Non-Regularly Varying and Non-Periodic Oscillation of the On-Diagonal Heat Kernels on Self-Similar Fractals

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Non-regularly varying and non-periodic oscillation of the on-diagonal heat kernels on self-similar fractals

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Dedicated to my mother on the occasion of her 65th birthday

Abstract. Let $p_t(x,y)$ be the canonical heat kernel associated with a self-similar Dirichlet form on a self-similar fractal and let d_{s} denote the spectral dimension of the Dirichlet space, so that $t^{d_{\mathrm{s}}/2}p_t(x,x)$ is uniformly bounded from above and below by positive constants for $t \in (0,1]$. In this article it is proved that, under certain mild assumptions on $p_t(x,y)$, for a "generic" (in particular, almost every) point x of the fractal, $p_{(\cdot)}(x,x)$ *neither* varies regularly at 0 (and hence the limit $\lim_{t\downarrow 0} t^{d_{\mathrm{s}}/2}p_t(x,x)$ does *not* exist) *nor* admits a periodic function $G : \mathbb{R} \to \mathbb{R}$ such that $p_t(x,x) = t^{-d_{\mathrm{s}}/2}G(-\log t) + o(t^{-d_{\mathrm{s}}/2})$ as $t \downarrow 0$. This result is applicable to most typical nested fractals (but *not* to the d-dimensional standard Sierpiński gasket with $d \geq 2$ at this moment) and *all* generalized Sierpiński carpets, and the assertion of non-regular variation is established also for post-critically finite self-similar fractals (possibly without good symmetry) possessing a certain simple topological property.

Contents

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1. Introduction

Heat kernels on fractals are believed to exhibit highly oscillatory behavior as opposed to the classical case of Riemannian manifolds. For example, as a generalization of the results of [**12, 30, 7**] for the standard Sierpiński gasket, Lindstrøm [**32**] constructed canonical Brownian motion on a certain large class of self-similar fractals called *nested fractals*, and Kumagai [**29**] proved that its transition density (heat kernel) $p = p_t(x,y)$ satisfies the two-sided *sub-Gaussian estimate*

$$\frac{c_{1.1}}{t^{d_\mathrm{s}/2}} \exp\biggl(-\Bigl(\frac{\rho(x,y)^{d_\mathrm{w}}}{c_{1.1}t}\Bigr)^{\frac{1}{d_\mathrm{w}-1}}\biggr) \leq p_t(x,y) \leq \frac{c_{1.2}}{t^{d_\mathrm{s}/2}} \exp\biggl(-\Bigl(\frac{\rho(x,y)^{d_\mathrm{w}}}{c_{1.2}t}\Bigr)^{\frac{1}{d_\mathrm{w}-1}}\biggr). \tag{1.1}$$

Here $c_{1.1}, c_{1.2} \in (0,\infty)$ are constants, $d_\mathrm{s} \in [1,\infty)$ and $d_\mathrm{w} \in [2,\infty)$ are also constants called the *spectral dimension* and the *walk dimension* of the fractal, respectively, and ρ is a suitably constructed geodesic metric on the fractal which is comparable to some power of the Euclidean metric[1]. Later Fitzsimmons, Hambly and Kumagai [**10**] extended these results to a larger class of self-similar fractals called *affine nested fractals*. In particular, given an affine nested fractal K, for any $x \in K$ we have

$$c_{1.1} \leq t^{d_\mathrm{s}/2} p_t(x,x) \leq c_{1.2}, \qquad t \in (0,1], \tag{1.2}$$

and then it is natural to ask how $t^{d_\mathrm{s}/2} p_t(x,x)$ behaves as $t \downarrow 0$ and especially whether the limit

$$\lim_{t\downarrow 0} t^{d_\mathrm{s}/2} p_t(x,x) \tag{1.3}$$

exists or not. As Barlow and Perkins conjectured in [**7**, Problem 10.5] in the case of the Sierpiński gasket, this limit was believed *not* to exist for most self-similar fractals, but this problem had remained open until the author's recent paper [**21**].

It was proved in [**21**] that, under very weak assumptions on the affine nested fractal K, the limit (1.3) does not exist for "generic" (hence almost every) $x \in K$, and that the same is true for *any* $x \in K$ when K is either the d-dimensional standard (level-2) Sierpiński gasket with $d \geq 2$ or the N-polygasket with $N \geq 3$, $N/4 \notin \mathbb{N}$ (see Figure 2 below). The proofs of these facts, however, heavily relied on the two important features of affine nested fractals — they are *finitely ramified* (i.e. can be made disconnected by removing finitely many points) and *highly symmetric*. In particular, the results of [**21**] were not applicable to self-similar fractals without these properties like Hata's tree-like set, which admits no isometric symmetry as shown in Proposition 4.17 below, and the Sierpiński carpet, which is *infinitely ramified* (see Figure 1).

The purpose of this paper is twofold. First, we replace the assumptions of finite ramification and symmetry of the fractal with certain properties of the heat kernel which are expected to be much robuster in many cases. In particular, our main results imply the non-existence of the limit (1.3) for "generic" points x in the cases of Hata's tree-like set and of the Sierpiński carpet. Secondly, we establish not only the non-existence of the limit (1.3) but also more detailed descriptions of the oscillation of $p_t(x,x)$ as $t \downarrow 0$ for "generic" points x of the self-similar fractal.

More specifically, let K be the self-similar set determined by a finite family $\{F_i\}_{i\in S}$ of injective contraction maps on a complete metric space, so that K is a

[1] To be precise, the heat kernel estimate in [**29**] had been presented in terms of the Euclidean metric, and the geodesic metric ρ was constructed later in [**10**].

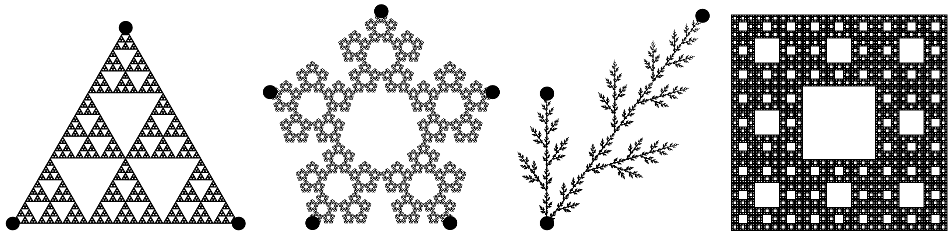

FIGURE 1. Examples of self-similar fractals within the reach of the main results of this paper. From the *left*, two-dimensional level-3 Sierpiński gasket, pentagasket (5-polygasket), Hata's tree-like set and Sierpiński carpet

compact metrizable topological space satisfying $K = \bigcup_{i\in S} F_i(K)$, and let V_0 be the set of boundary points of K (see Definition 2.3 for the precise definition of V_0). Assume $K \neq \overline{V_0}$, let μ be a Borel measure on K satisfying $\mu(F_{w_1} \circ \cdots \circ F_{w_m}(K)) = \mu_{w_1} \cdots \mu_{w_m}$ for any $w_1 \ldots w_m \in \bigcup_{n\in\mathbb{N}} S^n$ for some $(\mu_i)_{i\in S} \in (0,1)^S$ with $\sum_{i\in S} \mu_i = 1$, and assume that $(\mathcal{E}, \mathcal{F})$ is a *self-similar* symmetric regular Dirichlet form on $L^2(K, \mu)$ with resistance scaling factor $\mathbf{r}$ given by $\mathbf{r} = \bigl(\mu_i^{2/d_{\mathrm{s}}-1}\bigr)_{i\in S}$ for some $d_{\mathrm{s}} \in (0,\infty)$ (see Definition 2.7 for details). Further assuming that $(K, \mu, \mathcal{E}, \mathcal{F})$ admits a continuous heat kernel $p = p_t(x,y)$ and that the upper inequality of (1.1) holds for $t \in (0,1]$ for some $d_{\mathrm{w}} \in (1,\infty)$ and a suitable metric ρ on K satisfying $\mu(B_s(x,\rho)) \leq c_{1.3} s^{d_{\mathrm{s}} d_{\mathrm{w}}/2}$, $(s,x) \in (0,1] \times K$, $B_s(x,\rho) := \{y \in K \mid \rho(x,y) < s\}$, we establish the following assertions as the main results of this paper:

(NRV) $p_{(\cdot)}(x,x)$ *does* ***not*** *vary regularly at* 0 *for "generic"* $x \in K$*, if*

$$\limsup_{t\downarrow 0} \frac{p_t(y,y)}{p_t(z,z)} > 1 \qquad \text{for some } y, z \in K \setminus \overline{V_0}. \tag{1.4}$$

(NP) *"Generic"* $x \in K$ *does* ***not*** *admit a periodic function* $G : \mathbb{R} \to \mathbb{R}$ *such that*

$$p_t(x,x) = t^{-d_{\mathrm{s}}/2} G(-\log t) + o(t^{-d_{\mathrm{s}}/2}) \qquad \text{as } t \downarrow 0, \text{ if} \tag{1.5}$$

$$\liminf_{t\downarrow 0} \frac{p_t(y,y)}{p_t(z,z)} > 1 \qquad \text{for some } y, z \in K \setminus \overline{V_0}. \tag{1.6}$$

