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Anisotropic power-law k-inflation

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It is known that power-law k-inflation can be realized for the Lagrangian $P = Xg(Y)$, where $X = -(\partial\phi)^2/2$ is the kinetic energy of a scalar field ϕ and g is an arbitrary function in terms of $Y = Xe^{\lambda\phi/M_{\rm pl}}$ (λ is a constant and $M_{\rm pl}$ is the reduced Planck mass). In the presence of a vector field coupled to the inflaton with an exponential coupling $f(\phi) \propto e^{\mu\phi/M_{\rm pl}}$, we show that the models with the Lagrangian $P = Xg(Y)$ generally give rise to anisotropic inflationary solutions with $\Sigma/H =$ constant, where Σ is an anisotropic shear and H is an isotropic expansion rate. Provided these anisotropic solutions exist in the regime where the ratio Σ/H is much smaller than 1, they are stable attractors irrespective of the forms of $g(Y)$. We apply our results to concrete models of k-inflation such as the generalized dilatonic ghost condensate and the Dirac-Born-Infeld model and we numerically show that the solutions with different initial conditions converge to the anisotropic power-law inflationary attractors. Even in the de Sitter limit ($\lambda \to 0$) such solutions can exist, but in this case the null energy condition is generally violated. The latter property is consistent with the Wald's cosmic conjecture stating that the anisotropic hair does not survive on the de Sitter background in the presence of matter respecting the dominant/strong energy conditions.

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I. INTRODUCTION

The inflationary paradigm, which was originally proposed in [1], is now widely accepted as a viable phenomenology describing the cosmic acceleration in the very early Universe. The simplest inflationary scenario based on a single scalar field predicts the generation of nearly scale-invariant and adiabatic density perturbations [2]. This prediction is in agreement with the temperature fluctuations of the cosmic microwave background (CMB) observed by the WMAP [3] and Planck [4] satellites.

The WMAP data showed that there is an anomaly associated with the broken rotational invariance of the CMB perturbations [5]. This implies that the statistical isotropy of the power spectrum of curvature perturbations is broken, which is difficult to address in the context of the simplest single-field inflationary scenario. Although we cannot exclude the possibility that some systematic effects cause this anisotropy [6], it is worth exploring the primordial origin of such a broken rotational invariance.

If the inflaton field ϕ couples to a vector kinetic term $F_{\mu\nu}F^{\mu\nu}$, an anisotropic hair can survive during inflation for a suitable choice of the coupling $f^2(\phi)$ [7]. In such cases, the presence of the vector field gives rise to the anisotropic power spectrum consistent with the broken rotational invariance of the CMB perturbations [8,9] (see also Refs. [10–23] for related works). In addition, the models predict the detectable level of non-Gaussianities for the local shape averaged over all directions with respect to a squeezed wave number [24,25]. In the two-form field models where the inflaton couples to the kinetic term $H_{\mu\nu\lambda}H^{\mu\nu\lambda}$ the anisotropic hair can also survive [26], but their observational signatures imprinted in CMB are different from those in the vector model [27].

For a canonical inflaton field with the potential $V(\phi)$, the energy density of a vector field can remain nearly constant for the coupling $f(\phi) = \exp[\int 2V/(M_{\rm pl}^2 V_{,\phi})d\phi]$ [7], where $V_{,\phi} = dV/d\phi$. For the exponential potential $V(\phi) = ce^{-\lambda\phi/M_{\rm pl}}$ the coupling is of the exponential form $f(\phi) = e^{-2\phi/(\lambda M_{\rm pl})}$, as it often appears in string theory and supergravity [28]. In this case there exists an anisotropic power-law inflationary attractor along which the ratio Σ/H is constant [29], where Σ is an anisotropic shear and H is an isotropic expansion rate. For general slow-roll models in which the cosmic acceleration comes to an end, the solution with an anisotropic hair corresponds to a temporal attractor during inflation [7].

There exists another inflationary scenario based on the scalar-field kinetic energy $X = -(\partial\phi)^2/2$ with the Lagrangian $P(\phi, X)$, dubbed k-inflation [30]. The representative models of k-inflation are the (dilatonic) ghost condensate [31,32] and the Dirac-Born-Infeld (DBI) model [33]. In such cases the evolution of the inflaton can be faster than that of the standard slow-roll inflation, so the coupling $f(\phi)$ with the vector field can vary more significantly. It remains to see whether the anisotropic hair survives in k-inflation. This is important to show the generality of anisotropic inflation.

In Refs. [32,34] it was found that in the presence of a scalar field and a barotropic perfect fluid the condition for the existence of scaling solutions restricts the Lagrangian of the form $P(\phi, X) = Xg(Y)$, where g is an arbitrary function in terms of $Y = Xe^{\lambda\phi/M_{\rm pl}}$ and λ is a constant. On the flat Friedmann-Lemaître-Robertson-Walker (FLRW)

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background there exists a scalar-field dominated attractor responsible for inflation under the condition $\lambda^2 < 2\partial P/\partial X$ [35,36]. In fact, the Lagrangian $P(\phi, X) = Xg(Y)$ covers a wide class of power-law inflationary scenarios such as the canonical scalar field with the exponential potential $[g(Y) = 1 - cM_{\rm pl}^4/Y]$, the dilatonic ghost condensate $[g(Y) = -1 + cY/M_{\rm pl}^4]$, and the DBI model $[g(Y) = -(m^4/Y)\sqrt{1 - 2Y/m^4} - M^4/Y]$. There is also another power-law inflationary scenario studied in Ref. [37].

In the presence of a vector kinetic term $F_{\mu\nu}F^{\mu\nu}$ with the coupling $f(\phi) = f_0 e^{-\mu\phi/M_{\rm pl}}$, the canonical scalar field with the exponential potential $V(\phi) = ce^{-\lambda\phi/M_{\rm pl}}$ gives rise to stable anisotropic inflationary solutions under the condition $\lambda^2 + 2\mu\lambda - 4 > 0$ [29]. For the power-law DBI inflation it was shown that the anisotropic hair can survive under certain conditions [38] (see also Ref. [39] for the power-law tachyon inflation). In this paper we study the existence and the stability of anisotropic fixed points for the general Lagrangian $P(\phi, X) = Xg(Y)$. Remarkably, if anisotropic inflationary fixed points exist, they are stable irrespective of the forms of $g(Y)$ in the regime where the anisotropy is small ($\Sigma/H \ll 1$).

This paper is organized as follows. In Sec. II we derive the equations of motion for the Lagrangian $P(\phi, X)$ on the anisotropic cosmological background. In Sec. III we obtain anisotropic fixed points for the Lagrangian $P = Xg(Y)$ and discuss the stability of them against the homogenous perturbations. In Sec. IV we apply our general results to concrete models of power-law inflation and numerically confirm the existence of stable anisotropic solutions. Section V is devoted to conclusions.

