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CANONICAL BUNDLE FORMULA AND BASE CHANGE

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Abstract. We study invariants of an elliptic fibration over a complete discrete valuation ring with algebraically closed residue field. The invariants are given by the relative dualizing sheaf and the first direct image sheaf of the structure sheaf. In the studies of an elliptic surface over an algebraically closed field, the invariants appear as local invariants that determine important global invariants such as its plurigenera. We determine the invariants by investigating the change of the invariants by base change.

1. Introduction

In this paper, we study invariants of an elliptic fibration over a complete discrete valuation ring with algebraically closed residue field. In order to determine the invariants, we investigate the change of the invariants by base change.

1.1. **Canonical Bundle Formula and Invariants of Elliptic Fibrations.** Kodaira gave a canonical bundle formula for a complex analytic elliptic surface $f\colon X \to C$ that expresses its canonical bundle ω_X in terms of its singular fibers $\{F_s\}_{s\in S\subset C}$ [15, p. 772]:

$$\omega_X \cong f^*\mathcal{L} \otimes \mathcal{O}_X\left(\sum_{s\in S} \frac{a_s}{m_s} F_s\right)$$

where $\mathcal{L}$ is a line bundle on C, m_s is the multiplicity of F_s, and the following equalities hold: (1) $\deg \mathcal{L} = \chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_C)$; (2) $a_s = m_s - 1$. The integer $\chi(\mathcal{O}_X)$ may be determined by the configurations of the irreducible components of the singular fibers of f. Thus, the singular fibers of f determine important invariants of X such as its plurigenera.

Bombieri and Mumford gave an analog of this formula in the algebraic case [2]. In the characteristic zero case, no difference appears. However, in the positive characteristic case, neither (1) nor (2) holds in general. For a closed point s on C, we denote the length of the torsion of the $\mathcal{O}_{C,s}$-module $(R^1 f_*\mathcal{O}_X)_s$ by l_s. Then the followings hold: (1') $\deg \mathcal{L} = \chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_C) + \sum_{s\in S} l_s$; (2') $0 \le a_s < m_s$. The integer $\chi(\mathcal{O}_X)$ may be computed by means of the Jacobian and the discriminants ([6, 5.3.6] and [22, p. 20]). In order to apply the formula, we have to determine the invariants l_s and a_s. Nevertheless, only partial results and few examples are known for these invariants (see, e.g., [2], [13], [14], [6], and [18]).

In the present paper, we study the invariants l_s and a_s in a comprehensive and systematic way. To this end, we consider the case where the base space is an analog of a disk, i.e., the base space is the spectrum of a discrete valuation ring with algebraically closed residue field, where we may apply results and techniques in both the arithmetic geometry and the birational geometry. Our main results give formulas for the local invariants l_s and a_s (§1.2). Finally, we remark that our local results are applicable to the global studies (7.5.5).

1.2. **Main Results.** Let us explain our main results (§7). Let R be a complete discrete valuation ring with algebraically closed residue field k of characteristic p and field of fractions K. Put $C := \operatorname{Spec} R$. An *elliptic fibration* over C is a proper flat C-scheme that is regular and whose generic fiber is a geometrically connected smooth curve of genus one. An elliptic fibration is said to be *minimal* if the special fiber does not contain a (-1)-curve. Let $f\colon X \to C$ be a minimal elliptic fibration with generic fiber X_K. By ${}_mT$ we denote the type of the special fiber of f where m is the multiplicity and T is the type (Kodaira's symbol) of the configuration of the irreducible components. In the case $p = 0$, if $m > 1$, then $T = \mathrm{I}_n$ $(n \geq 0)$. However, in the case $p > 0$, this statement does not hold in general (see, e.g., [18]). We define integers $u(T)$ and $v(T)$ by Table 1.

T	I_n	I_n^*	II	II*	III	III*	IV	IV*
$u(T)$	1	2	6	6	4	4	3	3
$v(T)$	0	0	4	0	2	0	1	0

Table 1. The definition of $u(T)$ and $v(T)$.

By ω_f we denote the relative dualizing sheaf of f. Take the divisor D on X so that mD is equal to the special fiber of f. We study the following invariants (l, a) of f (3.3.6 and 4.1.5).

(1) The length l of the torsion of the $\mathcal{O}_C$-module $R^1 f_* \mathcal{O}_X$.
(2) The integer a in the isomorphism $\omega_f \cong f^* f_* \omega_f \otimes \mathcal{O}_X(aD)$ induced by the canonical injective $\mathcal{O}_X$-module homomorphism $f^* f_* \omega_f \to \omega_f$.

The inequalities $0 \leq a < m$ hold. If $m = 1$, then $(l, a) = (0, 0)$. Take a finite separable field extension K'/K of degree d. Take the normalization R' of R in K'. Put $C' := \operatorname{Spec} R'$ and $X'_{K'} := X_K \times_K K'$. Take the minimal regular C'-model $f'\colon X' \to C'$ of $X'_{K'}$. By ${}_{m'}T'$ we denote the type of the special fiber of f'. In the same way, we define the invariants (l', a') for f'. We study a relationship between (l, a) and (l', a'). The study of (l, a) may be reduced to the case $T = \mathrm{I}_n$ $(n \geq 0)$ whenever $p \not| \, u(T)$:

Theorem 1.2.1. *Assume that $p \not| \, u(T)$ and $d = u(T)$. If $T = \mathrm{I}_n$ $(n \geq 0)$ or I_n^* $(n \geq 0)$, then we put $n' := dn$. Otherwise, we put $n' := 0$. Then the equalities ${}_{m'}T' = {}_m\mathrm{I}_{n'}$ and*

$$u(T)(ml + a) = m'l' + a' + v(T)(m' - 1)$$

hold.

By $d_{C'/C}$ we denote the valuation of the different of R'/R [23, IV, §1]. The case $T = \mathrm{I}_n$ $(n > 0)$ was obtained:

Theorem 1.2.2 (Liu–Lorenzini–Raynaud [18, §8])**.** *Assume that $T = \mathrm{I}_n$ $(n > 0)$. Suppose that C'/C is the unique covering such that $d = m$ and $m' = 1$ (5.3.1). Then the equality*

$$ml + a = d_{C'/C}$$

holds.

Assume that $T = \mathrm{I}_0$. Then there exists a finite morphism $\pi_X \colon X' \to X$ that is an extension of the canonical projection $X'_{K'} \to X_K$. Localizing $\mathcal{O}_X$ and $\mathcal{O}_{X'}$ at the generic points of the special fibers, we obtain a finite extension of discrete valuation rings. By $d_{X'/X}$ we denote the valuation of the different of this extension (2.3.10). The remaining case $T = \mathrm{I}_0$ is settled:

Theorem 1.2.3. *Assume that $T = \mathrm{I}_0$. Put $d' := dm'/m$. Then the equality*

$$d'(ml + a) = m'l' + a' + m'd_{C'/C} - d_{X'/X}$$

holds.

We may take a covering C'/C so that $d = m$ and $m' = 1$ (5.1.2). Then $(l', a') = (0, 0)$. In order to determine l and a, we have to compute $d_{X'/X}$. The morphism $\pi_X \colon X' \to X$ factors as the composite of the étale part and the non-étale part (5.2.4; see also 7.5.2). If π_X is the étale part, then $d_{X'/X} = 0$ (7.5.1). Thus, we may assume that π_X is the non-étale part. In general, the generic fiber X_K corresponds to the unique element η of the Galois cohomology group $H^1(K, E_K)$ where E_K is the Jacobian of X_K. We determine $d_{X'/X}$ by the size of a cocycle that represents η (7.5.3 and 7.5.5). In this way, we finally obtain the desired invariants (l, a).

1.3. **Ideas of Proofs.** The above three theorems follow from a unified theorem including the case $p \mid u(T)$ (7.2.2). Take the minimal regular models $g \colon E \to C$ and $g' \colon E' \to C'$ of the Jacobians of X_K and $X'_{K'}$, respectively. The difference of the invariants of f and f' may be described in terms of the ramifications of the finite morphisms $\pi_C \colon C' \to C$, $\pi_X \colon X' \to X$, and $\pi_E \colon E' \to E$ if both π_X and π_E exist in the same way as in the case of (1.2.3). However, in general, there do not exist such finite morphisms π_X and π_E even if we replace the regular models of the generic fibers (7.3.6). In order to overcome this difficulty, we study a finite morphism between singular models of the generic fibers, which always exists (§§3–5). The invariants of the singular models may be compared by means of reflexive sheaves, which are useful for studies on singular models (§6). Comparing the singular models with the original fibrations, we determine the change of the invariants of f and f' in terms of the ramifications of finite morphisms (§7).

Let us give more details on the generalization of (1.2.3) to the case of singular models. We use the same notation as in (1.2.3). Put $e_{X'/X} := (d_{X'/X} + d'a - a')/m'$. The canonical homomorphisms $f^* f_* \omega_f \to \omega_f$ and $(f')^* f'_* \omega_{f'} \to \omega_{f'}$ define divisors D_f on X and $D_{f'}$ on X', respectively (4.1.1). By $D_{X'/X}$ we denote the ramification divisor of π_X (2.3.10). By s' we denote the unique prime divisor on C'. Then $e_{X'/X}$ satisfies

$$D_{X'/X} + \pi_X^* D_f - D_{f'} = (f')^*(e_{X'/X} s'). \tag{1}$$

Since g is smooth [18, 6.6], the equality $e_{E'/E} = d_{C'/C}$ holds (7.2.1). Thus, the equality in (1.2.3) may be expressed as

$$e_{X'/X} - e_{E'/E} + dl - l' = 0. \tag{2}$$

We generalize this equality to the case where X, X', E, and E' are normal (6.3.10). In the following, we assume that both π_X and π_E exist and explain the proof of the generalized equality.

In the general case, we define $e_{X'/X}$ by Equality (1) (6.3.8). By $L_{X'/X}$ we denote the length of the cokernel of the double dual of the canonical $\mathcal{O}_{C'}$-module homomorphism $\psi\colon \pi_C^* R^1 f_* \mathcal{O}_X \to R^1 f'_* \mathcal{O}_{X'}$. Comparing f and f', we first prove that the equality

$$e_{X'/X} = d_{C'/C} - L_{X'/X} \tag{3}$$

holds (6.3.7). Let us explain the proof of this equality. Put $\pi := f \circ \pi_X$. Since we have the canonical $\mathcal{O}_{X'}$-module homomorphisms

$$(f')^* \pi_C^* f_* \omega_f \xrightarrow{\cong} \pi_X^* f^* f_* \omega_f \longrightarrow \pi_X^* \omega_f \longrightarrow \pi_X^* \omega_f \otimes_{\mathcal{O}_{X'}} \omega_{\pi_X},$$

$$(f')^* f'_* \omega_{f'} \longrightarrow \omega_{f'} \longrightarrow \omega_{f'} \otimes_{\mathcal{O}_{X'}} (f')^* \omega_{\pi_C},$$

and

$$\pi_X^* \omega_f \otimes_{\mathcal{O}_{X'}} \omega_{\pi_X} \longrightarrow \omega_\pi \longleftarrow \omega_{f'} \otimes_{\mathcal{O}_{X'}} (f')^* \omega_{\pi_C}$$

(2.3.8 and 6.2.1), we may compare the images of the coherent $\mathcal{O}_{X'}$-modules $(f')^* \pi_C^* f_* \omega_f$ and $(f')^* f'_* \omega_{f'}$ in ω_π. The difference may be expressed by $e_{X'/X}$ and $d_{C'/C}$. On the other hand, the Grothendieck duality gives isomorphisms $f_* \omega_f \cong (R^1 f_* \mathcal{O}_X)^\vee$ and $f'_* \omega_{f'} \cong (R^1 f'_* \mathcal{O}_{X'})^\vee$. Thus, by the base change compatibility for trace maps (6.2.2), the above difference may be determined by means of ψ. As a result, we obtain Equality (3). In order to show Equality (2) in the general case, we compare f, f', g, and g'. If f and g are minimal elliptic fibrations, Liu–Lorenzini–Raynaud's result [18, 3.8] gives a canonical $\mathcal{O}_C$-module homomorphism $\tau\colon R^1 f_* \mathcal{O}_X \to R^1 g_* \mathcal{O}_E$ the length of whose cokernel is equal to l. We may generalize τ to the case of singular models by means of their minimal regular models. The four modules are connected by the four homomorphisms in the following way:

$$\begin{array}{ccccc} & R^1 f_* \mathcal{O}_X & \xleftarrow{\quad} \boxed{\tau \text{ for } f \text{ and } g} \xrightarrow{\quad} & R^1 g_* \mathcal{O}_E & \\ \boxed{\psi \text{ for } f \text{ and } f'} & \updownarrow & & \updownarrow & \boxed{\psi \text{ for } g \text{ and } g'} \\ & R^1 f'_* \mathcal{O}_{X'} & \xleftarrow{\quad} \boxed{\tau \text{ for } f' \text{ and } g'} \xrightarrow{\quad} & R^1 g'_* \mathcal{O}_{E'}. & \end{array}$$

This diagram and Equality (3) give Equality (2) in the general case (6.3.10).

2. Notation, Terminology, and Preliminaries

2.1. Reflexive Sheaves.

Definition 2.1.1. Let A be a Noetherian ring and M be a finite A-module. We define the *dual* of M by $M^\vee := \mathrm{Hom}_A(M, A)$. The A-module M is said to be *reflexive* if the homomorphism $M \to M^{\vee\vee}$ defined by $m \mapsto (\phi \mapsto \phi(m))$ is bijective. The A-module M is said to be *of rank* n if $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module of rank n for any associated prime ideal $\mathfrak{p}$ of A. The definitions are local with respect to the Zariski topology of $\operatorname{Spec} A$. Thus, we use the same notation and terminology for coherent sheaves on locally Noetherian schemes.

Lemma 2.1.2. *Let X be a locally Noetherian reduced scheme. Then the following statements hold.* (1) *The dual of any coherent $\mathcal{O}_X$-module is reflexive.* (2) *For any coherent $\mathcal{O}_X$-module $\mathcal{F}$ and any reflexive coherent $\mathcal{O}_X$-module $\mathcal{G}$, the coherent $\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is reflexive.*

Proof. Statement (1) follows from [8, 5.8.5] and [1, 6.1]. Statement (2) follows from Statement (1) and the $\mathcal{O}_X$-module isomorphisms $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}^{\vee\vee}) \cong (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}^\vee)^\vee$. ☐

Lemma 2.1.3. *Let X be a locally Noetherian normal scheme and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module with* $\operatorname{Supp}\mathcal{F} = X$. *Then the following conditions are equivalent*: (1) *$\mathcal{F}$ is reflexive*; (2) *$\mathcal{F}$ is $Z^{(2)}$-closed and $Z^{(1)}$-pure* [8, 5.9.9 and 5.10.13]; (3) *$\mathcal{F}$ is S_2* [8, 5.7.2].

Proof. The equivalence of (2) and (3) follows from [8, 5.10.14]. The equivalence of (1) and (3) follows from [8, 5.8.6] and [3, 1.4.1]. □

Since any connected component of any locally Noetherian normal scheme is integral [19, Ex. 9.11], we obtain the followings (use 2.1.3 for the proofs of 2.1.4–2.1.6; use [19, 15.1 (ii)], [8, 5.7.11 (i)], [9, 5.2 (c), 5.5 (a), and 6.1] for the proofs of 2.1.5, 2.1.6, and 2.1.7, respectively):

Lemma 2.1.4. *Let $Z \subset Y$ be two closed subsets of a locally Noetherian normal scheme X. Assume that Y and Z are of codimension at least one and at least two, respectively. Put $U := X \setminus Y$ and $V := X \setminus Z$. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules and $\phi_U : \mathcal{F}_U \to \mathcal{G}_U$ be an $\mathcal{O}_U$-module homomorphism. Suppose that $\mathcal{G}$ is reflexive and ϕ_U extends to an $\mathcal{O}_V$-module homomorphism. Then ϕ_U uniquely extends to the $\mathcal{O}_X$-module homomorphism.*

Lemma 2.1.5. *Let $f : X \to Y$ be a proper flat morphism between locally Noetherian normal schemes and $\mathcal{F}$ be a reflexive coherent $\mathcal{O}_X$-module. Then the coherent $\mathcal{O}_Y$-module $f_*\mathcal{F}$ is reflexive.*

Lemma 2.1.6. *Let $f : X \to Y$ be a finite morphism between locally Noetherian normal schemes and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume that the coherent $\mathcal{O}_Y$-module $f_*\mathcal{F}$ is reflexive. Then $\mathcal{F}$ is reflexive.*

Lemma 2.1.7. *Let X be a locally Noetherian scheme and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume that X is normal* (*resp. regular*). *Then $\mathcal{F}$ is reflexive of rank one if and only if $\mathcal{F}$ is induced by a Weil divisor on X* (*resp. a line bundle on X*).

2.2. **Weil Divisors Associated to Cokernels of Double Duals.**

Definition 2.2.1. Let R be a discrete valuation ring and $\phi\colon M \to N$ be an R-module homomorphism between finite R-modules. By $\phi^{\vee\vee}\colon M^{\vee\vee} \to N^{\vee\vee}$ we denote the double dual of ϕ. We put $L_R(\phi) := \operatorname{length}_R \operatorname{Coker}(\phi^{\vee\vee})$. We use the same notation $L_R(\phi)$ when ϕ is a homomorphism between coherent sheaves on $\operatorname{Spec} R$.

Definition 2.2.2. Let X be a locally Noetherian scheme and ϕ be an $\mathcal{O}_X$-module homomorphism between coherent $\mathcal{O}_X$-modules. Then ϕ is said to be *generically surjective* (resp. *generically injective*) if ϕ is surjective (resp. injective) at any associated point of X. Assume that X is normal. We define a formal sum $D(\phi)$ of prime divisors on X with integral coefficients by $D(\phi) := \sum_x a_x[x]$ where x runs through all points on X of codimension one, $[x]$ is the prime divisor corresponding to x, and $a_x := L_{\mathcal{O}_{X,x}}(\phi_x)$. If ϕ is generically surjective, then $D(\phi)$ is a Weil divisor.