Note that we still have the on-diagonal estimate (1.2) in this situation as shown in Proposition 2.16 below, and recall (see e.g. [**9**, Section VIII.8]) that a Borel measurable function $f : (0,\infty) \to (0,\infty)$ is said to *vary regularly at* 0 if and only if the limit $\lim_{t\downarrow 0} f(\alpha t)/f(t)$ exists in $(0,\infty)$ for any $\alpha \in (0,\infty)$. In particular, if $x \in K$ and $p_{(\cdot)}(x,x)$ does not vary regularly at 0, then it also follows that the limit (1.3) does not exist. Note also that a log-periodic behavior of the form (1.5) is valid when x is the fixed point of $F_{w_1} \circ \cdots \circ F_{w_m}$ for some $w_1 \ldots w_m \in \bigcup_{n\in\mathbb{N}} S^n$ by Proposition 3.7 below, which is a slight generalization of [**16**, Theorems 4.6 and 5.3]. Such a log-periodic behavior has been observed in various contexts of analysis on fractals such as Laplacian eigenvalue asymptotics on self-similar sets discussed in [**27, 16, 19**] and long time asymptotics of the transition probability of the simple random walk on self-similar graphs treated in [**13, 28**]. Contrary to these existent results, the combination of (NRV) and (NP) asserts that $p_t(x,x)$ oscillates as $t \downarrow 0$ in a *non*-log-*periodic* but still non-regularly varying way for "generic" $x \in K$ as long as the assumption (1.6) is satisfied.

In fact, for (NP) we will actually prove the following stronger result: *if* (1.6) *is satisfied, then for "generic" $x \in K$ and any periodic function $G : \mathbb{R} \to \mathbb{R}$,*

$$\limsup_{t\downarrow 0}\Bigl|t^{d_{\mathrm{s}}/2}p_t(x,x) - G(-\log t)\Bigr| \geq \frac{M_{y,z}}{2}, \tag{1.7}$$

where $M_{y,z} := \liminf_{t\downarrow 0} t^{d_{\mathrm{s}}/2}\bigl(p_t(y,y) - p_t(z,z)\bigr) \in (0,\infty)$ *with* y, z *as in* (1.6).

The proof of (NRV) and (NP) relies only on the self-similarity of the Dirichlet space, the joint continuity of the heat kernel and its sub-Gaussian upper bound, which are all known to hold quite in general, and is free of extra a priori assumptions. Instead, however, we still need certain topological properties of the fractal K to verify (1.4) or (1.6). Roughly speaking, (1.4) can be verified if the local geometry of K around $F_{w_1} \circ \cdots \circ F_{w_m}(x)$ is not the same for all $x \in V_0$ and $w_1 \dots w_m \in \bigcup_{n\in\mathbb{N}} S^n$ with $F_{w_1} \circ \cdots \circ F_{w_m}(x) \notin V_0$, and so can (1.6) if in addition the fractal K (or more precisely, the Dirichlet space $(K, \mu, \mathcal{E}, \mathcal{F})$) has good symmetry.

For example, when K is the two-dimensional level-3 Sierpiński gasket in Figure 1, the barycenter is contained in *three* of the cells $\{F_i(K) \mid i \in S\}$ but each of the other points of $\bigl(\bigcup_{i\in S} F_i(V_0)\bigr) \setminus V_0$ is contained only in *two* of them, which and the dihedral symmetry of K together imply (1.6). The pentagasket also satisfies (1.6) for exactly the same reason, whereas only (1.4) can be verified for Hata's tree-like set due to the lack of symmetry although (1.6) could actually be the case. For the Sierpiński carpet, and its generalizations called *generalized Sierpiński carpets*, (1.6) is proved by using their symmetry under the isometries of the unit cube and the fact that some faces of the cells $\{F_i(K) \mid i \in S\}$ are contained only in *one* cell but the others in *two* cells.

Unfortunately, actually *the author does* ***not*** *have any idea whether* (1.4) *and* (1.6) *are valid for the d-dimensional standard (level-2) Sierpiński gasket with $d \geq 2$*; the argument in the previous paragraph does not work in this case since any $x \in \bigl(\bigcup_{m\in\mathbb{N}}\bigcup_{w_1\dots w_m \in S^m} F_{w_1} \circ \cdots \circ F_{w_m}(V_0)\bigr) \setminus V_0$ has exactly two neighboring cells (see Figure 2 below). In fact, it will be proved in a forthcoming paper [**22**] that $p_{(\cdot)}(x,x)$ *does not vary regularly at* 0 *for any* $x \in K$ for certain specific post-critically finite self-similar fractals K where very detailed information on the eigenvalues of the Laplacian is known, including the d-dimensional standard Sierpiński gasket. This result alone, however, does not exclude the possibility that (1.4) is not valid.

This article is organized as follows. In Section 2, we introduce our framework of self-similar Dirichlet forms on self-similar sets and give the precise statements of our main results (NRV) and (NP) in Theorems 2.17 and 2.18, respectively. Section 3 is devoted to the proof of Theorems 2.17 and 2.18, and then they are applied to post-critically finite self-similar fractals and generalized Sierpiński carpets in Sections 4 and 5, respectively. In Section 4, after recalling basics of self-similar Dirichlet forms on post-critically finite self-similar fractals in Subsection 4.1, we verify (1.6) for those with good symmetry such as affine nested fractals in Subsection 4.2, and (1.4) for those possibly without good symmetry such as Hata's tree-like set in Subsection 4.3. Finally in Section 5, we first collect important facts concerning generalized Sierpiński carpets and their canonical self-similar Dirichlet form and then verify (1.6) for them.

Notation. In this paper, we adopt the following notation and conventions.
(1) $\mathbb{N} = \{1, 2, 3, \dots\}$, i.e. $0 \notin \mathbb{N}$.
(2) The cardinality (the number of elements) of a set A is denoted by $\#A$.

(3) We set $\sup\emptyset := 0$, $\inf\emptyset := \infty$ and set $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$ for $a, b \in [-\infty, \infty]$. All functions in this paper are assumed to be $[-\infty, \infty]$-valued.
(4) For $d \in \mathbb{N}$, $\mathbb{R}^d$ is always equipped with the Euclidean norm $|\cdot|$.
(5) Let E be a topological space. The Borel σ-field of E is denoted by $\mathcal{B}(E)$. We set $C(E) := \{u \mid u : E \to \mathbb{R},\ u \text{ is continuous}\}$, $\mathrm{supp}_E[u] := \overline{\{x \in E \mid u(x) \neq 0\}}$ and $\|u\|_\infty := \sup_{x\in E} |u(x)|$ for $u \in C(E)$. For $A \subset E$, $\mathrm{int}_E A$ denotes its interior in E.
(6) Let E be a set, $\rho : E \times E \to [0, \infty)$ and $x \in E$. We set $\mathrm{dist}_\rho(x, A) := \inf_{y\in A} \rho(x, y)$ for $A \subset E$ and $B_r(x, \rho) := \{y \in E \mid \rho(x, y) < r\}$ for $r \in (0, \infty)$.

2. Framework and main results

In this section, we first introduce our framework of a self-similar set and a self-similar Dirichlet form on it, and then state the main theorems of this paper.

Let us start with standard notions concerning self-similar sets. We refer to [**23**, Chapter 1], [**25**, Section 1.2] and [**19**, Subsection 2.2] for details. Throughout this and the next sections, we fix a compact metrizable topological space K with $\#K \geq 2$, a non-empty finite set S and a continuous injective map $F_i : K \to K$ for each $i \in S$. We set $\mathcal{L} := (K, S, \{F_i\}_{i\in S})$.

Definition 2.1. (1) Let $W_0 := \{\emptyset\}$, where $\emptyset$ is an element called the *empty word*, let $W_m := S^m = \{w_1 \ldots w_m \mid w_i \in S \text{ for } i \in \{1, \ldots, m\}\}$ for $m \in \mathbb{N}$ and let $W_* := \bigcup_{m\in\mathbb{N}\cup\{0\}} W_m$. For $w \in W_*$, the unique $m \in \mathbb{N} \cup \{0\}$ satisfying $w \in W_m$ is denoted by $|w|$ and called the *length of* w. For $i \in S$ and $n \in \mathbb{N} \cup \{0\}$ we write $i^n := i \ldots i \in W_n$.
(2) We set $\Sigma := S^{\mathbb{N}} = \{\omega_1\omega_2\omega_3 \ldots \mid \omega_i \in S \text{ for } i \in \mathbb{N}\}$, which is always equipped with the product topology, and define the *shift map* $\sigma : \Sigma \to \Sigma$ by $\sigma(\omega_1\omega_2\omega_3 \ldots) := \omega_2\omega_3\omega_4 \ldots$. For $i \in S$ we define $\sigma_i : \Sigma \to \Sigma$ by $\sigma_i(\omega_1\omega_2\omega_3 \ldots) := i\omega_1\omega_2\omega_3 \ldots$. For $\omega = \omega_1\omega_2\omega_3 \ldots \in \Sigma$ and $m \in \mathbb{N} \cup \{0\}$, we write $[\omega]_m := \omega_1 \ldots \omega_m \in W_m$.
(3) For $w = w_1 \ldots w_m \in W_*$, we set $F_w := F_{w_1} \circ \cdots \circ F_{w_m}$ ($F_\emptyset := \mathrm{id}_K$), $K_w := F_w(K)$, $\sigma_w := \sigma_{w_1} \circ \cdots \circ \sigma_{w_m}$ ($\sigma_\emptyset := \mathrm{id}_\Sigma$) and $\Sigma_w := \sigma_w(\Sigma)$, and if $w \neq \emptyset$ then $w^\infty \in \Sigma$ is defined by $w^\infty := www \ldots$ in the natural manner.