II. BACKGROUND EQUATIONS OF MOTION

Let us consider the theories described by the action

$$S = \int d^4x\sqrt{-g_M}\left[\frac{M_{\rm pl}^2}{2}R + P(\phi, X) - \frac{1}{4}f(\phi)^2F_{\mu\nu}F^{\mu\nu}\right], \tag{1}$$

where g_M is the determinant of the metric $g_{\mu\nu}$, R is the scalar curvature, and $P(\phi, X)$ is a function with respect to the inflaton ϕ and its derivative $X = -(1/2)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. The field ϕ couples to a vector kinetic term $F_{\mu\nu}F^{\mu\nu}$, where the vector field A_μ is related to $F_{\mu\nu}$ as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Choosing the gauge $A_0 = 0$, we can take the x axis for the direction of the vector field, i.e., $A_\mu = (0, v(t), 0, 0)$, where $v(t)$ is a function of the cosmic time t. Since there is the rotational symmetry in the (y, z) plane, we take the line element of the form

$$ds^2 = -\mathcal{N}(t)^2dt^2 + e^{2\alpha(t)}[e^{-4\sigma(t)}dx^2 + e^{2\sigma(t)}(dy^2 + dz^2)], \tag{2}$$

where $\mathcal{N}(t)$ is the Lapse function, $e^\alpha \equiv a$ and σ are the isotropic scale factor and the spatial shear, respectively. For this metric the action (1) reads

$$S = \int d^4x\frac{e^{3\alpha}}{\mathcal{N}}\left[3M_{\rm pl}^2(\dot\sigma^2 - \dot\alpha^2) + \mathcal{N}^2P(\phi, X(\mathcal{N})) + \frac{1}{2}f(\phi)^2e^{-2\alpha+4\sigma}\dot v^2\right], \tag{3}$$

where a dot represents a derivative with respect to t, and $X(\mathcal{N}) = \dot\phi^2\mathcal{N}^{-2}/2$. The field equation of motion for the field v following from the action (3) is integrated to give

$$\dot v = p_A f(\phi)^{-2}e^{-\alpha-4\sigma}, \tag{4}$$

where p_A is an integration constant. Varying the action (3) with respect to $\mathcal{N}$, α, σ, ϕ, and setting $\mathcal{N} = 1$, it follows that

$$H^2 = \dot\sigma^2 + \frac{1}{3M_{\rm pl}^2}\left[2XP_{,X} - P + \frac{p_A^2}{2}f(\phi)^{-2}e^{-4\alpha-4\sigma}\right], \tag{5}$$

$$\ddot\alpha = -3\dot\sigma^2 - \frac{1}{M_{\rm pl}^2}\left[XP_{,X} + \frac{p_A^2}{3}f(\phi)^{-2}e^{-4\alpha-4\sigma}\right], \tag{6}$$

$$\ddot\sigma = -3\dot\alpha\,\dot\sigma + \frac{p_A^2}{3M_{\rm pl}^2}f(\phi)^{-2}e^{-4\alpha-4\sigma}, \tag{7}$$

$$(P_{,X} + 2XP_{,XX})\ddot\phi + 3P_{,X}\dot\alpha\,\dot\phi + P_{,X\phi}\dot\phi^2 - P_{,\phi} - p_A^2f(\phi)^{-3}f_{,\phi}(\phi)e^{-4\alpha-4\sigma} = 0, \tag{8}$$

where $H \equiv \dot\alpha = \dot a/a$ is the Hubble expansion rate, and $P_{,X} \equiv \partial P/\partial X$ etc. We define the energy densities of the inflaton and the vector field, respectively, as

$$\rho_\phi \equiv 2XP_{,X} - P, \qquad \rho_A \equiv \frac{p_A^2}{2}f(\phi)^{-2}e^{-4\alpha-4\sigma}. \tag{9}$$

In order to sustain inflation, we require the condition $\rho_\phi \gg \rho_A$. Since the shear term $\Sigma \equiv \dot\sigma$ should be suppressed relative to H, Eq. (5) reads

$$H^2 \simeq \frac{\rho_\phi}{3M_{\rm pl}^2}. \tag{10}$$

On using the slow-roll parameter $\epsilon \equiv -\dot H/H^2$, Eq. (6) can be written as

$$\epsilon = \frac{3\dot\sigma^2}{H^2} + \frac{XP_{,X}}{M_{\rm pl}^2H^2} + \frac{2\rho_A}{3M_{\rm pl}^2H^2}. \tag{11}$$

Each term on the r.h.s. of this equation needs to be much smaller than unity. In particular, if the contributions of the shear and the vector-field energy density are negligible, Eq. (11) reduces to the standard relation $\epsilon \simeq XP_{,X}/(M_{\rm pl}^2H^2)$ of k-inflation [30].

From Eq. (7) the shear term obeys

$$\dot{\Sigma} = -3H\Sigma + \frac{2\rho_A}{3M_{\rm pl}^2}. \tag{12}$$

If Σ converges to a constant value, it follows that

$$\frac{\Sigma}{H} \simeq \frac{2\rho_A}{3\rho_\phi}, \tag{13}$$

where we used Eq. (10). If the evolution of ρ_A is proportional to ρ_ϕ, the ratio Σ/H remains constant. This actually happens for anisotropic inflationary attractors discussed in the next section.

III. POWER-LAW k-INFLATION AND THE STABILITY OF ANISOTROPIC FIXED POINTS

On the flat isotropic FLRW background the power-law k-inflation can be realized by the following general Lagrangian [35,36]:

$$P(\phi, X) = Xg(Y), \qquad Y \equiv Xe^{\lambda\phi/M_{\rm pl}}, \tag{14}$$

where g is an arbitrary function of Y, and λ is a constant. Originally, the Lagrangian (14) was derived for the existence of scaling solutions in the presence of a barotropic perfect fluid [32,34]. Under the condition $\lambda^2 < 2P_{,X}$ there exists a power-law inflationary solution for any function of $g(Y)$ [35].

For the choice $g(Y) = 1 - cM_{\rm pl}^4/Y$, where c is a constant, the Lagrangian (14) reduces to $P = X - cM_{\rm pl}^4 e^{-\lambda\phi/M_{\rm pl}}$ [40], in which case the dynamics of anisotropic inflation was studied in Ref. [29]. The dilatonic ghost condensate model $P = -X + ce^{\lambda\phi/M_{\rm pl}}X^2/M_{\rm pl}^4$ [32] corresponds to the choice $g(Y) = -1 + cY/M_{\rm pl}^4$. If we choose the function $g(Y) = -(m^4/Y)\sqrt{1 - 2Y/m^4} - M^4/Y$, we recover the DBI Lagrangian $P = -h(\phi)^{-1}\sqrt{1 - 2h(\phi)X} + h(\phi)^{-1} - V(\phi)$ with $h(\phi)^{-1} = m^4 e^{-\lambda\phi/M_{\rm pl}}$ and $V(\phi) = (M^4 + m^4)e^{-\lambda\phi/M_{\rm pl}}$.

In the following we study inflationary solutions for the Lagrangian (14) on the anisotropic background given by the metric (2).

A. Anisotropic fixed points

For the Lagrangian (14) the field equation of motion (8) reads

$$\ddot{\phi} + 3HA(Y)P_{,X}(Y)\dot{\phi} + \frac{\lambda X}{M_{\rm pl}}\{1 - [g(Y) + 2g_1(Y)]A(Y)\} - 2\frac{f_{,\phi}}{f}\rho_A A(Y) = 0, \tag{15}$$

where

$$g_n(Y) = Y^n \frac{dg^n(Y)}{dY^n}, \qquad P_{,X}(Y) = g(Y) + g_1(Y), \qquad A(Y) = [g(Y) + 5g_1(Y) + 2g_2(Y)]^{-1}. \tag{16}$$

The quantity $A = (P_{,X} + 2XP_{,XX})^{-1}$ is related to the sound speed c_s, as $c_s^2 = P_{,X}A$ [32,41].