Remark 2.2.3. Assume that $\phi\colon \mathcal{F} \to \mathcal{G}$ is generically surjective, $\mathcal{F}$ is a line bundle, and $\mathcal{G}$ is reflexive of rank one. By $\psi\colon \mathcal{O}_X \to \mathcal{O}_X(D(\phi))$ we denote the canonical injective $\mathcal{O}_X$-module homomorphism. Then there exists the unique $\mathcal{O}_X$-module isomorphism $\gamma\colon \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D(\phi)) \to \mathcal{G}$ such that $\phi = \gamma \circ (\mathcal{F} \otimes \psi)$ (2.1.7).

Since any finite module over a discrete valuation ring is the direct sum of a free module and a torsion module, we obtain the following:

Lemma 2.2.4. *Let X be a locally Noetherian normal scheme. Then the following statements hold.* (1) *For $i = 1$ and 2, let $\phi_i \colon \mathcal{F}_i \to \mathcal{F}_{i+1}$ be an $\mathcal{O}_X$-module homomorphism between coherent $\mathcal{O}_X$-modules. Assume that ϕ_2 is generically injective. Then $D(\phi_2 \circ \phi_1) = D(\phi_1) + D(\phi_2)$.* (2) *Let ϕ be an $\mathcal{O}_X$-module homomorphism between coherent $\mathcal{O}_X$-modules. Then $D(\phi^{\vee\vee}) = D(\phi)$ and $D(\phi \otimes \mathcal{F}) = nD(\phi)$ for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ of rank n.* (3) *For $i = 1$ and 2, let $\phi_i \colon \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-module homomorphism between coherent $\mathcal{O}_X$-modules. Assume that ϕ_1 is equal to ϕ_2 at any associated point of X. Then $D(\phi_1) = D(\phi_2)$.*

Definition 2.2.5. Let $f \colon X \to Y$ be a dominant morphism between locally Noetherian normal integral schemes and D be an effective Weil divisor on Y. We denote the canonical injective $\mathcal{O}_Y$-module homomorphism by $\phi \colon \mathcal{O}_Y \to \mathcal{O}_Y(D)$. Then $f^*\phi$ is a generically surjective. The Weil divisor $D(f^*\phi)$ is called the *pull-back* of D via f and denoted by f^*D.

Definition 2.2.6. Let $f \colon X \to Y$ be a morphism between locally Noetherian schemes and $\mathcal{F}$ be a coherent $\mathcal{O}_Y$-module. We denote the coherent $\mathcal{O}_X$-module $(f^*\mathcal{F})^{\vee\vee}$ by $f^{[*]}\mathcal{F}$. Let $\phi \colon \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_Y$-module homomorphism between coherent $\mathcal{O}_Y$-modules. We denote the induced $\mathcal{O}_X$-module homomorphism $f^{[*]}\mathcal{F} \to f^{[*]}\mathcal{G}$ by $f^{[*]}\phi$.

Lemma 2.2.7. *Let $f \colon X \to Y$ be a dominant morphism between locally Noetherian normal integral schemes, $\mathcal{F}$ be a line bundle on Y, $\mathcal{G}$ be a reflexive coherent $\mathcal{O}_Y$-module of rank one, and $\phi \colon \mathcal{F} \to \mathcal{G}$ be a generically surjective $\mathcal{O}_Y$-module homomorphism. Then $D(f^{[*]}\phi) = D(f^*\phi) = f^*D(\phi)$.*

Proof. We have only to show that $D(f^*\phi) = f^*D(\phi)$ (2.2.4 (2)). By $\psi \colon \mathcal{O}_Y \to \mathcal{O}_Y(D(\phi))$ we denote the canonical injective $\mathcal{O}_Y$-module homomorphism. We may assume that $\phi = \mathcal{F} \otimes \psi$ (2.2.3). Then $D(f^*\phi) = D(f^*\mathcal{F} \otimes f^*\psi) = D(f^*\psi) = f^*D(\phi)$, which concludes the proof. □

2.3. **Relative Dualizing Sheaves of Finite Morphisms.**

Definition 2.3.1. Let $f \colon X \to Y$ be a finite morphism of finite presentation between schemes (see [8, 1.4.7]). We define the *relative dualizing sheaf* ω_f of f and the *trace map* $\mathrm{tr}_f \colon f_*\omega_f \to \mathcal{O}_Y$ of ω_f as the following quasi-coherent $\mathcal{O}_X$-module and the following $\mathcal{O}_Y$-module homomorphism, respectively. We first assume that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Put $\omega_{B/A} := \operatorname{Hom}_A(B, A)$. We equip $\omega_{B/A}$ with a B-module structure by $b \cdot \phi := (b' \mapsto \phi(bb'))$. We define an A-module homomorphism $\mathrm{tr}_{B/A} \colon \omega_{B/A} \to A$ by $\phi \mapsto \phi(1_B)$ where 1_B is the multiplicative identity of B. We define ω_f and tr_f as the module and the homomorphism induced by $\omega_{B/A}$ and $\mathrm{tr}_{B/A}$, respectively. The formation of the pair $(\omega_f, \mathrm{tr}_f)$ commutes with any flat base change with respect to Y [19, 7.11]. In the general case, we define ω_f and tr_f by gluing the local pieces.

Remark 2.3.2. For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the trace map tr_f induces an $\mathcal{O}_Y$-module isomorphism $f_* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_f) \cong \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{F}, \mathcal{O}_Y)$.

Remark 2.3.3. Let L/K be a finite field extension and $T \colon L \to K$ be a non-zero K-module homomorphism. Then the L-module homomorphism $\lambda_{L/K,T} \colon L \to \omega_{L/K}$ defined by $c \mapsto (b \mapsto T(bc))$ is bijective since $\lambda_{L/K,T}$ is injective and the equality $\dim_K L = \dim_K \operatorname{Hom}_K(L, K)$ holds.

Proposition 2.3.4. *Let $f\colon X \to Y$ be a finite morphism between locally Noetherian schemes. Then ω_f is a coherent $\mathcal{O}_X$-module. If X and Y are integral and f is dominant, then ω_f is of rank one. If X and Y are normal, then ω_f is reflexive.*

Proof. We may assume that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. The A-module B may be presented as a quotient of a finite free A-module. Thus, the A-module $\omega_{B/A}$ is an A-submodule of a finite free A-module. Since A is a Noetherian ring, the A-module $\omega_{B/A}$ is finite, which implies that $\omega_{B/A}$ is finite as a B-module. The second statement follows from (2.3.3). Let us show the last statement. Assume that A and B are normal. Then the A-module $\omega_{B/A}$ is reflexive (2.1.2 (1)). Thus, the B-module $\omega_{B/A}$ is reflexive (2.1.6). □

Lemma 2.3.5. *Let A be a ring, B be a finite A-algebra of finite presentation, and K be a ring extension of A. Put $L := B \otimes_A K$. Let $T\colon L \to K$ be a K-module homomorphism. Assume that the L-module homomorphism $\lambda_{L/K,T}\colon L \to \omega_{L/K}$ defined by $c \mapsto (b \mapsto T(bc))$ is bijective (see 2.3.3). We define a B-module by $C_{B/A,T} := \{c \in L \mid T(Bc) \subset A\}$ and a B-module homomorphism $\mu_{B/A,T}\colon C_{B/A,T} \to \omega_{B/A}$ by $c \mapsto (b \mapsto T(bc))$. Then the following statements hold. (1) The B-module homomorphism $\mu_{B/A,T}$ is bijective. (2) Let A' be a flat A-algebra. By $T'\colon L' \to K'$ we denote the base change of $T\colon L \to K$ via A'/A. Then $\lambda_{L'/K',T'}$ is bijective, and the formations of $C_{B/A,T}$ and $\mu_{B/A,T}$ commute with the base change of B/A via A'/A.*

Proof. By $\phi\colon \omega_{B/A} \to \omega_{L/K}$ we denote the B-module homomorphism induced by the base change via K/A. Then ϕ is injective since $A \subset K$. We regard $\omega_{B/A}$ as a B-submodule of $\omega_{L/K}$ by ϕ. Then $C_{B/A,T} = \lambda_{L/K,T}^{-1}(\omega_{B/A})$ and $\mu_{B/A,T} = \lambda_{L/K,T}|_{C_{B/A,T}}$. Thus, Statement (1) holds. Statement (2) follows from the base change compatibility in (2.3.1). □

Definition 2.3.6. Let B/A be a finite extension of integral domains of finite presentation. By K we denote the field of fractions of A. Put $L := B \otimes_A K$. Suppose that the finite field extension L/K is separable. Then the trace $T_{L/K}$ of L/K is non-zero. Put $T := T_{L/K}$, $C_{B/A} := C_{B/A,T}$, $\mu_{B/A} := \mu_{B/A,T}$ (2.3.5), and $D_{B/A} := \{d \in L \mid dC_{B/A} \subset B\}$. The B-module $C_{B/A}$ (resp. $D_{B/A}$) is called the *codifferent* (resp. *different*) of B/A.

Remark 2.3.7. If A is normal, then $D_{B/A} \subset B \subset C_{B/A}$ since $T_{L/K}(B) \subset A$ [19, 9.2]. If A and B are Dedekind domains, then the definition coincides with that in [23, III, §3]. Let S be a multiplicatively closed subset of A. Then $S^{-1}C_{B/A} = C_{S^{-1}B/S^{-1}A}$ in L (2.3.5 (2)). If $C_{B/A}$ is finite as a B-module (see 2.3.4), then $S^{-1}D_{B/A} = D_{S^{-1}B/S^{-1}A}$ in L.

Definition 2.3.8. Let $f\colon X \to Y$ be a finite dominant morphism of finite presentation between integral schemes. Assume that Y is normal and that the function field extension induced by f is separable. We define an injective $\mathcal{O}_X$-module homomorphism $\lambda_f\colon \mathcal{O}_X \to \omega_f$ in the following way. We first assume that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. We define a B-module homomorphism $\lambda_{B/A}\colon B \to \omega_{B/A}$ as the composite of the inclusion B-module homomorphism $B \to C_{B/A}$ (2.3.7) and the B-module isomorphism $\mu_{B/A}\colon C_{B/A} \to \omega_{B/A}$ (2.3.5 (1)). We define λ_f as the homomorphism induced by $\lambda_{B/A}$. The formation of λ_f commutes with any localization with respect to the Zariski topology of Y. In the general case, we define λ_f by gluing the local pieces.

Proposition 2.3.9. *Let $f\colon X \to Y$ be a finite étale dominant morphism between integral schemes. Assume that Y is normal. Then the $\mathcal{O}_X$-module homomorphism λ_f introduced in* (2.3.8) *is an isomorphism.*

Proof. We have to show that $\lambda_{B/A}$ in (2.3.8) is bijective. After localization and strict Henselization, we may assume that A is strictly Henselian (2.3.5 (2)). Then the A-algebra B is isomorphic to a finite direct product of copies of A. In that case, the trace of B/A is given by the summation for all direct factors of B, which concludes the proof. □

Definition 2.3.10. Let $f\colon X \to Y$ be a finite dominant morphism between locally Noetherian normal integral schemes. Suppose that the function field extension induced by f is separable. Then λ_f introduced in (2.3.8) is generically surjective (2.2.2; see also 2.3.9). The Weil divisor $D(\lambda_f)$ is called the *ramification divisor* of f (see 2.2.2 for $D(\bullet)$; see also 2.3.4).

2.4. **Relative Dualizing Sheaves of Cohen–Macaulay Morphisms.** Let $f\colon X \to Y$ be a CM morphism of pure relative dimension r between locally Noetherian schemes. By ω_f we denote the relative dualizing sheaf of f. The coherent $\mathcal{O}_X$-module ω_f is $\mathcal{O}_Y$-flat [4, 3.5.1]. The morphism f is Gorenstein if and only if ω_f is a line bundle [4, 3.5.1]. If f is l.c.i. (resp. smooth), then ω_f is canonically isomorphic to the relative canonical sheaf $\omega_{X/Y}$ (resp. the sheaf of relative differentials $\Omega^r_{X/Y}$) [4, p. 157]. When f is proper, we denote the trace map of ω_f by $\mathrm{tr}_f\colon R^r f_*\omega_f \to \mathcal{O}_Y$ [4, 3.6.6]. The formation of the pair $(\omega_f, \mathrm{tr}_f)$ commutes with the base change via any morphism between locally Noetherian schemes [4, 3.6.6]. If f is finite, then the dualizing pair $(\omega_f, \mathrm{tr}_f)$ of f as a CM morphism is given by that of f as a finite morphism (§2.3).

Proposition 2.4.1. *Let $f\colon X \to Y$ be a CM morphism of pure relative dimension between CM schemes. Then the coherent $\mathcal{O}_X$-module ω_f is maximal CM. If X is reduced, then ω_f is of rank one. If X is normal, then ω_f is reflexive.*

Proof. By r we denote the relative dimension of f. We may assume that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Let us show the first statement. We have to show that the $\mathcal{O}_{X,x}$-module $\omega_{f,x}$ is maximal CM for any point x on X. Since ω_f is $\mathcal{O}_Y$-flat, we may assume that A is a field [19, 23.3]. Choose a presentation $B = S/I$ where $S = A[s_1, \ldots, s_n]$. Then there exists a B-module isomorphism $\Gamma(X, \omega_f) \cong \mathrm{Ext}^{n-r}_S(B, S)$ [4, p. 157]. We replace B and S by the localizations at x and its image, respectively. Then we obtain a B-module isomorphism $\omega_{f,x} \cong \mathrm{Ext}^{n-r}_S(B, S)$. By ω_A, ω_B, and ω_S we denote the canonical modules of A, B, and S, respectively [3, 3.3.1]. Since $\omega_A \cong A$ [3, 3.3.7 (a)] and $\omega_S \cong \omega_A \otimes_A S$ [3, 3.3.16 and 3.3.21], we obtain an S-module isomorphism $\omega_S \cong S$. Since $\omega_B \cong \mathrm{Ext}^{n-r}_S(B, \omega_S)$ [3, 3.3.7 (b)], we obtain a B-module isomorphism $\omega_{f,x} \cong \omega_B$. Since ω_B is maximal CM by definition [3, 3.3.1], the $\mathcal{O}_X$-module ω_f is maximal CM. Let us show the second statement. Assume that X is reduced. Take an associated point η of X. Put $L := \mathcal{O}_{X,\eta}$. Then L is a field [8, 5.8.5] and $\omega_{f,\eta} \cong \omega_L$ by the proof of the first statement. Since $\omega_L \cong L$ [3, 3.3.7 (a)], the L-module $\omega_{f,\eta}$ is of rank one. Thus, the $\mathcal{O}_X$-module ω_f is of rank one. The last statement follows from the equivalence of (1) and (3) in (2.1.3). □

3. Fibered Surfaces

Let g be a non-negative integer.

3.1. **Pre-Genus-g Fibrations and Genus-g Fibrations.**

Definition 3.1.1. Let S be a scheme. An *S-curve* is a separated faithfully flat S-scheme of finite type of pure relative dimension one. A triple (X, C, f) is called a *pre-genus-g fibration* if the following conditions are satisfied:

(1) X and C are excellent integral schemes of dimension two and one, respectively;
(2) $f: X \to C$ is a proper morphism;
(3) the homomorphism $\mathcal{O}_C \to f_*\mathcal{O}_X$ associated to f is an isomorphism;
(4) the generic fiber of f is a proper smooth curve of genus g.

A pre-genus-g fibration is said to be *normal* (resp. *regular*) if X is normal (resp. regular). A regular pre-genus-g fibration is also called a *genus-g fibration.* A pre-genus-one (resp. genus-one) fibration is also called a *pre-elliptic* (resp. *elliptic*) *fibration.* A genus-g fibration (X, C, f) is said to be *minimal* if any fiber of f does not contain a (-1)-curve.

Lemma 3.1.2. *Let (X, C, f) be a pre-genus-g fibration. Then the following statements hold.*

(1) *A point s on C is closed if and only if s is of codimension one.*
(2) *A point x on X is closed if and only if x is of codimension two.*
(3) *The morphism f is surjective and of pure relative dimension one.*
(4) *Any fiber of f is geometrically connected. In particular, the generic fiber of f is geometrically integral.*
(5) *If C is normal, then C is regular and f is faithfully flat. In particular, the C-scheme X is a C-curve.*
(6) *If X is normal, then X is CM, C is regular, and f is CM.*

Proof. Since C is integral and of dimension one, Statement (1) holds. Statements (2) and (3) follow from [8, 5.6.5] and [8, 5.5.2], respectively. By the assumption on f, Statement (4) holds. Statement (5) follows from [17, 4.3.10] and Statement (3). Let us show Statement (6). Assume that X is normal. Then X is CM [8, 5.8.6] and C is normal [17, 4.1.5]. Since f is flat (Statement (5)), the morphism f is CM [19, 23.3], which concludes the proof of Statement (6). □

Lemma 3.1.3. *Let (X, C, f) be a triple satisfying Conditions* (1)*,* (2)*, and* (4) *in* (3.1.1)*. Assume that X and C are normal. Then Condition* (3) *is satisfied if and only if the generic fiber of f is geometrically connected.*

Proof. The only-if part follows from (3.1.2 (4)). Let us show the converse. Take the Stein factorization $\tau \circ g: X \to C' \to C$ of f. We have only to show that τ is an isomorphism. Since the generic fiber of f is geometrically integral and X is normal and integral, the morphism τ is a birational morphism between normal integral schemes [17, 3.2.14 (c) and 4.1.5]. Since τ is finite, the morphism τ is an isomorphism, which concludes the proof. □

Lemma 3.1.4. *Let (X, C, f) be a pre-genus-g fibration. Let $\pi_C: C' \to C$ be a finite dominant morphism between regular integral schemes of dimension one. By $\pi_1: X \times_C C' \to X$ and $\pi_2: X \times_C C' \to C'$ we denote the base change of π_C via f and the base change of f via π_C, respectively. Then $X \times_C C'$ is integral. Take the normalization $\pi_0: X' \to X \times_C C'$ of $X \times_C C'$. Put $\pi_X := \pi_1 \circ \pi_0$ and $f' := \pi_2 \circ \pi_0$.*

Then π_X is finite and surjective, and (X', C', f') is a normal pre-genus-g fibration:

$$\begin{array}{ccccc} & & \overset{\pi_X}{\frown} & & \\ X' & \xrightarrow{\pi_0} & X \times_C C' & \xrightarrow{\pi_1} & X \\ & \searrow^{f'} & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle f} \\ & & C' & \xrightarrow{\pi_C} & C. \end{array}$$

Proof. Since π_C and f are flat dominant morphisms between locally Noetherian integral schemes [17, 4.3.10], any associated point of the C'-scheme $X \times_C C'$ is contained in the generic fiber [8, 3.3.6]. Since the generic fiber of f is geometrically integral (3.1.2 (4)), the scheme $X \times_C C'$ is integral. Since π_C and π_1 are finite, the schemes C' and $X \times_C C'$ are excellent, which implies that π_0 is finite. Thus, the morphism π_X is finite. Since π_X is finite and dominant, the morphism π_X is surjective. Finally, we show that (X', C', f') satisfies the conditions in (3.1.1). Since π_0 is finite and π_2 is proper, Condition (2) holds. Since π_X is finite, the scheme X' is of dimension at most two. By definition, the generic fiber of f' is a proper smooth geometrically connected curve of genus g (3.1.2 (4)). Thus, Conditions (1) and (4) hold. Condition (3) follows from (3.1.3), which concludes the proof of the last statement. □

Definition 3.1.5. We use the notation introduced in (3.1.4). We say that (X, C, f) and $\pi_C \colon C' \to C$ *induce* (X', C', f') (and $\pi_X \colon X' \to X$).