Definition 2.2. $\mathcal{L}$ is called a *self-similar structure* if and only if there exists a continuous surjective map $\pi : \Sigma \to K$ such that $F_i \circ \pi = \pi \circ \sigma_i$ for any $i \in S$. Note that such π, if exists, is unique and satisfies $\{\pi(\omega)\} = \bigcap_{m\in\mathbb{N}} K_{[\omega]_m}$ for any $\omega \in \Sigma$.

In what follows we always assume that $\mathcal{L}$ is a self-similar structure, so that $\#S \geq 2$ by $\#K \geq 2$ and $\pi(\Sigma) = K$. For $A \subset K$, the closure of A in K is denoted by $\overline{A}$.

Definition 2.3. (1) We define the *critical set* $\mathcal{C}$ and the *post-critical set* $\mathcal{P}$ *of* $\mathcal{L}$ by

$$\mathcal{C} := \pi^{-1}\bigl(\textstyle\bigcup_{i,j\in S,\, i\neq j} K_i \cap K_j\bigr) \qquad \text{and} \qquad \mathcal{P} := \bigcup_{n\in\mathbb{N}} \sigma^n(\mathcal{C}). \tag{2.1}$$

$\mathcal{L}$ is called *post-critically finite*, or *p.c.f.* for short, if and only if $\mathcal{P}$ is a finite set.
(2) We set $V_0 := \pi(\mathcal{P})$, $V_m := \bigcup_{w\in W_m} F_w(V_0)$ for $m \in \mathbb{N}$ and $V_* := \bigcup_{m\in\mathbb{N}} V_m$.
(3) We set $K^I := K \setminus \overline{V_0}$, $K^I_w := F_w(K^I)$ for $w \in W_*$ and $V_{**} := \bigcup_{w\in W_*} F_w(\overline{V_0})$.

V_0 should be considered as the *"boundary"* of the self-similar set K; recall that $K_w \cap K_v = F_w(V_0) \cap F_v(V_0)$ for any $w, v \in W_*$ with $\Sigma_w \cap \Sigma_v = \emptyset$ by [**23**, Proposition 1.3.5-(2)]. Note that $F_w(V_0) = \bigcup_{n\in\mathbb{N}} \pi(\sigma_w \circ \sigma^n(\mathcal{C})) \in \mathcal{B}(K)$ for any $w \in W_*$ by the compactness of Σ. According to [**23**, Lemma 1.3.11], $V_{m-1} \subset V_m$ for any $m \in \mathbb{N}$, and if $V_0 \neq \emptyset$ then V_* is dense in K. Furthermore by [**19**, Lemma 2.11], K^I_w is open in K and $K^I_w \subset K^I$ for any $w \in W_*$.

Definition 2.4. Let $(\mu_i)_{i\in S} \in (0,1)^S$ satisfy $\sum_{i\in S}\mu_i = 1$. A Borel probability measure μ on K is called a *self-similar measure on $\mathcal{L}$ with weight $(\mu_i)_{i\in S}$* if and only if the following equality (of Borel measures on K) holds:

$$\mu = \sum_{i\in S} \mu_i \mu \circ F_i^{-1}. \tag{2.2}$$

Let $(\mu_i)_{i\in S} \in (0,1)^S$ satisfy $\sum_{i\in S}\mu_i = 1$. Then there exists a self-similar measure on $\mathcal{L}$ with weight $(\mu_i)_{i\in S}$. Indeed, if ν is the Bernoulli measure on Σ with weight $(\mu_i)_{i\in S}$, then $\nu \circ \pi^{-1}$ is such a self-similar measure on $\mathcal{L}$; see [**23**, Section 1.4] for details. Moreover by [**25**, Theorem 1.2.7 and its proof], if $K \neq \overline{V_0}$ and μ is a self-similar measure on $\mathcal{L}$ with weight $(\mu_i)_{i\in S}$, then $\mu(K_w) = \mu_w$ and $\mu(F_w(\overline{V_0})) = 0$ for any $w \in W_*$, where $\mu_w := \mu_{w_1}\cdots\mu_{w_m}$ for $w = w_1 \dots w_m \in W_*$ ($\mu_\emptyset := 1$). In particular, a self-similar measure on $\mathcal{L}$ with given weight is unique if $K \neq \overline{V_0}$.

The following lemmas are immediate from the above-mentioned facts.

Lemma 2.5. *Assume $K \neq \overline{V_0}$, let μ be a self-similar measure on $\mathcal{L}$ with weight $(\mu_i)_{i\in S}$ and let $w \in W_*$. Then $\int_K |u\circ F_w| d\mu = \mu_w^{-1}\int_{K_w}|u|d\mu$ for any Borel measurable $u : K \to [-\infty,\infty]$. In particular, if we set $F_w^* u := u\circ F_w$ for $u : K \to [-\infty,\infty]$, then F_w^* defines a bounded linear operator $F_w^* : L^2(K,\mu) \to L^2(K,\mu)$.*

Lemma 2.6. *Let $w \in W_*$. For $u : K \to [-\infty,\infty]$, define $(F_w)_* u : K \to [-\infty,\infty]$ by*

$$(F_w)_* u := \begin{cases} u \circ F_w^{-1} & \text{on } K_w, \\ 0 & \text{on } K \setminus K_w. \end{cases} \tag{2.3}$$

If u is Borel measurable then so is $(F_w)_* u$, and if $K \neq \overline{V_0}$ in addition then $\int_K |(F_w)_* u| d\mu = \mu_w \int_K |u| d\mu$. In particular, if $K \neq \overline{V_0}$, then $(F_w)_*$ defines a bounded linear operator $(F_w)_* : L^2(K,\mu) \to L^2(K,\mu)$.

Next we define the notion of a homogeneously scaled self-similar Dirichlet space and state its basic properties. The following definition is a special case of [**19**, Definition 3.3]. See [**11**, Section 1.1] for basic notions concerning regular Dirichlet forms.

Definition 2.7 (Homogeneously scaled self-similar Dirichlet space). Assume $K \neq \overline{V_0}$. Let μ be a self-similar measure on $\mathcal{L}$ with weight $(\mu_i)_{i\in S}$, let $d_{\mathrm{s}} \in (0,\infty)$ and set $r_i := \mu_i^{2/d_{\mathrm{s}}-1}$ for $i \in S$. $(\mathcal{E},\mathcal{F})$ is called a *homogeneously scaled self-similar Dirichlet form on $L^2(K,\mu)$ with spectral dimension* d_{s} if and only if it is a non-zero symmetric regular Dirichlet form on $L^2(K,\mu)$ satisfying the following conditions:

(SSDF1) $u \circ F_i \in \mathcal{F}$ for any $i \in S$ and any $u \in \mathcal{F}\cap C(K)$.

(SSDF2) For any $u \in \mathcal{F}\cap C(K)$,

$$\mathcal{E}(u,u) = \sum_{i\in S}\frac{1}{r_i}\mathcal{E}(u\circ F_i, u\circ F_i). \tag{2.4}$$

(SSDF3) $(F_i)_* u \in \mathcal{F}$ for any $i \in S$ and any $u \in \mathcal{F}\cap C(K)$ with $\mathrm{supp}_K[u] \subset K^I$.

If $(\mathcal{E},\mathcal{F})$ is a homogeneously scaled self-similar Dirichlet form on $L^2(K,\mu)$ with spectral dimension d_{s}, then $(\mathcal{L},\mu,\mathcal{E},\mathcal{F})$ is called a *homogeneously scaled self-similar Dirichlet space with spectral dimension* d_{s}, and we call $(\mu_i)_{i\in S}$ its *weight*.

In the rest of this section, we assume that $(\mathcal{L},\mu,\mathcal{E},\mathcal{F})$ is a homogeneously scaled self-similar Dirichlet space with weight $(\mu_i)_{i\in S}$ and spectral dimension d_s. Then by [**19**, Lemma 5.5], (SSDF1) and (SSDF2) still hold if $\mathcal{F}\cap C(K)$ is replaced with $\mathcal{F}$.

LEMMA 2.8. *$(\mathcal{E},\mathcal{F})$ is conservative (i.e. $\mathbf{1}\in\mathcal{F}$ and $\mathcal{E}(\mathbf{1},\mathbf{1})=0$) and strongly local. Moreover, $V_0\neq\emptyset$.*

PROOF. Since K is compact and $(\mathcal{E},\mathcal{F})$ is regular, $\mathcal{F}\cap C(K)$ is dense in $(C(K),\|\cdot\|_\infty)$, so that there exists $u\in\mathcal{F}\cap C(K)$ such that $\|\mathbf{1}-u\|_\infty\leq 1/2$. Thus $\mathbf{1}=\min\{2u,\mathbf{1}\}\in\mathcal{F}$, and then it easily follows from (SSDF2) and $\sum_{i\in S}r_i^{-1}=\sum_{i\in S}\mu_i^{1-2/d_\mathrm{s}}>1$ that $\mathcal{E}(\mathbf{1},\mathbf{1})=0$. Moreover, $(\mathcal{E},\mathcal{F})$ is local by [**19**, Lemma 3.4], and it is also easily seen to be strongly local by virtue of its conservativeness.

