyset$	exceptional I
Case ix.	$\{\Psi_5, \Psi_4\}$	$\{\Psi_4\}$	$\emptyset$	$\{\Psi_5, \Psi_4\}$	exceptional I
Case x.	$\{\Psi_5, \Psi_4\}$	$\{\Psi_4\}$	$\emptyset$	$\{\Psi_4\}$	exceptional I
Case xi.	$\{\Psi_3, \Psi_6\}$	$\{\Psi_3\}$	$\emptyset$	$\emptyset$	exceptional I
Case xii.	$\{\Psi_3, \Psi_4\}$	$\{\Psi_3, \Psi_4\}$	$\emptyset$	$\{\Psi_4\}$	exceptional I

Table 6: Bases of the resulting eigenspaces (Proposition 4.13).

Proposition 4.14. *Suppose that $\beta \geq 4$ in Assumption 1.1, and that P, m_0 and q_0 are not invertible in $\mathcal{B}(\mathcal{K})$, $\mathcal{B}(Q\mathcal{K})$ and $\mathcal{B}(S\mathcal{K})$, respectively. Then*

$$r_0 = -Tz^*zT, \tag{4.30}$$

and r_0 is always invertible in $\mathcal{B}(T\mathcal{K})$. Bases of the eigenspaces are given as in Table 8 according to the Cases defined in Table 7, where the entries with parentheses are automatically determined by the assumption and those without.

The finiteness of the inversion iteration discussed in Section 3.2 also follows directly from Proposition 4.14. We state it as a corollary here. The proof is straightforward, if we combine Proposition 4.14 with (4.29) and (4.16).

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	Φ_1	Φ_2	Φ_3	Φ_4	Δ
Case i.	$= 0$	$= 0$	$(= 0)$	$= 0$	$(= 0)$
Case ii.	$\neq 0$	$= 0$	$= 0$	$= 0$	$= 0$
Case iii.	$\neq 0$	$= 0$	$= 0$	$= 0$	$\neq 0$
Case iv.	$= 0$	$\neq 0$	$(= 0)$	$= 0$	$(= 0)$
Case v.	$\neq 0$	$\neq 0$	$= 0$	$= 0$	$= 0$
Case vi.	$\neq 0$	$\neq 0$	$= 0$	$= 0$	$\neq 0$
Case vii.	$= 0$	$(\neq 0)$	$(= 0)$	$\neq 0$	$(= 0)$
Case viii.	$(\neq 0)$	$= 0$	$\neq 0$	$= 0$	arbitrary
Case ix.	$\neq 0$	$(\neq 0)$	$= 0$	$(\neq 0)$	$= 0$
Case x.	$\neq 0$	$(\neq 0)$	$= 0$	$\neq 0$	$\neq 0$
Case xi.	$(\neq 0)$	$\neq 0$	$\neq 0$	$= 0$	arbitrary
Case xii.	$(\neq 0)$	$(\neq 0)$	$\neq 0$	$\neq 0$	arbitrary

Table 7: Definitions of Cases (Proposition 4.14).

Corollary 4.15. *Under the assumption of Proposition 4.14 the identities $r_0^\dagger r_0 = r_0 r_0^\dagger = T$ hold, i.e., r_0 is invertible in $\mathcal{B}(T\mathcal{K})$, and hence the pseudo-inverse $r(\kappa)^\dagger$ can be computed by the Neumann series.*

In the remainder of this subsection we prove Propositions 4.11–4.14.

Proof of Proposition 4.11. The first part of the assertion is obvious.

Assume P is invertible in $\mathcal{B}(\mathcal{K})$, Then by Corollary 4.6 it immediately follows that

$$E = \{0\}.$$

In order to identify the remaining eigenspaces $\widetilde{\mathcal{E}}$, $\mathcal{E}$ and $\widetilde{\mathcal{E}}_{\mathrm{qs}}$ we separate the cases and formally apply the block decomposition argument for M_0. Here, in fact, the decomposition argument is not necessary, but this goes along with the following proofs. As in Remark 4.10, we can write the corresponding block matrix in the following manner. Then noting Corollary 4.6 and Lemma 4.9, we can complete Table 2 with ease.

Case i.

$$\mathcal{K} = \{0\}, \quad M_0 = 0.$$

Cases ii and iii.

$$\mathcal{K} = \mathbb{C}\Phi_1, \quad M_0 = \begin{pmatrix} ? \end{pmatrix}.$$

$\square$

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	$\widetilde{\mathcal{E}}/E$	$\mathcal{E}/E$	E	$\widetilde{\mathcal{E}}_{\mathrm{qs}}/E$	type
Case i.	$\{\Psi_1^0, \Psi_2^0\}$	$\{\Psi_1^0\}$	$0 < \# < \infty$	$\emptyset$	exceptional III
Case ii.	$\{\Psi_5, \Psi_2^0\}$	$\emptyset$	$0 < \# < \infty$	$\{\Psi_5\}$	exceptional II
Case iii.	$\{\Psi_5, \Psi_2^0\}$	$\emptyset$	$0 < \# < \infty$	$\emptyset$	exceptional II
Case iv.	$\{\Psi_1^0, \Psi_6\}$	$\{\Psi_1^0\}$	$0 < \# < \infty$	$\emptyset$	exceptional III
Case v.	$\{\Psi_5, \Psi_6\}$	$\emptyset$	$0 < \# < \infty$	$\{\Psi_5\}$	exceptional II
Case vi.	$\{\Psi_5, \Psi_6\}$	$\emptyset$	$0 < \# < \infty$	$\emptyset$	exceptional II
Case vii.	$\{\Psi_1^0, \Psi_4\}$	$\{\Psi_1^0, \Psi_4\}$	$0 < \# < \infty$	$\{\Psi_4\}$	exceptional III
Case viii.	$\{\Psi_3, \Psi_2^0\}$	$\{\Psi_3\}$	$0 < \# < \infty$	$\emptyset$	exceptional III
Case ix.	$\{\Psi_5, \Psi_4\}$	$\{\Psi_4\}$	$0 < \# < \infty$	$\{\Psi_5, \Psi_4\}$	exceptional III
Case x.	$\{\Psi_5, \Psi_4\}$	$\{\Psi_4\}$	$0 < \# < \infty$	$\{\Psi_4\}$	exceptional III
Case xi.	$\{\Psi_3, \Psi_6\}$	$\{\Psi_3\}$	$0 < \# < \infty$	$\emptyset$	exceptional III
Case xii.	$\{\Psi_3, \Psi_4\}$	$\{\Psi_3, \Psi_4\}$	$0 < \# < \infty$	$\{\Psi_4\}$	exceptional III

Table 8: Bases of the resulting eigenspaces (Proposition 4.14).

Proof of Proposition 4.12. The first identity of (4.27) is due to the definition $m_0 = QM_0Q$. The latter is also already known by the isomorphism (4.15) and the injectivity of z on $\mathrm{Ker}\,\widetilde{Q}M_0$ due to Proposition 4.4.

Let us assume that the operator m_0 is invertible in $\mathcal{B}(Q\mathcal{K})$. By Corollary 4.6 it follows

$$E = \{0\}.$$

We investigate the remaining eigenspaces employing the block decomposition argument as in the previous proof. We write

$$\mathcal{K} = \mathbb{C}\Phi_1 \oplus S^\perp Q\mathcal{K}, \quad M_0 = \begin{pmatrix} ? & ? \\ ? & * \end{pmatrix},$$

and proceed to Cases i–vi. We note that for Cases i–iii, compared to the previous proof, there appear extra column and row concerning $S^\perp Q\mathcal{K}$-factor, but these do not essentially affect the argument and we obtain the same list of eigenfunctions as before. Hence, we omit Cases i–iii, and consider only Cases iv–vi without distinguishing whether $S^\perp Q\mathcal{K}$ is trivial by the same reason. Once we write down the matrix expressions for M_0, then the assertion is straightforward by Corollary 4.6, Lemma 4.9 and (4.25).

Case iv.

$$\mathcal{K} = S^\perp Q\mathcal{K}, \quad M_0 = \begin{pmatrix} * \end{pmatrix}.$$

Cases v and vi.

$$\mathcal{K} = \mathbb{C}\Phi_1 \oplus S^\perp Q\mathcal{K}, \quad M_0 = \begin{pmatrix} ? & ? \\ ? & * \end{pmatrix}.$$

$\square$

Proof of Proposition 4.13. In order to prove (4.28) we use the expressions of q_0 and m_1 in Lemma 3.3. Noting that QM_0S and SM_0Q are trivial by the definition of S, we can write

$$q_0 = SM_1S - \gamma SM_0PM_0S.$$

We substitute (2.7) for M_0 and M_1 above, and use the kernel expressions (2.5) for G_0^0 and G_1^0, respectively. Then by $Sv^*\mathbf{1} = Qv^*\mathbf{1} = 0$ we obtain

$$q_0 = -\tfrac{1}{2}\langle Sv^*\mathbf{n}, \cdot\rangle Sv^*\mathbf{n} - \tfrac{\gamma^2}{2}\langle SM_0v^*\mathbf{1}, \cdot\rangle SM_0v^*\mathbf{1}.$$

Thus (4.28) is verified.

Next, we prove (4.29). First note that

$$S\mathcal{K} = \operatorname{Ker} m_0 = \operatorname{Ker} P \cap \operatorname{Ker} QM_0 \supset \operatorname{Ker} \widetilde{P} \cap \operatorname{Ker} M_0,$$

and hence it suffices to consider only $\Phi \in S\mathcal{K}$. By (4.28) it follows that, for $\Phi \in S\mathcal{K}$, the identity $q_0(\Phi) = 0$ holds if and only if

$$\langle SM_0v^*\mathbf{1}, \Phi\rangle = \langle Sv^*\mathbf{n}, \Phi\rangle = 0. \tag{4.31}$$

Note that this equivalence holds whether $Sv^*\mathbf{n}$ and $SM_0v^*\mathbf{1}$ are linearly independent or not. The condition (4.31) for $\Phi \in S\mathcal{K}$ is equivalent to

$$\Phi \in \operatorname{Ker} \widetilde{P} \cap \operatorname{Ker} M_0,$$

which implies the first identity of (4.29). The second isomorphism is due to (4.16).

Now we assume q_0 is invertible in $\mathcal{B}(S\mathcal{K})$. By (4.29) it follows

$$E = \{0\}.$$

The remaining eigenspaces are studied by the block decomposition argument, again. Note that by the above argument we have a further decomposition

$$S\mathcal{K} = \mathbb{C}\Phi_3 \oplus \mathbb{C}\Phi_4,$$

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and also that

$$M_0 \Phi_4 = 0. \tag{4.32}$$

By this and Lemma 4.9 we can calculate the matrix expressions for M_0, and complete Table 6 as in the previous proofs. Note that we do not have to distinguish whether $S^\perp Q \mathcal{K}$ is trivial or not, as before.