In order to study the dynamics of anisotropic power-law k-inflation, it is convenient to introduce the following dimensionless variables:

$$x_1 = \frac{\dot{\phi}}{\sqrt{6}HM_{\rm pl}}, \qquad x_2 = \frac{M_{\rm pl}e^{-\frac{\lambda\phi}{2M_{\rm pl}}}}{\sqrt{3}H}, \qquad x_3 = \frac{\dot{\sigma}}{H}, \qquad x_4 = \frac{\sqrt{\rho_A}}{\sqrt{3}HM_{\rm pl}}. \tag{17}$$

The variable Y is related to x_1 and x_2 via

$$Y/M_{\rm pl}^4 = x_1^2/x_2^2. \tag{18}$$

From Eq. (5) there is the constraint equation

$$x_4^2 = 1 - x_3^2 - x_1^2(P_{,X} + g_1), \tag{19}$$

whereas Eq. (6) gives

$$\frac{\dot{H}}{H^2} = -2 - x_3^2 - x_1^2(P_{,X} - 2g_1). \tag{20}$$

On using Eqs. (7), (15), (19), and (20), we obtain the following autonomous equations:

$$x_1'(N) = \frac{1}{2}x_1[4 + 2x_3^2 - \sqrt{6}\lambda x_1 + 2x_1^2(P_{,X} - 2g_1)] - \frac{\sqrt{6}A}{2}[\sqrt{6}x_1P_{,X} - x_1^2(P_{,X} + g_1)(\lambda + 2\mu) + 2(1 - x_3^2)\mu], \tag{21}$$

$$x_2'(N) = \frac{1}{2}x_2[4 + 2x_3^2 - \sqrt{6}\lambda x_1 + 2x_1^2(P_{,X} - 2g_1)], \tag{22}$$

$$x_3'(N) = (x_3^2 - 1)(x_3 - 2) + x_1^2[P_{,X}(x_3 - 2) - 2g_1(x_3 + 1)], \tag{23}$$

where $N = \ln a$, $x_i'(N) = dx_i(N)/dN$ $(i = 1, 2, 3)$, and $\mu = -M_{\rm pl}f_{,\phi}/f$. In the following we focus on the case of constant μ, i.e., the coupling

$$f(\phi) = f_0 e^{-\mu\phi/M_{\rm pl}}, \tag{24}$$

where f_0 is a constant.

The fixed points responsible for the cosmic acceleration correspond to nonzero values of x_1 and x_2. Setting the r.h.s. of Eqs. (21)–(23) to be 0, we obtain the following two fixed points:

(i) Isotropic fixed point

$$P_{,X}(Y)=\frac{\lambda}{\sqrt{6}x_1},\qquad g_1(Y)=\frac{6-\sqrt{6}\lambda x_1}{6x_1^2},\qquad x_3=0,\qquad x_4=0. \tag{25}$$

(ii) Anisotropic fixed point

$$\begin{aligned}P_{,X}(Y)&=\frac{(\lambda+2\mu)[2\sqrt{6}-(\lambda+6\mu)x_1]}{8x_1},\\ g_1(Y)&=\frac{[2\sqrt{6}-(\lambda+2\mu)x_1](\sqrt{6}-\lambda x_1)}{8x_1^2},\\ x_3&=\frac{\sqrt{6}}{4}(\lambda+2\mu)x_1-1,\\ x_4^2&=\frac{1}{8}[3(\lambda+2\mu)x_1-2\sqrt{6}](\sqrt{6}-\lambda x_1).\end{aligned} \tag{26}$$

Provided $g(Y)$ is given, the quantities Y and x_1 are known by solving the first two equations of (25) or (26). In the Appendix we discuss more explicit expressions of isotropic and anisotropic solutions corresponding to the fixed points (25) and (26), respectively. For both the isotropic and anisotropic fixed points, the slow-roll parameter is simply given by

$$\epsilon=-\frac{\dot{H}}{H^2}=\frac{\sqrt{6}}{2}\lambda x_1, \tag{27}$$

where we used Eq. (20). If $\lambda x_1>0$, the power-law inflation $a\propto t^{2/(\sqrt{6}\lambda x_1)}$ is realized. Violation of the condition $\lambda x_1>0$ means that the fixed points correspond to superinflationary solutions with $\dot{H}>0$. Then, the condition for the cosmic acceleration with a decreasing Hubble parameter is given by

$$0<\lambda x_1<\frac{\sqrt{6}}{3}. \tag{28}$$

The presence of the anisotropic fixed point (ii) implies that $x_4^2>0$. This translates to

$$3(\lambda+2\mu)x_1>2\sqrt{6}, \tag{29}$$

where we used (28). Under the condition (29) we also have $x_3>0$.

In the absence of the vector field coupled to ϕ, the ghost is absent for $P_{,X}>0$. For the anisotropic fixed point (ii), the condition $P_{,X}>0$ corresponds to

$$(\lambda+6\mu)x_1<2\sqrt{6}, \tag{30}$$

where we employed the fact that, from Eq. (29), the signs of x_1 and $\lambda+2\mu$ are the same.

From Eqs. (5) and (6) the total energy density and pressure are given by $\rho_t=2XP_{,X}-P+p_A^2f(\phi)^{-2}e^{-4\alpha-4\sigma}/2$ and $P_t=P+p_A^2f(\phi)^{-2}e^{-4\alpha-4\sigma}/6$, respectively. Then we have

$$\rho_t+P_t=2H^2M_{\rm pl}^2(3x_1^2P_{,X}+2x_4^2). \tag{31}$$

If $P_{,X}>0$, then the null energy condition (NEC) $\rho_t+P_t>0$ is automatically satisfied. At the anisotropic fixed point (ii), it is also possible to satisfy the NEC even for $P_{,X}<0$. Substituting Eq. (26) into Eq. (31), it follows that

$$\rho_t+P_t=2H^2M_{\rm pl}^2[\sqrt{6}(3\mu+2\lambda)x_1-9(\lambda+2\mu)^2x_1^2/8-3]. \tag{32}$$

Then, the NEC translates to

$$\frac{2\sqrt{6}}{9}\frac{4\lambda+6\mu-\sqrt{\lambda(7\lambda+12\mu)}}{(\lambda+2\mu)^2}<x_1<\frac{2\sqrt{6}}{9}\frac{4\lambda+6\mu+\sqrt{\lambda(7\lambda+12\mu)}}{(\lambda+2\mu)^2}, \tag{33}$$

whose existence requires that $\lambda(7\lambda+12\mu)>0$. Let us consider the case where $\lambda>0$ and $\mu>0$. As long as the upper bound of Eq. (33) is larger than the value $2\sqrt{6}/[3(\lambda+2\mu)]$, there are some values of x_1 consistent with both (29) and (33). This is interpreted as the condition $\lambda+\sqrt{\lambda(7\lambda+12\mu)}>0$, which is in fact satisfied for $\lambda>0$.

In the limit that $\lambda\to0$ the condition (29) reduces to $x_1>\sqrt{6}/(3\mu)$, while the region (33) shrinks to the point $x_1=\sqrt{6}/(3\mu)$. When $\lambda=0$, Eq. (32) reads

$$\rho_t+P_t=-H^2M_{\rm pl}^2(\sqrt{6}-3\mu x_1)^2, \tag{34}$$

which is negative for $x_1>\sqrt{6}/(3\mu)$. Notice that, from Eq. (27), the limit $\lambda\to0$ corresponds to the de Sitter solution with constant H. Hence the NEC is generally violated on the de Sitter solution. The violation of the NEC means that the dominant energy condition (DEC; $\rho_t\geq|P_t|$) and the strong energy condition (SEC; $\rho_t+P_t\geq0$ and $\rho_t+3P_t\geq0$) are not satisfied [42]. This property is consistent with Wald's cosmic no-hair conjecture [43] stating that, in the presence of an energy-momentum tensor satisfying both DEC and SEC, the anisotropic hair does not survive on the de Sitter background.

In summary, for $\lambda>0$ and $\mu>0$, the anisotropic fixed points satisfying both $P_{,X}>0$ and the NEC exist in the regime

$$\frac{2\sqrt{6}}{3(\lambda+2\mu)}<x_1<\frac{2\sqrt{6}}{\lambda+6\mu}, \tag{35}$$

whose upper bound (which comes from $P_{,X}>0$) gives a tighter constraint than that in Eq. (33) (which comes from the NEC). As long as $\lambda>0$, there are some allowed values of x_1 which exist in the region (35). Under the condition (35) the anisotropic parameter $x_3=\Sigma/H$ is in the range

$$0<x_3<\frac{2}{1+6\mu/\lambda}, \tag{36}$$

whose upper limit is determined by the ratio μ/λ. For compatibility of the two conditions (28) and (29) we

require that $\mu/\lambda > 1/2$. Hence the anisotropic parameter is generally constrained to be $x_3 < 1/2$.