3.2. **Models of Fibered Surfaces.**

Definition 3.2.1. Let X be an integral scheme. A *desingularization* of X is a proper birational morphism from a regular integral scheme to X. A desingularization $Y \to X$ of X is said to be *minimal* if, for any desingularization $Z \to X$ of X, there exists an X-morphism $Z \to Y$.

Remark 3.2.2. Any desingularization of X factors through the normalization of X. If X admits a minimal desingularization, then it is unique up to unique X-isomorphism. The total space Y of any pre-genus-g fibration admits the minimal desingularization [17, 9.3.32]. Further, a desingularization $g \colon Z \to Y$ of Y is minimal if and only if no (-1)-curve on Z is contained in the exceptional locus of g [17, 9.3.32].

Definition 3.2.3 ([16, 1.1 and §13])**.** A point x of codimension two on a locally Noetherian scheme X is said to be *rational* (resp. *of type A_n*) if $U := \operatorname{Spec} \mathcal{O}_{X,x}$ is normal and there exists a desingularization $f \colon V \to U$ of U such that the equality $R^1 f_* \mathcal{O}_V = 0$ holds (resp. the reduction of $f^{-1}(x)$ consists of n smooth rational curves over the residue field at x whose intersection matrix is given by the Cartan matrix of type A_n).

Definition 3.2.4. Let C be an excellent regular integral scheme of dimension one with function field K. Let X_K be a proper smooth geometrically connected K-curve of genus g. A pre-genus-g fibration (X, C, f) with K-isomorphism between X_K and the generic fiber f is called a *C-model* of X_K. If (X, C, f) is normal (resp. regular, resp. regular and minimal), then the C-model is said to be *normal* (resp. *regular*, resp. *regular and minimal*).

Remark 3.2.5. Since X_K is projective over K, we may take a normal C-model of X_K as the normalization of an integral closed subscheme of a projective space over C (3.1.3). Thus, we may take a minimal regular C-model of X_K [17, 8.3.44 and 9.3.19]. If $g > 0$, then a minimal regular C-model of X_K is unique up to unique C-isomorphism [17, 9.3.21].

Lemma 3.2.6. *We use the notation introduced in* (3.2.4). *Let* (X_1, C, f_1) *and* (X_2, C, f_2) *be regular* C*-models of* X_K. *Then there exist a regular* C*-model* (X_0, C, f_0) *of* X_K *and proper birational* C*-morphisms* $u_1 : X_0 \to X_1$ *and* $u_2 : X_0 \to X_2$ *that are extensions of the* K*-isomorphisms between the generic fibers.*

Proof. Take the graph Γ of the K-isomorphism between the generic fibers of f_1 and f_2. Then any desingularization of the closure of Γ in $X_1 \times_C X_2$ with the reduced structure gives desired C-model and C-morphisms. □

Proposition 3.2.7. *Let* $\pi_C : C' \to C$ *be a finite Galois covering of excellent regular integral schemes of dimension one with Galois group* G. *Let* (X', C', f') *be a genus-*g *fibration. Assume that* G *equivariantly acts on* X'/C'. *Then there exists the quotient* X *of the action of* G *on* X'. *By* $\pi_X : X' \to X$ *and* $f : X \to C$ *we denote the quotient morphism and the unique morphism satisfying* $f \circ \pi_X = \pi_C \circ f'$, *respectively. Then the following statements hold*: (1) *the formation of* X *commutes with the restriction to the generic fiber*; (2) (X, C, f) *is a normal pre-genus-*g *fibration*; (3) (X, C, f) *and* π_C *induce* (X', C', f') *and* π_X (3.1.5).

Proof. Let us show the first statement. If C is affine, then $\pi_C \circ f'$ is projective [17, 8.3.16], which implies that any G-orbit of a point on X is contained in an affine open subset of X. Thus, in that case, the quotient exists [7, V, 4.1]. Since the formation of the quotient commutes with the base change via any flat morphism to C, the quotient exists in the general case. Further, Statement (1) holds, and the generic fiber of f is a proper smooth geometrically connected curve of genus g (3.1.2 (4)). Proposition 4 in [20] and [12, 32.7] imply that π_X is finite and surjective, f is proper, and X is normal and integral. Thus, the scheme X is excellent and of dimension two [19, 9.4 (ii) and 15.1]. Therefore, Statement (2) follows from (3.1.3). Since X' is normal and π_X is finite, Statement (3) holds. □

Proposition 3.2.8. *Let* $\pi_C : C' \to C$ *be a finite Galois covering of excellent regular integral schemes of dimension one with function field extension* K'/K. *Let* X_K *be a proper smooth geometrically connected* K*-curve. Put* $X'_{K'} := X_K \times_K K'$. *Then there exists a normal* C*-model* (X, C, f) *of* X_K *satisfying the following conditions*: (1) *any closed point on* X *is rational*; (2) (X, C, f) *and* π_C *induce a regular* C'*-model* (X', C', f') *of* $X'_{K'}$ (3.1.5).

Proof. By G we denote the Galois group of K'/K. Take a regular C-model (Y, C, h) of X_K (3.2.5). The triple (Y, C, h) and the morphism π_C induce a normal C'-model (Y', C', h') of $X'_{K'}$ and a finite morphism $\pi_Y : Y' \to Y$ (3.1.5). Take the minimal desingularization $h'_0 : X' \to Y'$ of Y' (3.2.2). Put $f' := h' \circ h'_0$. Then the triple (X', C', f') is a regular C'-model of $X'_{K'}$. By the uniqueness of the minimal desingularization, we obtain an equivariant action of G on X'/C'. The quotient of this action gives a normal C-model (X, C, f) of X_K satisfying Condition (2) (3.2.7). By $\pi_X : X' \to X$ and $h_0 : X \to Y$ we denote the quotient morphism and the unique morphism satisfying $h_0 \circ \pi_X = \pi_Y \circ h'_0$, respectively. Since X is normal, Y is regular, and h_0 is proper and birational, Condition (1) is satisfied [16, 1.2]. □

Lemma 3.2.9. *Let (X, C, f) be a minimal genus-g fibration with generic fiber $f_K: X_K \to C_K$. Assume that $g > 0$. Let τ be an automorphism of C. By τ_K we denote the restriction of τ to C_K. Let σ_K be an automorphism of X_K. Suppose that $\tau_K \circ f_K = f_K \circ \sigma_K$. Then σ_K uniquely extends to the automorphism σ of X satsfying $\tau \circ f = f \circ \sigma$.*

Proof. Since $g > 0$, the lemma follows from the uniqueness of the minimal regular C-model of X_K (3.2.5). □

3.3. **Invariants.**

Definition 3.3.1. The *multiplicity* of a non-zero Weil divisor D on a locally Noetherian normal scheme X is the maximum integer m satisfying $D = mD'$ where D' is a Weil divisor on X. Let (X, C, f) be a normal pre-genus-g fibration and s be a closed point on C. We write $f^{-1}(s) = m_{f,s} V_{f,s}$ where $m_{f,s}$ is the multiplicity of $f^{-1}(s)$ and $V_{f,s}$ is a Weil divisor on X.

Lemma 3.3.2 ([17, 9.2.2] and [10, 8.3 (1)]; use 3.2.6)**.** *We use the notation introduced in* (3.2.4)*. Let s be a closed point on C. Then the multiplicity of the fiber of a regular C-model of X_K over s does not depend on the choice of the regular C-model of X_K.*

Definition 3.3.3. Let $f: X \to Y$ and $g: Y \to Z$ be proper morphisms between locally Noetherian schemes. Put $h := g \circ f$. We define an $\mathcal{O}_Z$-module homomorphism $\eta_{f,g}$ as the composite of the canonical $\mathcal{O}_Z$-module homomorphisms $R^1 g_* \mathcal{O}_Y \to R^1 g_* f_* \mathcal{O}_X \to R^1 h_* \mathcal{O}_X$.

Lemma 3.3.4. *We use the notation introduced in* (3.3.3)*. Assume that f is a proper birational morphism between locally Noetherian normal integral schemes. Suppose that any point on Y of codimension two is closed and rational. Then the $\mathcal{O}_Z$-module homomorphism $\eta_{f,g}$ is an isomorphism.*

Proof. The lemma follows from [16, 1.2] and the Grothendieck spectral sequence for $h_* = g_* \circ f_*$ and $\mathcal{O}_X$. □

Definition 3.3.5. We use the notation $\eta_{\bullet,\bullet}$ introduced in (3.3.3) and the same notation as in (3.2.6). Then η_{u_i,f_i} is an isomorphism for $i = 1$ and 2 (3.3.4). If there exists a proper birational C-morphism $u_{12}: X_1 \to X_2$ satisfying $u_{12} \circ u_1 = u_2$, then $\eta_{u_1,f_1} \circ \eta_{u_{12},f_2} = \eta_{u_2,f_2}$. Thus, we may identify the $\mathcal{O}_C$-modules $R^1 \widehat{f}_* \mathcal{O}_{\widehat{X}}$ for all regular C-models $(\widehat{X}, C, \widehat{f})$ of X_K. Let (X, C, f) be a C-model of X_K. We define an $\mathcal{O}_C$-module homomorphism $\eta_f: R^1 f_* \mathcal{O}_X \to R^1 \widehat{f}_* \mathcal{O}_{\widehat{X}}$ in the following way. Take a desingularization $f_0: \widehat{X} \to X$ of X. Put $\widehat{f} := f \circ f_0$. Then the triple $(\widehat{X}, C, \widehat{f})$ is a regular C-model of X_K. We put $\eta_f := \eta_{f_0,f}$. The definition of η_f does not depend on the choice of the regular C-model $(\widehat{X}, C, \widehat{f})$ of X_K.

Definition 3.3.6. We use the notation introduced in (3.3.5). Let s be a closed point on C. We put $l_{f,s} := \mathrm{length}_{\mathcal{O}_{C,s}} \mathrm{torsion}\,(R^1 \widehat{f}_* \mathcal{O}_{\widehat{X}})_s + L_{\mathcal{O}_{C,s}}(\eta_{f,s})$ (see 2.2.1 for $L_{\mathcal{O}_{C,s}}(\bullet)$). The definition of $l_{f,s}$ does not depend on the choice of the regular C-model $(\widehat{X}, C, \widehat{f})$ of X_K (see 3.3.5).

Remark 3.3.7. If any closed point on X is rational, then η_f is an isomorphism (3.3.4) and the equalities $l_{f,s} = \mathrm{length}_{\mathcal{O}_{C,s}} \mathrm{torsion}\,(R^1 \widehat{f}_* \mathcal{O}_{\widehat{X}})_s = \mathrm{length}_{\mathcal{O}_{C,s}} \mathrm{torsion}\,(R^1 f_* \mathcal{O}_X)_s$ hold.

Proposition 3.3.8. *Let (X, C, f) be a normal pre-genus-g fibration. By ω_f we denote the relative dualizing sheaf of f* (3.1.2 (3) and (6))*. We denote the canonical $\mathcal{O}_X$-module homomorphism by $\gamma_f\colon f^*f_*\omega_f \to \omega_f$. Then the following statements hold.* (1) *If $g = 1$* (*resp. $g \geq 1$, resp. $g = 0$*)*, then the restriction of γ_f to the generic fiber is an isomorphism* (*resp. surjective, resp. zero-map*)*.* (2) *The coherent $\mathcal{O}_X$-module ω_f is reflexive of rank one.* (3) *The coherent $\mathcal{O}_C$-module $f_*\omega_f$ is a vector bundle of rank g.*

Proof. Since the generic fiber of f is a proper smooth geometrically connected curve of genus g (3.1.2 (4)), Statement (1) holds [17, 7.4.10] and the $\mathcal{O}_C$-module $f_*\omega_f$ is of rank g. Statement (2) follows from (2.4.1). Statement (3) follows from (3.1.2 (6) and 2.1.5). □

Proposition 3.3.9. *Let (X, C, f) be a normal pre-genus-g fibration. Take a desingularization $f_0\colon Y \to X$ of X. Put $h := f \circ f_0$. By ω_f and ω_h we denote the relative dualizing sheaves of f and h, respectively. Then the coherent $\mathcal{O}_Y$-modules $f_0^{[*]}\omega_f$ and ω_h are line bundles* (*see* 2.2.6 *for $f_0^{[*]}$*)*. By E we denote the exceptional locus of f_0. Put $U := Y \setminus E$. By $\phi_U\colon f_0^{[*]}\omega_f|_U \to \omega_h|_U$ we denote the $\mathcal{O}_U$-module isomorphism induced by the isomorphism $f_0|_U\colon U \to X \setminus f_0(E)$. Then there exists the unique divisor D on Y such that $D|_U = 0$ and the $\mathcal{O}_U$-module isomorphism ϕ_U extends to an $\mathcal{O}_Y$-module isomorphism $f_0^{[*]}\omega_f \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(D) \to \omega_h$.*

Proof. Since $f_0^{[*]}\omega_f$ and ω_h are reflexive of rank one (2.1.2 (1) and 2.4.1) and Y is regular, the coherent $\mathcal{O}_Y$-modules $f_0^{[*]}\omega_f$ and ω_h are line bundles (2.1.7). We may take a divisor D_1 on Y so that $(f_0^{[*]}\omega_f)^\vee \otimes_{\mathcal{O}_Y} \omega_h \cong \mathcal{O}_Y(D_1)$. Since $\mathcal{O}_Y(D_1)|_U \cong \mathcal{O}_U$, the divisor $D_1|_U$ on U is principal. We denote the function field of an integral scheme Z by $K(Z)$. Since the inclusion morphism $U \to Y$ induces an isomorphism $K(Y) \cong K(U)$, we may take a principal divisor D_2 on Y so that $D_1|_U = D_2|_U$. Put $D := D_1 - D_2$. Then the divisor D satisfies the desired condition.

Let us show the uniqueness of D. We have only to show the following: any principal divisor D' on Y with $D'|_U = 0$ is equal to zero. Take $F \in K(Y)$ so that F defines D'. By $h^*\colon K(C) \to K(Y)$ we denote the injective homomorphism induced by the dominant morphism h between integral schemes. Since $f_0(E)$ is a closed subset of codimension at least two, the following statements hold: (1) $h(E)$ is a proper closed subset of C; (2) $h(U) = C$. Since the homomorphism $\mathcal{O}_C \to h_*\mathcal{O}_Y$ associated to h is an isomorphism, Statement (1) implies that we may take $G \in K(C)$ so that $F = h^*G$. Statement (2) shows that $G \in \mathcal{O}_C(C)^\times$, which implies that $F \in \mathcal{O}_Y(Y)^\times$. Thus, the equality $D' = 0$ holds, which proves the uniqueness of D. □

4. Pre-Elliptic Fibrations and Elliptic Fibrations

4.1. **Invariants.**

Definition 4.1.1. We use the notation introduced in (3.3.8). Suppose that $g = 1$. We put $D_f := D(\gamma_f)$ (see 2.2.2 for $D(\bullet)$).

Remark 4.1.2. The Weil divisor D_f is effective and vertical with respect to f (3.3.8 (1)). The $\mathcal{O}_X$-module homomorphism γ_f induces an $\mathcal{O}_X$-module isomorphism $f^*f_*\omega_f \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_f) \cong \omega_f$ (2.2.3 and 3.3.8).