Suppose $V_0=\emptyset$, so that $\pi:\Sigma\to K$ is a homeomorphism by [**23**, Proposition 1.3.5-(3)]. Then since K_w is compact and open, we easily see from the conservativeness of $(\mathcal{E},\mathcal{F})$ and [**11**, Theorem 1.4.2-(ii) and Exercise 1.4.1] that $\mathbf{1}_{K_w}\in\mathcal{F}$ and $\mathcal{E}(\mathbf{1}_{K_w},\mathbf{1}_{K_w})=0$ for any $w\in W_*$. This fact together with the denseness of the linear span of $\{\mathbf{1}_{K_w}\}_{w\in W_*}$ in $L^2(K,\mu)$ yields $\mathcal{F}=L^2(K,\mu)$ and $\mathcal{E}=0$, contradicting the assumption that $(\mathcal{E},\mathcal{F})$ is non-zero. □

We need to introduce several geometric notions to formulate the assumption of a sub-Gaussian heat kernel upper bound which is required for our main results. We refer the reader to [**25**, Sections 1.1 and 1.3] and [**19**, Section 2] for further details.

DEFINITION 2.9. (1) Let $w,v\in W_*$, $w=w_1\dots w_m$, $v=v_1\dots v_n$. We define $wv\in W_*$ by $wv:=w_1\dots w_m v_1\dots v_n$ ($w\emptyset:=w$, $\emptyset v:=v$). We write $w\leq v$ if and only if $w=v\tau$ for some $\tau\in W_*$. Note that $\Sigma_w\cap\Sigma_v=\emptyset$ if and only if neither $w\leq v$ nor $v\leq w$.
(2) A finite subset Λ of W_* is called a *partition of* Σ if and only if $\Sigma_w\cap\Sigma_v=\emptyset$ for any $w,v\in\Lambda$ with $w\neq v$ and $\Sigma=\bigcup_{w\in\Lambda}\Sigma_w$.
(3) Let Λ_1,Λ_2 be partitions of Σ. We say that Λ_1 is a *refinement of* Λ_2, and write $\Lambda_1\leq\Lambda_2$, if and only if for each $w^1\in\Lambda_1$ there exists $w^2\in\Lambda_2$ such that $w^1\leq w^2$.

DEFINITION 2.10. (1) Set $\gamma_w:=\mu_w^{1/d_\mathrm{s}}$ for $w\in W_*$. We define $\Lambda_1:=\{\emptyset\}$,

$$\Lambda_s:=\{w\mid w=w_1\dots w_m\in W_*\setminus\{\emptyset\},\ \gamma_{w_1\dots w_{m-1}}>s\geq\gamma_w\}\tag{2.5}$$

for each $s\in(0,1)$, and $\mathcal{S}:=\{\Lambda_s\}_{s\in(0,1]}$. We call $\mathcal{S}$ the *scale on* Σ *associated with* $(\mathcal{L},\mu,\mathcal{E},\mathcal{F})$.
(2) For each $(s,x)\in(0,1]\times K$, we define $\Lambda^0_{s,x}:=\{w\in\Lambda_s\mid x\in K_w\}$, $U^{(0)}_s(x):=\bigcup_{w\in\Lambda^0_{s,x}}K_w$, and inductively for $n\in\mathbb{N}$,

$$\Lambda^n_{s,x}:=\{w\in\Lambda_s\mid K_w\cap U^{(n-1)}_s(x)\neq\emptyset\}\quad\text{and}\quad U^{(n)}_s(x):=\bigcup_{w\in\Lambda^n_{s,x}}K_w.\tag{2.6}$$

Clearly $\lim_{s\downarrow 0}\min\{|w|\mid w\in\Lambda_s\}=\infty$, and it is easy to see that Λ_s is a partition of Σ for any $s\in(0,1]$ and that $\Lambda_{s_1}\leq\Lambda_{s_2}$ for any $s_1,s_2\in(0,1]$ with $s_1\leq s_2$. These facts together with [**23**, Proposition 1.3.6] imply that for any $n\in\mathbb{N}\cup\{0\}$ and any $x\in K$, $\{U^{(n)}_s(x)\}_{s\in(0,1]}$ is non-decreasing in s and forms a fundamental system of neighborhoods of x in K. Note also that $\Lambda^n_{s,x}$ and $U^{(n)}_s(x)$ are non-decreasing in $n\in\mathbb{N}\cup\{0\}$ for any $(s,x)\in(0,1]\times K$.

We would like to consider $U_s^{(n)}(x)$ to be a "ball of radius s centered at x". The following definition formulates the situation where $U_s^{(n)}(x)$ may be thought of as actual balls with respect to a distance function on K.

DEFINITION 2.11. (1) Let $\rho : K \times K \to [0,\infty)$. For $\alpha \in (0,\infty)$, ρ is called an *α-qdistance on K* if and only if $\rho^\alpha := \rho(\cdot,\cdot)^\alpha$ is a distance on K. Moreover, ρ is called a *qdistance on K* if and only if it is an α-qdistance on K for some $\alpha \in (0,\infty)$.
(2) A qdistance ρ on K is called *adapted to* $\mathcal{S}$ if and only if there exist $\beta_1, \beta_2 \in (0,\infty)$ and $n \in \mathbb{N}$ such that for any $(s,x) \in (0,1] \times K$,

$$B_{\beta_1 s}(x,\rho) \subset U_s^{(n)}(x) \subset B_{\beta_2 s}(x,\rho). \tag{2.7}$$

If ρ is an α-qdistance on K adapted to $\mathcal{S}$, then ρ^α is compatible with the original topology of K, since $\{U_s^{(n)}(x)\}_{s\in(0,1]}$ is a fundamental system of neighborhoods of x in the original topology of K.

DEFINITION 2.12. We say that $\mathcal{S}$ is *locally finite with respect to $\mathcal{L}$*, or simply $(\mathcal{L},\mathcal{S})$ is *locally finite*, if and only if $\sup\{\#\Lambda_{s,x}^1 \mid (s,x) \in (0,1] \times K\} < \infty$.

Note that by [**23**, Lemma 1.3.6], $(\mathcal{L},\mathcal{S})$ is locally finite if and only if $\sup\{\#\Lambda_{s,x}^n \mid (s,x) \in (0,1] \times K\} < \infty$ for any $n \in \mathbb{N}$. The local finiteness of $(\mathcal{L},\mathcal{S})$ is closely related with local behavior of μ. In fact, we have the following proposition.

PROPOSITION 2.13. *Set $\gamma := \min_{i\in S}\gamma_i$ and let $n \in \mathbb{N}\cup\{0\}$. Then for any $(s,x) \in (0,1] \times K$,*

$$\gamma^{d_s} s^{d_s} \#\Lambda_{s,x}^n \le \mu\bigl(U_s^{(n)}(x)\bigr) \le s^{d_s}\#\Lambda_{s,x}^n. \tag{2.8}$$

In particular, for fixed $n \in \mathbb{N}$, $(\mathcal{L},\mathcal{S})$ is locally finite if and only if there exist $c_{\mathrm{V},n} \in (0,\infty)$ such that $\mu\bigl(U_s^{(n)}(x)\bigr) \le c_{\mathrm{V},n} s^{d_s}$ for any $(s,x) \in (0,1] \times K$,

PROOF. We easily see from the definition (2.5) of Λ_s that

$$\gamma^{d_s} s^{d_s} < \mu_w \le s^{d_s}, \qquad s \in (0,1],\ w \in \Lambda_s. \tag{2.9}$$

Since $\mu(K_w) = \mu_w$ and $\mu(F_w(\overline{V_0})) = 0$ for any $w \in W_*$ by the assumption that $K \ne \overline{V_0}$, (2.9) implies that for any $(s,x) \in (0,1] \times K$,

$$\gamma^{d_s} s^{d_s}\#\Lambda_{s,x}^n \le \sum_{w\in\Lambda_{s,x}^n} \mu_w = \sum_{w\in\Lambda_{s,x}^n} \mu(K_w) = \mu\bigl(U_s^{(n)}(x)\bigr) \le s^{d_s}\#\Lambda_{s,x}^n,$$

proving (2.8). The latter assertion is immediate from (2.8). □

Next we prepare fundamental conditions for our main results concerning the heat kernel of $(K,\mu,\mathcal{E},\mathcal{F})$.

