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(Citation)
Journal of Knot Theory and Its Ramifications,25(13):1671001-1671001

(Issue Date)
2016-11

(Resource Type)
journal article

(Version)
Accepted Manuscript

A NOTE ON THE OU SEQUENCES OF A 2-BRIDGE KNOT

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Abstract. An OU sequence is a cyclically ordered sequence in symbols O and U such that the number of O's is equal to that of U's. Every knot diagram defines an OU sequence by reading O and U at over- and under-crossings, respectively, appeared along the diagram. In this note, we determine the OU sequences for a 2-bridge knot arising from its diagrams with two over-bridges.

An *OU-sequence* is a cyclically ordered sequence in symbols O and U,

$$w = O^{a_1}U^{a_2}\dots O^{a_{2n-1}}U^{a_{2n}} = \underbrace{O\dots O}_{a_1}\underbrace{U\dots U}_{a_2}\dots\underbrace{O\dots O}_{a_{2n-1}}\underbrace{U\dots U}_{a_{2n}}$$

with $n \geq 0$, $a_1,\dots,a_{2n} \geq 1$, and $\sum_{i=1}^{n} a_{2i-1} = \sum_{i=1}^{n} a_{2i}$. For an oriented knot diagram D, we walk along D from some basepoint to the original position and read O or U when we meet an over- or under-crossing, respectively, so that we obtain an OU-sequence $w = f(D)$. We remark that the number n of blocks of O in $f(D)$ is coincident with that of longest over-bridges of D.

For an oriented knot K, we say that an OU sequence w is *K-realizable* if there is a diagram D of K with $f(D) = w$. Let 0_1 and 3_1 denote the trivial knot and the trefoil knot, respectively. In [1], we prove that any OU sequence is 0_1-realizable, and that an OU sequence w is 3_1-realizable if and only if

(i) w has $n \geq 2$ blocks of O,
(ii) for $n = 2$, $w = O^aU^bO^cU^d$ with $a, b, c, d \geq 2$, and
(iii) for $n \geq 3$, $w \neq OU^aO^bUO^cU^d$, $UO^aU^bOU^cO^d$ with $a \not\equiv b \pmod 2$.

The aim of this paper is to generalize the property (ii) for any 2-bridge knot as follows. Here, $\det(K)$ denotes the determinant of a knot K.

Theorem 1. *For a 2-bridge knot K and an OU sequence $w = O^aU^bO^cU^d$, the following are equivalent.*

(i) *w is K-realizable.*
(ii) *$a, b, c, d \geq \det(K) - 1$.*

For a 2-bridge knot K, let $\mathrm{c}_2(K)$ denote the minimum number of crossings for all diagrams of K with two over-bridges. By Theorem 1, we have the following immediately.

Corollary 2. *For any 2-bridge knot K, it holds that $\mathrm{c}_2(K) = 2\det(K) - 2$.* □

To prove (i)⇒(ii) in Theorem 1, we prepare a lemma concerning a Schubert normal form of a 2-bridge knot. Let D be a knot diagram with two over-bridges u_1 and u_2, and α_i the number of over-crossings on u_i plus one ($i = 1, 2$). Assume that

The third author is partially supported by JSPS KAKENHI Grant Number 25400090.
2010 *Mathematics Subject Classification.* 57M25.
Key words and phrases. OU sequence, 2-bridge knot, diagram, Schubert normal form, contraction.

D is located in a 2-sphere S^2, which can be divided into two disks E_1, E_2 and an annulus A such that $D \cap E_i$ consists of the over-bridge u_i and $\alpha_i - 1$ under-arcs. See the left of Figure 1. We label the $2\alpha_i$ points of $D \cap \partial E_i$ by $0, 1, \ldots, 2\alpha_i - 1 \pmod{2\alpha_i}$ as shown in the figure, where 0 and α_i are the endpoints of u_i. If $\alpha_1 = \alpha_2 = \alpha$ and any arc of $D \cap A$ connects between ∂E_1 and ∂E_2, then D is called a *Schubert normal form* of K. Since there is an integer β such that j on $D \cap E_1$ and $j + \beta$ on $D \cap E_2$ are connected by an arc of $D \cap A$ for any j, we denote it by $S(\alpha, \beta)$ (cf. [2]). The right of the figure shows $S(5, 3)$.