B. Stability of the anisotropic fixed point

We study the stability of the anisotropic inflationary solution by considering small perturbations δx_1, δx_2, and δx_3 about the anisotropic critical point (ii) given by $(x_1^{(c)}, x_2^{(c)}, x_3^{(c)})$, i.e.,

$$x_i = x_i^{(c)} + \delta x_i \qquad (i = 1, 2, 3). \tag{37}$$

We expand the function $g(Y)$ around $Y_c = (x_1^{(c)}/x_2^{(c)})^2 M_{\rm pl}^4$, i.e.,

$$g(Y) = g_c + g'(Y_c)(Y - Y_c) + \frac{g''(Y_c)}{2}(Y - Y_c)^2 + \cdots, \tag{38}$$

where $g_c \equiv g(Y_c)$ and $g'(Y) = dg(Y)/dY$. Taking the terms up to the second order of $Y - Y_c$, we have $\delta P_{,X} = (2g_c' + Y_c g_c'')\delta Y$ and $\delta g_1 = (g_c' + Y_c g_c'')\delta Y$. Note that g_c'' and δY can be expressed as $g_c'' = (A^{-1} - g_c - 5Y g_c')/(2Y_c^2)$ and $\delta Y/M_{\rm pl}^4 = 2[x_1^{(c)}\delta x_1/(x_2^{(c)})^2 - (x_1^{(c)})^2\delta x_2/(x_2^{(c)})^3]$. In the following we omit the subscripts "c" and "(c)" for the background quantities.

Perturbing Eqs. (21)–(23) around the critical point (ii), we can write the resulting perturbation equations in the form

$$\frac{d}{dN}\begin{pmatrix}\delta x_1\\ \delta x_2\\ \delta x_3\end{pmatrix} = \mathcal{M}\begin{pmatrix}\delta x_1\\ \delta x_2\\ \delta x_3\end{pmatrix}, \tag{39}$$

where $\mathcal{M}$ is the 3×3 matrix expressed in terms of x_1, Y, A, λ, and μ. Using the relations (26), the three eigenvalues of the matrix $\mathcal{M}$, which determine the stability of the anisotropic point (ii), are

$$\gamma_1 = \frac{\sqrt{6}}{2}\lambda x_1 - 3, \qquad \gamma_2 = \frac{\sqrt{6}}{4}\lambda x_1 - \frac{3}{2} + \frac{1}{8}\sqrt{\mathcal{D}},$$
$$\gamma_3 = \frac{\sqrt{6}}{4}\lambda x_1 - \frac{3}{2} - \frac{1}{8}\sqrt{\mathcal{D}}, \tag{40}$$

where

$$\mathcal{D} = 16[9 - \sqrt{6}(2\lambda + 3\mu)x_1]^2 + 3A(\lambda + 2\mu)[3(\lambda + 2\mu)x_1 - 2\sqrt{6}][(\lambda^2 + 28\mu\lambda + 36\mu^2)x_1 - 2\sqrt{6}(\lambda + 14\mu)]. \tag{41}$$

As long as the condition (28) of the cosmic acceleration is satisfied, we have that $\gamma_1 < 0$. The term $\sqrt{6}\lambda x_1/4 - 3/2$ inside γ_2 and γ_3 is also negative under the same condition. If $\mathcal{D}$ is negative, then the anisotropic fixed point is a stable spiral. For positive $\mathcal{D}$ the eigenvalue γ_3 is negative. When $x_1 = 2\sqrt{6}/[3(\lambda + 2\mu)]$, the eigenvalue γ_2 vanishes for the same signs of λ and μ. In order to see this more precisely, we substitute $x_1 = 2\sqrt{6}/[3(\lambda + 2\mu)] + \delta$ into the eigenvalue γ_2, where δ is a small parameter. It then follows that

$$\gamma_2 = -\frac{3\sqrt{6}}{16}(\lambda + 2\mu)[4 + A(\lambda + 2\mu)(\lambda + 4\mu)]\delta + O(\delta^2). \tag{42}$$

Provided that $A > 0$, we have $\gamma_2 < 0$ either for $\lambda > 0$, $\mu > 0$, $\delta > 0$ or $\lambda < 0$, $\mu < 0$, $\delta < 0$. Then the anisotropic fixed point is stable for $3(\lambda + 2\mu)x_1 > 2\sqrt{6}$, which is exactly equivalent to the condition (29). Plugging $x_1 = 2\sqrt{6}/[3(\lambda + 2\mu)] + \delta$ into $P_{,X}$ of Eq. (26), we obtain

$$P_{,X} = \frac{1}{4}\lambda(\lambda + 2\mu) - \frac{3\sqrt{6}}{32}(\lambda + 2\mu)^3\delta + O(\delta^2), \tag{43}$$

which is positive at $x_1 = 2\sqrt{6}/[3(\lambda + 2\mu)]$ for the same signs of λ and μ. If $P_{,X} > 0$ and $A > 0$ then the sound speed squared $c_s^2 = P_{,X}A$ is positive, so that the Laplacian instability of small-scale perturbations can be avoided. For x_1 away from $2\sqrt{6}/[3(\lambda + 2\mu)]$, the quantity $P_{,X}$ can be negative. In order to avoid this, we require the condition (30). We also note that A can change its sign at some value of x_1. Since this depends on the forms of the function $g(Y)$, we shall study this property in several different models in Sec. IV.

We recall that x_3 and x_4^2 exactly vanish at $3(\lambda + 2\mu)x_1 = 2\sqrt{6}$. In order to keep the small level of anisotropies ($x_3 \ll 1$ and $x_4^2 \ll 1$), it is required that x_1 is only slightly larger than the critical value $2\sqrt{6}/[3(\lambda + 2\mu)]$ for positive λ and μ. In this regime the stability of the anisotropic fixed point is ensured for $A > 0$.

IV. CONCRETE MODELS OF POWER-LAW INFLATION

In this section we study the existence of anisotropic fixed points as well as their stabilities in concrete models of power-law inflation. For simplicity we shall focus on the case of the positive values of λ and μ.

A. Canonical field with an exponential potential

Let us first consider the model

$$P = X - cM_{\rm pl}^4 e^{-\lambda\phi/M_{\rm pl}} \qquad (c = \text{constant}), \tag{44}$$

i.e., the function $g(Y) = 1 - cM_{\rm pl}^4/Y$. Solving the first two equations of (26) for this function, we obtain the following anisotropic fixed point:

$$x_1 = \frac{2\sqrt{6}(\lambda + 2\mu)}{\lambda^2 + 8\mu\lambda + 12\mu^2 + 8},$$
$$cx_2^2 = \frac{6(2 + 2\mu^2 + \mu\lambda)(8 + 12\mu^2 + 4\mu\lambda - \lambda^2)}{(\lambda^2 + 8\mu\lambda + 12\mu^2 + 8)^2},$$
$$x_3 = \frac{2(\lambda^2 + 2\mu\lambda - 4)}{\lambda^2 + 8\mu\lambda + 12\mu^2 + 8},$$
$$x_4^2 = \frac{3(\lambda^2 + 2\mu\lambda - 4)(8 + 12\mu^2 + 4\mu\lambda - \lambda^2)}{(\lambda^2 + 8\mu\lambda + 12\mu^2 + 8)^2}, \tag{45}$$