Lemma 4.1.3. *We use the notation introduced in* (3.3.9)*. Suppose that* $g = 1$ *and that any closed point on* X *is rational* (3.2.3)*. Take* D_f *and* D_h *introduced in* (4.1.1)*. Then* $f_0^* D_f + D = D_h$ *(see* 2.2.5 *for* f_0^**).*

Proof. We write $D = D_1 - D_2$ where D_1 and D_2 are effective divisors on Y with $D_1|_U = 0$ and $D_2|_U = 0$. Then ϕ_U extends to an $\mathcal{O}_Y$-module isomorphism $\phi\colon f_0^{[*]}\omega_f \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(D_1) \to \omega_h \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(D_2)$. For $i = 1$ and 2, we denote the canonical injective $\mathcal{O}_Y$-module homomorphism by $\psi_i\colon \mathcal{O}_Y \to \mathcal{O}_Y(D_i)$. We define an $\mathcal{O}_C$-module isomorphism χ as the composite of the $\mathcal{O}_C$-module isomorphisms $f_*\omega_f \to (R^1 f_*\mathcal{O}_X)^\vee \to (R^1 h_*\mathcal{O}_Y)^\vee \to h_*\omega_h$ where the middle arrow is the dual of the $\mathcal{O}_C$-module isomorphism given by (3.3.4) and the other arrows are induced by the Grothendieck duality. The $\mathcal{O}_C$-module isomorphism χ induces an $\mathcal{O}_Y$-module isomorphism $h^*\chi\colon f_0^* f^* f_*\omega_f \to h^* h_*\omega_h$. Since the canonical $\mathcal{O}_Y$-module homomorphism $f_0^* f^* f_*\omega_f \to f_0^{[*]} f^* f_*\omega_f$ is an isomorphism (3.3.8 (3)), the $\mathcal{O}_X$-module homomorphism γ_f induces an $\mathcal{O}_Y$-module homomorphism $f_0^{[*]}\gamma_f\colon f_0^* f^* f_*\omega_f \to f_0^{[*]}\omega_f$. The above homomorphisms give the following diagram of $\mathcal{O}_Y$-modules and $\mathcal{O}_Y$-module homomorphisms:

$$(*)\qquad \begin{array}{ccccc} f_0^* f^* f_*\omega_f & \xrightarrow{f_0^{[*]}\gamma_f} & f_0^{[*]}\omega_f & \xrightarrow{f_0^{[*]}\omega_f\otimes\psi_1} & f_0^{[*]}\omega_f \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(D_1) \\ {\scriptstyle h^*\chi}\downarrow{\scriptstyle \cong} & & & & {\scriptstyle \phi}\downarrow{\scriptstyle \cong} \\ h^* h_*\omega_h & \xrightarrow{\gamma_h} & \omega_h & \xrightarrow{\omega_h\otimes\psi_2} & \omega_h \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(D_2). \end{array}$$

The diagram obtained by the restriction of the modules and homomorphisms in Diagram $(*)$ to $Y \setminus h^{-1}(h(E))$ is commutative. Since $\omega_h \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(D_2)$ is reflexive, Diagram $(*)$ is commutative (2.1.4). Since $D(f_0^{[*]}\omega_f \otimes \psi_1) = D_1$, $D(\omega_h \otimes \psi_2) = D_2$, $D(f_0^{[*]}\gamma_f) = f_0^* D_f$, and $D(\gamma_h) = D_h$ (2.2.4 (2) and 2.2.7; see 2.2.2 for $D(\bullet)$), the equality $f_0^* D_f + D_1 = D_h + D_2$ holds (2.2.4 (1)), which concludes the proof. □

Lemma 4.1.4. *We use the notation introduced in* (3.3.9)*. Suppose that* $g = 1$ *and that* f_0 *is the blowing-up at a regular point. We equip* E *with the reduced structure. Take* D_f *and* D_h *introduced in* (4.1.1)*. Then* $f_0^* D_f + E = D_h$*.*

Proof. We have only to show that $D = E$ (4.1.3). Since X is regular by assumption, the lemma follows from [17, 9.2.24]. □

Definition 4.1.5. Let (X, C, f) be a minimal elliptic fibration and s be a closed point on C with algebraically closed residue field. We use the notation $f^{-1}(s) = m_{f,s} V_{f,s}$ introduced in (3.3.1) and D_f introduced in (4.1.1). The vertical divisor over s appearing in D_f is equal to $a_{f,s} V_{f,s}$, where the integer $a_{f,s}$ satisfies the inequalities $0 \le a_{f,s} < m_{f,s}$ [18, 5.7]. The configurations of the irreducible components of $V_{f,s}$ are classified into three categories and eight types (Kodaira's symbol): (1) *good*: I_0; (2) *multiplicative*: I_n $(n > 0)$; (3) *additive*: I_n^* $(n \ge 0)$, II, II*, III, III*, IV, and IV*. The fiber $f^{-1}(s)$ is said to be *of type* ${}_mT$ where $m := m_{f,s}$ and T is the type of $V_{f,s}$.

4.2. **Notation for Local Base Spaces.** In this subsection, we introduce the notation used in §4.3 and §§5–7. Let R be a complete discrete valuation ring with algebraically closed residue field k of characteristic p and field of fractions K. We fix an algebraic closure $\overline{K}$ of K. Choose a finite field extension K'/K of degree

d in $\overline{K}$. Take the normalization R' of R in K'. Put $C := \operatorname{Spec} R$, $C' := \operatorname{Spec} R'$, $C_K := \operatorname{Spec} K$, and $C'_{K'} := \operatorname{Spec} K'$. The extensions R'/R and K'/K induce finite morphisms $\pi_C \colon C' \to C$ and $\pi_{C_K} \colon C'_{K'} \to C_K$, respectively. By s and s' we denote the closed points on C and C', respectively.

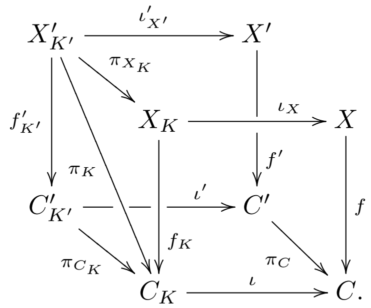
4.3. **Quotient.** We use the notation introduced in §4.2. Assume that K'/K is a finite Galois extension with Galois group G. By $\rho_{C'_{K'}/C_K}$ (resp. $\rho_{C'/C}$) we denote the C_K-action of G on $C'_{K'}$ (resp. the C-action of G on C'). Let E_K be an elliptic curve over K. Put $E'_{K'} := E_K \times_K K'$. Take the minimal regular models (E, C, g) and (E', C', g') of E_K and $E'_{K'}$, respectively. By $\operatorname{Aut}(E'/C')$ we denote the group of all C'-automorphisms of E'. The translation of $E'_{K'}$ by the addition of any element of $E'_{K'}(K')$ induces a C'-automorphism of E' (3.2.9). Thus, we may regard $E'(K')$ as a subgroup of $\operatorname{Aut}(E'/C')$. Choose a cocycle $\{a_g\}_{g \in G} \in Z^1(G, \operatorname{Aut}(E'/C'))$. The base change of the action $\rho_{C'_{K'}/C_K}$ via the structure morphism $E_K \to C_K$ uniquely extends to the action $\rho_{E'}$ on E' (3.2.9). We define an action ρ on E' by $\rho(g) := a_g \circ \rho_{E'}(g)$ (left action) for $g \in G$. Then $g' \colon E' \to C'$ is equivariant with respect to ρ and $\rho_{C'/C}$. We may take the quotient $\pi_X \colon E' \to X$ of the action ρ (3.2.7). By $f \colon X \to C$ we denote the unique morphism satisfying $f \circ \pi_X = \pi_C \circ g'$.

Proposition 4.3.1. *Assume that $\{a_g\}_{g \in G} \in Z^1(G, E'(K'))$. Then the Jacobian of the generic fiber of f is K-isomorphic to E_K. If $g \colon E \to C$ is smooth, then the triple (X, C, f) is a minimal elliptic fibration with special fiber of type ${}_m\mathrm{I}_0$. In that case, the quotient morphism π_X is finite and flat.*

Proof. The first statement follows from (3.2.7 (1)). Let us show the second statement. The triple (X, C, f) is a normal pre-elliptic fibration (3.2.7 (2)). Thus, in order to show that f is an elliptic fibration, we have only to show that X is regular. Suppose that X admits a non-regular point. By T we denote the base changes of the K-rational points of E_K via $C'_{K'}/C_K$. Then the group G trivially acts on the subgroup T of $E'(K')$. Thus, the translation of E' by the addition of any element of T descends to a C-automorphism of X. Since $g \colon E \to C$ is smooth, the restriction of the specialization homomorphism $E'(K') \to E'(k)$ to T is surjective. Thus, the action of T on X induces a transitive action on $X(k)$. Since the special fiber is a curve over the algebraically closed field k, the set $X(k)$ is dense in the special fiber. Thus, the singular locus of X is equal to the special fiber, which contradicts the fact that X is normal. Therefore, the quotient X is regular. Since the number of the irreducible components of the special fiber of f is equal to one, the elliptic fibration (X, C, f) is minimal. Since the special fiber of f is of type ${}_m\mathrm{I}_0$ [18, 6.6], the second statement holds. Since E' and X are regular and π_X is finite (3.2.7 (3)), the morphism π_X is flat [19, 23.1]. □

4.4. **Notation for Base Change of Local Pre-Elliptic Fibrations.** In this subsection, we introduce the notation used in §§5–7. We use the notation introduced in §4.2. Let (X, C, f) be a normal pre-elliptic fibration with generic fiber $f_K \colon X_K \to C_K$. By $\pi_{X_K} \colon X'_{K'} \to X_K$ and $f'_{K'} \colon X'_{K'} \to C'_{K'}$ we denote the base change of π_{C_K} via f_K and the base change of f_K via π_{C_K}, respectively. Put $\pi_K := f_K \circ \pi_{X_K}$. Let (X', C', f') be a normal C'-model of $X'_{K'}$. By $\iota \colon C_K \to C$, $\iota' \colon C'_{K'} \to C'$,

$\iota_X : X_K \to X$, and $\iota'_{X'} : X'_{K'} \to X'$ we denote the inclusion morphisms:



By ${}_mT$ (resp. ${}_{m'}T'$) we denote the type of the special fiber of the minimal regular C-model of X_K (resp. the minimal regular C'-model of $X'_{K'}$) (4.1.5).

In §§5–7, we compare the invariants of (X, C, f) and (X', C', f') by means of the following normal pre-elliptic fibrations $(\widetilde{X}, C, \widetilde{f})$ and $(\widetilde{X}', C', \widetilde{f}')$ and the following finite morphism $\pi_{\widetilde{X}} : \widetilde{X}' \to \widetilde{X}$. Let $(\widetilde{X}, C, \widetilde{f})$ be a normal C-model of X_K. The triple $(\widetilde{X}, C, \widetilde{f})$ and the morphism $\pi_C : C' \to C$ induce a normal pre-elliptic fibration $(\widetilde{X}', C', \widetilde{f}')$ and a finite morphism $\pi_{\widetilde{X}} : \widetilde{X}' \to \widetilde{X}$ in the following way (3.1.5). By $\pi_1 : \widetilde{X} \times_C C' \to \widetilde{X}$ and $\pi_2 : \widetilde{X} \times_C C' \to C'$ we denote the base change of π_C via $\widetilde{f}$ and the base change of $\widetilde{f}$ via π_C, respectively. Take the normalization $\pi_0 : \widetilde{X}' \to \widetilde{X} \times_C C'$ of $\widetilde{X} \times_C C'$. Put $\pi_{\widetilde{X}} := \pi_1 \circ \pi_0$, $\widetilde{f}' := \pi_2 \circ \pi_0$, and $\pi := \widetilde{f} \circ \pi_{\widetilde{X}}$. In §§5–6, we consider the case where $\widetilde{X} = X$ and $\widetilde{X}' = X'$. In §7, we consider the case where (X, C, f) and (X', C', f') are minimal elliptic fibrations.

By E_K and $E'_{K'}$ we denote the Jacobians of X_K and $X'_{K'}$, respectively. Let (E, C, g) and $(\widetilde{E}, C, \widetilde{g})$ be two normal C-models of E_K, and (E', C', g') be a normal C'-model of $E'_{K'}$. We define a normal pre-elliptic fibration $(\widetilde{E}', C', \widetilde{g}')$ in the same way as in the case of $(\widetilde{X}', C', \widetilde{f}')$.

In §§6–7, we use the following notation. Assume that the finite field extension K'/K is separable. By $D_{C'/C}$ and $D_{\widetilde{X}'/\widetilde{X}}$ we denote the ramification divisors of π_C and $\pi_{\widetilde{X}}$, respectively (2.3.10). The support of $D_{\widetilde{X}'/\widetilde{X}}$ is contained in the special fiber of $\widetilde{f}'$ (2.3.9). By $d_{C'/C}$ we denote the degree of $D_{C'/C}$. By $D_{\widetilde{f}}$ and $D_{\widetilde{f}'}$ we denote the Weil divisors introduced in (4.1.1). Put $l := l_{f,s}$, $l' := l_{f',s'}$, $l_g := l_{g,s}$, $l_{g'} := l_{g',s'}$, $l_{\widetilde{f}} := l_{\widetilde{f},s}$, $l_{\widetilde{g}} := l_{\widetilde{g},s}$, $l_{\widetilde{f}'} := l_{\widetilde{f}',s'}$, and $l_{\widetilde{g}'} := l_{\widetilde{g}',s'}$ (3.3.6).

5. Base Change

We use the notation introduced in §4.4. Suppose that (X, C, f) is an elliptic fibration and that $\widetilde{X} = X$ and $\widetilde{X}' = X'$.

5.1. **Multiplicities.**

Lemma 5.1.1. *There exists a prime divisor on X whose coefficient in $V_{f,s}$ (3.3.1) is equal to one.*

Proof. We may assume that (X, C, f) is minimal (3.3.2). Then the lemma follows from the classification of singular fibers (4.1.5). □

Proposition 5.1.2. *There exists a finite separable field extension K'/K of degree m such that $X(K') \neq \emptyset$. The relations $m' \mid m$ and $m \mid dm'$ hold* (*see* 3.3.2). *If π_X is étale, then $m = dm'$.*

Proof. The first statement follows from [10, 8.4] and (5.1.1). Let us show that $m' \mid m$. By η and η' we denote the cohomology classes of X_K and $X'_{K'}$ in the Galois cohomology groups $H^1(K, E_K)$ and $H^1(K', E'_{K'})$, respectively. The group homomorphism $H^1(K, E_K) \to H^1(K', E'_{K'})$ induced by K'/K maps η to η'. Since the orders of η and η' are equal to m and m', respectively [18, 6.6], the relation $m' \mid m$ holds. Next, we show that $m \mid dm'$. Take a desingularization $f'_0 \colon Y \to X'$ of X'. Put $h := f' \circ f'_0$ and $\tau := \pi_X \circ f'_0$. Then the triple (Y, C', h) is an elliptic fibration. We use the notation introduced in (3.3.1): $f^{-1}(s) = mV_{f,s}$ and $h^{-1}(s') = m'V_{h,s'}$. Since R'/R is totally ramified of degree d, the equality $f \circ \tau = \pi_C \circ h$ gives $dm'V_{h,s'} = m\tau^{-1}(V_{f,s})$. Applying (5.1.1) to h, we obtain the relation $m \mid dm'$. Finally, we show the last statement. Assume that π_X is étale. Then X' is regular. Thus, we may assume that τ is étale. Applying (5.1.1) to f, we obtain the relation $dm' \mid m$, which proves the last statement. □

5.2. **Étale Parts and Non-Étale Parts.**

Definition 5.2.1. If π_X is étale, then we say that the field extension K'/K *induces a* (*finite*) *étale covering π_X of X.*

Lemma 5.2.2. *The following statements hold.* (1) *If K'/K induces an étale covering of X, then K'/K is separable.* (2) *If two finite field extensions L_1/K and L_2/K in $\overline{K}$ induce étale coverings of X, then the composite field L_1L_2/K induces an étale covering of X.*

Proof. Statement (1) follows from the fact that π_{X_K} is étale. Statement (2) follows from Statement (1) and the following fact: for any two finite étale coverings $X_1 \to X$ and $X_2 \to X$, the canonical projection $X_1 \times_X X_2 \to X$ is a finite étale covering. □

Lemma 5.2.3. *There exists the maximum one M/K among all finite field extensions in $\overline{K}$ inducing étale coverings of X. The field extension M/K is Galois and does not depend on the choice of the regular C-model X of X_K.*

Proof. The existence of M/K follows from (5.1.2 and 5.2.2 (2)). Since M/K is separable (5.2.2 (1)) and any conjugate field of M/K in $\overline{K}$ induces an étale covering of X, the field extension M/K is Galois. The last statement follows from (3.2.6) and Zariski–Nagata's purity [11, X, 3.1]. □

Definition 5.2.4. The field extension M/K in $\overline{K}$ given by (5.2.3) is called the *étale part* (*of the resolutions of the multiple fibers of regular C-models*) of X_K. Assume that $X(K') \neq \emptyset$. By (5.2.5) below, the relation $M \subset K'$ holds. The field extension K'/M is called the *non-étale part* of X_K in K'.