Case vii.

$$\mathcal{K} = S^\perp Q \mathcal{K} \oplus \mathbb{C}\Phi_4, \quad M_0 = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}.$$

Cases viii and xi.

$$\mathcal{K} = \mathbb{C}\Phi_1 \oplus S^\perp Q \mathcal{K} \oplus \mathbb{C}\Phi_3, \quad M_0 = \begin{pmatrix} ? & ? & * \\ ? & * & 0 \\ * & 0 & 0 \end{pmatrix}.$$

Case ix and x.

$$\mathcal{K} = P\mathcal{K} \oplus S^\perp Q \mathcal{K} \oplus \mathbb{C}\Phi_4, \quad M_0 = \begin{pmatrix} ? & ? & 0 \\ ? & * & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Case xii.

$$\mathcal{K} = \mathbb{C}\Phi_1 \oplus S^\perp Q \mathcal{K} \oplus \mathbb{C}\Phi_3 \oplus \mathbb{C}\Phi_4, \quad M_0 = \begin{pmatrix} ? & ? & * & 0 \\ ? & * & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\square$

Before the proof of Proposition 4.14 we show the following lemma:

Lemma 4.16. *Let $x_j \in \mathcal{L}^4$ for $j = 1, 2$ satisfy*

$$\langle \mathbf{1}, x_j \rangle = \langle \mathbf{n}, x_j \rangle = 0. \tag{4.33}$$

Then $G_0^0 x_j \in \mathcal{L}^2$ for $j = 1, 2$, and

$$\langle x_1, G_2^0 x_2 \rangle = -\langle G_0^0 x_1, G_0^0 x_2 \rangle. \tag{4.34}$$

Proof. The assertion $G_0^0 x_j \in \mathcal{L}^2$ for $j = 1, 2$ follows from Lemma 4.1, so that the right-hand side of (4.34) makes sense. By the kernel expressions (2.5) and the assumption (4.33) we have

$$\langle x_1, G_{-1}^0 x_2 \rangle = \langle x_1, G_1^0 x_2 \rangle = 0,$$

and hence by Proposition 2.1 with $N = 2$, where the large weight 4 is required,

$$\langle x_1, G_2^0 x_2 \rangle = \lim_{\kappa \to +0} \tfrac{1}{\kappa^2} \langle x_1, (R_0(\kappa) - G_0^0) x_2 \rangle. \qquad (4.35)$$

We now use the Fourier transform to compute the right-hand side. We already know the Fourier transform of $R(\kappa)$ by (2.1), and thus consider that of G_0^0. We claim

$$\langle x_1, G_0^0 x_2 \rangle = \int_{\mathbb{T}} \frac{\overline{\hat{x}_1(\theta)} \hat{x}_2(\theta)}{4 \sin^2(\theta/2)} \, d\theta. \qquad (4.36)$$

Note that the right-hand side is actually a convergent integral due to

$$\hat{x}_j \in C^4(\mathbb{T}), \quad \hat{x}_j(0) = 0, \quad \hat{x}_j'(0) = 0. \qquad (4.37)$$

We consider the distribution $f \in \mathcal{D}'(\mathbb{T})$ defined by

$$f(\theta) = -(8\pi)^{-1/2} \tfrac{d}{d\theta} \text{p.v.} \cot \tfrac{\theta}{2},$$

and then, using Cauchy's integral formula, we can verify

$$\mathcal{F}^{-1}(f)[n] = -\tfrac{1}{2} |n| = G_0^0(n).$$

Hence, for rapidly decreasing sequences y_j, $j = 1, 2$, or equivalently, $\hat{y}_j \in C^\infty(\mathbb{T})$, we obtain

$$\langle y_1, G_0^0 y_2 \rangle = \sqrt{2\pi} \langle \hat{y}_1, \hat{y}_2 \cdot f \rangle = \tfrac{1}{2} \text{p.v.} \int_{\mathbb{T}} \left(\overline{\hat{y}_1(\theta)} \hat{y}_2(\theta) \right)' \cot \tfrac{\theta}{2} \, d\theta. \qquad (4.38)$$

By the density argument the identity (4.38) extends to $y_j \in \mathcal{L}$, $j = 1, 2$. If we set $y_j = x_j$ of the lemma, then we obtain the claim (4.36) by integration by parts. We note that we can drop 'p.v.' using (4.37).

We note that from the claim (4.36) and the density argument a slightly general identity follows :

$$\mathcal{F}(G_0^0 x_j)(\theta) = \frac{\hat{x}_j(\theta)}{4 \sin^2(\theta/2)}. \qquad (4.39)$$

By (4.35), (2.1) and (4.36) we can write

$$\langle x_1, G_2^0 x_2 \rangle = - \lim_{\kappa \to +0} \int_{\mathbb{T}} \frac{\overline{\hat{x}_1(\theta)} \hat{x}_2(\theta)}{4 \sin^2(\theta/2)(4 \sin^2(\theta/2) + \kappa^2)} \, d\theta.$$

By (4.37) we can apply the dominated convergence theorem, and then by (4.39) we obtain the assertion. □

Proof of Proposition 4.14. We first note

$$TM_0 = 0, \quad TM_1 Q = 0. \tag{4.40}$$

In fact, we can rewrite using $1_{\mathcal{K}} = P + Q$, (2.7) and (2.5)

$$TM_0 = \tfrac{\gamma}{2} \langle v^* \mathbf{1}, \cdot \rangle TM_0 v^* \mathbf{1} + TM_0 Q, \quad TM_1 Q = -\tfrac{1}{2} \langle Qv^* \mathbf{n}, \cdot \rangle Tv^* \mathbf{n},$$

and then (4.40) immediately follow. By Lemma 3.3, (4.40) and its adjoint we can compute

$$r_0 = Tv^* G_2^0 vT.$$

Since the image $vT(\mathcal{K}) \subset \mathcal{L}^\beta$ is orthogonal to both $\mathbf{1}$ and $\mathbf{n}$ with respect to the ℓ^2-inner product, we can apply Lemma 4.16 and then obtain

$$r_0 = -Tv^* G_0^0 G_0^0 vT = -Tz^* zT.$$

Here we replaced $-G_0^0 v$ by z using the first identity of (4.40). Thus we obtain (4.30).

Now we investigate the eigenspaces. We already know

$$E \neq \{0\}, \quad 0 < d < \infty.$$

As in the previous proofs, we apply the block decomposition argument. We decompose

$$\mathcal{K} = \mathbb{C}\Phi_1 \oplus S^\perp Q\mathcal{K} \oplus \mathbb{C}\Phi_3 \oplus \mathbb{C}\Phi_4 \oplus T\mathcal{K},$$

and

$$M_0 = \begin{pmatrix} ? & ? & * & 0 & 0 \\ ? & * & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.41}$$

accordingly. The representation (4.41) can be verified, e.g., by Lemma 4.9, (4.32) and (4.40). Fortunately, the space $T\mathcal{K}$ does not have any effects under the action of M_0, and hence we can reduce Cases i–xii to those of the previous propositions. Thus we are done. □

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5 Expansion of $R(\kappa)$

5.1 The case: P is invertible

Theorem 5.1. *Suppose that $\beta \geq 1$ in Assumption 1.1, and that P defined by (3.5) is invertible in $\mathcal{B}(\mathcal{K})$. Let $N \in [-1, \beta - 2]$ be any integer. Then as $\kappa \to 0$ with $\mathrm{Re}\,\kappa > 0$, the resolvent $R(\kappa)$ has the expansion in $\mathcal{B}^{N+2}$:*

$$R(\kappa) = \sum_{j=-1}^{N} \kappa^j G_j + \mathcal{O}(\kappa^{N+1}), \quad G_j \in \mathcal{B}^{j+1}, \tag{5.1}$$

and the coefficients G_j can be computed explicitly.

Proof. Since the leading operator of the expansion of $M(\kappa)$ is invertible, we can expand $M(\kappa)^{-1}$ employing the Neumann series. Using Proposition 2.2 for any integer $N \in [-1, \beta - 2]$, we have the expansion in $\mathcal{B}(\mathcal{K})$:

$$M(\kappa)^{-1} = \sum_{j=1}^{N+2} \kappa^j A_j + \mathcal{O}(\kappa^{N+3}), \quad A_j \in \mathcal{B}(\mathcal{K}). \tag{5.2}$$

The coefficients A_j can be written explicitly in terms of M_j:

$$A_1 = \gamma, \quad A_j = -\sum_{k=1}^{j-1}(-\gamma)^{k+1} \sum_{\substack{j_1 \geq 0, \dots, j_k \geq 0 \\ j_1 + \cdots + j_k = j-k-1}} \prod_{l=1}^{k} M_{j_l} \quad \text{for } j \geq 2. \tag{5.3}$$

Let us substitute the expansions (5.2) and (2.4) into (1.9). Then for the same N as above we have the expansion in $\mathcal{B}^{N+2}$:

$$R(\kappa) = \sum_{j=-1}^{N} \kappa^j G_j^0 - \sum_{j=-1}^{N} \kappa^j \sum_{\substack{j_1 \geq -1, j_2 \geq 1, j_3 \geq -1 \\ j_1 + j_2 + j_3 = j}} G_{j_1}^0 v A_{j_2} v^* G_{j_3}^0 + \mathcal{O}(\kappa^{N+1}).$$

Hence, we obtain the expansion of $R(\kappa)$ by putting for $j \geq -1$

$$G_j = G_j^0 - \sum_{\substack{j_1 \geq -1, j_2 \geq 1, j_3 \geq -1 \\ j_1 + j_2 + j_3 = j}} G_{j_1}^0 v A_{j_2} v^* G_{j_3}^0 \in \mathcal{B}^{j+1}. \tag{5.4}$$

Hence, we are done. $\qquad\square$

We next discuss the relation between the coefficients and the eigenspaces. By (5.4) and (5.3)

$$\begin{aligned}
G_{-1} &= G_{-1}^0 - G_{-1}^0 v A_1 v^* G_{-1}^0 \\
&= G_{-1}^0 - \gamma G_{-1}^0 v v^* G_{-1}^0,
\end{aligned} \tag{5.5}$$

$$G_0 = G_0^0 - G_{-1}^0 v A_2 v^* G_{-1}^0 - G_{-1}^0 v A_1 v^* G_0^0 - G_0^0 v A_1 v^* G_{-1}^0$$
$$= G_0^0 + \gamma^2 G_{-1}^0 v M_0 v^* G_{-1}^0 - \gamma G_{-1}^0 v v^* G_0^0 - \gamma G_0^0 v v^* G_{-1}^0. \quad (5.6)$$

Theorem 5.2. *Under the assumption of Theorem 5.1 the coefficient G_{-1} is of the form (1.5) (up to the factor i due to a different convention).*

Remark 5.3. If $V \neq 0$ is local, then we always have $G_{-1} = 0$, see Section B.2.