DEFINITION 2.14 (CHK). We say that $(K,\mu,\mathcal{E},\mathcal{F})$ *satisfies* (CHK), or simply (CHK) *holds*, if and only if the Markovian semigroup $\{T_t\}_{t\in(0,\infty)}$ on $L^2(K,\mu)$ associated with $(\mathcal{E},\mathcal{F})$ admits a *continuous integral kernel* p, i.e. a continuous function $p = p_t(x,y) : (0,\infty) \times K \times K \to \mathbb{R}$ such that for any $u \in L^2(K,\mu)$ and any $t \in (0,\infty)$,

$$T_t u = \int_K p_t(\cdot,y)u(y)d\mu(y) \qquad \mu\text{-a.e.} \tag{2.10}$$

Such p, if exists, is unique and satisfies $p_t(x,y) = p_t(y,x) \ge 0$ for any $(t,x,y) \in (0,\infty)\times K\times K$ by a standard monotone class argument. p is called the *(continuous) heat kernel of* $(K,\mu,\mathcal{E},\mathcal{F})$.

Definition 2.15 (CUHK). We say that $(\mathcal{L}, \mu, \mathcal{E}, \mathcal{F})$ *satisfies* (CUHK), or simply (CUHK) *holds*, if and only if $(\mathcal{L}, \mathcal{S})$ is locally finite, $(K, \mu, \mathcal{E}, \mathcal{F})$ satisfies (CHK) and there exist $d_{\mathrm{w}} \in (1, \infty)$, a $(2/d_{\mathrm{w}})$-qdistance ρ on K adapted to $\mathcal{S}$ and $c_{2.1}, c_{2.2} \in (0, \infty)$ such that for any $(t, x, y) \in (0, 1] \times K \times K$,

$$p_t(x, y) \leq c_{2.1} t^{-d_{\mathrm{s}}/2} \exp\biggl(-c_{2.2}\Bigl(\frac{\rho(x, y)^2}{t}\Bigr)^{\frac{1}{d_{\mathrm{w}}-1}}\biggr). \tag{2.11}$$

Note that (CUHK) remains the same if we replace $t^{-d_{\mathrm{s}}/2}$ with $1/\mu\bigl(B_{\sqrt{t}}(x, \rho)\bigr)$ in (2.11) and omit the condition that $(\mathcal{L}, \mathcal{S})$ is locally finite; indeed, this equivalence easily follows from Definition 2.11-(2), Proposition 2.13 and [**19**, Proposition 5.8].

Proposition 2.16. *Suppose that* (CUHK) *holds. Then there exist* $c_{2.3}, c_{2.4} \in (0, \infty)$ *such that for any* $x \in K$*,*

$$c_{2.3} \leq t^{d_{\mathrm{s}}/2} p_t(x, x) \leq c_{2.4}, \qquad t \in (0, 1]. \tag{2.12}$$

Proof. $t^{d_{\mathrm{s}}/2} p_t(x, x) \leq c_{2.1}$ for any $(t, x) \in (0, 1] \times K$ by (2.11). For the lower bound we follow [**24**, Proof of Theorem 2.13]. Let ρ be the qdistance on K as in Definition 2.15. Since $(\mathcal{L}, \mathcal{S})$ is assumed to be locally finite, Definition 2.11-(2) and Proposition 2.13 easily imply that $\mu(B_r(x, \rho)) \leq c_{2.5} r^{d_{\mathrm{s}}}$ for any $(r, x) \in (0, \infty) \times K$ for some $c_{2.5} \in (0, \infty)$, and the same calculation as [**24**, Proof of Lemma 4.6-(1)] shows that $\int_{K \setminus B_{\delta\sqrt{t}}(x, \rho)} p_t(x, y) d\mu(y) \leq 1/2$ for any $(t, x) \in (0, 1] \times K$ for some $\delta \in (0, \infty)$. Now for $(t, x) \in (0, 1] \times K$, the conservativeness of $(\mathcal{E}, \mathcal{F})$ yields $\int_K p_t(x, y) d\mu(y) = 1$, and hence

$$\begin{aligned} \frac{1}{2} &\leq 1 - \int_{K \setminus B_{\delta\sqrt{t}}(x, \rho)} p_t(x, y) d\mu(y) = \int_{B_{\delta\sqrt{t}}(x, \rho)} p_t(x, y) d\mu(y) \\ &\leq \sqrt{\mu\bigl(B_{\delta\sqrt{t}}(x, \rho)\bigr) \int_K p_t(x, y)^2 d\mu(y)} \leq \sqrt{c_{2.5} \delta^{d_{\mathrm{s}}} t^{d_{\mathrm{s}}/2} p_{2t}(x, x)} \end{aligned}$$

by the symmetry and the semigroup property of the heat kernel p, proving the lower inequality in (2.12). □

Now we are in the stage of stating the main theorems of this paper. Note that any Borel measure on K vanishing on $V_*(\in \mathcal{B}(K))$ is of the form $\nu \circ \pi^{-1}$ with ν a Borel measure on Σ, since $\pi|_{\Sigma \setminus \pi^{-1}(V_*)} : \Sigma \setminus \pi^{-1}(V_*) \to K \setminus V_*$ is a homeomorphism. Recall the following notions: a Borel measure ν on Σ is called σ*-ergodic* if and only if $\nu \circ \sigma^{-1} = \nu$ and $\nu(A)\nu(\Sigma \setminus A) = 0$ for any $A \in \mathcal{B}(\Sigma)$ with $\sigma^{-1}(A) = A$, and it is said to have *full support* if and only if $\nu(U) > 0$ for any non-empty open subset U of Σ. Recall also that we set $K^I := K \setminus \overline{V_0}$ and $V_{**} := \bigcup_{w \in W_*} F_w(\overline{V_0})$.

Theorem 2.17. *Suppose that* (CUHK) *holds and that*

$$\limsup_{t \downarrow 0} \frac{p_t(y, y)}{p_t(z, z)} > 1 \qquad \text{for some } y, z \in K^I. \tag{2.13}$$

Then there exists $N_{\mathrm{RV}} \in \mathcal{B}(K)$ *satisfying* $V_{**} \subset N_{\mathrm{RV}}$ *and* $\nu \circ \pi^{-1}(N_{\mathrm{RV}}) = 0$ *for any* σ*-ergodic finite Borel measure* ν *on* Σ *with full support, such that* $p_{(\cdot)}(x, x)$ *does* ***not*** *vary regularly at* 0 *for any* $x \in K \setminus N_{\mathrm{RV}}$*. In particular, the limit* $\lim_{t \downarrow 0} t^{d_{\mathrm{s}}/2} p_t(x, x)$ *does* ***not*** *exist for any* $x \in K \setminus N_{\mathrm{RV}}$*.*

Note that (2.13) does not hold if and only if $\lim_{t \downarrow 0} p_t(y, y)/p_t(z, z) = 1$ for any $y, z \in K^I$.

THEOREM 2.18. *Suppose that* (CUHK) *holds and that*

$$\liminf_{t\downarrow 0}\frac{p_t(y,y)}{p_t(z,z)}>1 \qquad \text{for some } y,z\in K^I. \tag{2.14}$$

Then there exists $N_{\mathrm{P}}\in\mathcal{B}(K)$ *satisfying* $V_{**}\subset N_{\mathrm{P}}$ *and* $\nu\circ\pi^{-1}(N_{\mathrm{P}})=0$ *for any* σ*-ergodic finite Borel measure* ν *on* Σ *with full support, such that for any* $x\in K\setminus N_{\mathrm{P}}$ *and any periodic function* $G:\mathbb{R}\to\mathbb{R}$,

$$\limsup_{t\downarrow 0}\Bigl|t^{d_{\mathrm{s}}/2}p_t(x,x)-G(-\log t)\Bigr|\geq\frac{M_{y,z}}{2}, \tag{2.15}$$

where $M_{y,z}:=\liminf_{t\downarrow 0}t^{d_{\mathrm{s}}/2}\bigl(p_t(y,y)-p_t(z,z)\bigr)\in(0,\infty)$ *with* y,z *as in* (2.14).

Note that by (2.12), for each $y,z\in K$, $\liminf_{t\downarrow 0}p_t(y,y)/p_t(z,z)>1$ if and only if $\liminf_{t\downarrow 0}t^{d_{\mathrm{s}}/2}\bigl(p_t(y,y)-p_t(z,z)\bigr)\in(0,\infty)$.

REMARK 2.19. Let $y,z\in K^I$ be as in (2.13) or (2.14). Then *the sets* N_{RV} *in Theorem* 2.17 *and* N_{P} *in Theorem* 2.18 *can be given explicitly in terms of (and hence can be determined solely by)* y, z *and* π; see (3.8), Lemmas 3.10 and 3.12 below.

The proof of Theorems 2.17 and 2.18 is given in the next section. As we will see in Sections 4 and 5, the conditions (2.13) and (2.14) are satisfied for many typical examples such as most nested fractals and *all* generalized Sierpiński carpets.

3. Proof of Theorems 2.17 and 2.18

Throughout this section, we fix a homogeneously scaled self-similar Dirichlet space $(\mathcal{L}=(K,S,\{F_i\}_{i\in S}),\mu,\mathcal{E},\mathcal{F})$ with weight $(\mu_i)_{i\in S}$ and spectral dimension d_{s} and assume that (CUHK) holds with d_{w} and ρ as in Definition 2.15.