Lemma 5.2.5. *Take the étale parts M/K and M'/K' of X_K and $X'_{K'}$ in $\overline{K}$, respectively* (5.2.4). *Then $K'M \subset M'$. In particular, if $X(K') \neq \emptyset$, then $M \subset K'$.*

Proof. Let us show the first statement. Take a desingularization $f'_0 \colon Y \to X'$ of X'. By definition, the extension M/K induces a finite étale covering τ of X. Since M/K is separable (5.2.3) and the base change of τ via $\pi_X \circ f'_0$ is a finite étale covering of Y, the relation $K'M \subset M'$ holds. Let us show the last statement. Assume that $X(K') \neq \emptyset$. Then $K' = M'$ (5.1.2). Thus, the first statement shows that $M \subset K'$. □

5.3. **Type $_m\mathrm{I}_n$ $(n>0)$.**

Proposition 5.3.1 ([18, §8]; use 5.2.5 for the proof of (1)). *Assume that (X,C,f) is a minimal elliptic fibration with special fiber of type $_m\mathrm{I}_n$ $(n>0)$. Take the étale part M/K of X_K* (5.2.4). *Then the following statements hold.* (1) *The field extension M/K is cyclic of degree m and minimum among all finite field extensions L/K in $\overline{K}$ satisfying $X(L)\neq\emptyset$.* (2) *If $K'\subset M$, then (X',C',f') is a minimal elliptic fibration with special fiber of type $_{m/d}\mathrm{I}_{dn}$.*

5.4. **Type $_m\mathrm{I}_0$.**

Lemma 5.4.1. *Assume that (X,C,f) is a minimal elliptic fibration. Let (Y,C,h) be a normal pre-elliptic fibration satisfying the following conditions.* (1) *There exists a K-isomorphism between the generic fibers of f and h.* (2) *For any irreducible component D of the special fiber of h with the reduced structure, there exist an elliptic curve E_D over k and a dominant k-morphism $D\to E_D$. Then the K-isomorphism in Condition* (1) *uniquely extends to the C-isomorphism $X\to Y$.*

Proof. Take the minimal desingularization $h_0\colon \widehat{Y}\to Y$ of Y (3.2.2). Put $\widehat{h}:=h\circ h_0$. Then the triple $(\widehat{Y},C,\widehat{h})$ is an elliptic fibration. By $\widehat{Y}_k$ we denote the special fiber of $\widehat{h}$. Take the normalization Z of $\widehat{Y}_k$. The morphism h_0 induces a morphism $\tau\colon Z\to Y$. The classification of singular fibers implies that the connected components of Z consist of (a) a positive number of projective lines over k or (b) a non-negative number of projective lines over k and one elliptic curve over k. Thus, Condition (2) implies that the following statements hold: (i) there exists the unique connected component D of Z that is k-isomorphic to an elliptic curve over k; (ii) τ maps the other connected components of Z to closed points on Y. Therefore, the minimality of the desingularization h_0 implies that $Z=D$, which shows that $(\widehat{Y},C,\widehat{h})$ is a minimal regular C-model of the generic fiber of h. Since $\widehat{Y}_k$ is irreducible, the morphism h_0 is quasi-finite. Since h_0 is a quasi-finite proper birational morphism between normal integral schemes, Zariski's main theorem [8, 8.12.10] implies that h_0 is an isomorphism. Thus, the uniqueness of the minimal regular C-model (3.2.5) concludes the proof. □

Proposition 5.4.2. *Assume that (X,C,f) is a minimal elliptic fibration with special fiber of type $_m\mathrm{I}_0$. Then (X',C',f') is a minimal elliptic fibration.*

Proof. By $X_{k,\mathrm{red}}$ and $X'_{k,\mathrm{red}}$ we denote the reductions of the special fibers of f and f', respectively. Then $X_{k,\mathrm{red}}$ is k-isomorphic to an elliptic curve E_0 over k. Since π_X induces a finite k-morphism $X'_{k,\mathrm{red}}\to X_{k,\mathrm{red}}$, any irreducible component of $X'_{k,\mathrm{red}}$ admits a dominant k-morphism to the elliptic curve E_0 (3.1.2 (5)). Thus, the proposition follows from (5.4.1). □

6. Invariants of Pre-Elliptic Fibrations

6.1. **Notation.** We use the notation introduced in §4.4. Assume that the finite field extension K'/K is separable. Suppose that $\widetilde{X}=X$, $\widetilde{E}=E$, $\widetilde{X}'=X'$, and $\widetilde{E}'=E'$. For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, by $\psi_{X'/X,\mathcal{F}}$ we denote the composite of the canonical $\mathcal{O}_{C'}$-module homomorphisms $\pi_C^*R^1f_*\mathcal{F}\cong R^1\pi_{2*}\pi_1^*\mathcal{F}\to R^1\pi_{2*}\pi_{0*}\pi_0^*\pi_1^*\mathcal{F}\to R^1f'_*\pi_X^*\mathcal{F}$. For a quasi-coherent $\mathcal{O}_{X_K}$-module $\mathcal{F}_K$, the flat base change theorem for cohomology gives a canonical $\mathcal{O}_{C'_{K'}}$-module isomorphism

$\psi_{X'_{K'}/X_K,\mathcal{F}_K}\colon \pi^*_{C_K}R^1f_{K*}\mathcal{F}_K \to R^1f'_{K'*}\pi^*_{X_K}\mathcal{F}_K$. If $\mathcal{F}_K = \iota_X^*\mathcal{F}$, then the pull-back $(\iota')^*\psi_{X'/X,\mathcal{F}}$ induces $\psi_{X'_{K'}/X_K,\mathcal{F}_K}$. Put $\psi_{X'/X} := \psi_{X'/X,\mathcal{O}_X}$ and $\psi_{X'_{K'}/X_K} := \psi_{X'_{K'}/X_K,\mathcal{O}_{X_K}}$.

6.2. **Extension of Base Change Compatibility for Trace Maps.** The morphisms f and f' are CM (3.1.2 (6)). Since π_C and f' are CM, the morphism π is CM. Note that the finite morphism π_X is not necessarily flat. We refer to §§2.3–2.4 for relative dualizing sheaves (see also 3.3.8). Put $\omega := \pi_X^*\omega_f \otimes_{\mathcal{O}_{X'}} \omega_{\pi_X}$, $\omega' := \omega_{f'} \otimes_{\mathcal{O}_{X'}} (f')^*\omega_{\pi_C}$, $\omega_{K'} := (\iota'_{X'})^*\omega$, and $\omega'_{K'} := (\iota'_{X'})^*\omega'$. Take the canonical $\mathcal{O}_{X'}$-module isomorphism $\xi_{f',\pi_C}\colon \omega' \to \omega_\pi$ and the canonical $\mathcal{O}_{X'_{K'}}$-module isomorphism $\xi_{\pi_{X_K},f_K}\colon \omega_{K'} \to \omega_{\pi_K}$ [4, 4.3.3].

Lemma 6.2.1. *There exists the minimum closed subset Z of X such that both $U := X \setminus E$ and $V := \pi_X^{-1}(U)$ are regular. The closed subsets $\pi_X^{-1}(Z)$ and Z are of codimension at least two. The restriction of π to V is l.c.i. Further, the $\mathcal{O}_{X'_{K'}}$-module isomorphism $\xi_{\pi_{X_K},f_K}$ uniquely extends to the $\mathcal{O}_{X'}$-module homomorphism $\xi_{\pi_X,f}\colon \omega \to \omega_\pi$, which is an isomorphism on V.*

Proof. Since π_X is finite and X and X' are excellent and normal, the first three statements follow from [19, 9.4 (ii), 15.1, 21.2 (ii), and 23.1]. By $\iota_V\colon V \to X'$ we denote the inclusion morphism. Then there exists an $\mathcal{O}_V$-module isomorphism $\xi_V\colon \iota_V^*\omega \to \iota_V^*\omega_\pi$ that is an extension of $\xi_{\pi_{X_K},f_K}$ to V [4, 4.3.3]. Since ω_π is reflexive (2.4.1), the lemma follows from (2.1.4). □

Put $\xi := \xi_{f',\pi_C}^{-1} \circ \xi_{\pi_X,f}\colon \omega \to \omega'$. Take the injective $\mathcal{O}_{C'}$-module homomorphism $\lambda_{\pi_C}\colon \mathcal{O}_{C'} \to \omega_{\pi_C}$ and the injective $\mathcal{O}_{X'}$-module homomorphism $\lambda_{\pi_X}\colon \mathcal{O}_{X'} \to \omega_{\pi_X}$ introduced in (2.3.8). By $\pi_C^\flat\colon \pi_C^*\mathcal{O}_C \to \mathcal{O}_{C'}$ we denote the canonical $\mathcal{O}_{C'}$-module isomorphism. The above homomorphisms give the following diagram of $\mathcal{O}_{C'}$-modules and $\mathcal{O}_{C'}$-module homomorphisms:

$$(**)\qquad \begin{array}{ccccc}
R^1f'_*\pi_X^*\omega_f & \xleftarrow{\psi_{X'/X,\omega_f}} & \pi_C^*R^1f_*\omega_f & \xrightarrow{\pi_C^*\operatorname{tr}_f} & \pi_C^*\mathcal{O}_C \\
\Big\downarrow {\scriptstyle R^1f'_*(\pi_X^*\omega_f\otimes\lambda_{\pi_X})} & & & & \\
R^1f'_*\omega & & & & {\scriptstyle \pi_C^\flat}\Big\downarrow \cong \\
\Big\downarrow {\scriptstyle R^1f'_*\xi} & & & & \\
R^1f'_*\omega' & \xleftarrow[R^1f'_*(\omega_{f'}\otimes(f')^*\lambda_{\pi_C})]{} & R^1f'_*\omega_{f'} & \xrightarrow[\operatorname{tr}_{f'}]{} & \mathcal{O}_{C'}.
\end{array}$$

The pull-back of any arrow via ι' is an isomorphism between line bundles.

Lemma 6.2.2. *Diagram* (**) *is commutative modulo torsion* (*i.e., the diagram obtained by the pull-back of the modules and homomorphisms in Diagram* (**) *via ι' is commutative*).

Proof. Since π_{C_K} and f_K are smooth, we obtain the $\mathcal{O}_{X'_{K'}}$-module isomorphisms $\beta_{f_K,\pi_{C_K}}\colon \pi_{X_K}^*\omega_{f_K} \to \omega_{f'_{K'}}$ and $\beta_{\pi_{C_K},f_K}\colon (f'_{K'})^*\omega_{\pi_{C_K}} \to \omega_{\pi_{X_K}}$ by the base change property of the sheaves of relative differentials. The pull-back $(\iota'_{X'})^*\xi$ induces an $\mathcal{O}_{X'_{K'}}$-module isomorphism $\xi_{K'}\colon \omega_{K'} \to \omega'_{K'}$. Further, the equality $\xi_{K'} =$

$\beta_{f_K,\pi_{C_K}} \otimes \beta^{-1}_{\pi_{C_K},f_K}$ holds. The pull-backs $(\iota')^*\lambda_{\pi_C}$ and $(\iota'_{X'})^*\lambda_{\pi_X}$ induce the canonical isomorphisms $\lambda_{\pi_{C_K}}$ and $\lambda_{\pi_{X_K}}$, respectively (2.3.9). Then the diagram

$$\begin{array}{ccc} \mathcal{O}_{X'_{K'}} & \xrightarrow[\cong]{\lambda_{\pi_{X_K}}} & \omega_{\pi_{X_K}} \\ {\scriptstyle (f'_{K'})^\flat}\uparrow\cong & & {\scriptstyle \beta_{\pi_{C_K},f_K}}\uparrow\cong \\ (f'_{K'})^*\mathcal{O}_{C'_{K'}} & \xrightarrow[\cong]{(f'_{K'})^*\lambda_{\pi_{C_K}}} & (f'_{K'})^*\omega_{\pi_{C_K}} \end{array}$$

is commutative (2.3.5 (2)) where $(f'_{K'})^\flat$ is the canonical $\mathcal{O}_{C'_{K'}}$-module isomorphism. Thus, tensoring the inverses of the vertical arrows with $\beta_{f_K,\pi_{C_K}}$, we obtain the commutative diagram

$$\begin{array}{ccc} \pi^*_{X_K}\omega_{f_K} & \xrightarrow[\cong]{\pi^*_{X_K}\omega_{f_K}\otimes\lambda_{\pi_{X_K}}} & \omega_{K'} \\ {\scriptstyle \beta_{f_K,\pi_{C_K}}}\downarrow\cong & & {\scriptstyle \xi_{K'}}\downarrow\cong \\ \omega_{f'_{K'}} & \xrightarrow[\cong]{\omega_{f'_{K'}}\otimes(f'_{K'})^*\lambda_{\pi_{C_K}}} & \omega'_{K'}. \end{array}$$

Therefore, pulling back the modules and homomorphisms in Diagram (∗∗) via ι', we obtain the diagram

$$\begin{array}{ccccc} R^1 f'_{K'*}\pi^*_{X_K}\omega_{f_K} & \xleftarrow[\cong]{\psi_{X'_{K'}/X_K,\omega_{f_K}}} & \pi^*_{C_K} R^1 f_{K*}\omega_{f_K} & \xrightarrow[\cong]{\pi^*_{C_K}\operatorname{tr}_{f_K}} & \pi^*_{C_K}\mathcal{O}_{C_K} \\ & \searrow^{\cong}_{R^1 f'_{K'*}\beta_{f_K,\pi_{C_K}}} & & & {\scriptstyle \pi^\flat_{C_K}}\downarrow\cong \\ & & R^1 f'_{K'*}\omega_{f'_{K'}} & \xrightarrow[\cong]{\operatorname{tr}_{f'_{K'}}} & \mathcal{O}_{C'_{K'}}. \end{array}$$

The base change compatibility for trace maps [4, 3.6.5] shows that the diagram is commutative. Therefore, Diagram (∗∗) is commutative modulo torsion. □

6.3. **Invariants and Base Change.** We use the notation $L_R(\bullet)$ introduced in (2.2.1). Let (Y, C, h) be a normal pre-elliptic fibration. We denote the canonical $\mathcal{O}_Y$-module homomorphism by $\gamma_h \colon h^*h_*\omega_h \to \omega_h$ (see 3.3.8).

Lemma 6.3.1. *For $i = 1$ and 2, let $\mathcal{F}_i$ be a coherent $\mathcal{O}_Y$-module whose restriction to the generic fiber is a trivial line bundle. Assume that $\mathcal{F}_2$ is reflexive. Then the R-module $\operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{F}_1, \mathcal{F}_2)$ is free of rank one.*

Proof. The lemma follows from (2.1.2 (2) and 2.1.5). □

Lemma 6.3.2. *For $i = 1$ and 2, let $\mathcal{F}_i$ be a coherent $\mathcal{O}_Y$-module such that $\iota^*R^1h_*\mathcal{F}_i$ is a line bundle. Then the R-module $\operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{F}_i, \omega_h)$ is free of rank one for $i = 1$ and 2. Let $\kappa\colon \mathcal{F}_1 \to \mathcal{F}_2$ be an $\mathcal{O}_Y$-module homomorphism. Then $L_R(R^1h_*\kappa) = L_R(\operatorname{Hom}_{\mathcal{O}_Y}(\kappa, \omega_h))$.*

Proof. The two pairings of $\mathcal{O}_C$-modules in the diagram

$$\begin{array}{ccc} h_*\,\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}_1,\omega_h)\otimes_{\mathcal{O}_C} R^1h_*\mathcal{F}_1 & & \\ {\scriptstyle h_*\,\mathcal{H}om_{\mathcal{O}_Y}(\kappa,\omega_h)}\uparrow \qquad {\scriptstyle R^1h_*\kappa}\downarrow & \searrow & \\ & & R^1h_*\omega_h \\ & \nearrow & \\ h_*\,\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}_2,\omega_h)\otimes_{\mathcal{O}_C} R^1h_*\mathcal{F}_2 & & \end{array}$$

are compatible with $h_* \mathcal{H}om_{\mathcal{O}_Y}(\kappa, \omega_h)$ and $R^1 h_* \kappa$. Thus, the trace map tr_h induces a commutative diagram

$$\begin{array}{ccc} h_* \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}_1, \omega_h) & \longrightarrow & (R^1 h_* \mathcal{F}_1)^\vee \\ {\scriptstyle h_* \mathcal{H}om_{\mathcal{O}_Y}(\kappa,\omega_h)} \uparrow & & \uparrow {\scriptstyle (R^1 h_* \kappa)^\vee} \\ h_* \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}_2, \omega_h) & \longrightarrow & (R^1 h_* \mathcal{F}_2)^\vee. \end{array}$$

The Grothendieck duality shows that the horizontal arrows are isomorphisms. By assumption, any coherent $\mathcal{O}_C$-module in the above diagram is a line bundle. Thus, the above commutative diagram shows the lemma. □

Lemma 6.3.3. *Let $\mathcal{L}$ be a trivial line bundle on Y. Then the R-module homomorphism $\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{L}, \gamma_h)$ is bijective.*

Proof. The injectivity follows from that of γ_h (3.3.8 (1) and (3)). Thus, we have only to show the surjectivity. By $\phi\colon h^* h_* \mathcal{L} \to \mathcal{L}$ we denote the canonical $\mathcal{O}_Y$-module homomorphism. Take an $\mathcal{O}_Y$-module homomorphism $\kappa\colon \mathcal{L} \to \omega_h$. Put $\psi := h^* h_* \kappa\colon h^* h_* \mathcal{L} \to h^* h_* \omega_h$. Then $\kappa \circ \phi = \gamma_h \circ \psi$. Since $\mathcal{L} \cong \mathcal{O}_Y$ and the homomorphism $\mathcal{O}_C \to h_* \mathcal{O}_Y$ associated to h is an isomorphism, the $\mathcal{O}_Y$-module homomorphism ϕ is an isomorphism. Thus, the equality $\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{L}, \gamma_h)(\psi \circ \phi^{-1}) = \kappa$ concludes the proof. □

Lemma 6.3.4. *For $i = 1$ and 2, let $\mathcal{L}_i$ be a trivial line bundle on Y. For $i = 1$ and 2, we fix a trivialization $\kappa_i\colon \mathcal{O}_Y \to \mathcal{L}_i$ of $\mathcal{L}_i$. Let $\kappa\colon \mathcal{L}_1 \to \mathcal{L}_2$ be an $\mathcal{O}_Y$-module homomorphism. By κ_0 we denote the $\mathcal{O}_Y$-module endomorphism $\kappa_2^{-1} \circ \kappa \circ \kappa_1$ of $\mathcal{O}_Y$. Then there exists the unique element r of R such that κ_0 is given by the multiplication of $\mathcal{O}_Y$ by r. Moreover, the equality $L_R(R^1 h_* \kappa) = \mathrm{length}_R\, R/rR$ holds.*

Proof. The diagram of R-modules and R-module homomorphisms

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{L}_2, h^* h_* \omega_h) & \xrightarrow{\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{L}_2, \gamma_h)} & \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{L}_2, \omega_h) \\ {\scriptstyle \phi := \mathrm{Hom}_{\mathcal{O}_Y}(\kappa, h^* h_* \omega_h)} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}_{\mathcal{O}_Y}(\kappa, \omega_h) =: \psi} \\ \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{L}_1, h^* h_* \omega_h) & \xrightarrow{\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{L}_1, \gamma_h)} & \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{L}_1, \omega_h) \end{array}$$

is commutative. Since $h^* h_* \omega_h \cong \mathcal{O}_Y$ and the horizontal arrows are bijective (6.3.3), the equalities $L_R(h_*(\kappa_0^\vee)) = L_R(\phi) = L_R(\psi) = L_R(R^1 h_* \kappa)$ hold (6.3.2). Thus, the lemma follows from the fact that the homomorphism $\mathcal{O}_C \to h_* \mathcal{O}_Y$ associated to h is an isomorphism. □

Lemma 6.3.5. $L_R(R^1 h_* \gamma_h) = 0$.