Proof. We already have the list of the eigenfunctions in Table 2 for all Cases i–iii in Table 1. Then since $G_{-1} = 2^{-1}\langle \Psi_1^0, \cdot \rangle \Psi_1^0$ for Case i and $G_{-1} = 0$ for Cases ii and iii, the assertion follows immediately. $\qquad \square$

Let us consider G_0. It is obvious that we have

$$G_0 \equiv G_0^0 + \langle \Psi_5, \cdot \rangle \Psi_1^0 + \langle \Psi_1^0, \cdot \rangle \Psi_5 \mod \mathcal{B}^0.$$

By a direct computation, e.g., with $v^* G_{-1}^0 v = \gamma^\dagger$, $P = 1_{\mathcal{K}}$ and $\Phi_1 = \Phi_5$, we obtain

$$
\begin{aligned}
(1_{\mathcal{L}^*} + G_0^0 V)G_0 &= G_0^0 + \gamma^2 G_{-1}^0 v M_0 v^* G_{-1}^0 - \gamma G_{-1}^0 v v^* G_0^0 - \gamma G_0^0 v v^* G_{-1}^0 \\
&\quad + G_0^0 V G_0^0 + \gamma G_0^0 v U P M_0 v^* G_{-1}^0 - G_0^0 v U P v^* G_0^0 \\
&\quad - \gamma G_0^0 V G_0^0 v v^* G_{-1}^0 \\
&= G_0^0 + \langle \Psi_5, \cdot \rangle \Psi_1^0. \quad (5.7)
\end{aligned}
$$

By (5.7) and Lemma 4.2 it immediately follows that

$$HG_0 = 1_{\mathcal{L}}.$$

A similar adjoint computation applies to $G_0 H$ with Lemma 4.2, and verifies $G_0 H = 1_{\mathcal{L}}$. We can also prove $HG_0 = G_0 H = 1_{\mathcal{L}}$ by using (1.1). If $\widetilde{\mathcal{E}}_{\mathrm{qs}} = \{0\}$, then $1_{\mathcal{L}^*} + G_0^0 V$ is invertible and it follows by (5.7) and (4.26) that

$$G_0 = (1_{\mathcal{L}^*} + G_0^0 V)^{-1} G_0^0 + \Delta^\dagger \langle \Psi_5, \cdot \rangle \Psi_5.$$

Even if $\widetilde{\mathcal{E}}_{\mathrm{qs}} \neq \{0\}$, we can still modify the argument. Set

$$\pi_1 = 1_{\mathcal{L}} - \langle \Psi_1^0, \cdot \rangle V \Psi_5, \quad \pi_2 = 1_{\mathcal{L}} - \langle \Psi_5, \cdot \rangle V \Psi_1^0.$$

Since $\langle \Psi_1^0, V \Psi_5 \rangle = \langle \Psi_5, V \Psi_1^0 \rangle = 1 \neq 0$, the operators π_1 and π_2 are in fact projections on the Banach space $\mathcal{L}$. Then noting $\mathrm{Ran}(1_{\mathcal{L}} + G_0^0 V) = \pi_1^* \mathcal{L}$ and $\mathrm{Ker}(1_{\mathcal{L}^*} + G_0^0 V) = \mathbb{C}\Psi_5$ by (4.20) and Table 2, we can write (5.7) as

$$(1_{\mathcal{L}^*} + G_0^0 V)\pi_2^* G_0 = \pi_1^* G_0^0. \quad (5.8)$$

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Now recall the Riesz–Schauder theorem. Since $\langle \Psi_5, V\Psi_1^0 \rangle \neq 0$, we can invert $\pi_1^*(1_{\mathcal{L}^*} + G_0^0 V)\pi_2^*$ between the projected spaces. We may write (5.8) as

$$\pi_2^* G_0 = [\pi_1^*(1_{\mathcal{L}^*} + G_0^0 V)\pi_2^*]^{-1}\pi_1^* G_0^0.$$

On the other hand, we can compute from (5.6) with $M_0 = 0$ and $U = \pm 1_{\mathcal{K}}$ that

$$(1 - \pi_2^*)G_0 = \langle \Psi_1^0, \cdot \rangle \Psi_5.$$

Hence we obtain the expression

$$G_0 = [\pi_1^*(1_{\mathcal{L}^*} + G_0^0 V)\pi_2^*]^{-1}\pi_1^* G_0^0 + \langle \Psi_1^0, \cdot \rangle \Psi_5.$$

We gather the above results as a theorem. The adjoint statements are obtained similarly.

Theorem 5.4. *Under the assumption of Theorem 5.1 the coefficient G_0 satisfies*

$$G_0 \equiv G_0^0 + \langle \Psi_5, \cdot \rangle \Psi_1^0 + \langle \Psi_1^0, \cdot \rangle \Psi_5 \mod \mathcal{B}^0$$

and

$$HG_0 = G_0 H = 1_{\mathcal{L}}.$$

Moreover, with the above notation, if $\widetilde{\mathcal{E}}_{\mathrm{qs}} = \{0\}$, then

$$\begin{aligned} G_0 &= (1_{\mathcal{L}^*} + G_0^0 V)^{-1} G_0^0 + \Delta^\dagger \langle \Psi_5, \cdot \rangle \Psi_5 \\ &= G_0^0 (1_{\mathcal{L}} + V G_0^0)^{-1} + \Delta^\dagger \langle \Psi_5, \cdot \rangle \Psi_5, \end{aligned}$$

and, if $\widetilde{\mathcal{E}}_{\mathrm{qs}} \neq \{0\}$, then

$$\begin{aligned} G_0 &= [\pi_1^*(1_{\mathcal{L}^*} + G_0^0 V)\pi_2^*]^{-1}\pi_1^* G_0^0 + \langle \Psi_1^0, \cdot \rangle \Psi_5 \\ &= G_0^0 \pi_1 [\pi_2 (1_{\mathcal{L}} + V G_0^0)\pi_1]^{-1} + \langle \Psi_5, \cdot \rangle \Psi_1^0. \end{aligned} \tag{5.9}$$

Remarks 5.5. 1. The coefficient G_0 appears in the expansion (5.1) for $N \geq 0$, so that we seemingly need $\beta \geq 2$, but do not in fact, because that is required to dominate the error term. If we take (5.6) as the definition of G_0, we only need $\beta \geq 1$. Theorem 5.4 reads in this sense.

2. As in Lemma 4.2, the composition $G_0 H$ makes sense also on the extended space $\mathbb{C}\mathbf{n} \oplus \mathbb{C}|\mathbf{n}| \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\boldsymbol{\sigma} \oplus \mathcal{L}$, and we can compute the action on $\mathbb{C}\mathbf{n} \oplus \mathbb{C}|\mathbf{n}| \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\boldsymbol{\sigma}$ by a direct computation employing (5.6). We here write down only the consequences, omitting the detail:

$$\begin{aligned} G_0 H \Psi_1^0 &= \|\Phi_1\|^{\dagger 2}\|\Phi_1\|^2 \Psi_1^0, & G_0 H \Psi_2^0 &= 0, \\ G_0 H \boldsymbol{\sigma} &= \boldsymbol{\sigma}, & G_0 H |\mathbf{n}| &= |\mathbf{n}| - 2\Psi_5. \end{aligned}$$

5.2 The case: P is not invertible, and m_0 is invertible

Theorem 5.6. *Suppose that $\beta \geq 2$ in Assumption 1.1, that P defined by (3.5) is not invertible in $\mathcal{B}(\mathcal{K})$, and that m_0 defined by (3.9) is invertible in $\mathcal{B}(Q\mathcal{K})$. Let $N \in [0, \beta-2]$ be any integer. Then as $\kappa \to 0$ with $\operatorname{Re}\kappa > 0$, the resolvent $R(\kappa)$ has the expansion in $\mathcal{B}^{N+2}$:*

$$R(\kappa) = \sum_{j=-1}^{N} \kappa^j G_j + \mathcal{O}(\kappa^{N+1}), \quad G_j \in \mathcal{B}^{j+1}, \tag{5.10}$$

and the coefficients G_j can be computed explicitly.

Proof. By Proposition 1.13 and (3.8) we can write

$$\begin{aligned}
R(\kappa) = R_0(\kappa) &- \kappa R_0(\kappa)v(Q + \kappa M(\kappa))^{-1}v^*R_0(\kappa) \\
&- R_0(\kappa)v(Q + \kappa M(\kappa))^{-1}m(\kappa)^\dagger(Q + \kappa M(\kappa))^{-1}v^*R_0(\kappa).
\end{aligned} \tag{5.11}$$

We compute the expansions of $m(\kappa)^\dagger$ and $(Q + \kappa M(\kappa))^{-1}$ employing the Neumann series. Let $N \in [0, \beta-2]$ be any integer as in the assertion. Then we have the expansions in $\mathcal{B}(\mathcal{K})$ and $\mathcal{B}(Q\mathcal{K})$:

$$(Q + \kappa M(\kappa))^{-1} = \sum_{j=0}^{N+1} \kappa^j A_j + \mathcal{O}(\kappa^{N+2}), \quad A_j \in \mathcal{B}(\mathcal{K}), \tag{5.12}$$

$$m(\kappa)^\dagger = \sum_{j=0}^{N} \kappa^j B_j + \mathcal{O}(\kappa^{N+1}), \quad B_j \in \mathcal{B}(Q\mathcal{K}), \tag{5.13}$$

where

$$A_0 = Q + \gamma P, \quad A_j = \sum_{k=1}^{j} \sum_{\substack{j_1 \geq 0, \dots, j_k \geq 0 \\ j_1 + \cdots + j_k = j-k}} A_0 \prod_{l=1}^{k} (-M_{j_l} A_0) \quad \text{for } j \geq 1, \tag{5.14}$$

$$B_0 = m_0^\dagger, \qquad B_j = \sum_{k=1}^{j} \sum_{\substack{j_1 \geq 1, \dots, j_k \geq 1 \\ j_1 + \cdots + j_k = j}} B_0 \prod_{l=1}^{k} (-m_{j_l} B_0) \quad \text{for } j \geq 1, \tag{5.15}$$

respectively. We substitute (2.4), (5.12) and (5.13) into (5.11), and then in the topology of $\mathcal{B}^{N+2}$

$$\begin{aligned}
R(\kappa) = \sum_{j=-1}^{N} \kappa^j G_j^0 &- \sum_{j=-1}^{N} \kappa^j \sum_{\substack{j_1 \geq -1, j_2 \geq 0, j_3 \geq -1 \\ j_1 + j_2 + j_3 = j-1}} G_{j_1}^0 v A_{j_2} v^* G_{j_3}^0 \\
&- \sum_{j=-2}^{N} \kappa^j \sum_{\substack{j_1 \geq -1, j_2 \geq 0, j_3 \geq 0, j_4 \geq 0, j_5 \geq -1 \\ j_1 + j_2 + j_3 + j_4 + j_5 = j}} G_{j_1}^0 v A_{j_2} B_{j_3} A_{j_4} v^* G_{j_5}^0 + \mathcal{O}(\kappa^{N+1}).
\end{aligned} \tag{5.16}$$