DEFINITION 3.1. Let U be a non-empty open subset of K. We define $\mu|_U:=\mu|_{\mathcal{B}(U)}$,

$$\mathcal{F}_U:=\overline{\{u\in\mathcal{F}\cap C(K)\mid \operatorname{supp}_K[u]\subset U\}} \qquad\text{and}\qquad \mathcal{E}^U:=\mathcal{E}|_{\mathcal{F}_U\times\mathcal{F}_U}, \tag{3.1}$$

where the closure is taken in the Hilbert space $\mathcal{F}$ with inner product $\mathcal{E}_1(u,v):=\mathcal{E}(u,v)+\int_K uv\,d\mu$. $(\mathcal{E}^U,\mathcal{F}_U)$ is called the *part of the Dirichlet form* $(\mathcal{E},\mathcal{F})$ *on* U.

Since $u=0$ μ-a.e. on $K\setminus U$ for any $u\in\mathcal{F}_U$, we can regard $\mathcal{F}_U$ as a linear subspace of $L^2(U,\mu|_U)$ in the natural manner. Under this identification, we have the following lemma.

LEMMA 3.2. *Let* U *be a non-empty open subset of* K. *Then* $(\mathcal{E}^U,\mathcal{F}_U)$ *is a strongly local regular Dirichlet form on* $L^2(U,\mu|_U)$ *whose associated Markovian semigroup* $\{T^U_t\}_{t\in(0,\infty)}$ *admits a unique continuous integral kernel* $p^U=p^U_t(x,y):(0,\infty)\times U\times U\to\mathbb{R}$, *called the* Dirichlet heat kernel on U, *similarly to* (2.10). *Moreover,* $0\leq p^U_t(x,y)=p^U_t(y,x)\leq p_t(x,y)$ *for any* $(t,x,y)\in(0,\infty)\times U\times U$.

PROOF. Recall Lemma 2.8. The regularity of $(\mathcal{E},\mathcal{F})$ yields that of $(\mathcal{E}^U,\mathcal{F}_U)$ by (3.1) and [**11**, Lemma 1.4.2-(ii)], and the strong locality of $(\mathcal{E},\mathcal{F})$ implies that of $(\mathcal{E}^U,\mathcal{F}_U)$. Since $(\mathcal{E},\mathcal{F})$ is conservative, a continuous integral kernel p^U of $\{T^U_t\}_{t\in(0,\infty)}$ exists by [**19**, Lemma 7.11-(2)] and (CUHK), and a monotone class argument immediately shows the uniqueness of such p^U. Finally, the last assertion easily follows from [**25**, (C.2)] and a monotone class argument again. □

LEMMA 3.3. *Let U be a non-empty open subset of K. Then for any $(t,x,y) \in (0,\infty)\times U\times U$,*

$$p_t(x,y) - p^U_t(x,y) \leq \sup_{s\in[t/2,\,t]} \sup_{z\in\overline{U}\setminus U} p_s(x,z) + \sup_{s\in[t/2,\,t]} \sup_{z\in\overline{U}\setminus U} p_s(z,y). \tag{3.2}$$

PROOF. This is immediate from [**15**, Theorem 5.1] (or [**14**, Theorem 10.4]), the continuity of the heat kernels $p_t(x,y)$ and $p^U_t(x,y)$ and the compactness of $\overline{U}$. □

LEMMA 3.4. *Let $w \in W_*$. Then for any $(t,x,y) \in (0,\infty)\times K^I\times K^I$,*

$$(\gamma_w^2 t)^{d_s/2} p^{K^I_w}_{\gamma_w^2 t}\big(F_w(x),F_w(y)\big) = t^{d_s/2} p^{K^I}_t(x,y). \tag{3.3}$$

PROOF. $F_w|_{K^I} : K^I \to K^I_w$ is clearly a homeomorphism, and F^*_w defines a bijection $F^*_w : L^2(K^I_w, \mu|_{K^I_w}) \to L^2(K^I, \mu|_{K^I})$ such that $F^*_w(\mathcal{F}_{K^I_w}) = \mathcal{F}_{K^I}$ by [**19**, Lemma 5.5]. Moreover, $\gamma_w^{d_s}\int_{K^I}(F^*_w u)^2 d\mu = \int_{K^I_w} u^2 d\mu$ for any $u \in L^2(K^I_w, \mu|_{K^I_w})$ by Lemma 2.5 and $\gamma_w^{d_s-2}\mathcal{E}(F^*_w u, F^*_w u) = \mathcal{E}(u,u)$ for any $u \in \mathcal{F}_{K^I_w}$ by (SSDF2). It easily follows from these facts and [**11**, Lemma 1.3.4-(i)] that $F^*_w T^{K^I_w}_{\gamma_w^2 t} = T^{K^I}_t F^*_w$ for any $t \in (0,\infty)$, which and the uniqueness of the continuous heat kernels $p^{K^I_w}$ and p^{K^I} imply (3.3). □

LEMMA 3.5. *There exists $c_{3.1} \in (0,\infty)$ such that for any $x \in K$ and any $w \in W_*$,*

$$\operatorname{dist}_\rho\big(F_w(x), F_w(V_0)\big) \geq c_{3.1}\gamma_w \operatorname{dist}_\rho(x, V_0). \tag{3.4}$$

PROOF. Let $\beta_1, \beta_2 \in (0,\infty)$ and $n \in \mathbb{N}$ be as in Definition 2.11-(2) for the qdistance ρ, let $x \in K$, $w \in W_*$ and set $\delta := \operatorname{dist}_\rho(x,V_0)$. The assertion is obvious for $x \in \overline{V_0}$. Assuming $x \in K\setminus\overline{V_0} = K^I$, by $K = U^{(n)}_1(x) \subset B_{\beta_2}(x,\rho)$ we have $\rho(x,y) < \beta_2$ for any $y \in K$ and hence $\delta \in (0,\beta_2)$ (recall that $V_0 \neq \emptyset$ by Lemma 2.8), and $U^{(n)}_{\delta/\beta_2}(x) \cap \overline{V_0} = \emptyset$ since $U^{(n)}_{\delta/\beta_2}(x) \subset B_\delta(x,\rho) \subset K^I$. Then an induction in k easily shows that $\Lambda^k_{\gamma_w\delta/\beta_2, F_w(x)} = \{wv \mid v \in \Lambda^k_{\delta/\beta_2, x}\}$ for any $k \in \{0,\ldots,n\}$, and hence

$$B_{\gamma_w\delta\beta_1/\beta_2}(F_w(x),\rho) \subset U^{(n)}_{\gamma_w\delta/\beta_2}(F_w(x)) = F_w\big(U^{(n)}_{\delta/\beta_2}(x)\big) \subset K^I_w = K_w \setminus F_w(\overline{V_0}).$$

Thus $\rho(F_w(x),y) \geq \gamma_w\delta\beta_1/\beta_2 = (\beta_1/\beta_2)\gamma_w \operatorname{dist}_\rho(x,V_0)$ for any $y \in F_w(\overline{V_0})$ and (3.4) follows with $c_{3.1} := \beta_1/\beta_2$. □

LEMMA 3.6. *There exist $c_{3.2}, c_{3.3} \in (0,\infty)$ such that for any $(t,x) \in (0,1]\times K^I$ and any $w \in W_*$,*

$$\begin{aligned}&\Big|(\gamma_w^2 t)^{d_s/2} p_{\gamma_w^2 t}\big(F_w(x),F_w(x)\big) - t^{d_s/2}p_t(x,x)\Big|\\ &\qquad\qquad \leq c_{3.2}\exp\Big(-c_{3.3}\operatorname{dist}_\rho(x,V_0)^{\frac{2}{d_w-1}} t^{-\frac{1}{d_w-1}}\Big).\end{aligned} \tag{3.5}$$

PROOF. We easily see from (2.11), Lemmas 3.3 and 3.5 that, with $c_{3.2} := 2^{1+d_s/2}c_{2.1}$ and $c_{3.3} := c_{2.2}c_{3.1}^{\frac{2}{d_w-1}}$, for any $w \in W_*$ and any $(t,x) \in (0,\gamma_w^{-2}]\times K^I$,

$$\begin{aligned}0 &\leq (\gamma_w^2 t)^{d_s/2}\Big(p_{\gamma_w^2 t}\big(F_w(x),F_w(x)\big) - p^{K^I_w}_{\gamma_w^2 t}\big(F_w(x),F_w(x)\big)\Big)\\ &\qquad\qquad \leq c_{3.2}\exp\Big(-c_{3.3}\operatorname{dist}_\rho(x,V_0)^{\frac{2}{d_w-1}} t^{-\frac{1}{d_w-1}}\Big),\end{aligned} \tag{3.6}$$

which together with Lemma 3.4 immediately shows (3.5). □

PROPOSITION 3.7. *Let $w \in W_* \setminus \{\emptyset\}$, set $x_w := \pi(w^\infty)$ and suppose $x_w \in K^I$. Then there exist constants $c_{3.4}, c_{3.5} \in (0,\infty)$ independent of w and a continuous $\log(\gamma_w^{-2})$-periodic function $G_w : \mathbb{R} \to (0,\infty)$ such that for any $t \in \bigl(0, 1 \wedge \mathrm{dist}_\rho(x_w, V_0)^2\bigr]$,*

$$\left| t^{d_s/2} p_t(x_w, x_w) - G_w(-\log t) \right| \le c_{3.4} \exp\Bigl(-c_{3.5}\, \mathrm{dist}_\rho(x_w, V_0)^{\frac{2}{d_w - 1}} t^{-\frac{1}{d_w - 1}} \Bigr). \tag{3.7}$$