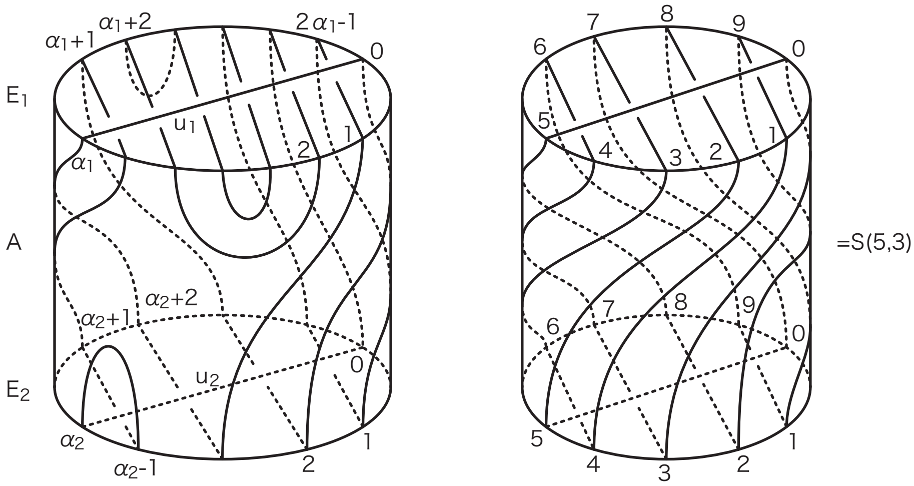

FIGURE 1

Lemma 3. *Let D be a diagram of a 2-bridge knot K. If D has two over-bridges, then there is a finite sequence of diagrams $D = D_0, D_1, \ldots, D_m$ such that*

(i) *D_k is obtained from D_{k-1} by a Reidemeister move* I *or* II *which reduces the number of crossings ($k = 1, 2, \ldots, m$),*
(ii) *each D_k has two over-bridges, and*
(iii) *D_m is a Schubert normal form $S(\alpha, \beta)$ for $\alpha = \det(K)$ and some β.*

Proof. If D is a Schubert normal form $S(\alpha, \beta)$, then we have $\alpha = \det(K)$. Now we assume that D is not a Schubert normal form. Since A is an annulus, the innermost argument induces the existence of an arc t of $D \cap A$ such that

(i) the endpoints of t are both on the same boundary ∂E_i, and
(ii) the disk component of $A \setminus t$ misses any arcs of $D \cap A$.

If one of the endpoints of t is 0 or α_i, then we perform a Reidemeister move I containing t to remove a crossing from u_i. If the endpoints of t are neither 0 nor α_i, then we perform a Reidemeister move II containing t to cancel a pair of crossings from u_i. In any case, we can reduce the number of crossings with keeping a diagram having two over-bridges. By repeating this process, we obtain a Schubert normal form finally. □

Proof of Theorem 1(i)⇒(ii). Assume that there is a diagram D of K with $f(D) = O^a U^b O^c U^d$. Since D has two over-bridges, there is a finite sequence of diagrams of K, $D = D_0, D_1, \ldots, D_m$, as in Lemma 3. Put $f(D_k) = O^{a_k} U^{b_k} O^{c_k} U^{d_k}$.

If D_k is obtained from D_{k-1} by a Reidemeister move I, then $f(D_k)$ is obtained from $f(D_{k-1})$ by removing a subsequence OU or UO. If D_k is obtained from D_{k-1} by a Reidemeister move II, then $f(D_k)$ is obtained from $f(D_{k-1})$ by removing a pair of subsequences O^2 and U^2. In any case, we may assume that

$$a_{k-1} \geq a_k, \ b_{k-1} \geq b_k, \ c_{k-1} \geq c_k, \text{ and } d_{k-1} \geq d_k.$$

Since $D_m = S(\alpha, \beta)$ is a Schubert normal form with $\alpha = \det(K)$, it holds that

$$a_m = b_m = c_m = d_m = \alpha - 1.$$

Therefore, we have $a, b, c, d \geq \alpha - 1 = \det(K) - 1$. □

We say that an OU sequence w' is obtained from w by a *contraction* if w' is obtained by deleting a subsequence OU or UO in w. To prove (ii)⇒(i) in Theorem 1, we use the following lemma.

Lemma 4 ([1])**.** *Let K be an oriented knot, and w and w' OU sequences. Suppose that w' is obtained from w by a finite sequence of contractions. If w' is K-realizable, then so is w.* □

Proof of Theorem 1(ii)⇒(i). Since $w = O^aU^bO^cU^d$ is cyclically ordered and satisfies $a + c = b + d$, we may assume that $a \leq b$. By contractions $a - (\alpha - 1)$ times between O^a and U^b, we obtain

$$w_1 = O^{\alpha-1}U^{b-a+(\alpha-1)}O^cU^d,$$

where $\alpha = \det(K) - 1$. Next, by contractions $b - a (\geq 0)$ times between the second and third blocks of w_1, we obtain

$$w_2 = O^{\alpha-1}U^{\alpha-1}O^{c-b+a}U^d = O^{\alpha-1}U^{\alpha-1}O^dU^d.$$

Finally, by contractions $d - (\alpha - 1)$ times between O^d and U^d, we obtain

$$w' = O^{\alpha-1}U^{\alpha-1}O^{\alpha-1}U^{\alpha-1}.$$

Since K is presented by a Schubert normal form $S(\alpha, \beta)$ for some β, w' is K-realizable. Therefore, w is also K-realizable by Lemma 4. □

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