Proof. Put $L := \mathrm{Hom}_{\mathcal{O}_Y}(\omega_h, \omega_h)$, $M := \mathrm{Hom}_{\mathcal{O}_C}(h_* \omega_h, h_* \omega_h)$, and $N := \mathrm{Hom}_{\mathcal{O}_Y}(h^* h_* \omega_h, \omega_h)$. Then the R-modules L, M, and N are free of rank one (6.3.1). The image of the identity of $h_* \omega_h$ under the canonical R-module isomorphism $M \to N$ is equal to γ_h. Since the identity of $h_* \omega_h$ generates M, the image γ_h generates N. Since the image of the identity of ω_h under $\mathrm{Hom}_{\mathcal{O}_Y}(\gamma_h, \omega_h)\colon L \to N$ is equal to γ_h, the R-module homomorphism $\mathrm{Hom}_{\mathcal{O}_Y}(\gamma_h, \omega_h)$ is surjective, which implies that $\mathrm{Hom}_{\mathcal{O}_Y}(\gamma_h, \omega_h)^{\vee\vee}$ is surjective. Thus, the lemma follows from (6.3.2). □

Lemma 6.3.6. $L_{R'}(\psi_{X'/X, \omega_f}) = L_{R'}(\psi_{X'/X}) + L_{R'}(R^1 f'_* \pi_X^* \gamma_f)$.

Proof. The diagram of $\mathcal{O}_{C'}$-modules and $\mathcal{O}_{C'}$-module homomorphisms

$$\begin{array}{ccc} \pi_C^* R^1 f_* f^* f_* \omega_f & \xrightarrow{\pi_C^* R^1 f_* \gamma_f} & \pi_C^* R^1 f_* \omega_f \\ {\scriptstyle \psi_{X'/X, f^* f_* \omega_f}} \downarrow & & \downarrow {\scriptstyle \psi_{X'/X, \omega_f}} \\ R^1 f'_* \pi_X^* f^* f_* \omega_f & \xrightarrow{R^1 f'_* \pi_X^* \gamma_f} & R^1 f'_* \pi_X^* \omega_f \end{array}$$

is commutative. Since $L_{R'}(\psi_{X'/X, f^* f_* \omega_f}) = L_{R'}(\psi_{X'/X})$ by the projection formula, the lemma follows from (2.2.4 (1) and 6.3.5). $\square$

Theorem 6.3.7. *The Weil divisor $D_{X'/X} + \pi_X^* D_f - D_{f'}$ is equal to the pull-back of an effective divisor D on C' via f' (see 2.2.5 for π_X^*). Moreover, the degree of D is equal to $d_{C'/C} - L_{R'}(\psi_{X'/X})$.*

Proof. We define coherent $\mathcal{O}_{X'}$-modules $\mathcal{F}_i$ in the following way:

$\mathcal{F}_1 := \pi_X^* f^* f_* \omega_f$; $\mathcal{F}_2 := \pi_X^* \omega_f$;
$\mathcal{F}_3 := \omega = \pi_X^* \omega_f \otimes_{\mathcal{O}_{X'}} \omega_{\pi_X}$; $\mathcal{F}_4 := \omega' = \omega_{f'} \otimes_{\mathcal{O}_{X'}} (f')^* \omega_{\pi_C}$;
$\mathcal{F}_5 := (f')^* f'_* \omega_{f'} \otimes_{\mathcal{O}_{X'}} (f')^* \omega_{\pi_C}$; $\mathcal{F}_6 := \omega_{f'}$.

For any i, the pull-back $(\iota'_{X'})^* \mathcal{F}_i$ is a trivial line bundle. We define $\mathcal{O}_{X'}$-module homomorphisms $\kappa_{ij} \colon \mathcal{F}_i \to \mathcal{F}_j$ in the following way:

$\kappa_{12} := \pi_X^* \gamma_f$; $\kappa_{23} := \pi_X^* \omega_f \otimes \lambda_{\pi_X}$;
$\kappa_{34} := \xi$; $\kappa_{24} := \kappa_{34} \circ \kappa_{23}$;
$\kappa_{14} := \kappa_{24} \circ \kappa_{12}$; $\kappa_{54} := \gamma_{f'} \otimes (f')^* \omega_{\pi_C}$;
$\kappa_{64} := \omega_{f'} \otimes (f')^* \lambda_{\pi_C}$.

Take the unique $\mathcal{O}_{X'}$-module homomorphism $\kappa_{15} \colon \mathcal{F}_1 \to \mathcal{F}_5$ satisfying $\kappa_{14} = \kappa_{54} \circ \kappa_{15}$ (6.3.3). By definition, the equality $\kappa_{jk} \circ \kappa_{ij} = \kappa_{ik}$ holds for any κ_{ij} and any κ_{jk}. Thus, the following equalities hold:

$$\begin{array}{ll} D_{X'/X} + \pi_X^* D_f - D_{f'} & \\ = D(\lambda_{\pi_X}) + \pi_X^* D(\gamma_f) - D(\gamma_{f'}) & \text{(by definition)} \\ = D(\kappa_{23}) + \pi_X^* D(\gamma_f) - D(\kappa_{54}) & \text{(2.2.4 (2))} \\ = D(\kappa_{23}) + D(\kappa_{12}) - D(\kappa_{54}) & \text{(2.2.7)} \\ = D(\kappa_{15}) - D(\kappa_{34}) & \text{(2.2.4 (1))} \\ = D(\kappa_{15}) & \text{(6.2.1).} \end{array}$$

Since both $\mathcal{F}_1$ and $\mathcal{F}_5$ are trivial line bundles, the Weil divisor $D(\kappa_{15})$ is equal to the pull-back of the divisor on C' of degree $L_{R'}(R^1 f'_* \kappa_{15})$ via f' (6.3.4). Thus, we have only to show that the equality

$$(***) \qquad d_{C'/C} - L_{R'}(\psi_{X'/X}) = L_{R'}(R^1 f'_* \kappa_{15})$$

holds. The projection formula gives $d_{C'/C} = L_{R'}(R^1 f'_* \kappa_{64})$. The diagram

$$\begin{array}{ccccc} R^1 f'_* \mathcal{F}_2 & \xleftarrow{\psi_{X'/X, \omega_f}} & \pi_C^* R^1 f_* \omega_f & \xrightarrow{\pi_C^* \operatorname{tr}_f} & \pi_C^* \mathcal{O}_C \\ \downarrow {\scriptstyle R^1 f'_* \kappa_{24}} & & & & {\scriptstyle \pi_C^\flat} \downarrow {\scriptstyle \cong} \\ R^1 f'_* \mathcal{F}_4 & \xleftarrow{R^1 f'_* \kappa_{64}} & R^1 f'_* \mathcal{F}_6 & \xrightarrow{\operatorname{tr}_{f'}} & \mathcal{O}_{C'} \end{array}$$

is commutative module torsion (6.2.2). Since $(\pi_C^* \operatorname{tr}_f)^{\vee\vee}$ and $\operatorname{tr}_{f'}^{\vee\vee}$ are isomorphisms [4, 4.4.5], the equality $d_{C'/C} - L_{R'}(\psi_{X'/X}) = L_{R'}(R^1 f'_* \kappa_{12}) + L_{R'}(R^1 f'_* \kappa_{24})$ holds (2.2.4 and 6.3.6). The following equalities hold:

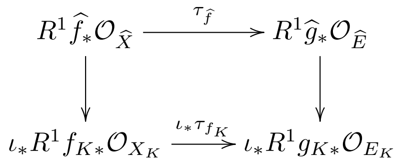
$$\begin{aligned}
&L_{R'}(R^1 f'_* \kappa_{12}) + L_{R'}(R^1 f'_* \kappa_{24}) && \\
&= L_{R'}(R^1 f'_* \kappa_{15}) + L_{R'}(R^1 f'_* \kappa_{54}) && (2.2.4\ (1)) \\
&= L_{R'}(R^1 f'_* \kappa_{15}) + L_{R'}(R^1 f'_* \gamma_{f'}) && (\text{the projection formula}) \\
&= L_{R'}(R^1 f'_* \kappa_{15}) && (6.3.5).
\end{aligned}$$

Therefore, Equality $(***)$ holds, which concludes the proof. □

Definition 6.3.8. By $e_{X'/X}$ we denote the degree of the divisor D on C' satisfying $(f')^* D = D_{X'/X} + \pi_X^* D_f - D_{f'}$ (6.3.7).

Proposition 6.3.9. $L_{R'}(\psi_{X'/X}) - L_{R'}(\psi_{E'/E}) = d(l_f - l_g) - (l_{f'} - l_{g'})$.

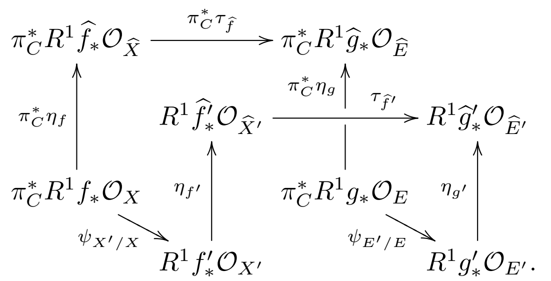
Proof. Take the minimal regular models $(\widehat{X}, C, \widehat{f})$, $(\widehat{E}, C, \widehat{g})$, $(\widehat{X}', C', \widehat{f}')$, and $(\widehat{E}', C', \widehat{g}')$ of the generic fibers X_K, E_K, $X'_{K'}$, and $E'_{K'}$, respectively. There exist an $\mathcal{O}_C$-module homomorphism $\tau_{\widehat{f}}\colon R^1 \widehat{f}_* \mathcal{O}_{\widehat{X}} \to R^1 \widehat{g}_* \mathcal{O}_{\widehat{E}}$ and an $\mathcal{O}_{C_K}$-module isomorphism $\tau_{f_K}\colon R^1 f_{K*} \mathcal{O}_{X_K} \to R^1 g_{K*} \mathcal{O}_{E_K}$ satisfying the following conditions [18, 3.8]: (1) the diagram



is commutative where the vertical arrows are induced by the base change via ι; (2) the formation of τ_{f_K} commutes with the base change via any field extension. In the same way, we define $\tau_{\widehat{f}'}$ and $\tau_{f'_{K'}}$.

Take η_f, η_g, $\eta_{f'}$, and $\eta_{g'}$ introduced in (3.3.5). We have the following diagram of $\mathcal{O}_{C'}$-modules and $\mathcal{O}_{C'}$-module homomorphisms:

$(****)$



The pull-back of any arrow via ι' is an isomorphism between line bundles. By the vertical arrows, we identify the pull-backs of the four top modules with the pull-backs of the four bottom modules, respectively. Then the other arrows induce a commutative diagram of $\mathcal{O}_{C'_{K'}}$-modules and $\mathcal{O}_{C'_{K'}}$-module isomorphisms. Thus, Diagram $(****)$ gives the equality

$$\begin{aligned}
L_{R'}(\psi_{X'/X}) - L_{R'}(\psi_{E'/E}) = d\big(&L_R(\tau_{\widehat{f}}) + L_R(\eta_f) - L_R(\eta_g)\big) \\
&- \big(L_{R'}(\tau_{\widehat{f}'}) + L_{R'}(\eta_{f'}) - L_{R'}(\eta_{g'})\big).
\end{aligned}$$

Put $l_{\widehat{f}} := l_{\widehat{f},s}$, $l_{\widehat{g}} := l_{\widehat{g},s}$, $l_{\widehat{f}'} := l_{\widehat{f}',s'}$, and $l_{\widehat{g}'} := l_{\widehat{g}',s'}$ (3.3.6). The equality

$$d(l_f - l_g) - (l_{f'} - l_{g'}) = d\big(l_{\widehat{f}} + L_R(\eta_f) - l_{\widehat{g}} - L_R(\eta_g)\big) \\ - \big(l_{\widehat{f}'} + L_{R'}(\eta_{f'}) - l_{\widehat{g}'} - L_{R'}(\eta_{g'})\big)$$

holds by definition. Note that $l_{\widehat{g}} = 0$ and $l_{\widehat{g}'} = 0$ (Proposition 1 in [20]). Since $l_{\widehat{f}} = L_R(\tau_{\widehat{f}})$ and $l_{\widehat{f}'} = L_{R'}(\tau_{\widehat{f}'})$ [18, 3.8], the desired equality holds. □

Corollary 6.3.10 (use 6.3.7 and 6.3.9)**.** *Take $e_{X'/X}$ and $e_{E'/E}$ introduced in* (6.3.8)*. Then $e_{X'/X} - e_{E'/E} + d(l_f - l_g) - (l_{f'} - l_{g'}) = 0$.*

7. Invariants of Elliptic Fibrations

7.1. **Notation.** We use the notation introduced in §4.4. Assume that the finite field extension K'/K is separable. Suppose that (X, C, f), (E, C, g), (X', C', f'), and (E', C', g') are minimal elliptic fibrations. By X_k, X'_k, $\widetilde{X}_k$, and $\widetilde{X}'_k$ we denote the special fibers of f, f', $\widetilde{f}$, and $\widetilde{f}'$, respectively. Put $d' := dm'/m$. Then d' is an integer (5.1.2). Put $a := a_{f,s}$ and $a' := a_{f',s'}$ (4.1.5). We define $\mathbb{Q}$-Cartier divisors by $V_{\widetilde{f}} := \widetilde{X}_k/m$ and $V_{\widetilde{f}'} := \widetilde{X}'_k/m'$. Put $F_{\widetilde{X}'/\widetilde{X}} := D_{\widetilde{X}'/\widetilde{X}} + \pi^*_{\widetilde{X}}(D_{\widetilde{f}} - aV_{\widetilde{f}}) - (D_{\widetilde{f}'} - a'V_{\widetilde{f}'})$. Take $e_{\widetilde{X}'/\widetilde{X}}$ introduced in (6.3.8). Put $d_{\widetilde{X}'/\widetilde{X}} := m'e_{\widetilde{X}'/\widetilde{X}} - d'a + a'$. By definition, the equality $d_{\widetilde{X}'/\widetilde{X}}V_{\widetilde{f}'} = F_{\widetilde{X}'/\widetilde{X}}$ holds. In the same way, we define $e_{\widetilde{E}'/\widetilde{E}}$ and $d_{\widetilde{E}'/\widetilde{E}}$.

7.2. Invariants and Base Change.

Lemma 7.2.1. *Assume that $m = 1$. Then $d_{\widetilde{X}'/\widetilde{X}} = e_{\widetilde{X}'/\widetilde{X}}$. Further, if $\widetilde{X}' = \widetilde{X} \times_C C'$, then $d_{\widetilde{X}'/\widetilde{X}} = d_{C'/C}$.*

Proof. Since $m' = 1$ (5.1.2), the equalities $a = a' = 0$ hold (4.1.5), which gives the first equality. Suppose that $\widetilde{X}' = \widetilde{X} \times_C C'$. The flat base change theorem for cohomology gives $L_{R'}(\psi_{\widetilde{X}'/\widetilde{X}}) = 0$ (see 2.2.1 and §6.1 for $L_{R'}(\bullet)$ and $\psi_{\widetilde{X}'/\widetilde{X}}$, respectively). Thus, the last equality follows from (6.3.7). □

Theorem 7.2.2. *Put $\widetilde{l} := l_{\widetilde{f}} - l_{\widetilde{g}}$ and $\widetilde{l}' := l_{\widetilde{f}'} - l_{\widetilde{g}'}$. Then $d'(m\widetilde{l} + a) = m'\widetilde{l}' + a' + m'd_{\widetilde{E}'/\widetilde{E}} - d_{\widetilde{X}'/\widetilde{X}}$. Moreover, if any closed point on $\widetilde{X}$, $\widetilde{E}$, $\widetilde{X}'$, and $\widetilde{E}'$ is rational* (*see* 3.2.8)*, then $\widetilde{l} = l$ and $\widetilde{l}' = l'$.*

Proof. The equality $d_{\widetilde{E}'/\widetilde{E}} = e_{\widetilde{E}'/\widetilde{E}}$ holds (7.2.1). Thus, the equality $d_{\widetilde{X}'/\widetilde{X}} + d'a - a' + m'(-d_{\widetilde{E}'/\widetilde{E}} + d\widetilde{l} - \widetilde{l}') = 0$ holds (6.3.10), which proves the first statement. Since $l_g = 0$ and $l_{g'} = 0$ (Proposition 1 in [20]), the last statement follows from (3.3.7). □

Proposition 7.2.3. *Assume that $\widetilde{X} = X$ and $\widetilde{X}' = X'$. Suppose that π_X is étale. Then $d' = 1$ and $d_{X'/X} = 0$.*

Proof. Since π_X is étale, the equalities $D_{X'/X} = 0$ and $d' = 1$ hold (2.3.9 and 5.1.2). Since $D_f = aV_f$ and $D_{f'} = a'V_{f'}$, the equality $F_{X'/X} = D_{X'/X}$ holds, which gives $d_{X'/X} = 0$. □

7.3. **Reduction to Type ${}_m\mathrm{I}_n$.** We define integers $u(T)$ and $v(T)$ by Table 1 in §1.2.

Lemma 7.3.1. *Assume that $p \nmid u(T)$. Then the following statements hold.* (1) *If $d \mid u(T)$, then $m' = m$.* (2) *Suppose that $u(T) \mid d$. If $T = \mathrm{I}_n$ $(n \geq 0)$ or I_n^* $(n \geq 0)$, then $T' = \mathrm{I}_{dn}$. Otherwise, the equality $T' = \mathrm{I}_0$ holds.*

Proof. Let us show Statement (1). We may assume that $u(T) > 1$. Then T is additive, which implies that m is a power of p [18, 7.4]. Since $p \nmid d$ by assumption, the equality $m' = m$ holds (5.1.2). Let us show Statement (2). We have only to show the case where $X = E$ and $X' = E'$ [18, 6.6]. Since $p \nmid u(T)$ and $u(T) \mid d$ by assumption, Statement (2) follows from Equalities and Tables in [25, §2 and §4]. □

When $p \nmid d$, we denote the Galois group of the cyclic extension K'/K by G. The group G equivariantly acts on X'/C' (3.2.9).