PROOF. Note that $F_w(x_w) = x_w \in K_w^I$. Since $p^{K_w^I} \le p^{K^I} \le p$ on $(0,\infty) \times K_w^I \times K_w^I$ by $K_w^I \subset K^I$, [**25**, (C.2)] and a monotone class argument, we see from (3.6) and Lemma 3.4 that for any $t \in (0, \gamma_w^{-2}]$, $(\gamma_w^2 t)^{d_s/2} p^{K_w^I}_{\gamma_w^2 t}(x_w, x_w) - t^{d_s/2} p_t^{K^I}(x_w, x_w)$ is subject to the same upper and lower bounds as those in (3.6) with $x = x_w$. On the other hand, by [**8**, Theorem 2.1.4], the generator Δ_{K^I} of $\{T_t^{K^I}\}_{t \in (0,\infty)}$ has compact resolvent, p^{K^I} admits the eigenfunction expansion [**8**, (2.1.4)], and it easily follows from these facts that $p_t^{K^I}(x,y) \le \bigl(2^{d_s} c_{2.4}^2 e^{\lambda_1^0}\bigr) e^{-\lambda_1^0 t}$ for any $t \in [1,\infty)$, where λ_1^0 denotes the smallest eigenvalue of $-\Delta_{K^I}$. Moreover, we have $\lambda_1^0 > 0$; indeed, if $\mathcal{E}(u,u) = 0$ for some $u \in \mathcal{F}_{K^I} \setminus \{0\}$, then for any $w \in W_*$, $(F_w)_* u \in \mathcal{F}_{K_w^I} \subset \mathcal{F}_{K^I}$ by [**19**, Lemma 5.5] and $\mathcal{E}\bigl((F_w)_* u, (F_w)_* u\bigr) = 0$ by (SSDF2), contradicting $\dim\ker \Delta_{K^I} = \dim\{v \in \mathcal{F}_{K^I} \mid \mathcal{E}(v,v) = 0\} < \infty$. Now exactly the same argument as [**16**, Proof of Theorem 4.6] easily shows the existence of a continuous $\log(\gamma_w^{-2})$-periodic function $G_w : \mathbb{R} \to \mathbb{R}$ satisfying (3.7) with $p_t^{K^I}$ in place of p_t (see also [**23**, Proof of Theorem B.4.3] for the remainder estimate). Then (3.7) follows by using (3.6) with x_w and $\emptyset$ in place of x and w, respectively, and Proposition 2.16 implies that G_w is $(0,\infty)$-valued. □

In the rest of this section, we fix the following setting:

(3.8) Let $y, z \in K^I$, $\omega^y \in \pi^{-1}(y)$ and $\omega^z \in \pi^{-1}(z)$. Define $w_n := [\omega^y]_n$, $v_n := [\omega^z]_n$, $x_n := \pi((w_n v_n)^\infty)$ and $\tilde{x}_n := \pi((v_n w_n)^\infty)$ for $n \in \mathbb{N}$. Also set $n_0 := 1 + \sup\{n \in \mathbb{N} \mid \{x_n, \tilde{x}_n\} \not\subset K^I\}$ and for $n \ge n_0$ let G_n denote the periodic function $G_{w_n v_n}$ given in Proposition 3.7 for $w = w_n v_n$. Finally, set $D := \inf_{n \ge n_0} \mathrm{dist}_\rho(x_n, V_0) \wedge \mathrm{dist}_\rho(\tilde{x}_n, V_0)$.

Note that $n_0 < \infty$ and $D > 0$ by $\lim_{n\to\infty} x_n = y \in K^I$ and $\lim_{n\to\infty} \tilde{x}_n = z \in K^I$.

LEMMA 3.8. *Let $\varepsilon \in (0,\infty)$, $\delta \in (0, 1 \wedge D^2]$ and $\alpha \in (0,1]$, and suppose that $c_{3.4} \exp\bigl(-c_{3.5} (D^2/\delta)^{\frac{1}{d_w - 1}}\bigr) \le \varepsilon/2$. Then there exists $n_1 \ge n_0$ such that for any $n \ge n_1$ and any $t \in [\alpha\delta, \delta]$,*

$$\left| t^{d_s/2} p_t(y,y) - G_n(-\log t) \right| \le \varepsilon, \quad \left| t^{d_s/2} p_t(z,z) - G_n\bigl(-\log(\gamma_{w_n}^2 t)\bigr) \right| \le \varepsilon. \tag{3.9}$$

PROOF. Let $n \ge n_0$ and let $\tilde{G}_n$ be the periodic function $G_{v_n w_n}$ given in Proposition 3.7 for $w = v_n w_n$. Then $\lim_{t\downarrow 0} \bigl| \tilde{G}_n(-\log t) - G_n\bigl(-\log(\gamma_{w_n}^2 t)\bigr) \bigr| = 0$ by Lemma 3.6 and Proposition 3.7, and hence $\tilde{G}_n = G_n\bigl(\cdot - \log(\gamma_{w_n}^2)\bigr)$ in view of the fact that G_n and $\tilde{G}_n$ are both $\log(\gamma_{w_n v_n}^{-2})$-periodic.

Since $\lim_{n\to\infty} x_n = y$, $\lim_{n\to\infty} \tilde{x}_n = z$ and the heat kernel p is uniformly continuous on $[\alpha\delta, \delta] \times K \times K$, we can choose $n_1 \ge n_0$ so that

$$t^{d_s/2} \bigl| p_t(x_n, x_n) - p_t(y,y) \bigr| \le \frac{\varepsilon}{2} \quad \text{and} \quad t^{d_s/2} \bigl| p_t(\tilde{x}_n, \tilde{x}_n) - p_t(z,z) \bigr| \le \frac{\varepsilon}{2} \tag{3.10}$$

for any $n \geq n_1$ and any $t \in [\alpha\delta, \delta]$. Now for such n and t, Proposition 3.7, (3.10), $c_{3.4} \exp\bigl(-c_{3.5}(D^2/\delta)^{\frac{1}{d_{\mathrm{w}}-1}}\bigr) \leq \varepsilon/2$ and $\tilde{G}_n = G_n\bigl(\cdot - \log(\gamma_{w_n}^2)\bigr)$ together immediately yield (3.9). □

LEMMA 3.9. *Assume* $\liminf_{t\downarrow 0} p_t(y,y)/p_t(z,z) > 1$. *Let* $\varepsilon \in (0,\infty)$, $\delta_0 \in (0, 1\wedge D^2]$, $\alpha \in (0,1]$ *and set* $M_{y,z} := \liminf_{t\downarrow 0} t^{d_{\mathrm{s}}/2}\bigl(p_t(y,y) - p_t(z,z)\bigr)$. *Then there exist* $\delta \in (0,\delta_0]$ *and* $n_2 \geq n_0$ *such that for any* $n \geq n_2$ *and any* $t \in [\alpha\delta, \delta]$,

$$G_n(-\log t) \geq \min_{\mathbb{R}} G_n + M_{y,z} - \varepsilon. \tag{3.11}$$

PROOF. Choose $\delta \in (0,\delta_0]$ so that $\inf_{t\in(0,\delta]} t^{d_{\mathrm{s}}/2}\bigl(p_t(y,y) - p_t(z,z)\bigr) \geq M_{y,z} - \varepsilon/3$ and $c_{3.4}\exp\bigl(-c_{3.5}(D^2/\delta)^{\frac{1}{d_{\mathrm{w}}-1}}\bigr) \leq \varepsilon/6$. Also let $n_1 \geq n_0$ be as in Lemma 3.8 for $\varepsilon/3, \delta, \alpha$ and set $n_2 := n_1$. Then by (3.9), for any $n \geq n_2$ and any $t \in [\alpha\delta,\delta]$,

$$\begin{aligned} G_n(-\log t) &\geq t^{d_{\mathrm{s}}/2} p_t(y,y) - \varepsilon/3 \geq t^{d_{\mathrm{s}}/2} p_t(z,z) + M_{y,z} - 2\varepsilon/3 \\ &\geq G_n\bigl(-\log(\gamma_{w_n}^2 t)\bigr) + M_{y,z} - \varepsilon \geq \min_{\mathbb{R}} G_n + M_{y,z} - \varepsilon, \end{aligned}$$

completing the proof. □

LEMMA 3.10. *Let* $q \in K^I$ *and define* $N_q \subset K$ *by*

$$N_q := \left\{ x \in K \;\middle|\; \begin{array}{l} \pi(\sigma^{m_k}(\omega)) \text{ does not converge to } q \text{ as } k\to\infty \text{ for any} \\ \omega \in \pi^{-1}(x) \text{ and any strictly increasing } \{m_k\}_{k\in\mathbb{N}} \subset \mathbb{N} \end{array} \right\}. \tag{3.12}$$

Then $N_q \in \mathcal{B}(K)$, $V_{**} \subset N_q$ *and* $\nu \circ \pi^{-1}(N_q) = 0$ *for any* σ*-ergodic finite Borel measure* ν *on* Σ *with full support.*