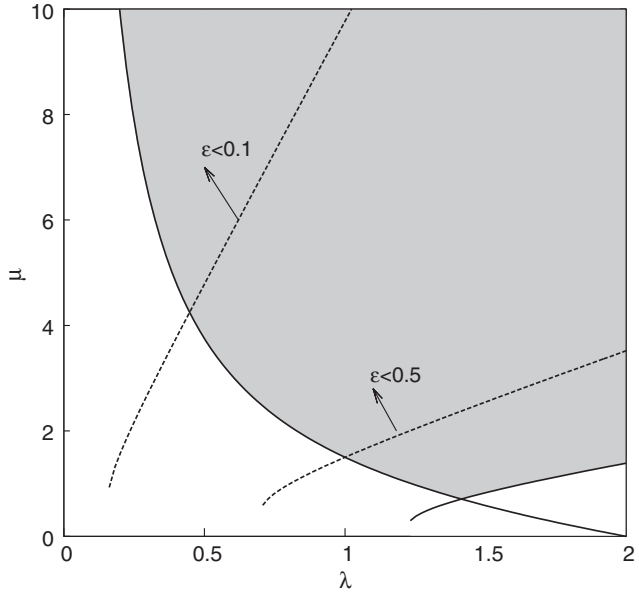

FIG. 1. The parameter space in the (λ, μ) plane for the model $P = X - cM_{\rm pl}^4 e^{-\lambda\phi/M_{\rm pl}}$. The two solid curves, which determine the minimum values of μ for large and small λ, correspond to the bounds (46) and (47), respectively. The two dotted curves correspond to $\epsilon = 0.1$ and $\epsilon = 0.5$. In order to realize $\epsilon \ll 1$, we require that $\mu/\lambda \gg 1$.

which agree with those derived in Ref. [29]. The upper bound of Eq. (28) translates to

$$8 + 12\mu^2 - 4\mu\lambda - 5\lambda^2 > 0, \tag{46}$$

which is satisfied for $\mu \gg \lambda$. The condition (29) for the existence of the anisotropic fixed point is interpreted as

$$\lambda^2 + 2\mu\lambda - 4 > 0. \tag{47}$$

Since $P_{,X} = 1 > 0$, x_1 is smaller than the upper bound of Eq. (35). In this model the quantity A is 1, so that the stability of the anisotropic inflationary solution is ensured under the condition (47) in the regime where x_1 is not far away from the value $2\sqrt{6}/[3(\lambda + 2\mu)]$. Even for $x_1 \gg 2\sqrt{6}/[3(\lambda + 2\mu)]$ the determinant $\mathcal{D}$ appearing in γ_2 of Eq. (40) becomes negative and hence the fixed point is a stable spiral. This means that the anisotropic inflationary solution is an attractor under the condition (47) [29].

In Fig. 1 we show the viable parameter space in the (λ, μ) plane satisfying the two bounds (46) and (47). The stable anisotropic inflation can be realized for the parameters in the shaded region. We also plot the two curves corresponding to $\epsilon = 0.1$ and $\epsilon = 0.5$. For λ and μ satisfying the conditions $\mu \gg \lambda$ and $\mu \gg 1$, we approximately have $x_1 \simeq \sqrt{6}/(3\mu)$ from Eq. (45) and hence $\epsilon \simeq \lambda/\mu$ from Eq. (27). The slow-roll parameter ϵ of the order of 10^{-2} can be realized for $\mu/\lambda = O(10^2)$. If $\mu/\lambda = 10^2$, for example, the condition $\lambda^2 + 2\mu\lambda - 4 > 0$ translates to $\mu = 10^2\lambda > 14$.

In the limit that $\lambda \to 0$, the condition $\lambda^2 + 2\mu\lambda - 4 > 0$ is not fulfilled. Hence, in this model, the stable anisotropic solution does not exist on the de Sitter background. This comes from the fact that the field is frozen in the slow-roll limit ($\epsilon \to 0$), so that there is no variation of the coupling $f(\phi)$ in Eq. (24) to give rise to anisotropic solutions.

B. Generalized ghost condensate

The second model is the generalized ghost condensate given by the Lagrangian

$$P = -X + \frac{c}{M_{\rm pl}^{4n}} e^{n\lambda\phi/M_{\rm pl}} X^{n+1} \qquad (c, n = \text{constant with } n \geq 1), \tag{48}$$

in which case $g(Y) = -1 + c(Y/M_{\rm pl}^4)^n$. The dilatonic ghost condensate model [32] corresponds to the case $n = 1$. From the first two equations of (26) we find that

$$x_1 = \frac{-\sqrt{6}[3\lambda + 2\mu + (5\lambda + 6\mu)n] \pm \sqrt{6[3\lambda + 2\mu + (5\lambda + 6\mu)n]^2 + 48(n+1)[2(4 - \lambda^2 - 5\lambda\mu - 6\mu^2)n - \lambda(\lambda + 2\mu)]}}{2[2(4 - \lambda^2 - 5\lambda\mu - 6\mu^2)n - \lambda(\lambda + 2\mu)]}. \tag{49}$$

Since the plus sign of Eq. (49) can give positive values of x_1, we use this solution in the following discussion. Then the anisotropic parameter x_3 reads

$$x_3 = \frac{3\lambda + 10\mu + (\lambda + 6\mu)n - \sqrt{(9\lambda^2 - 20\lambda\mu - 60\mu^2 + 64)n^2 + 2(3\lambda^2 - 20\lambda\mu - 36\mu^2 + 32)n + (\lambda - 2\mu)^2}}{3\lambda + 2\mu + (5\lambda + 6\mu)n + \sqrt{(9\lambda^2 - 20\lambda\mu - 60\mu^2 + 64)n^2 + 2(3\lambda^2 - 20\lambda\mu - 36\mu^2 + 32)n + (\lambda - 2\mu)^2}}. \tag{50}$$

The condition (35) translates to

$$\mu_1 < \mu < \mu_2, \quad \text{where } \mu_1 \equiv \frac{\sqrt{(2n+1)^2\lambda^2 + 24n(n+1)} - (n+2)\lambda}{6(n+1)}, \qquad \mu_2 \equiv \frac{1}{12}\left(\lambda + \sqrt{\lambda^2 + \frac{96n}{n+1}}\right). \tag{51}$$

From the condition (28) the variable μ is bounded to be

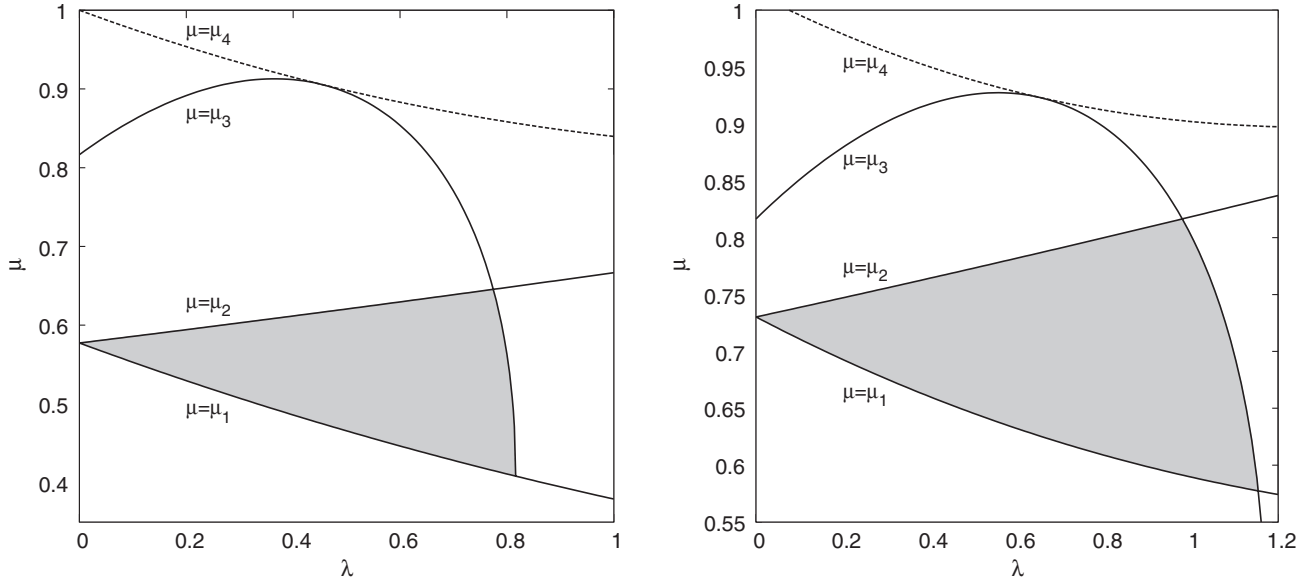

FIG. 2. The parameter space in the (λ, μ) plane for the generalized ghost condensate model with $n = 1$ (left) and $n = 4$ (right). The four curves correspond to the borders given in Eqs. (51)–(53). In the shaded region, all the conditions (51)–(53) are satisfied.