Here we note that the error term is not $\mathcal{O}(\kappa^{N-1})$ or $\mathcal{O}(\kappa^N)$, because the possible error terms of order κ^{N-1} or κ^N contain the factors $G_{-1}v^*A_0Q$ or $QA_0vG^0_{-1}$, both of which vanish:

$$G_{-1}vA_0Q = 0, \quad QA_0v^*G^0_{-1} = 0. \tag{5.17}$$

The same fact (5.17) also guarantees that the coefficient of κ^{-2} to the right of (5.16) vanishes, and furthermore that, if we set for $j \geq -1$

$$
\begin{aligned}
G_j = G^0_j &- \sum_{\substack{j_1 \geq -1, j_2 \geq 0, j_3 \geq -1 \\ j_1 + j_2 + j_3 = j-1}} G^0_{j_1} v A_{j_2} v^* G^0_{j_3} \\
&- \sum_{\substack{j_1 \geq -1, j_2 \geq 0, j_3 \geq 0, j_4 \geq 0, j_5 \geq -1 \\ j_1 + j_2 + j_3 + j_4 + j_5 = j}} G^0_{j_1} v A_{j_2} B_{j_3} A_{j_4} v^* G^0_{j_5},
\end{aligned}
\tag{5.18}
$$

then $G_j \in \mathcal{B}^{j+1}$. Hence we obtain the desired expansion (5.10). $\qquad \square$

We next investigate the coefficients. We can write by (5.18), (5.17), (5.14) and (5.15)

$$
\begin{aligned}
G_{-1} &= G^0_{-1} - G^0_{-1} v A_0 v^* G^0_{-1} \\
&= G^0_{-1} - \gamma G^0_{-1} v v^* G^0_{-1},
\end{aligned}
\tag{5.19}
$$

$$
\begin{aligned}
G_0 &= G^0_0 - G^0_{-1} v A_1 v^* G^0_{-1} - G^0_{-1} v A_0 v^* G^0_0 - G^0_0 v A_0 v^* G^0_{-1} \\
&\quad - (G^0_{-1} v A_1 + G^0_0 v A_0) B_0 (A_1 v^* G^0_{-1} + A_0 v^* G^0_0) \\
&= G^0_0 + \gamma^2 G^0_{-1} v M_0 v^* G^0_{-1} - \gamma G^0_{-1} v v^* G^0_0 - \gamma G^0_0 v v^* G^0_{-1} \\
&\quad - (\gamma G^0_{-1} v M_0 - G^0_0 v) m^\dagger_0 (\gamma M_0 v^* G^0_{-1} - v^* G^0_0).
\end{aligned}
\tag{5.20}
$$

We note that parts of expressions (5.19), (5.20) coincide with those of (5.5), (5.6), respectively, because parts of (5.18) and (5.4) are the same. But we have to be careful since the definitions (5.14) and (5.3) of A_j are different.

Theorem 5.7. *Under the assumption of Theorem 5.6 the coefficient G_{-1} is of the form (1.5) (up to the factor i due to a different convention).*

Proof. With the bases listed in Table 4 for Cases i–vi in Table 3 the assertion is obvious as that of Theorem 5.2. $\qquad \square$

Theorem 5.8. *Under the assumption of Theorem 5.6 the coefficient G_0 satisfies*

$$G_0 \equiv G^0_0 + \langle \Psi_5, \cdot \rangle \Psi^0_1 + \langle \Psi^0_1, \cdot \rangle \Psi_5 \mod \mathcal{B}^0 \tag{5.21}$$

and

$$HG_0 = G_0H = 1_{\mathcal{L}}. \tag{5.22}$$

Moreover, if $\widetilde{\mathcal{E}}_{\mathrm{qs}} = \{0\}$, *then*

$$
\begin{aligned}
G_0 &= (1_{\mathcal{L}^*} + G_0^0 V)^{-1} G_0^0 + \Delta^\dagger \langle \Psi_5, \cdot \rangle \Psi_5 \\
&= G_0^0 (1_{\mathcal{L}} + V G_0^0)^{-1} + \Delta^\dagger \langle \Psi_5, \cdot \rangle \Psi_5,
\end{aligned}
\tag{5.23}
$$

and, if $\widetilde{\mathcal{E}}_{\mathrm{qs}} \neq \{0\}$, *then*

$$
\begin{aligned}
G_0 &= [\pi_1^*(1_{\mathcal{L}^*} + G_0^0 V)\pi_2^*]^{-1} \pi_1^* G_0^0 + \langle \Psi_1^0, \cdot \rangle \Psi_5 \\
&= G_0^0 \pi_1 [\pi_2(1_{\mathcal{L}} + V G_0^0)\pi_1]^{-1} + \langle \Psi_5, \cdot \rangle \Psi_1^0,
\end{aligned}
\tag{5.24}
$$

where π_1, π_2 *are projections in* $\mathcal{L}$ *defined by*

$$\pi_1 = 1_{\mathcal{L}} - \langle \Psi_1^0, \cdot \rangle V \Psi_5, \quad \pi_2 = 1_{\mathcal{L}} - \langle \Psi_5, \cdot \rangle V \Psi_1^0, \tag{5.25}$$

and $[\pi_1^*(1_{\mathcal{L}^*} + G_0^0 V)\pi_2^*]^{-1}$, $[\pi_2(1_{\mathcal{L}} + V G_0^0)\pi_1]^{-1}$ *are inverses in the projected spaces given by the Riesz–Schauder theorem.*

Proof. Note that by Lemma 4.1 $G_0^0 v m_0^\dagger v^* G_0^0 \in \mathcal{B}^0$. Then by the expression (5.20) the formula (5.21) follows.

We compute directly using the expression (5.20):

$$
\begin{aligned}
&(1_{\mathcal{L}^*} + G_0^0 V) G_0 \\
&= G_0^0 + \gamma^2 G_{-1}^0 v M_0 v^* G_{-1}^0 - \gamma G_{-1}^0 v v^* G_0^0 - \gamma G_0^0 v v^* G_{-1}^0 - G_0^0 v m_0^\dagger v^* G_0^0 \\
&\quad + \gamma G_0^0 v m_0^\dagger M_0 v^* G_{-1}^0 + \gamma G_{-1}^0 v M_0 m_0^\dagger v^* G_0^0 - \gamma^2 G_{-1}^0 v M_0 m_0^\dagger M_0 v^* G_{-1}^0 \\
&\quad + \gamma G_0^0 v v^* G_{-1}^0 + G_0^0 v m_0^\dagger v^* G_0^0 - \gamma G_0^0 v m_0^\dagger M_0 v^* G_{-1}^0 \\
&= G_0^0 - \gamma G_{-1}^0 v(1_{\mathcal{K}} - M_0 m_0^\dagger) v^* G_0^0 + \gamma^2 G_{-1}^0 v(M_0 - M_0 m_0^\dagger M_0) v^* G_{-1}^0.
\end{aligned}
$$

If we note $Q(M_0 - M_0 m_0^\dagger M_0) = 0$, then we obtain

$$(1_{\mathcal{L}^*} + G_0^0 V) G_0 = G_0^0 + \langle \Psi_5, \cdot \rangle \Psi_1^0. \tag{5.26}$$

Thus, it follows that $HG_0 = 1_{\mathcal{L}}$ by (5.26) and Lemma 4.2, and that $G_0 H|_{\mathcal{L}} = 1_{\mathcal{L}}$ by the adjoint argument. These show (5.22).

If $\widetilde{\mathcal{E}}_{\mathrm{qs}} = \{0\}$, then $1_{\mathcal{L}^*} + G_0^0 V$ is invertible, and (5.23) can be easily verified by (5.26) and (4.26). Now assume $\widetilde{\mathcal{E}}_{\mathrm{qs}} \neq \{0\}$. We note, since $\langle \Psi_1^0, V \Psi_5 \rangle = \langle \Psi_5, V \Psi_1^0 \rangle = 1 \neq 0$, the operators π_1 and π_2 defined by (5.25) are in fact

projections. Repeating the arguments in the previous subsection, we can write using the Riesz–Schauder theorem that

$$\pi_2^* G_0 = [\pi_1^*(1_{\mathcal{L}^*} + G_0^0 V)\pi_2^*]^{-1}\pi_1^* G_0^0.$$

Hence, it suffices to compute $(1-\pi_2^*)G_0$. Let us note by a direct computation with (5.20) that

$$V G_0 = \langle \Psi_1^0, \cdot \rangle v\Phi_5 + vm_0^\dagger v^* G_0^0.$$

Then it follows that

$$(1 - \pi_2^*)G_0 = \langle \Psi_1^0, \cdot \rangle \Psi_5,$$

and hence we obtain the former identity of (5.24). By the adjoint computation we can verify the latter, and we are done. $\qquad\square$

Remarks 5.9.   1. We take (5.20) as the definition of G_0, cf. Remark 5.5.

    2. The action of $G_0 H$ on the extended space $\mathbb{C}\mathbf{n} \oplus \mathbb{C}|\mathbf{n}| \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\boldsymbol{\sigma} \oplus \mathcal{L}$ is more complicated than before.

5.3   The case: P, m_0 are not invertible, and q_0 is invertible

Theorem 5.10. *Suppose that $\beta \geq 3$ in Assumption 1.1, that P and m_0 defined by (3.5) and (3.9) are not invertible in $\mathcal{B}(\mathcal{K})$ and $\mathcal{B}(Q\mathcal{K})$, respectively, and that q_0 defined by (3.12) is invertible in $\mathcal{B}(S\mathcal{K})$. Let $N \in [-1, \beta - 4]$ be any integer. Then as $\kappa \to 0$ with $\mathrm{Re}\,\kappa > 0$, the resolvent $R(\kappa)$ has the expansion in $\mathcal{B}^{N+3}$:*

$$R(\kappa) = \sum_{j=-1}^{N} \kappa^j G_j + \mathcal{O}(\kappa^{N+1}), \quad G_j \in \mathcal{B}^{j+2}, \tag{5.27}$$

and the coefficients G_j can be computed explicitly.