Lemma 7.3.2. *Assume that $p \nmid u(T)$ and $d = u(T) > 1$. Then the fixed locus of the action of G on X' does not intersect the singular locus of the reduction of X'_k. Further, the actions of the non-trivial stabilizer subgroups of closed points on X' are given by Table 2.*

I_n^*	II	II^*
$2, 2, 2, 2$	$2, 3^+, 6^+$	$2, 3^-, 6^-$

III	III^*	IV	IV^*
$2, 4^+, 4^+$	$2, 4^-, 4^-$	$3^+, 3^+, 3^+$	$3^-, 3^-, 3^-$

TABLE 2. The actions of the non-trivial stabilizer subgroups of closed points on X'. The actions in the table are given for each orbit by $2(x_1, x_2) = (-x_1, -x_2)$, $n^+(x_1, x_2) = (\zeta_n x_1, \zeta_n x_2)$, and $n^-(x_1, x_2) = (\zeta_n x_1, \zeta_n^{-1} x_2)$ where $\{x_1, x_2\}$ is a system of local parameters and ζ_n is a primitive n-th root of unity.

Proof. Put $Y := X'/G$. By Y_k we denote the special fiber of Y/C. We may determine T' from T (7.3.1 (2)).

First, let us show the case $T = \mathrm{I}_n^*$ $(n > 0)$. In that case, the equality $T' = \mathrm{I}_{2n}$ holds. Take the generator σ of the group G of order two. Suppose that σ fixes a singular point on the reduction of X'_k. Then any irreducible component of X'_k is stable under the action of G (A.4), which implies that Y is regular (A.5 and A.6 (1)). Since Y_k contains a cycle of projective line, the special fiber X_k contains a cycle of projective lines. This contradicts the assumption on T. Thus, the element σ fixes no singular point on the reduction of X'_k. Suppose that no irreducible component of X'_k is stable under the action of G. Then Y is regular. Since Y_k contains a cycle of projective line, the special fiber X_k contains a cycle of projective lines. This contradicts the assumption on T. Thus, there exists an irreducible component of X'_k that is stable under the action of G. Since $T' = \mathrm{I}_{2n}$ $(n > 0)$, there exist exactly two irreducible components of X'_k that are stable under the action of G. Further, on each of the two irreducible components, there exist exactly two fixed points. Therefore, the case $T = \mathrm{I}_n^*$ $(n > 0)$ follows from (A.4).

Let us show the other cases. In those cases, the equality $T' = \mathrm{I}_0$ holds. Thus, we have only to show the last statement. By H we denote the maximum normal subgroup of G that trivially acts on $X'(k)$. Put $e := \sharp(G/H)$. Then X'/H is a minimal elliptic fibration over C'/H with special fiber of type ${}_{m'}\mathrm{I}_0$ (A.6 (1)). By ϕ we denote the finite dominant k-morphism from the reduction of the special fiber of X'/H to the normalization of Y_k. By the definitions of H and e, the k-morphism ϕ is separable of degree e. Since T is additive, the k-morphism ϕ is not étale. Since G/H is cyclic, the k-morphism ϕ is a cyclic covering of the projective line over k by an elliptic curve over k. By n we denote the number of the branch points of ϕ. Over each branch point of ϕ, the ramification indexes are equal to each other. We denote the combination of these numbers by $I = (e_1 \dots, e_n)$ where $e_i \leq e_j$ if $i \leq j$. The Riemann–Hurwitz formula gives $\sum_{i=1}^{n} e_i^{-1} = n - 2$, which implies that $n = 3$ or 4. If $n = 3$, then $I = (2,3,6)$, $(2,4,4)$, or $(3,3,3)$. If $n = 4$, then $I = (2,2,2,2)$. If $I = (2,2,2,2)$ (resp. $(2,3,6)$, $(2,4,4)$, $(3,3,3)$), then $u(T) = 2$ (resp. 6, 4, 3) (A.4 and A.6 (2) and (3)). Since $e_i \mid e$, $e \mid d$, and $d = u(T)$, the equality $e = d$ holds, which implies that H is trivial. Thus, the lemma follows from (A.4 and A.6 (2) and (3)). □

Lemma 7.3.3. *Assume that $p \not| u(T)$ and $d = u(T)$. Suppose that $T = \mathrm{I}_n^*$ $(n \geq 0)$, II^*, III^*, or IV^*. Put $\widetilde{X}' := X'$ and $\widetilde{X} := \widetilde{X}'/G$ (3.2.7). Then any closed point on $\widetilde{X}$ is l.c.i. and rational. The scheme X may be given by the minimal desingularization of $\widetilde{X}$ (3.2.1). Moreover, the equalities $m = m'$, $D_{\widetilde{f}} = aV_{\widetilde{f}}$, $D_{\widetilde{f}'} = a'V_{\widetilde{f}'}$, $D_{\widetilde{X}'/\widetilde{X}} = 0$, and $F_{\widetilde{X}'/\widetilde{X}} = 0$ hold.*

Proof. The first two statements follow from (3.2.2, 7.3.2, and A.6 (2)). The equality $m = m'$ follows from (7.3.1 (1)). The equalities $D_{\widetilde{f}} = aV_{\widetilde{f}}$ and $D_{\widetilde{f}'} = a'V_{\widetilde{f}'}$ follow from (4.1.3). Since $\pi_{\widetilde{X}}$ is étale in codimension one, the equality $D_{\widetilde{X}'/\widetilde{X}} = 0$ holds. These equalities give $F_{\widetilde{X}'/\widetilde{X}} = 0$. □

Lemma 7.3.4. *Assume that $p \not| u(T)$ and $d = u(T)$. Suppose that $T = \mathrm{II}$, III, or IV. By $f_0' \colon \widetilde{X}' \to X'$ we denote the blowing-up of X' along the non-free locus of the action of G. By the universal property of blowing-up, the action of G on X' induces that on $\widetilde{X}'$. Put $\widetilde{X} := \widetilde{X}'/G$ (3.2.7). Then $\widetilde{X}$ is regular. We define integers (e_1, e_2, e_3) by Table 3. Take the orbits (P_1, P_2, P_3) of the action of G on X' where the orders of the stabilizer subgroups are given by (e_1, e_2, e_3), respectively. For each i $(1 \leq i \leq 3)$, by D_i' we denote the union of (-1)-curves $(f_0')^{-1}(P_i)$. By D_0' we denote the strict transform of the reduction of X_k' via f_0'. For each i $(0 \leq i \leq 3)$, by D_i we denote the image of D_i' under the quotient morphism $\pi_{\widetilde{X}}$ with the reduced structure. If $T = \mathrm{II}$ (resp. $T = \mathrm{III}$, resp. $T = \mathrm{IV}$), then X may be given by the successively blowing-downs of D_0, D_1, and D_2 (resp. D_0 and D_1, resp. D_0). Moreover, the equality $m = m'$ holds, and we obtain Table 3.*

Proof. The first two statements follow from (7.3.2 and A.6 (3)). The equality $m = m'$ follows from (7.3.1 (1)). Let us show that the equalities given by Table 3 hold. Put $e_0 := 1$. For any i $(0 \leq i \leq 3)$, the ramification index of $\pi_{\widetilde{X}}$ along D_i is equal to e_i (A.6 (3)). Thus, the equalities for $D_{\widetilde{X}'/\widetilde{X}}$ hold. Further, for any i $(0 \leq i \leq 3)$, the equality $\pi_{\widetilde{X}}^* D_i = e_i D_i'$ holds. Thus, the equalities for $\pi_{\widetilde{X}}^*(D_{\widetilde{f}} - aV_{\widetilde{f}})$ and $D_{\widetilde{f}'} - a'V_{\widetilde{f}'}$ follow from (4.1.4). These equalities give the equalities for $F_{\widetilde{X}'/\widetilde{X}}$. □

T	II	III	IV
e_1, e_2, e_3	$2,3,6$	$2,4,4$	$3,3,3$
$D_{\widetilde{X}'/\widetilde{X}}$	$0,1,2,5$	$0,1,3,3$	$0,2,2,2$
$\pi^*_{\widetilde{X}}(D_{\widetilde{f}} - aV_{\widetilde{f}})$	$4,4,3,0$	$2,2,0,0$	$1,0,0,0$
$D_{\widetilde{f}'} - a'V_{\widetilde{f}'}$	$0,1,1,1$	$0,1,1,1$	$0,1,1,1$
$F_{\widetilde{X}'/\widetilde{X}}$	$4,4,4,4$	$2,2,2,2$	$1,1,1,1$

TABLE 3. The definition of (e_1, e_2, e_3) and the coefficients (a_0, a_1, a_2, a_3) of the divisors $\sum_{i=0}^{3} a_i D'_i$ on $\widetilde{X}'$.

Corollary 7.3.5 (use 7.3.3 and 7.3.4). *Assume that $p \nmid u(T)$ and $d = u(T)$. Then there exists $\widetilde{X}$ such that any closed point on both $\widetilde{X}$ and $\widetilde{X}'$ is l.c.i. and rational, and the equalities $m' = m$ and $d_{\widetilde{X}'/\widetilde{X}} = v(T)$ hold.*

Proof of (1.2.1). The first equality follows from (7.3.1). The last equality follows from (7.2.2 and 7.3.5). □

Example 7.3.6. In general, we cannot take a C-model $\widetilde{X}$ so that both $\widetilde{X}$ and $\widetilde{X}'$ are regular. For simplicity, we assume that R is equi-characteristic. Suppose that $p \neq 2$, m is odd, $T = \mathrm{III}^*$, and $d = 4$ (see 7.3.3). Then X admits a closed point where the completion of the local ring is isomorphic to $k[[u, x, y]]/(x^m y^{2m} - u)$ where u defines X_k. Take a regular C-model $\widetilde{X}$ of X_K. Then there exists a proper birational morphism $\widetilde{X} \to X$, which factors as a finite succession of blowing-downs of (-1)-curves [17, 9.2.2]. Thus, the scheme $\widetilde{X}$ admits a closed point where the completion of the local ring is isomorphic to $k[[u, x, z]]/(x^{2mn+m} z^{2m} - u)$ for a non-negative integer n (z corresponds to y/x^n). Therefore, the base change $\widetilde{X} \times_C C'$ admits a closed point where the completion of the local ring is isomorphic to $k[[v, x, z]]/(x^{2mn+m} z^{2m} - v^4)$ where $u = v^4$. The normalization of this ring is isomorphic to $k[[s, t, z]]/(sz - t^2)$ (s and t correspond to $v^2/x^{(2mn+m-1)/2} z^m$ and $v/s^{(2mn+m-1)/2} z^{(m-1)/2}$, respectively; $s^2 = x$ and $t^4 = xz^2$), which is not regular.

7.4. **Type** ${}_m\mathrm{I}_n$ **(**$n > 0$**).**

Proof of (1.2.2). Recall the following [5, 2.3 and 8.4]: any closed point on any minimal Weierstrass model over R is rational (3.2.3). Thus, we may assume that $\widetilde{E}$ is a minimal Weierstrass model of the generic fiber of g. Then $\widetilde{E}' = \widetilde{E} \times_C C'$ since $\widetilde{E} \times_C C'$ is a minimal Weierstrass model of the generic fiber of g' (Table 2 in [25, §4]). Thus, the equality $d_{\widetilde{E}'/\widetilde{E}} = d_{C'/C}$ holds (7.2.1). The equalities $m' = 1$, $l' = 0$, and $a' = 0$ hold. By (5.3.1), we may assume that $\widetilde{X} = X$ and $\widetilde{X}' = X'$. Since π_X is étale, the equalities $d' = 1$ and $d_{\widetilde{X}'/\widetilde{X}} = 0$ hold (7.2.3). Therefore, the theorem follows from (7.2.2). □

7.5. **Type** ${}_m\mathrm{I}_0$**.** In this subsection, we assume that $T = \mathrm{I}_0$. We may assume that $\widetilde{X} = X$, $\widetilde{E} = E$, $\widetilde{X}' = X'$, and $\widetilde{E}' = E'$ (5.4.2).

Proof of (1.2.3). Since $E' = E \times_C C'$, the equality $d_{E'/E} = d_{C'/C}$ holds (7.2.1). Thus, the theorem follows from (7.2.2). □

Corollary 7.5.1 (use 1.2.3 and 7.2.3)**.** *Assume that* $T = \mathrm{I}_0$*. Suppose that* π_X *is étale. Then* $m = dm'$*,* $d_{X'/X} = 0$*, and* $ml + a = m'l' + a' + m'd_{C'/C}$*.*

Lemma 7.5.2. *Assume that* $T = \mathrm{I}_0$*. Take the separable closure* K^{sep} *of* K *in* $\overline{K}$*. By* G_K *we denote the Galois group of* K^{sep}/K*. By* $\mathfrak{n}$ *we denote the maximal ideal of the valuation ring of* K^{sep}*. By* $\widehat{E}$ *we denote the formal group law over* R *associated to* E/C*. Then the exact sequence of* G_K*-modules* $0 \to \widehat{E}(\mathfrak{n}) \to E(K^{\mathrm{sep}}) \to E(k) \to 0$ [24, VII] *induces an exact sequence*

$$0 \longrightarrow H^1(G_K, \widehat{E}(\mathfrak{n})) \xrightarrow{\lambda} H^1(K, E_K) \xrightarrow{\mu} \mathrm{Hom}(G_K, E(k)).$$

Take the element $\eta \in H^1(K, E_K)$ *corresponding to* X_K*. Then* $\mu(\eta) = 0$ *if and only if the étale part of* X_K *is trivial* (5.2.4)*. Assume that* $\mu(\eta) = 0$*. Suppose that* $d = m$ *and* $m' = 1$ *(see* 5.1.2*). Take a finite Galois extension* L/K *in* K^{sep} *so that* $K' \subset L$*. By* G_L*,* G*, and* H *we denote the Galois groups of* K^{sep}/L*,* L/K*, and* L/K'*, respectively. By* $\mathfrak{m}$ *we denote the maximal ideal of the valuation ring of* L*. The inflation-restriction exact sequence*

$$0 \longrightarrow H^1(G, \widehat{E}(\mathfrak{m})) \xrightarrow{\mathrm{Inf}_{L/K}} H^1(G_K, \widehat{E}(\mathfrak{n})) \xrightarrow{\mathrm{Res}_{L/K}} H^1(G_L, \widehat{E}(\mathfrak{n}))$$

gives an element $\xi \in H^1(G, \widehat{E}(\mathfrak{m}))$ *satisfying* $\lambda \circ \mathrm{Inf}_{L/K}(\xi) = \eta$*. Then there exists a representative* $\{a_g\}_{g \in G} \in Z^1(G, \widehat{E}(\mathfrak{m}))$ *of* ξ *such that* $a_h = 0$ *for any* $h \in H$*. In particular, the map* $G \to \widehat{E}(\mathfrak{m})$ *defined by* $g \mapsto a_g$ *factors through the canonical projection* $G \to G/H$*.*

Proof. Since G_K trivially acts on $E(k)$, we obtain a canonical isomorphism between abelian groups $H^1(G_K, E(k)) \cong \mathrm{Hom}(G_K, E(k))$. Since g is smooth, the specialization homomorphism $E(K) \to E(k)$ is surjective, which concludes the proof of the first statement. The second statement follows from (3.2.7 and 4.3.1). Assume that $\mu(\eta) = 0$. Since the image of ξ under the restriction homomorphism $H^1(G, \widehat{E}(\mathfrak{m})) \to H^1(H, \widehat{E}(\mathfrak{m}))$ is equal to zero, we obtain a desired representative $\{a_g\}_{g \in G}$ of ξ. Thus, the last statement follows from the equalities $a_{gh} = a_g + g a_h = a_g$ for any $g \in G$ and any $h \in H$. □

Theorem 7.5.3. *Assume that* $T = \mathrm{I}_0$*. Suppose that the étale part of* X_K *is trivial* (5.2.4) *and that* $m = d$ *and* $m' = 1$ *(see* 5.1.2*). Then* $ml + a = d_{C'/C} - d_{X'/X}$*. Take* L*,* $H \subset G$*, and* $\{a_g\}_{g \in G} \in Z^1(G, \widehat{E}(\mathfrak{m}))$ *in the same way as in* (7.5.2)*. By* v_L *we denote the valuation of* L *whose value group is equal to the additive group of integers. Then* $d_{X'/X} = \sum_g v_L(a_g)/[L : K']$ *where* g *runs through all representatives of* $(G/H) \setminus \{H\}$*.*

Proof. Since $m = d$, $m' = 1$, $l' = 0$, and $a' = 0$, the first equality follows from (1.2.3). We use the notation introduced in (7.5.2). Take the normalization S of R in L. Put $C_S := \mathrm{Spec}\, S$ and $E_S := E \times_C C_S$. The elliptic fibrations (X, C, f) and (X', C', f') are given by the quotients of the equivariant actions on E_S/C_S induced by the cocycles $\{a_g\}_{g \in G}$ and $\{a_g\}_{g \in H}$, respectively (4.3.1). The actions fix the special fiber of E_S/C_S since $a_g \in \widehat{E}(\mathfrak{m})$ for any $g \in G$. Since $a_h = 0$ for any $h \in H$, we may identify X' with E'. By y_0 we denote the origin of the special fiber of E_S. Note that $E' = E \times_C C'$. We denote the images of y_0 on X, X', E, and E' by x, x', y, and y', respectively. Then $\mathcal{O}_{X,x} \subset \mathcal{O}_{X',x'} = \mathcal{O}_{E',y'} \supset \mathcal{O}_{E,y}$ and $\mathcal{O}_{E',y'} \subset \mathcal{O}_{E_S,y_0}$.