$$\mu < \mu_3, \quad \text{where } \mu_3 \equiv \frac{(2n+1)\lambda + \sqrt{24n^2 - (11n^2 + 26n - 1)\lambda^2}}{6n}. \tag{52}$$

For the determinant of Eq. (49) to be positive, we require that

$$\mu < \mu_4, \quad \text{where } \mu_4 \equiv \frac{4\sqrt{2n(5n+1)(n+1)^2\lambda^2 + 4n(n+1)(15n^2+18n-1)} - (5n^2+10n+1)\lambda}{2(15n^2+18n-1)}. \tag{53}$$

In Fig. 2 we plot the parameter space in the (λ, μ) plane satisfying the conditions (51)–(53) for $n = 1$ and $n = 4$. In the limit that $\lambda \to 0$, the region described by (51) shrinks to the point $\mu = \sqrt{2n/[3(n+1)]}$. As we see in Fig. 2, the region (51) tends to be wider for larger λ. The condition (52) gives upper bounds of λ and μ. The intersection point of the curves $\mu = \mu_1$ and $\mu = \mu_3$ is given by $(\lambda, \mu) = (\sqrt{2n/(n+2)}, \sqrt{n/[2(n+2)]})$, whereas the curves $\mu = \mu_2$ and $\mu = \mu_3$ intersect at the point $(\lambda, \mu) = (\sqrt{6n/[5(n+1)]}, \sqrt{5n/[6(n+1)]})$. For $n \geq 1$ the parameters λ and μ are in the range

$$0 < \lambda < \sqrt{\frac{2n}{n+2}}, \qquad \sqrt{\frac{n}{2(n+2)}} < \mu < \sqrt{\frac{5n}{6(n+1)}}. \tag{54}$$

We note that the condition (53) does not provide an additional bound. From Eq. (54) the parameter μ is of the order of 0.1 (with the maximum value $\mu = \sqrt{5/6}$ in the limit $n \to \infty$).

In Fig. 3 we plot the phase space trajectories in the two-dimensional plane (x_3, x_4) for $n = 1$, $\lambda = 0.35$, and $\mu = 0.5$. The trajectories with different initial conditions converge to the anisotropic fixed point (ii) and hence the fixed point is stable. As long as μ is close to the lower bound $\mu = \mu_1$, the anisotropic parameter $x_3 = \Sigma/H$ is much smaller than 1. For increasing μ the anisotropy gets larger. In the numerical simulation of Fig. 3 the slow-roll parameter is $\epsilon = 0.521$ along the anisotropic attractor. In order to realize ϵ of the order of 10^{-2}, we require that $\lambda = O(10^{-2})$.

For μ close to its upper bound, it can happen that the stability of the anisotropic fixed point is subject to change. In fact, the parameter $A = 1/[c(n+1)(2n+1) \times (Y/M_{\rm pl}^4)^n - 1]$ diverges at $c(Y/M_{\rm pl}^4)^n = 1/[(n+1)(2n+1)]$. This leads to the sign change of the determinant (41) from negative to positive by passing the singular point at $\mu = \mu_5$. If $n = 1$ then we have $\mu_5 = \sqrt{\lambda^2 + 8}/3 - \lambda/6$, so the anisotropic fixed point is stable for

$$\mu < \sqrt{\lambda^2 + 8}/3 - \lambda/6. \tag{55}$$

This does not give an additional bound to those given in Eqs. (51)–(53). When $n > 1$ the condition (55) is more involved, but the situation is similar to that discussed for $n = 1$. It is worth mentioning that, for $n = 1$, the condition (55) is equivalent to $x_3 < 1$.

The maximum value of x_3 is reached for $(\lambda, \mu) = (\sqrt{6n/[5(n+1)]}, \sqrt{5n/[6(n+1)]})$. Substituting these values into Eq. (50) we have $x_3 = 1/3$, which corresponds to the upper bound of (36) with $\mu/\lambda = 5/6$. Hence the anisotropic parameter is constrained to be

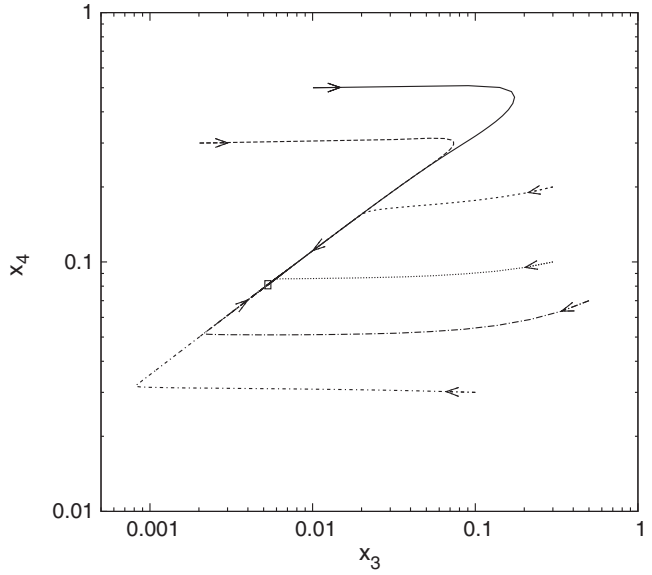

FIG. 3. The phase space in the two-dimensional plane (x_3, x_4) for the dilatonic ghost condensate model with the Lagrangian $P = -X + e^{\lambda\phi/M_{\rm pl}}X^2/M_{\rm pl}^4$. The model parameters are chosen to be $\lambda = 0.35$ and $\mu = 0.5$ with the initial condition $x_1 = 1.0$ and several different initial values of x_2 and x_3. The solutions finally converge to the anisotropic fixed point $(x_3, x_4) = (5.306\times10^{-3}, 8.109\times10^{-2})$ with $x_1 = 1.216$, $x_2 = 1.629$, and $\epsilon = 0.521$.

$$\Sigma/H < 1/3, \tag{56}$$

which holds independent of n. This bound comes from the combination of the conditions $P_{,X} > 0$ and $\lambda x_1 < \sqrt{6}/3$. If we impose the NEC $\rho_t + P_t > 0$ instead of $P_{,X} > 0$, the upper bound (56) gets larger. However, such a large anisotropy is not accepted observationally.

In summary, for $n \geq 1$, there exist the allowed parameter spaces satisfying all the conditions (51)–(53). In order to realize the sufficient amount of inflation ($\epsilon \ll 1$) with the suppressed anisotropy ($x_3 \ll 1$), we require that $\lambda \ll 1$ and that μ is close to the lower bound μ_1.

C. DBI model

The DBI model is characterized by the Lagrangian [33]

$$P = -h(\phi)^{-1}\sqrt{1 - 2h(\phi)X} + h(\phi)^{-1} - V(\phi), \tag{57}$$

where $h(\phi)$ and $V(\phi)$ are functions of ϕ. For the choice $g(Y) = -(m^4/Y)\sqrt{1 - 2Y/m^4} - M^4/Y$, where m and M are constants having a dimension of mass, we obtain the Lagrangian (57) with $h(\phi)^{-1} = m^4 e^{-\lambda\phi/M_{\rm pl}}$ and $V(\phi) = (M^4 + m^4)e^{-\lambda\phi/M_{\rm pl}}$. The ultrarelativistic regime corresponds to the case where the quantity Y/m^4 is close to $1/2$. In order to sustain a sufficient amount of inflation in this regime, we require that the ratio $c_M \equiv M^4/m^4$ is much larger than 1 [36].