Proof. By Proposition 1.13, (3.8) and (3.11) we can write

$$
\begin{aligned}
R(\kappa) = {} & R_0(\kappa) - \kappa R_0(\kappa)v(Q + \kappa M(\kappa))^{-1}v^* R_0(\kappa) \\
& - R_0(\kappa)v(Q + \kappa M(\kappa))^{-1}(S + m(\kappa))^\dagger(Q + \kappa M(\kappa))^{-1}v^* R_0(\kappa) \\
& - \tfrac{1}{\kappa}R_0(\kappa)v(Q + \kappa M(\kappa))^{-1}(S + m(\kappa))^\dagger \\
& \qquad \cdot q(\kappa)^\dagger(S + m(\kappa))^\dagger(Q + \kappa M(\kappa))^{-1}v^* R_0(\kappa).
\end{aligned}
$$
$$\tag{5.28}$$

We compute the expansions of $(Q + \kappa M(\kappa))^{-1}$, $(S + m(\kappa))^{\dagger}$ and $q(\kappa)^{\dagger}$ employing the Neumann series. Let $N \in [-1, \beta - 4]$ be any integer. Then we have the expansions in $\mathcal{B}(\mathcal{K})$, $\mathcal{B}(Q\mathcal{K})$ and $\mathcal{B}(S\mathcal{K})$:

$$(Q + \kappa M(\kappa))^{-1} = \sum_{j=0}^{N+3} \kappa^j A_j + \mathcal{O}(\kappa^{N+4}), \quad A_j \in \mathcal{B}(\mathcal{K}), \tag{5.29}$$

$$(S + m(\kappa))^{\dagger} = \sum_{j=0}^{N+2} \kappa^j B_j + \mathcal{O}(\kappa^{N+3}), \quad B_j \in \mathcal{B}(Q\mathcal{K}), \tag{5.30}$$

$$q(\kappa)^{\dagger} = \sum_{j=0}^{N+1} \kappa^j C_j + \mathcal{O}(\kappa^{N+2}), \quad C_j \in \mathcal{B}(S\mathcal{K}), \tag{5.31}$$

where

$$A_0 = Q + \gamma P, \qquad A_j = \sum_{k=1}^{j} \sum_{\substack{j_1 \geq 0, \ldots, j_k \geq 0 \\ j_1 + \cdots + j_k = j - k}} A_0 \prod_{l=1}^{k} (-M_{j_l} A_0) \quad \text{for } j \geq 1,$$

$$B_0 = S + m_0^{\dagger}, \qquad B_j = \sum_{k=1}^{j} \sum_{\substack{j_1 \geq 1, \ldots, j_k \geq 1 \\ j_1 + \cdots + j_k = j}} B_0 \prod_{l=1}^{k} (-m_{j_l} B_0) \quad \text{for } j \geq 1,$$

$$C_0 = q_0^{\dagger}, \qquad C_j = \sum_{k=1}^{j} \sum_{\substack{j_1 \geq 1, \ldots, j_k \geq 1 \\ j_1 + \cdots + j_k = j}} C_0 \prod_{l=1}^{k} (-q_{j_l} C_0) \quad \text{for } j \geq 1,$$

respectively. We substitute (5.29)–(5.31) and

$$R_0(\kappa) = \sum_{j=-1}^{N+1} G_j^0 + \mathcal{O}(\kappa^{N+2}) \quad \text{in } \mathcal{B}^{N+3}$$

into (5.28), and then obtain in the topology of $\mathcal{B}^{N+3}$

$$R(\kappa) = \sum_{j=-1}^{N} \kappa^j G_j + \mathcal{O}(\kappa^{N+1}),$$

where for $j \geq -1$

$$
\begin{aligned}
G_j = G_j^0 &- \sum_{\substack{j_1 \geq -1, j_2 \geq 0, j_3 \geq -1 \\ j_1 + j_2 + j_3 = j-1}} G_{j_1}^0 v A_{j_2} v^* G_{j_3}^0 \\
&- \sum_{\substack{j_1 \geq -1, j_2 \geq 0, j_3 \geq 0, j_4 \geq 0, j_5 \geq -1 \\ j_1 + j_2 + j_3 + j_4 + j_5 = j}} G_{j_1}^0 v A_{j_2} B_{j_3} A_{j_4} v^* G_{j_5}^0 \\
&- \sum_{\substack{j_1 \geq -1, j_2 \geq 0, \ldots, j_6 \geq 0, j_7 \geq -1 \\ j_1 + \cdots + j_7 = j+1}} G_{j_1}^0 v A_{j_2} B_{j_3} C_{j_4} B_{j_5} A_{j_6} v^* G_{j_7}^0.
\end{aligned}
\tag{5.32}
$$

Here, as in the proof of Theorem 5.6, we have used the identities (5.17) to guarantee that the error term is $\mathcal{O}(\kappa^{N+1})$ and also that the coefficient of κ^{-2} vanishes. By (5.32) and (5.17) again we obtain $G_j \in \mathcal{B}^{j+2}$ and hence the expansion (5.27) is verified. $\qquad\square$

By (5.32) and (5.17) we can write

$$
\begin{aligned}
G_{-1} = G_{-1}^0 &- G_{-1}^0 v A_0 v^* G_{-1}^0 \\
&- (G_{-1}^0 v A_1 + G_0^0 v A_0) B_0 C_0 B_0 (A_1 v^* G_{-1}^0 + A_0 v^* G_0^0).
\end{aligned}
\tag{5.33}
$$

$$
\begin{aligned}
G_0 = G_0^0 &- G_{-1}^0 v A_1 v^* G_{-1}^0 - G_{-1}^0 v A_0 v^* G_0^0 - G_0^0 v A_0 v^* G_{-1}^0 \\
&- (G_{-1}^0 v A_1 + G_0^0 v A_0) B_0 (A_1 v^* G_{-1}^0 + A_0 v^* G_0^0) \\
&- (G_{-1}^0 v A_1 + G_0^0 v A_0) B_0 C_1 B_0 (A_1 v^* G_{-1}^0 + A_0 v^* G_0^0) \\
&- (G_{-1}^0 v A_1 + G_0^0 v A_0) B_0 C_0 B_1 (A_1 v^* G_{-1}^0 + A_0 v^* G_0^0) \\
&- (G_{-1}^0 v A_1 + G_0^0 v A_0) B_1 C_0 B_0 (A_1 v^* G_{-1}^0 + A_0 v^* G_0^0) \\
&- (G_{-1}^0 v A_1 + G_0^0 v A_0) B_0 C_0 B_0 (A_2 v^* G_{-1}^0 + A_1 v^* G_0^0 + A_0 v^* G_1^0) \\
&- (G_{-1}^0 v A_2 + G_0^0 v A_1 + G_1^0 v A_0) B_0 C_0 B_0 (A_1 v^* G_{-1}^0 + A_0 v^* G_0^0).
\end{aligned}
\tag{5.34}
$$

We can see G_0 is actually in $\mathcal{B}^2$. We rewrite the expressions (5.33) and (5.34) substituting

$$
\begin{aligned}
&A_0 = Q + \gamma P, \quad A_1 = -(Q + \gamma P) M_0 (Q + \gamma P), \\
&A_2 = -(Q + \gamma P) M_1 (Q + \gamma P) + (Q + \gamma P) M_0 (Q + \gamma P) M_0 (Q + \gamma P), \\
&B_0 = S + m_0^\dagger, \quad B_1 = -(S + m_0^\dagger) m_1 (S + m_0^\dagger), \quad C_0 = q_0^\dagger, \quad C_1 = -q_0^\dagger q_1 q_0^\dagger.
\end{aligned}
$$

We note that $S M_0 Q = Q M_0 S = 0$, $q_0 = S m_1 S$ and $z = \gamma G_{-1}^0 v M_0 - G_0^0 v$ on $S\mathcal{K}$. Then after some computations,

$$
G_{-1} = G_{-1}^0 - \gamma G_{-1}^0 v v^* G_{-1}^0 - z q_0^\dagger z^*,
\tag{5.35}
$$

$$
\begin{aligned}
G_0 = {} & G_0^0 + \gamma^2 G_{-1}^0 v M_0 v^* G_{-1}^0 - \gamma G_{-1}^0 v v^* G_0^0 - \gamma G_0^0 v v^* G_{-1}^0 \\
& - (\gamma G_{-1}^0 v M_0 - G_0^0 v) m_0^\dagger (\gamma M_0 v^* G_{-1}^0 - v^* G_0^0) + z S z^* + z q_0^\dagger q_1 q_0^\dagger z^* \\
& + z q_0^\dagger m_1 m_0^\dagger (\gamma M_0 v^* G_{-1}^0 - v^* G_0^0) + (\gamma G_{-1}^0 v M_0 - G_0^0 v) m_0^\dagger m_1 q_0^\dagger z^* \\
& + z q_0^\dagger v^* G_1^0 (1_{\mathcal{L}} - \gamma v v^* G_{-1}^0) + \gamma z q_0^\dagger M_0 P (\gamma M_0 v^* G_{-1}^0 - v^* G_0^0) \\
& + (1_{\mathcal{L}^*} - \gamma G_{-1}^0 v v^*) G_1^0 v q_0^\dagger z^* + \gamma (\gamma G_{-1}^0 v M_0 - G_0^0 v) P M_0 q_0^\dagger z^*.
\end{aligned}
\tag{5.36}
$$

In the proofs of the following theorems we need an explicit expression of $q_0^\dagger$. Let us compute it here. By (4.28) we have

$$
\begin{aligned}
q_0 = -\tfrac{1}{2} \Big[& \big(1 + |\langle \Phi_3^*, \Phi_2 \rangle|^2\big) \langle \Phi_3, \cdot \rangle \Phi_3 + \langle \Phi_4, \cdot \rangle \Phi_4 \\
& + \overline{\langle \Phi_3^*, \Phi_2 \rangle} \langle \Phi_3, \cdot \rangle \Phi_4 + \langle \Phi_3^*, \Phi_2 \rangle \langle \Phi_4, \cdot \rangle \Phi_3 \Big].
\end{aligned}
$$