Take a defining function $z \in \mathcal{O}_{E,y}$ of the zero section of E. Then $z \in \mathcal{O}_{X',x'} = \mathcal{O}_{E',y'}$ defines the zero section of E'. Take a defining function $\tau \in \mathcal{O}_{X,x}$ of the unique vertical prime divisor on X. Since $m = d$ and $m' = 1$, the element $\tau \in \mathcal{O}_{X',x'}$ is a defining function of the unique vertical prime divisor on X' (see the proof of (5.1.2)). Thus, the set $\{z, \tau\}$ is a system of parameters of $\mathcal{O}_{X',x'}$. Since π_X is a finite flat morphism of degree d (4.3.1), the quotient ring $\mathcal{O}_{X',x'}/(\tau)$ is a finite flat extension of $\mathcal{O}_{X,x}/(\tau)$ of degree d between discrete valuation rings with the same residue field k. Thus, the set $\{z^i\}_{i=0}^{d-1}$ is a basis of the k-vector space $\mathcal{O}_{X',x'} \otimes_{\mathcal{O}_{X,x}} k$, which implies that $\{z^i\}_{i=0}^{d-1}$ is a basis of the free $\mathcal{O}_{X,x}$-module $\mathcal{O}_{X',x'}$.

The base change of the C-action of G on C_S via E/C induces an $\mathcal{O}_{E,y}$-action ρ_{E_S} of G on $\mathcal{O}_{E_S,y_0}$. The cocycle $\{a_g\}_{g\in G}$ induces an $\mathcal{O}_{X,x}$-action ρ of G on $\mathcal{O}_{E_S,y_0}$ where $\rho(g)$ is the composite of the automorphism $\rho_{E_S}(g)$ and the automorphism $\rho'(g)$ induced by the translation of E_S by the addition of $a_g \in \widehat{E}(\mathfrak{m})$ (§4.3). Since $\rho_{E_S}(g)(z) = z$, the equality $\rho(g)(z) = \rho'(g)(z)$ holds. Put $F(T) := \prod_g (T - \rho(g)(z))$ where g runs through all representatives of G/H. Since $F(T) \in \mathcal{O}_{X,x}[T]$ and $\deg F(T) = d$, the polynomial $F(T)$ is the minimum polynomial of z over $\mathcal{O}_{X,x}$. Thus, the $\mathcal{O}_{X,x}$-algebra homomorphism $\mathcal{O}_{X,x}[T]/(F(T)) \to \mathcal{O}_{X',x'}$ defined by $T \mapsto z$ is bijective. Localizing the extensions $\mathcal{O}_{E_S,y_0}/\mathcal{O}_{X',x'}/\mathcal{O}_{X,x}$ at the generic points of the special fibers, we obtain finite extensions $V_S/V'/V$ of discrete valuation rings. Since $m' = 1$, a prime element π of R' generates the maximal ideal of V'. Thus, the equality $F'(z)V' = \pi^{d_{X'/X}} V'$ holds (Corollary 2 in [23, III, §6]). The formal group law $\widehat{E}$ is given by a formal power series $G(z_1, z_2) \in R[[z_1, z_2]]$ [24, IV]. Since there exists $H(z_1, z_2) \in R[[z_1, z_2]]$ such that $G(z_1, z_2) = z_1 + z_2 + z_1 z_2 H(z_1, z_2)$, the equalities $F'(z) = \prod_g (z - \rho(g)(z)) = u \prod_g a_g$ hold where $u \in V_S^\times$ and g runs through all representatives of $(G/H) \setminus \{H\}$. Thus, the last equality holds. □

Remark 7.5.4. The morphism π_X induces a finite flat morphism $\pi_{X,k}$ of degree d between the reductions of the special fibers since $m = d$ and $m' = 1$. The morphism $\pi_{X,k}$ is purely inseparable since the equivariant action on E_S/C_S induced by the cocycle $\{a_g\}_{g\in G}$ fixes the special fiber of E_S/C_S.

Remark 7.5.5. Vvedenskiĭ gave lots of examples of $\{v_L(a_g)\}_{g\in G}$ by explicit calculations ([26] and [27]). As an application of the above theorem, we obtain elliptic fibrations with lots of combinations of the invariants (l, a). By the same method as in [21], we may construct elliptic fibrations over proper smooth curves over an algebraically closed field with such multiple fibers.

Appendix A. Quotient Singularities

We describe quotient singularities of fibered surfaces for §7.

Lemma A.1. *Let n be an integer satisfying $n \geq 2$. Let S_0 be a regular local ring of dimension $d \geq 3$ with system of parameters $\{x_i\}_{i=1}^d$. Put $S := S_0/(x_1x_2 - x_3^n)$. Then S is l.c.i. and normal. If $d = 3$, then the singularity of S appears only at the closed point, which is of type A_{n-1} and rational* (3.2.3).

Proof. Since the localizations S_{x_1} and S_{x_2} are regular, the ring S is regular in codimension one. Since S is l.c.i., the ring S is CM. By Serre's criterion for normality [8, 5.8.6], the ring S is normal. Assume that $d = 3$. Then an explicit desingularization shows that the singularity is of type A_{n-1}, which concludes the proof [16, 27.1]. □

The proof of the following lemma is straightforward.

Lemma A.2. *Let S be a ring, I be an ideal of S, and σ be an automorphism of S of finite order d satisfying $\sigma I = I$. Let ζ and x be elements of S satisfying $\zeta^d = 1$, $\sigma\zeta = \zeta$, and $\sigma x \equiv \zeta x \bmod I$. Put $y := \sum_{i=0}^{d-1} \zeta^{-i}\sigma^i x$. Then $\sigma y = \zeta y$ and $y \equiv dx \bmod I$.*

Lemma A.3. *Let A be a ring, G be an affine A-group scheme, and S' be an A-algebra with A-action of G. Put $S := (S')^G$. Assume that S is a Noetherian ring and that S' is finite over S. Let I be an ideal of S. Put $I' := S'I$. By $\widehat{S}$ and $\widehat{S}'$ we denote the completions of S and S' with respect to I and I', respectively. Then the base change of the A-action of G on S' via $\widehat{S}/S$ gives an A-action of G on $\widehat{S}'$. Further, the equality $\widehat{S} = (\widehat{S}')^G$ holds where we regard $\widehat{S}$ as a subring of $\widehat{S}'$.*

Proof. By $\iota\colon S \to S'$ we denote the inclusion ring homomorphism. We write $G = \operatorname{Spec} B$. The A-action of G on S' is given by an A-algebra homomorphism $\rho\colon S' \to B \otimes_A S'$. Since $S = (S')^G$, the ring homomorphism ρ is S-linear. Tensoring ι and ρ with $\widehat{S}$ over S, we obtain a ring homomorphism $\widehat{\iota}\colon \widehat{S} \to \widehat{S}'$ and an $\widehat{S}$-algebra homomorphism $\widehat{\rho}\colon \widehat{S}' \to B \otimes_A \widehat{S}'$ [19, 8.7]. The $\widehat{S}$-algebra homomorphism $\widehat{\rho}$ defines an A-action of G on $\widehat{S}'$. Since $\widehat{S}$ is flat over S [19, 8.8] and the kernel of $\rho - 1$ is given by ι, the kernel of $\widehat{\rho} - 1$ is given by $\widehat{\iota}$, which proves the last statement. □

In the following, we concentrate ourselves to the case of complete local rings by (A.3) (see also [19, 21.2 and 32.2] and [16, 16.5]).

Lemma A.4. *Let R'/R be a totally ramified finite extension of complete discrete valuation rings of degree d. By p we denote the characteristic of the residue field of R. Assume that $p \nmid d$. Put $S'_0 := R'[[x'_1, x'_2]]$. Let S' be an R'-algebra that admits a presentation $S' \cong S'_0/(\pi' - (x'_1)^{m_1}(x'_2)^{m_2}\epsilon')$ as an R'-algebra where π' is a prime element of R', m_1 is a non-negative integer, m_2 is a positive integer, and ϵ' is a unit of S'_0. Let σ be an R-algebra automorphism of S' of order d. By G we denote the group of R-algebra automorphisms of S' generated by σ. Suppose that the subring R' of S' is stable under the action of G. Assume that $\sigma\zeta_d = \zeta_d$ and $\sigma\pi' = \zeta_d\pi'$ where ζ_d is a primitive d-th root of unity. Then we may replace the system of generators $\{x'_1, x'_2\}$ of S'_0 over R' and the unit ϵ' of S'_0 so that the above presentation of S' is preserved and there exist integers u_1 and u_2 satisfying $\sigma x'_1 = \zeta_d^{u_1} x'_1$, $\sigma x'_2 = \zeta_d^{u_2} x'_2$, $\sigma\epsilon' = \epsilon'$, and $u_1 m_1 + u_2 m_2 \equiv 1 \bmod d$.*

Proof. We first note that S' is a regular local ring with system of parameters $\{x'_1, x'_2\}$. By $\mathfrak{n}$ we denote the maximal ideal of S'. Put $k := S'/\mathfrak{n}$. The canonical ring homomorphism $R' \to S'$ induces an isomorphism $R'/(\pi') \to k$ between the residue fields. Since R'/R is totally ramified, the group G trivially acts on k. The elements x'_1 and x'_2 define prime divisors D_1 and D_2 on $\operatorname{Spec} S'$, respectively. Since the special fiber of $\operatorname{Spec} S'/\operatorname{Spec} R'$ is equal to $m_1D_1 + m_2D_2$, the divisor $m_1D_1 + m_2D_2$ is stable under the action of G. We first show the following: after replacing x'_1 and ϵ' if necessary, we may assume that both D_1 and D_2 are stable under the action of G. *Case* 1: $m_1 = 0$. The divisor D_2 is stable under the action of G. We may take a d-th root of unity ζ in R so that $\sigma x'_1 \equiv \zeta x'_1 \bmod \mathfrak{n}^2$. Thus, by (A.2), we may replace x'_1 and ϵ' so that the presentation of S' is preserved and both D_1 and D_2 are stable under the action of G. *Case* 2: $m_1 > 0$. Suppose that σ exchanges D_1 and D_2. Then d is even. Since $m_1D_1 + m_2D_2 = m_2D_1 + m_1D_2$, the equality $m_1 = m_2$

holds. Put $m := m_1$, $c := d/2$, and $\epsilon_{ij} := \sigma x'_i/x'_j$ for $(i,j) = (1,2)$ and $(2,1)$. The equalities $x'_1 = \sigma^d x'_1 \equiv \epsilon_{12}^c \epsilon_{21}^c x'_1 \bmod \mathfrak{n}^2$ give the equality $\epsilon_{12}^c\epsilon_{21}^c \equiv 1 \bmod \mathfrak{n}$. Since $\pi' = (x'_1 x'_2)^m \epsilon'$ in S', the equality $(\sigma\pi'/\pi')^c \equiv (\sigma\epsilon'/\epsilon')^c \bmod \mathfrak{n}$ holds, which contradicts the equalities $\sigma\pi' = \zeta_d \pi'$ and $\sigma\epsilon' \equiv \epsilon' \bmod \mathfrak{n}$. Thus, both D_1 and D_2 are stable under the action of G.

In the following, we consider the general case $m_1 \geq 0$. Put $\epsilon_i := \sigma x'_i/x'_i$ for $i = 1$ and 2. The equalities $x'_i = \sigma^d x'_i \equiv \epsilon_i^d x'_i \bmod \mathfrak{n}^2$ give $\epsilon_i^d \equiv 1 \bmod \mathfrak{n}$. Thus, by (A.2), we may replace x'_1, x'_2, and ϵ' so that the presentation of S' is preserved and the equalities $\sigma x'_1 = \zeta_d^{u_1} x'_1$ and $\sigma x'_2 = \zeta_d^{u_2} x'_2$ hold. Since $\pi' = (x'_1)^{m_1}(x'_2)^{m_2}\epsilon'$ in S', the equality $\sigma\pi'/\pi' \equiv \zeta_d^{u_1 m_1 + u_2 m_2}(\sigma\epsilon'/\epsilon') \bmod \mathfrak{n}$ holds. Since $\sigma\pi' = \zeta_d \pi'$ and $\sigma\epsilon' \equiv \epsilon' \bmod \mathfrak{n}$, the equality $u_1 m_1 + u_2 m_2 \equiv 1 \bmod d$ holds. Thus, the equality $(x'_1)^{m_1}(x'_2)^{m_2}(\sigma\epsilon' - \epsilon') = 0$ holds. Since both x'_1 and x'_2 are non-zero in the integral domain S', the equality $\sigma\epsilon' = \epsilon'$ holds. □

Lemma A.5. *We use the notation introduced in* (A.4). *We replace x'_1, x'_2, and ϵ' in the same way as in* (A.4). *Assume that $d = 2$ and $m_1 = m_2$. Put $m := m_1$, $S := (S')^G$, and $\pi := (\pi')^2$. After exchanging x'_1 and x'_2 if necessary, we may assume that $u_1 = 1$ and $u_2 = 0$. Put $R_2 := R[[x_1, x_2]]$. Then the R-algebra homomorphism $R_2 \to S$ defined by $x_1 \mapsto (x'_1)^2$ and $x_2 \mapsto x'_2$ induces an R-algebra isomorphism $R_2/(\pi - x_1^m x_2^{2m}\epsilon) \to S$ where $\epsilon \in R_2^\times$. In particular, the invariant ring S is regular.*

Proof. We define an R-algebra automorphism σ_0 of S'_0 by $\pi' \mapsto -\pi'$, $x'_1 \mapsto -x'_1$, and $x'_2 \mapsto x'_2$. The R-algebra automorphism σ_0 is a lifting of σ and induces an R-action of G on S'_0. Put $S_0 := (S'_0)^G$, $R_3 := R[[x_1, x_2, x_3]]$, and $T_0 := R_3/(x_3^2 - \pi x_1)$. We define an R-algebra homomorphism $\psi\colon T_0 \to S'_0$ by $x_1 \mapsto (x'_1)^2$, $x_2 \mapsto x'_2$, and $x_3 \mapsto \pi' x'_1$. Since both T_0 and S'_0 are integral domains of dimension three and ψ is finite, the ring homomorphism ψ is injective. We regard T_0 as a subring of S'_0 by ψ. Then we obtain finite extensions of integral domains $T_0 \subset S_0 \subset S'_0$. We denote the field of fractions of an integral domain A by $Q(A)$. Since $[Q(S'_0) : Q(T_0)] \leq 2$ and $[Q(S'_0) : Q(S_0)] \geq 2$, the equality $Q(T_0) = Q(S_0)$ holds in $Q(S'_0)$. Since both T_0 and S_0 are normal (A.1 and [12, 32.7]), the equality $T_0 = S_0$ holds.

Put $\epsilon := \epsilon'\sigma_0\epsilon'$ in S'_0. Then $\epsilon \in S_0$. By $\phi\colon S'_0 \to S'$ we denote the canonical surjective homomorphism. Since $\sigma\epsilon' = \epsilon'$, the equality $\phi(\epsilon) = (\epsilon')^2$ holds. Since G trivially acts on the residue field of S'_0, S_0 is Henselian, and $p \neq 2$, we may take $\eta \in S_0$ so that $\eta^2 = \epsilon$ and $\phi(\eta) = \epsilon'$. Take a lifting of η in R_3. We denote this lifting and its square by the same notation η and ϵ, respectively. Put $n := (m+1)/2$, $f := \pi - x_1^m x_2^{2m}\epsilon$, $g := x_3 - x_1^n x_2^m \eta$, and $T := S_0/(f, g)$. Then we obtain the R_3-algebra isomorphism $T \cong R_3/(f, g)$ since $x_3^2 - \pi x_1 = (x_3 + x_1^n x_2^m \eta)g - x_1 f$ in R_3, which implies that T is a regular local ring with system of parameters $\{x_1, x_2\}$. Note that $\phi(f) = (\pi' - (x'_1)^m(x'_2)^m\epsilon')(\pi' + (x'_1)^m(x'_2)^m\epsilon') = 0$ and $\phi(g) = x'_1(\pi' - (x'_1)^m(x'_2)^m\epsilon') = 0$. Thus, the restriction of ϕ to S_0 factors as the composite of the canonical surjective homomorphism $S_0 \to T$ and a ring homomorphism $\overline{\phi}\colon T \to S'$. Since both T and S' are integral domains of dimension two and $\overline{\phi}$ is finite, the ring homomorphism $\overline{\phi}$ is injective. We regard T as a subring of S' by $\overline{\phi}$. In the same way as in the proof of the equality $T_0 = S_0$, we may show that $T = S$, which concludes the proof. □

Lemma A.6. *We use the notation introduced in* (A.4)*. We replace x'_1, x'_2, and ϵ' in the same way as in* (A.4)*. Assume that $m_1 = 0$. Put $m := m_2$, $S := (S')^G$, and $\pi := (\pi')^d$. Then the following statements hold.*

(1) *Assume that $u_1 = 0$. Put $R_2 := R[[x_1, x_2]]$. Then the R-algebra homomorphism $R_2 \to S$ defined by $x_1 \mapsto x'_1$ and $x_2 \mapsto (x'_2)^d$ induces an R-algebra isomorphism $R_2/(\pi - x_2^m \epsilon) \to S$ where $\epsilon \in R_2^\times$. In particular, the invariant ring S is regular.*

(2) *Assume that $u_1 = -u_2$. Put $R_3 := R[[x_1, x_2, x_3]]$. Then the R-algebra homomorphism $R_3 \to S$ defined by $x_1 \mapsto (x'_1)^d$, $x_2 \mapsto (x'_2)^d$, and $x_3 \mapsto x'_1 x'_2$ induces an R-algebra isomorphism $R_3/(x_1 x_2 - x_3^d, \pi - x_2^m \epsilon) \to S$ where $\epsilon \in R_3^\times$. In particular, the singularity of S appears only at the closed point, which is l.c.i., of type A_{d-1}, and rational* (3.2.3)*.*

(3) *Assume that $u_1 = u_2$. Put $Y' := \operatorname{Spec} S'$. By Y'' and E we denote the blowing-up of Y' at the closed point and the exceptional locus on Y'', respectively. By the universal property of blowing-up, the action of G on Y' induces that on Y''. The fixed locus of the latter action is equal to E. Further, the quotient Y''/G is regular. The image D of E under the quotient morphism endowed with the reduced structure is isomorphic to the projective line over the residue field of R. The self-intersection number of D is equal to $-d$.*

Proof. The last statement of (2) follows from (A.1). The statements of (3) may be shown by using the equality $x'_1 \sigma x'_2 = x'_2 \sigma x'_1$. The other statements may be proved in the same way as in the proof of (A.5). □

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