Since $P_{,X} = [1 - 2h(\phi)X]^{-1/2} > 0$ in the DBI model, the upper bound of Eq. (35) and the NEC are automatically satisfied. We also have $A = (1 - 2Y/m^4)^{3/2}$, so that there is no divergence associated with the determinant (41) in the regime $Y/m^4 < 1/2$. From the first two equations of (26) we find that the anisotropic fixed point satisfies the fourth order equation of x_1, but it is not analytically solvable for general values of λ, μ, and c_M. However, substituting the lower bound of Eq. (35) into the fourth order equation of x_1, we obtain the following constraint:

$$\mu > \frac{2\sqrt{\lambda^4 + 12c_M\lambda^2 + 36} - \lambda^2}{6\lambda}. \tag{58}$$

In the ultrarelativistic regime the quantity Y/m^4 is close to $1/2$, so that $P_{,X} = (1 - 2Y/m^4)^{-1/2}$ is much larger than 1. Using the bound (29), the anisotropic fixed point of Eq. (26) satisfies the relation $P_{,X} < \lambda(\lambda + 2\mu)/4$, i.e.,

$$\sqrt{1 - \frac{2Y}{m^4}} > \frac{4}{\lambda(\lambda + 2\mu)}. \tag{59}$$

In order to realize the situation where Y/m^4 is close to $1/2$, we require that $\lambda(\lambda + 2\mu) \gg 1$. As x_1 is away from the value $2\sqrt{6}/[3(\lambda + 2\mu)]$, there is a tendency that the anisotropic fixed point deviates from the ultrarelativistic regime because of the decrease of $P_{,X}$. In the following we focus on the situation where x_1 is close to $2\sqrt{6}/[3(\lambda + 2\mu)]$, in which case the anisotropic fixed point is stable with a small anisotropy.

For $x_1 \simeq 2\sqrt{6}/[3(\lambda + 2\mu)]$, the slow-roll parameter is given by $\epsilon \simeq 2\lambda/(\lambda + 2\mu)$ from Eq. (27). In order to realize $\epsilon \ll 1$, we need the condition $\mu \gg \lambda$. Then, the condition $\lambda(\lambda + 2\mu) \gg 1$ discussed above can be interpreted as $\mu\lambda \gg 1$. From Eq. (58) the condition $\mu\lambda \gg 1$ can be satisfied for $c_M\lambda^2 \gg 10$, in which case Eq. (58) reduces to $\mu > 2\sqrt{3c_M}/3$. When $c_M = \mathcal{O}(100)$, for example, we have $\mu \gtrsim \mathcal{O}(10)$ and $\lambda \gtrsim \mathcal{O}(1)$.

For compatibility of the two conditions (28) and (29), we require that $\mu > \lambda/2$. If $\lambda > \lambda_m \equiv \sqrt{2c_M + 2\sqrt{c_M^2 + 3}}$, the condition $\mu > \lambda/2$ is stronger than the bound (58). In the regime $\lambda < \lambda_m$, as λ gets larger around the lower bound of μ given in Eq. (58), the slow-roll parameter also increases and it reaches the value $\epsilon = 1$ at $\lambda = \lambda_m$. Then, the realization of anisotropic inflation demands the condition

$$\lambda < \sqrt{2c_M + 2\sqrt{c_M^2 + 3}}. \tag{60}$$

When $c_M = 500$, for example, the condition (60) translates to $\lambda < 44.7$. As long as λ is much smaller than the upper bound of Eq. (60), anisotropic inflation with $\epsilon \ll 1$ occurs in the ultrarelativistic regime for μ close to the lower bound of Eq. (58).

In Fig. 4 we show the trajectories of solutions in the three-dimensional phase space $(Y/m^4, x_3, x_4)$ for $c_M = 500$, $\lambda = 1$, and $\mu = 26$. In this case the solutions with several different initial conditions converge to the

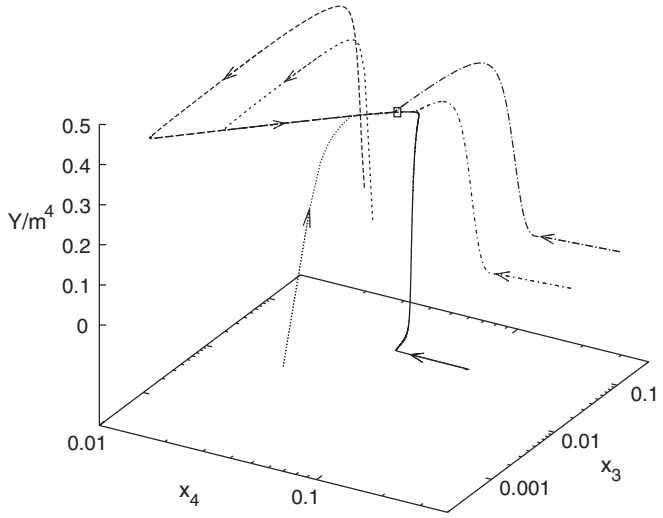

FIG. 4. The three-dimensional phase space $(Y/m^4, x_3, x_4)$ for the DBI model with the Lagrangian $P=-m^4 e^{-\lambda\phi/M_{\rm pl}}\times\sqrt{1-2Xe^{\lambda\phi/M_{\rm pl}}/m^4}-M^4e^{-\lambda\phi/M_{\rm pl}}$. The model parameters are chosen to be $c_M = M^4/m^4 = 500$, $\lambda = 1$, and $\mu = 26$ with the initial condition $Y/m^4 = 10^{-2}$ and several different initial values of x_3 and x_4. The trajectories with different initial conditions converge to the anisotropic fixed point $(Y/m^4, x_3, x_4) = (4.911\times10^{-1}, 5.494\times10^{-3}, 9.020\times10^{-2})$ with $x_1 = 3.098\times10^{-2}$ and $\epsilon = 3.794\times10^{-2}$.

anisotropic fixed point with constant values of x_3, x_4 satisfying $x_3 \ll 1$ and $x_4 \ll 1$. The attractor is in the ultrarelativistic regime (Y/m^4 close to $1/2$) with ϵ of the order of 0.01. It is also possible to realize stable anisotropic inflation for $\lambda = O(10)$ and $\mu = O(10)$, but in such cases the slow-roll parameter ϵ is not much smaller than 1.

In summary, the stable anisotropic DBI inflation can be realized in the ultrarelativistic regime under the conditions (58) and (60) for μ close to the lower bound (58).

V. CONCLUSIONS

We have studied the dynamics of anisotropic power-law k-inflation in the presence of a vector kinetic term $F_{\mu\nu}F^{\mu\nu}$ coupled to the inflaton field ϕ. Such a power-law k-inflation can be accommodated for the general Lagrangian $P = Xg(Y)$, where $Y = Xe^{\lambda\phi/M_{\rm pl}}$. The cosmological dynamics in the anisotropic cosmological background is known by solving the autonomous equations (21)–(23).

Without specifying the functional forms of $g(Y)$, we have shown that anisotropic inflationary solutions exist for the exponential coupling (24). The anisotropic fixed point satisfying Eq. (26) is present for $3(\lambda + 2\mu)x_1 > 2\sqrt{6}$, where $x_1 = \dot\phi/(\sqrt{6}HM_{\rm pl})$. The condition for the cosmic acceleration translates to $\lambda x_1 < \sqrt{6}/3$. Provided the conditions $3(\lambda + 2\mu)x_1 > 2\sqrt{6}$ and $A = (P_{,X} + 2XP_{,XX})^{-1} > 0$ are satisfied, the anisotropic inflationary fixed point is stable in the regime where x_1 is close to $2\sqrt{6}/[3(\lambda + 2\mu)]$. This property holds irrespective of the forms of $g(Y)$ and hence the anisotropic hair survives whenever the anisotropic power-law inflationary solutions are present.