Since Φ_3 and Φ_4 are orthogonal, we can compute the inverse $q_0^\dagger$, using the 2×2 matrix inverse if $\Phi_3 \neq 0$ and $\Phi_4 \neq 0$, and the 1×1 matrix inverse otherwise. At last after some computations we have the following two cases: If $\Phi_4 \neq 0$, then

$$
\begin{aligned}
q_0^\dagger = -2 \Big[& \langle \Phi_3^*, \cdot \rangle \Phi_3^* + \big(1 + |\langle \Phi_3^*, \Phi_2 \rangle|^2\big) \langle \Phi_4^*, \cdot \rangle \Phi_4^* \\
& - \overline{\langle \Phi_3^*, \Phi_2 \rangle} \langle \Phi_3^*, \cdot \rangle \Phi_4^* - \langle \Phi_3^*, \Phi_2 \rangle \langle \Phi_4^*, \cdot \rangle \Phi_3^* \Big],
\end{aligned}
\tag{5.37}
$$

and, if $\Phi_4 = 0$, then

$$
q_0^\dagger = -2 \big(1 + |\langle \Phi_3^*, \Phi_2 \rangle|^2\big)^{-1} \langle \Phi_3^*, \cdot \rangle \Phi_3^*.
\tag{5.38}
$$

Theorem 5.11. *Under the assumption of Theorem 5.10 the coefficient G_{-1} is of the form (1.5) (up to the factor i due to a different convention).*

Proof. The assertion is obvious by (5.35), (5.37), (5.38), Tables 5 and 6. $\qquad\square$

Theorem 5.12. *Under the assumption of Theorem 5.10 the coefficient G_0 satisfies*

$$
H G_0 = G_0 H = 1_{\mathcal{L}^2}.
\tag{5.39}
$$

Moreover, if $\widetilde{\mathcal{E}}_{\mathrm{qs}} = \{0\}$, then there exists a computable constant $\Delta_1 \in \mathbb{R}$ and $\Delta_2 \in \mathbb{C}$ such that

$$
\begin{aligned}
G_0 &= (1_{\mathcal{L}^*} + G_0^0 V)^{-1} G_0^0 + \Delta_1 \langle \Psi_3, \cdot \rangle \Psi_3 + \Delta_2 \langle \Psi_3, \cdot \rangle \Psi_6 + \overline{\Delta_2} \langle \Psi_6, \cdot \rangle \Psi_3 \\
&= G_0^0 (1_{\mathcal{L}} + V G_0^0)^{-1} + \Delta_1 \langle \Psi_3, \cdot \rangle \Psi_3 + \Delta_2 \langle \Psi_3, \cdot \rangle \Psi_6 + \overline{\Delta_2} \langle \Psi_6, \cdot \rangle \Psi_3.
\end{aligned}
\tag{5.40}
$$

Proof. Let us compute $(1_{\mathcal{L}^*} + G_0^0 V)G_0$ using (5.36). Here we omit the details, but describe the outline. We first use the identities

$$(1_{\mathcal{L}^*} + G_0^0 V)(\gamma G_{-1}^0 v M_0 - G_0^0 v) = -G_0^0 v U Q M_0 + \gamma G_{-1}^0 v M_0, \qquad (5.41)$$

$$(1_{\mathcal{L}^*} + G_0^0 V) z S = \gamma G_{-1}^0 v M_0 S, \qquad (5.42)$$

$$(1_{\mathcal{L}^*} + G_0^0 V)(1_{\mathcal{L}^*} - \gamma G_{-1}^0 v v^*) = 1_{\mathcal{L}^*} - \gamma G_{-1}^0 v v^* + G_0^0 v U Q v^*. \qquad (5.43)$$

Then the contributions from the first five terms of (5.36) can be directly computed. As for the contributions from the remaining eight terms, we use Lemma 3.3, e.g., $m_0 = Q M_0 Q$, $m_0 m_0^\dagger = Q - S$, $m_1 = Q M_1 Q - Q M_0 (Q + \gamma P) M_0 Q$, $S m_0 = 0$, $q_0 = S m_1 S$ and so on. At last we obtain

$$
\begin{aligned}
(1_{\mathcal{L}^*} + G_0^0 V)G_0 = {}& G_0^0 + \gamma G_{-1}^0 v (1_{\mathcal{K}} - M_0 m_0^\dagger)(\gamma M_0 v^* G_{-1}^0 - v^* G_0^0) \\
& + \gamma G_{-1}^0 v M_0 S z^* + \gamma G_{-1}^0 v M_0 q_0^\dagger q_1 q_0^\dagger z^* \\
& + \gamma G_{-1}^0 v M_0 q_0^\dagger m_1 m_0^\dagger (\gamma M_0 v^* G_{-1}^0 - v^* G_0^0) \\
& + \gamma G_{-1}^0 v M_0 m_0^\dagger m_1 q_0^\dagger z^* + \gamma G_{-1}^0 v M_0 q_0^\dagger v^* G_1^0 (1_{\mathcal{L}} - \gamma v v^* G_{-1}^0) \\
& + \gamma^2 G_{-1}^0 v M_0 q_0^\dagger M_0 P (\gamma M_0 v^* G_{-1}^0 - v^* G_0^0) \\
& + (1_{\mathcal{L}^*} - \gamma G_{-1}^0 v^* v) G_1^0 v q_0^\dagger z^* + \gamma^2 G_{-1}^0 v M_0 P M_0 q_0^\dagger z^*. \quad (5.44)
\end{aligned}
$$

The above computation can be followed rather easier if we focus on the terms with the factor G_0^0 to their left. Hence by (5.44) and Lemma 4.2 we can deduce (5.39).

Next, assume that $1_{\mathcal{L}^*} + G_0^0 V$ is invertible. In this case we further compute $(1_{\mathcal{L}^*} + G_0^0 V)G_0(1_{\mathcal{L}} + V G_0^0)$. We use (5.44) and the adjoints of (5.41)–(5.43), and proceed similarly to the above. After some computations, we obtain

$$
\begin{aligned}
(1_{\mathcal{L}^*} + G_0^0 V)G_0(1_{\mathcal{L}} + V G_0^0) = {}& G_0^0 + G_0^0 V G_0^0 \\
& + \gamma^2 G_{-1}^0 v (M_0 - M_0 m_0^\dagger M_0) v^* G_{-1}^0 \\
& + \gamma^2 G_{-1}^0 v M_0 S M_0 v^* G_{-1}^0 + \gamma^2 G_{-1}^0 v M_0 q_0^\dagger q_1 q_0^\dagger M_0 v^* G_{-1}^0 \\
& + \gamma^2 G_{-1}^0 v M_0 q_0^\dagger m_1 m_0^\dagger M_0 v^* G_{-1}^0 + \gamma^2 G_{-1}^0 v M_0 m_0^\dagger m_1 q_0^\dagger M_0 v^* G_{-1}^0 \\
& + \gamma^3 G_{-1}^0 v M_0 q_0^\dagger M_0 P M_0 v^* G_{-1}^0 + \gamma^3 G_{-1}^0 v M_0 P M_0 q_0^\dagger M_0 v^* G_{-1}^0 \\
& + \gamma G_{-1}^0 v M_0 q_0^\dagger v^* G_1^0 (1_{\mathcal{L}} - \gamma v v^* G_{-1}^0) \\
& + (1_{\mathcal{L}^*} - \gamma G_{-1}^0 v^* v) G_1^0 v q_0^\dagger (\gamma M_0 v^* G_{-1}^0).
\end{aligned}
$$

Hence we can write for some $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$

$$
\begin{aligned}
(1_{\mathcal{L}^*} + G_0^0 V)G_0(1_{\mathcal{L}} + V G_0^0) = {}& G_0^0 + G_0^0 V G_0^0 \\
& + \alpha \langle \Psi_1^0, \cdot \rangle \Psi_1^0 + \beta \langle \Psi_1^0, \cdot \rangle \Psi_2^0 + \overline{\beta} \langle \Psi_2^0, \cdot \rangle \Psi_1^0.
\end{aligned}
$$

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We note that by Tables 5 and 6 we necessarily have $\Phi_4 = 0$. Then by (4.26) we can verify obtain (5.40). $\qquad\square$

Remark 5.13. Remarks similar to Remarks 5.9 hold also for this case. By (5.36) we have

$$G_0 - G_0^0 \equiv -\gamma G_{-1}^0 v v^* G_0^0 + z q_0^\dagger v^* G_1^0 - \gamma z q_0^\dagger M_0 v^* G_0^0 + \text{h.c.} \mod \mathcal{B}^0,$$

and hence, using (5.37) and (5.38), we can express $G_0 - G_0^0$ modulo $\mathcal{B}^0$ as in the former subsections. However, the expression seems very long and we do not elaborate it here. We also note that we do not investigate the case $\widetilde{\mathcal{E}}_{\text{qs}} \neq \{0\}$.

5.4 The case: P, m_0, q_0 are not invertible

Theorem 5.14. *Suppose that $\beta \geq 4$ in Assumption 1.1, and that P, m_0 and q_0 defined by (3.5), (3.9) and (3.12) are not invertible in $\mathcal{B}(\mathcal{K})$, $\mathcal{B}(Q\mathcal{K})$ and $\mathcal{B}(S\mathcal{K})$, respectively. Let $N \in [-2, \beta - 6]$ be any integer. Then as $\kappa \to 0$ with $\operatorname{Re} \kappa > 0$, the resolvent $R(\kappa)$ has the expansion in $\mathcal{B}^{N+4}$:*

$$R(\kappa) = \sum_{j=-2}^{N} \kappa^j G_j + \mathcal{O}(\kappa^{N+1}), \quad G_j \in \mathcal{B}^{j+3}, \tag{5.45}$$

and the coefficients G_j can be computed explicitly.

Proof. By Proposition 1.13, (3.8), (3.11) and (3.14) we can write

$$
\begin{aligned}
R(\kappa) = {}& R_0(\kappa) - \kappa R_0(\kappa) v (Q + \kappa M(\kappa))^{-1} v^* R_0(\kappa) \\
& - R_0(\kappa) v (Q + \kappa M(\kappa))^{-1} Q (S + m(\kappa))^\dagger (Q + \kappa M(\kappa))^{-1} v^* R_0(\kappa) \\
& - \tfrac{1}{\kappa} R_0(\kappa) v (Q + \kappa M(\kappa))^{-1} (S + m(\kappa))^\dagger \\
& \quad\cdot (T + q(\kappa))^\dagger (S + m(\kappa))^\dagger (Q + \kappa M(\kappa))^{-1} v^* R_0(\kappa) \\
& - \tfrac{1}{\kappa^2} R_0(\kappa) v (Q + \kappa M(\kappa))^{-1} Q (S + m(\kappa))^\dagger (T + q(\kappa))^\dagger \\
& \quad\cdot r(\kappa)^\dagger (T + q(\kappa))^\dagger (S + m(\kappa))^\dagger (Q + \kappa M(\kappa))^{-1} v^* R_0(\kappa). \tag{5.46}
\end{aligned}
$$

We compute the expansions of $(Q + \kappa M(\kappa))^{-1}$, $(S + m(\kappa))^\dagger$, $(T + q(\kappa))^\dagger$ and $r(\kappa)^\dagger$ employing the Neumann series. Let $N \in [-2, \beta - 6]$ be any integer. Then we have the expansions in $\mathcal{B}(\mathcal{K})$, $\mathcal{B}(Q\mathcal{K})$ and $\mathcal{B}(S\mathcal{K})$:

$$(Q + \kappa M(\kappa))^{-1} = \sum_{j=0}^{N+5} \kappa^j A_j + \mathcal{O}(\kappa^{N+6}), \quad A_j \in \mathcal{B}(\mathcal{K}), \tag{5.47}$$

$$(S + m(\kappa))^\dagger = \sum_{j=0}^{N+4} \kappa^j B_j + \mathcal{O}(\kappa^{N+5}), \quad B_j \in \mathcal{B}(Q\mathcal{K}), \qquad (5.48)$$

$$(T + q(\kappa))^\dagger = \sum_{j=0}^{N+3} \kappa^j C_j + \mathcal{O}(\kappa^{N+4}), \quad C_j \in \mathcal{B}(S\mathcal{K}), \qquad (5.49)$$

$$r(\kappa)^\dagger = \sum_{j=0}^{N+2} \kappa^j D_j + \mathcal{O}(\kappa^{N+3}), \quad D_j \in \mathcal{B}(T\mathcal{K}), \qquad (5.50)$$

$$A_0 = Q + \gamma P, \qquad A_j = \sum_{k=1}^{j} \sum_{\substack{j_1 \geq 0, \ldots, j_k \geq 0 \\ j_1 + \cdots + j_k = j-k}} A_0 \prod_{l=1}^{k} (-M_{j_l} A_0) \quad \text{for } j \geq 1,$$

$$B_0 = S + m_0^\dagger, \qquad B_j = \sum_{k=1}^{j} \sum_{\substack{j_1 \geq 1, \ldots, j_k \geq 1 \\ j_1 + \cdots + j_k = j}} B_0 \prod_{l=1}^{k} (-m_{j_l} B_0) \quad \text{for } j \geq 1,$$

$$C_0 = T + q_0^\dagger, \qquad C_j = \sum_{k=1}^{j} \sum_{\substack{j_1 \geq 1, \ldots, j_k \geq 1 \\ j_1 + \cdots + j_k = j}} C_0 \prod_{l=1}^{k} (-q_{j_l} C_0) \quad \text{for } j \geq 1,$$

$$D_0 = r_0^\dagger, \qquad D_j = \sum_{k=1}^{j} \sum_{\substack{j_1 \geq 1, \ldots, j_k \geq 1 \\ j_1 + \cdots + j_k = j}} D_0 \prod_{l=1}^{k} (-r_{j_l} D_0) \quad \text{for } j \geq 1,$$

respectively. We substitute the expansions (5.47)–(5.50) and

$$R_0(\kappa) = \sum_{j=-1}^{N+2} G_j^0 + \mathcal{O}(\kappa^{N+3}) \quad \text{in } \mathcal{B}^{N+4}$$

into (5.46), and then obtain in the topology of $\mathcal{B}^{N+4}$

$$R(\kappa) = \sum_{j=-2}^{N} \kappa^j G_j + \mathcal{O}(\kappa^{N+1}),$$

where for $j \geq -2$ with $G^0_{-2} = 0$

$$G_j = G^0_j - \sum_{\substack{j_1 \geq -1, j_2 \geq 0, j_3 \geq -1 \\ j_1 + j_2 + j_3 = j-1}} G^0_{j_1} v A_{j_2} v^* G^0_{j_3}$$

$$- \sum_{\substack{j_1 \geq -1, j_2 \geq 0, j_3 \geq 0, j_4 \geq 0, j_5 \geq -1 \\ j_1 + j_2 + j_3 + j_4 + j_5 = j}} G^0_{j_1} v A_{j_2} B_{j_3} A_{j_4} v^* G^0_{j_5}$$

$$- \sum_{\substack{j_1 \geq -1, j_2 \geq 0, \ldots, j_6 \geq 0, j_7 \geq -1 \\ j_1 + \cdots + j_7 = j+1}} G^0_{j_1} v A_{j_2} B_{j_3} C_{j_4} B_{j_5} A_{j_6} v^* G^0_{j_7}.$$

$$- \sum_{\substack{j_1 \geq -1, j_2 \geq 0, \ldots, j_8 \geq 0, j_9 \geq -1 \\ j_1 + \cdots + j_9 = j+2}} G^0_{j_1} v A_{j_2} B_{j_3} C_{j_4} D_{j_5} C_{j_6} B_{j_7} A_{j_8} v^* G^0_{j_9}.$$
(5.51)

As in the proof of Theorem 5.6, we have used the identities in (5.17) to guarantee that the error term is $\mathcal{O}(\kappa^{N+1})$. By (5.51) and (5.17) we obtain $G_j \in \mathcal{B}^{j+3}$, and hence the expansion (5.45) is obtained. $\qquad\square$

We have by (5.51) and (5.17)

$$G_{-2} = -(G^0_{-1} v A_1 + G^0_0 v A_0) B_0 C_0 D_0 C_0 B_0 (A_1 v^* G^0_{-1} + A_0 v^* G^0_0), \qquad (5.52)$$

$$G_{-1} = G^0_{-1} - G^0_{-1} v A_0 v^* G^0_{-1}$$
$$\quad - (G^0_{-1} v A_1 + G^0_0 v A_0) B_0 C_0 B_0 (A_1 v^* G^0_{-1} + A_0 v^* G^0_0)$$
$$\quad - (G^0_{-1} v A_1 + G^0_0 v A_0)(B_1 C_0 D_0 C_0 B_0 + B_0 C_1 D_0 C_0 B_0 + B_0 C_0 D_1 C_0 B_0$$
$$\qquad + B_0 C_0 D_0 C_1 B_0 + B_0 C_0 D_0 C_0 B_1)(A_1 v^* G^0_{-1} + A_0 v^* G^0_0)$$
$$\quad - (G^0_{-1} v A_1 + G^0_0 v A_0) B_0 C_0 D_0 C_0 B_0 (A_0 v^* G^0_1 + A_1 v^* G^0_0 + A_2 v^* G^0_{-1})$$
$$\quad - (G^0_1 v A_0 + G^0_0 v A_1 + G^0_{-1} v A_2) B_0 C_0 D_0 C_0 B_0 (A_1 v^* G^0_{-1} + A_0 v^* G^0_0).$$
(5.53)

The coefficient G_0 will be considered separately. We proceed using Lemma 3.3 and

$$A_0 = Q + \gamma P, \quad A_1 = -(Q + \gamma P) M_0 (Q + \gamma P),$$
$$A_2 = -(Q + \gamma P) M_1 (Q + \gamma P) + (Q + \gamma P) M_0 (Q + \gamma P) M_0 (Q + \gamma P),$$
$$B_0 = S + m^\dagger_0, \quad B_1 = -(S + m^\dagger_0) m_1 (S + m^\dagger_0),$$
$$C_0 = T + q^\dagger_0, \quad C_1 = -(T + q^\dagger_0) q_1 (T + q^\dagger_0), \quad D_0 = r^\dagger_0, \quad D_1 = -r^\dagger_0 r_1 r^\dagger_0.$$
(5.54)

First we substitute (5.54) to (5.52) and (5.53), and then we compute it particularly noting

$$(G^0_{-1} v A_1 + G^0_0 v A_0) S = -zS, \quad Tm_1 = 0, \quad TQM_1 Q = 0, \quad TM_0 = 0.$$

After some computations we obtain the expressions

$$G_{-2} = -z r_0^\dagger z^*, \tag{5.55}$$

$$\begin{aligned} G_{-1} = {} & G_{-1}^0 - \gamma G_{-1}^0 v v^* G_{-1}^0 + z(T + r_0^\dagger r_1 r_0^\dagger) z^* + z(-q_0^\dagger + q_0^\dagger q_1 r_0^\dagger + r_0^\dagger q_1 q_0^\dagger) z^* \\ & + z r_0^\dagger v^* G_1^0 (1 - \gamma v v^* G_{-1}^0) + (1 - \gamma G_{-1}^0 v v^*) G_1^0 v r_0^\dagger z^*. \end{aligned} \tag{5.56}$$

Theorem 5.15. *Under the assumption of Theorem 5.14 the coefficients G_{-2} and G_{-1} are of the form (1.5) (up to the factors -1 and i, respectively, due to different conventions).*

Proof. By (5.55) and (4.30) we have

$$G_{-2} = zT(Tz^* zT)^\dagger T z^*.$$

This implies that G_{-2} on $\mathcal{H}$ is the orthogonal projection onto the eigenspace E, and hence the assertion for G_{-2} follows.

We rewrite (5.56) modulo $\langle E, \cdot \rangle E$ as follows: If $\Phi_1 \neq 0$, then

$$G_{-1} \equiv - z(S - r_0^\dagger q_1) q_0^\dagger (S - q_1 r_0^\dagger) z^*,$$

and, if $\Phi_1 = 0$, then

$$\begin{aligned} G_{-1} \equiv {} & - z(S - r_0^\dagger q_1) q_0^\dagger (S - q_1 r_0^\dagger) z^* \\ & + \tfrac{1}{2} \big\langle \big(\Psi_1^0 + \tfrac{1}{2} z r_0^\dagger v^* \mathbf{n}^2\big), \cdot \big\rangle \big(\Psi_1^0 + \tfrac{1}{2} z r_0^\dagger v^* \mathbf{n}^2\big) \end{aligned}$$

where $\mathbf{n}^2 \in (\mathcal{L}^2)^*$ is the sequence whose n-th entry is n^2. By (5.37), (5.38), Tables 7 and 8 we can see that the assertion for G_{-1} holds. $\square$