The quantity A is related to the sound speed c_s as $c_s^2 = AP_{,X}$, so that the Laplacian instability can be avoided for $A > 0$ and $P_{,X} > 0$. For the models in which $P_{,X}$ can be negative, it happens that the NEC $\rho_t + P_t > 0$ is not satisfied for some model parameters; see Eq. (31). In the de Sitter limit ($\lambda \to 0$) we found that the NEC is always violated for anisotropic solutions. This is consistent with Wald's cosmic no-hair conjecture. As long as λ is not 0, there are some parameter spaces in which the NEC is satisfied.

In Sec. IV we applied our general results to concrete models of k-inflation such as the generalized ghost condensate and the DBI model. In the generalized ghost condensate we showed that there are allowed parameter spaces in the (λ, μ) plane where stable anisotropic inflationary solutions with $P_{,X} > 0$ and $A > 0$ are present; see Fig. 2. The existence of such anisotropic attractors is confirmed in the numerical simulation of Fig. 3. In this model anisotropic inflation with $\lambda = O(0.1)$ and $\mu = O(0.1)$ occurs, but if the slow-roll parameter ϵ is of the order of 10^{-2}, it follows that $\lambda = O(10^{-2})$. In the DBI model there exists stable anisotropic inflationary solutions in the ultrarelativistic regime ($Y/m^4 \simeq 1/2$) for μ close to the lower bound of Eq. (58) and λ satisfying the bound (60) (see Fig. 4). The model parameters are typically of the order of $\lambda = O(1)$ and $\mu = O(10)$ to realize $\epsilon = O(10^{-2})$.

While we focused on the vector field coupled to the inflaton in this paper, we expect that the similar property should also hold for the two-form field models studied in Ref. [26] in the context of potential-driven slow-roll inflation. It is also known that in k-inflation the non-Gaussianities of scalar metric perturbations can be large for the equilateral shape due to the nonlinear field self-interactions inside the Hubble radius [44]. It will be of interest to study how the nonlinear estimator $f_{\rm NL}$ of the single-field k-inflation is modified by the interactions between inflaton and the vector/two-form fields. We leave these issues for future work.

ACKNOWLEDGMENTS

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APPENDIX

In this appendix we provide more explicit analysis for the properties of isotropic and anisotropic solutions given in Eqs. (25) and (26).

The power-law inflationary solution corresponds to $\dot\alpha = H = \zeta/t$, where ζ is a constant larger than 1. Since the quantities $x_3 = \dot\sigma/H$ and $x_1 = \dot\phi/(\sqrt{6}HM_{\rm pl})$ are constant along the fixed points, we have that $\dot\sigma = \eta/t$ and

$\dot{\phi}/M_{\rm pl} = \xi/t$, respectively, where $\eta = \zeta x_3$ and $\xi = \sqrt{6}x_1\zeta$. Then, the evolution of α, σ, and ϕ is characterized by

$$\alpha = \zeta \log\frac{t}{t_0}, \qquad \sigma = \eta \log\frac{t}{t_0}, \qquad \frac{\phi}{M_{\rm pl}} = \xi \log\frac{t}{t_0}, \tag{A1}$$

where t_0 is a constant. For $Y = Xe^{\lambda\phi/M_{\rm pl}}$ to be constant, we need to require

$$\lambda\xi = 2. \tag{A2}$$

For the solutions (A1) satisfying the relation (A2), the dimensionless variables defined in Eq. (17) read

$$x_1 = \frac{2}{\sqrt{6}\lambda\zeta}, \quad x_2 = \frac{M_{\rm pl}t_0}{\sqrt{3}\zeta}, \quad x_3 = \frac{\eta}{\zeta}, \quad x_4^2 = \frac{W^2}{6\zeta^2}, \tag{A3}$$

where $W^2 = p_A^2 t_0^2/(M_{\rm pl}^2 f_0^2)$. Substituting the solutions (A1) into Eqs. (5)–(8), we obtain

$$\mu\xi - 2\zeta - 2\eta = -1, \tag{A4}$$

$$\zeta^2 = \eta^2 + \frac{2P_{,X} - g}{6}\xi^2 + \frac{W^2}{6}, \tag{A5}$$

$$\zeta = 3\eta^2 + \frac{P_{,X}}{2}\xi^2 + \frac{W^2}{3}, \tag{A6}$$

$$\eta = 3\zeta\eta - \frac{W^2}{3}, \tag{A7}$$

$$\xi - 3AP_{,X}\zeta\xi - \frac{\lambda}{2}(1 - AP_X)\xi^2 + \frac{\lambda}{2}(P_{,X} - g)A\xi^2 - \mu AW^2 = 0. \tag{A8}$$

Notice that Eq. (A4) follows from the demand to have the time dependence t^{-2} for the last term of Eq. (5). Plugging the relation (A2) into Eq. (A8), it follows that

$$W^2 = \frac{2}{\lambda\mu}[P_{,X}(2 - 3\zeta) - g]. \tag{A9}$$

First, let us seek isotropic solutions. In this case, Eq. (A4) is absent and $\eta = W = 0$. From Eqs. (A6) and (A5) we obtain the following relations:

$$\zeta = \frac{2P_{,X}}{\lambda^2}, \qquad P_{,X}^2 - \frac{\lambda^2}{3}P_{,X} + \frac{\lambda^2}{6}g = 0, \tag{A10}$$

respectively. Note that these are consistent with Eq. (A9). On using the correspondence (A3), we find that the two relations (A10) are equivalent to the first two of Eq. (25).

Now, we move on to anisotropic power-law solutions. From Eq. (A4) we have $\zeta + \eta = 1/2 + \mu/\lambda$. Combining Eqs. (A6) and (A7), it follows that $\zeta + \eta = 3\eta(\zeta + \eta) + P_{,X}\xi^2/2$. Then we obtain

$$\zeta = \frac{(\lambda + 2\mu)(\lambda + 6\mu) + 8P_{,X}}{6\lambda(\lambda + 2\mu)}, \tag{A11}$$

$$\eta = \frac{\lambda^2 + 2\lambda\mu - 4P_{,X}}{3\lambda(\lambda + 2\mu)}, \tag{A12}$$

by which the anisotropy of the expansion is

$$\frac{\Sigma}{H} = \frac{\eta}{\zeta} = \frac{2(\lambda^2 + 2\lambda\mu - 4P_{,X})}{(\lambda + 2\mu)(\lambda + 6\mu) + 8P_{,X}}. \tag{A13}$$

Substituting Eqs. (A11) and (A12) into Eq. (A7), we have

$$W^2 = -\frac{(\lambda^2 + 2\lambda\mu - 4P_{,X})(\lambda^2 - 4\lambda\mu - 12\mu^2 - 8P_{,X})}{2\lambda^2(\lambda + 2\mu)^2}. \tag{A14}$$

From Eq. (A11) and the first of Eq. (A3) we can express $P_{,X}$ in terms of x_1. This exactly corresponds to the first relation of Eq. (26). Substituting this into Eqs. (A13) and (A14) and using the correspondence (A3), we obtain the third and fourth relations of Eq. (26). On using Eqs. (A9) and (A14) as well as the relation $P_{,X} = g + g_1$, we find that g_1 can be expressed as the second of Eq. (26). If we want to obtain the metric explicitly, we need to use Eq. (A3) to get the following relations:

$$\zeta = \frac{2}{\sqrt{6}\lambda x_1}, \qquad \eta = \frac{2x_3}{\sqrt{6}\lambda x_1}. \tag{A15}$$

The anisotropic power-law inflationary solutions are given by

$$ds^2 = -dt^2 + \left(\frac{t}{t_0}\right)^{2\zeta}\left[\left(\frac{t}{t_0}\right)^{-4\eta}dx^2 + \left(\frac{t}{t_0}\right)^{2\eta}(dy^2 + dz^2)\right]. \tag{A16}$$

Now it is easy to write down the metric corresponding to the solutions derived in Sec. IV.

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