Finally we consider G_0 with $\beta \geq 5$. We can see that G_0 actually belongs

to $\mathcal{B}^3$ by the expression following from (5.51):

$$
\begin{aligned}
G_0 = {}& G_0^0 - G_{-1}^0 v A_1 v^* G_{-1}^0 - G_{-1}^0 v A_0 v^* G_0^0 - G_0^0 v A_0 v^* G_{-1}^0 \\
& - (G_{-1}^0 v A_1 + G_0^0 v) B_0 (A_1 v^* G_{-1}^0 + v^* G_0^0) \\
& - (G_{-1}^0 v A_1 + G_0^0 v)(B_1 C_0 + C_1 + C_0 B_1)(A_1 v^* G_{-1}^0 + A_0 v^* G_0^0) \\
& - (G_{-1}^0 v A_1 + G_0^0 v) C_0 (A_2 v^* G_{-1}^0 + A_1 v^* G_0^0 + v^* G_1^0) \\
& - (G_{-1}^0 v A_2 + G_0^0 v A_1 + G_1^0 v) C_0 (A_1 v^* G_{-1}^0 + v^* G_0^0) \\
& - (G_{-1}^0 v A_1 + G_0^0 v)(B_2 D_0 + C_2 D_0 + D_2 + D_0 C_2 + D_0 B_2 \\
& \quad + B_1 C_1 D_0 + B_1 D_1 + B_1 D_0 C_1 + B_1 D_0 B_1 + C_1 D_1 + C_1 D_0 C_1 \\
& \quad + C_1 D_0 B_1 + D_1 C_1 + D_1 B_1 + D_0 C_1 B_1)(A_1 v^* G_{-1}^0 + v^* G_0^0) \\
& - (G_{-1}^0 v A_1 + G_0^0 v)(B_1 D_0 + C_1 D_0 + D_1 \\
& \quad + D_0 C_1 + D_0 B_1)(A_2 v^* G_{-1}^0 + A_1 v^* G_0^0 + v^* G_1^0) \\
& - (G_{-1}^0 v A_2 + G_0^0 v A_1 + G_1^0 v)(B_1 D_0 + C_1 D_0 + D_1 \\
& \quad + D_0 C_1 + D_0 B_1)(A_1 v^* G_{-1}^0 + v^* G_0^0) \\
& - (G_{-1}^0 v A_1 + G_0^0 v) D_0 (A_3 v^* G_{-1}^0 + A_2 v^* G_0^0 + A_1 v^* G_1^0 + v^* G_2^0) \\
& - (G_{-1}^0 v A_2 + G_0^0 v A_1 + G_1^0 v) D_0 (A_2 v^* G_{-1}^0 + A_1 v^* G_0^0 + v^* G_1^0) \\
& - (G_{-1}^0 v A_3 + G_0^0 v A_2 + G_1^0 v A_1 + G_2^0 v) D_0 (A_1 v^* G_{-1}^0 + v^* G_0^0),
\end{aligned}
$$

where we have removed unnecessary factors from the beginning. Then by (1.1) the following theorem is obtained. We omit the proof.

Theorem 5.16. *Under the assumption of Theorem 5.14 with $\beta \geq 5$ the coefficient G_0 satisfies*

$$
HG_0 = G_0 H = 1_{\mathcal{L}^3} - G_{-2}.
$$

Remark 5.17. A remark similar to Remarks 5.13 hold also for this case. We note that in this case we always have $\widetilde{\mathcal{E}}_{\mathrm{qs}} \neq \{0\}$.

A   The threshold $\lambda = 4$

The discrete Schrödinger operators considered in the paper have thresholds at both $\lambda = 0$ and $\lambda = 4$. There is a simple relation between the two thresholds that makes it possible to reduce the analysis at threshold $\lambda = 4$ to the one given for threshold $\lambda = 0$. We define for any sequence x

$$
(Jx)[n] = (-1)^n x[n].
$$

We see that J satisfies $J^2 = 1$ and is a bounded self-adjoint and unitary operator on $\mathcal{H}$. Furthermore, if a potential V satisfies Assumption 1.1, then

so does the conjugation $V_J = JVJ^{-1}$. A straightforward computation yields for such V

$$J(H_0 + V)J^{-1} = -(H_0 - V_J - 4). \tag{A.1}$$

Thus for the resolvents we have

$$J(H_0 + V - z)^{-1}J^{-1} = -(H_0 - V_J - (4 - z))^{-1}.$$

We note that for a multiplicative potential V we have $V_J = V$.

B Examples

B.1 Local and non-local potentials

We provide a prototype of the potentials satisfying Assumption 1.1. This obviously includes *non-local* operators of finite rank.

Proposition B.1. *Let $\beta \geq 1$ be any real number, and $v_j \in \mathcal{L}^\beta$, $j = 1, 2, \ldots,$ be at most a countable number of linearly independent vectors with*

$$\sum_j \|v_j\|_{\mathcal{L}^\beta}^2 < \infty. \tag{B.1}$$

Then for any $\sigma_j \in \{\pm 1\}$ the operator series

$$V = \sum_j \sigma_j \langle v_j, \cdot \rangle v_j$$

converge in the uniform topology of $\mathcal{B}((\mathcal{L}^\beta)^*, \mathcal{L}^\beta)$ and satisfy Assumption 1.1 with the same β.

The proof is omitted. We just have to take an abstract Hilbert space $\mathcal{K}$ with the same cardinality of a complete orthonormal system as $\#\{v_j\}$. A potential V given in Proposition B.1 is *local* in the following sense if and only if each v_j has support consisting of one point.

Definition B.2. A symmetric operator V on the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z})$ is said to be local, if it does not extend supports of functions.

Since our base space is discrete, a local operator V is always identified with a multiplication operator by some function, which we shall denote also by V. In particular, a local operator V satisfies Assumption 1.1, if

$$\sum_{n \in \mathbb{Z}} (1 + n^2)^\beta |V[n]| < \infty.$$

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B.2 Local potentials with P invertible

With the following computations we show that for $V \neq 0$ local only Case iii. in Proposition 4.11 occurs. If V is local and of rank one, then up to a translation we have for some $c \neq 0$ that

$$V = c\langle e_0, \cdot \rangle e_0, \tag{B.2}$$

where $e_0[n] = 1$ for $n = 0$, and $e_0[n] = 0$ for $n \neq 0$. The factorization is given as follows. We have $\mathcal{K} = \mathbb{C}$, $vz = |c|^{1/2}ze_0$, $z \in \mathcal{K}$, $v^*x = |c|^{1/2}\langle e_0, x \rangle$, $x \in (\mathcal{L}^s)^*$, any $s > 0$, and $Uz = \operatorname{sign} c \cdot z$.

Straightforward computations show that a basis for the solutions to $H\Psi = 0$ is given by $\{u_1, u_2\}$, where

$$u_1[n] = \begin{cases} 1, & n \leq 0 \\ 1 + cn, & n > 0. \end{cases}$$

$$u_2[n] = \begin{cases} 1 - cn, & n < 0 \\ 1, & n \geq 0. \end{cases}$$

To compare with the results in Table 2 we rewrite the basis as follows. We omit the details, which are straightforward.

$$u_1 + u_2 = 2\,\mathbf{1} + c\,|\mathbf{n}| = (2\operatorname{sign} c)\Psi_5,$$
$$u_1 - u_2 = c\,\mathbf{n} = \Psi_2^0.$$

B.3 Construction of threshold resonances

Here we give examples of discrete Schrödinger operator with a threshold resonance at $\lambda = 0$. The technique for finding such examples is very simple. We are looking for a multiplicative potential V such that there is a sequence $x \in (\mathcal{L}^0)^*$ satisfying

$$-(x[n+1] + x[n-1] - 2x[n]) + V[n]x[n] = 0.$$

We find such V by first choosing x and then taking V accordingly to

$$V[n] = \frac{x[n+1] + x[n-1]}{x[n]} - 2.$$

Here are some examples of this technique.

Examples B.3. We can take, e.g.,

1.
$$x[n] = \begin{cases} 2 & \text{if } n = 0, \\ 1 & \text{otherwise,} \end{cases} \qquad V[n] = \begin{cases} -1 & \text{if } n = 0, \\ 1 & \text{if } n = \pm 1, \\ 0 & \text{otherwise;} \end{cases}$$

2.
$$x[n] = \begin{cases} 3 & \text{if } n = 0, \\ 2 & \text{if } n = \pm 1, \\ 1 & \text{otherwise,} \end{cases} \qquad V[n] = \begin{cases} -2/3 & \text{if } n = 0, \\ 0 & \text{if } n = \pm 1, \\ 1 & \text{if } n = \pm 2, \\ 0 & \text{otherwise.} \end{cases}$$

Using the results from Appendix A, we can also get examples of threshold resonances at the threshold $\lambda = 4$. In fact, if we can construct nontrivial $x \in (\mathcal{L}^0)^*$ satisfying $(H_0 - V_J)x = 0$, then (A.1) implies that $y = J^{-1}x$ satisfies

$$(H_0 + V)y = 4y.$$

We note that these constructions work also in the continuous setting.

B.4 Construction of threshold eigenvalues

As stated in Proposition 4.8, for a multiplicative potential there does not exist a decaying eigenfunction at the threshold. However, we can construct an example of a non-local potential that possesses linearly independent such eigenfunctions as finitely many as we want. We construct it using Proposition B.1 in such way that M_0 has a nontrivial kernel, cf. Corollary 4.6.

Example B.4. Let us define the potential V by

$$V = -\sum_{j=1}^{N} \langle v_j, \cdot \rangle v_j; \qquad v_j[n] = \begin{cases} \sqrt{2} & \text{if } n = 3j, \\ -1/\sqrt{2} & \text{if } n = 3j \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then the linearly independent sequences $\Psi_j \in \mathcal{L}$, $j = 1, \ldots, N$, given by

$$\Psi_j[n] = \begin{cases} 1 & \text{if } n = 3j, \\ 0 & \text{otherwise,} \end{cases}$$

are obviously the decaying eigenfunctions for $H = H_0 + V$.

B.5 Construction of both threshold eigenvalues and resonances

We now give an example exhibiting the simultaneous occurrence of threshold eigenvalues and resonances (an exceptional point of the third kind).

Example B.5. We define the following sequences:

$$\phi_j[n] = \begin{cases} -1, & n = 4j, \\ 1, & n = 4j + 1, \\ 0, & \text{otherwise,} \end{cases} \qquad j = 0, 1, 2,$$

$$u_j[n] = \begin{cases} 1 & n \leq 4j, \\ -1 & n > 4j, \end{cases} \qquad j = 0, 1, 2.$$

Then we define

$$Vx = -\sum_{j=0}^{2} \langle \phi_j, x \rangle \phi_j.$$

With these definitions it is straightforward to verify that

$$(H_0 + V)u_j = 0, \quad j = 0, 1, 2.$$

If we define

$$w[n] = \begin{cases} 1 & n = 1, 2, 3, 4, \\ 0 & \text{otherwise,} \end{cases}$$

then we have $u_2 = \frac{1}{2}u_0 + \frac{1}{2}u_1 + w$.

This example should be compared with the discussion in the introduction. It is easy to see that the example can be modified to provide a threshold eigenvalue of any finite multiplicity, besides the two linearly independent resonance functions.

Acknowledgement

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