PROOF. $V_* \subset N_q$ since $\sigma^m(\pi^{-1}(V_m)) = \mathcal{P}$ for $m \in \mathbb{N}\cup\{0\}$ by [**23**, Proposition 1.3.5-(1)]. Noting that $\pi|_{\Sigma\setminus\pi^{-1}(V_*)} : \Sigma \setminus \pi^{-1}(V_*) \to K \setminus V_*$ is a homeomorphism, we get $N_q = V_* \cup \pi(\{\omega \in \Sigma\setminus\pi^{-1}(V_*) \mid \liminf_{m\to\infty} \rho(\pi(\sigma^m(\omega)), q) > 0\}) \in \mathcal{B}(K)$. Let $x \in V_{**}\setminus V_*$, so that $x = F_w(\pi(\omega))$ for some $w \in W_*$ and $\omega \in \pi^{-1}(\overline{V_0}\setminus V_*)$. By $\pi(\omega) \in \overline{V_0}$, $\pi^{-1}(V_0) = \mathcal{P}$ and the compactness of Σ, there exist $\tau \in \Sigma$ and $\{\omega_n\}_{n\in\mathbb{N}} \subset \mathcal{P}$ such that $\pi(\omega_n) \to \pi(\omega)$ and $\omega_n \to \tau$ as $n\to\infty$, but then $\pi(\omega) = \pi(\tau) \in K\setminus V_*$ and hence $\omega = \tau \in \overline{\mathcal{P}}$. Thus $\sigma^m(\omega) \in \overline{\sigma^m(\mathcal{P})} \subset \overline{\mathcal{P}}$ and $\pi(\sigma^m(\omega)) \in \overline{V_0}$ for any $m \in \mathbb{N}$ and therefore $x \in N_q$ on account of $\pi^{-1}(x) = \{\sigma_w(\omega)\}$, proving $V_{**} \subset N_q$. Finally, since N_q can be written as

$$\bigcup_{n\in\mathbb{N}} \Bigl\{ x \in K \;\Big|\; \lim_{m\to\infty} \mathrm{dist}_\rho\bigl(\pi(\sigma^m(\omega)), K\setminus B_{1/n}(q,\rho)\bigr) = 0 \text{ for any } \omega \in \pi^{-1}(x) \Bigr\},$$

the last assertion follows in exactly the same way as [**21**, Proposition 3.2]. □

LEMMA 3.11. *Let* $x \in K$, $\omega \in \pi^{-1}(x)$, $n \geq n_0$ *and let* $\{m_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ *be strictly increasing and satisfy* $\lim_{k\to\infty} \pi(\sigma^{m_k}(\omega)) = x_n$. *Let* $\varepsilon \in (0,\infty)$, *let* $\delta \in (0, 1\wedge D^2]$ *satisfy* $c_{3.6}\exp\bigl(-c_{3.7}(D^2/\delta)^{\frac{1}{d_{\mathrm{w}}-1}}\bigr) < \varepsilon/3$ *with* $c_{3.6} := c_{3.2} \vee c_{3.4}$ *and* $c_{3.7} := c_{3.3} \wedge c_{3.5}$, *and let* $\alpha \in (0,1]$. *Then there exists* $k_1 \in \mathbb{N}$ *such that for any* $k \geq k_1$ *and any* $t \in [\alpha\delta,\delta]$,

$$\Bigl| \bigl(\gamma_{[\omega]_{m_k}}^2 t\bigr)^{d_{\mathrm{s}}/2} p_{\gamma_{[\omega]_{m_k}}^2 t}(x,x) - G_n(-\log t) \Bigr| \leq \varepsilon. \tag{3.13}$$

Proof. Since $\lim_{k\to\infty}\pi(\sigma^{m_k}(\omega)) = x_n$ and the heat kernel p is uniformly continuous on $[\alpha\delta,\delta]\times K\times K$, we can choose $k_1\in\mathbb{N}$ so that for any $k\geq k_1$,

$$t^{d_{\mathrm{s}}/2}\bigl|p_t\bigl(\pi(\sigma^{m_k}(\omega)),\pi(\sigma^{m_k}(\omega))\bigr)-p_t(x_n,x_n)\bigr|\leq\frac{\varepsilon}{3},\quad t\in[\alpha\delta,\delta], \tag{3.14}$$

$$c_{3.6}\exp\Bigl(-c_{3.7}\operatorname{dist}_\rho\bigl(\pi(\sigma^{m_k}(\omega)),V_0\bigr)^{\frac{2}{d_{\mathrm{w}}-1}}\delta^{-\frac{1}{d_{\mathrm{w}}-1}}\Bigr)\leq\frac{\varepsilon}{3}. \tag{3.15}$$

Now for such k and t, noting that $\pi(\sigma^{m_k}(\omega))\in K\setminus N_{x_n}\subset K^I$ by Lemma 3.10, we obtain (3.13) from Lemma 3.6 with $\pi(\sigma^{m_k}(\omega))$ and $[\omega]_{m_k}$ in place of x and w, (3.15), (3.14), Proposition 3.7 with $w=w_nv_n$, and $c_{3.6}\exp\bigl(-c_{3.7}(D^2/\delta)^{\frac{1}{d_{\mathrm{w}}-1}}\bigr)<\varepsilon/3$. □

Lemma 3.12. *Recalling* (3.12), *set* $N:=\bigcup_{k\geq n_0}\bigcap_{n\geq k}N_{x_n}$ *and let* $x\in K\setminus N$.
(1) *If* $\limsup_{t\downarrow 0}p_t(y,y)/p_t(z,z)>1$, *then* $p_{(\cdot)}(x,x)$ *does not vary regularly at* 0.
(2) *If* $\liminf_{t\downarrow 0}p_t(y,y)/p_t(z,z)>1$, *then* (2.15) *holds for any periodic function* $G:\mathbb{R}\to\mathbb{R}$, *where* $M_{y,z}:=\liminf_{t\downarrow 0}t^{d_{\mathrm{s}}/2}\bigl(p_t(y,y)-p_t(z,z)\bigr)$.

Proof. Let $\omega\in\pi^{-1}(x)$ and let $c_{2.3},c_{2.4}\in(0,\infty)$ be as in Proposition 2.16.
(1) Set $M:=\limsup_{t\downarrow 0}p_t(y,y)/p_t(z,z)-1$ and $\varepsilon:=c_{2.3}M/10$. Choose $\delta\in(0,1\wedge D^2]$ so that $c_{3.6}\exp\bigl(-c_{3.7}(D^2/\delta)^{\frac{1}{d_{\mathrm{w}}-1}}\bigr)<\varepsilon/3$ and $p_\delta(y,y)/p_\delta(z,z)\geq 1+M/2$, and let $n_1\geq n_0$ be as in Lemma 3.8 for these ε,δ and $\alpha=1$. By $x\in K\setminus N$ we can take $n\geq n_1$ such that $x\in K\setminus N_{x_n}$, and then $\lim_{k\to\infty}\pi(\sigma^{m_k}(\omega))=x_n$ for some strictly increasing $\{m_k\}_{k\in\mathbb{N}}\subset\mathbb{N}$. Let $k_1\in\mathbb{N}$ be as in Lemma 3.11 for these $x,\omega,n,\{m_k\}_{k\in\mathbb{N}},\varepsilon,\delta$ and $\alpha=\gamma_{w_n}^2$. Then by (3.13), (3.9) and (2.12), for any $k\geq k_1$,

$$\begin{aligned}&\bigl(\gamma_{[\omega]_{m_k}}^2\delta\bigr)^{d_{\mathrm{s}}/2}p_{\gamma_{[\omega]_{m_k}}^2\delta}(x,x)-\bigl(\gamma_{[\omega]_{m_k}}^2\gamma_{w_n}^2\delta\bigr)^{d_{\mathrm{s}}/2}p_{\gamma_{[\omega]_{m_k}}^2\gamma_{w_n}^2\delta}(x,x)\\&\geq G_n(-\log\delta)-G_n\bigl(-\log(\gamma_{w_n}^2\delta)\bigr)-2\varepsilon\geq\delta^{d_{\mathrm{s}}/2}p_\delta(y,y)-\delta^{d_{\mathrm{s}}/2}p_\delta(z,z)-4\varepsilon\\&\geq(1+M/2-1)\delta^{d_{\mathrm{s}}/2}p_\delta(z,z)-4\varepsilon\geq c_{2.3}M/2-4\varepsilon=\varepsilon,\end{aligned}$$

which together with (2.12) yields, by letting $k\to\infty$,

$$\limsup_{t\downarrow 0}\frac{t^{d_{\mathrm{s}}/2}p_t(x,x)}{(\gamma_{w_n}^2t)^{d_{\mathrm{s}}/2}p_{\gamma_{w_n}^2t}(x,x)}\geq 1+\frac{\varepsilon}{c_{2.4}}>1. \tag{3.16}$$

Now suppose that $p_{(\cdot)}(x,x)$ varies regularly at 0, so that by [**9**, Section VIII.8, Lemma 1], $p_t(x,x)=t^\beta L(t)$ for any $t\in(0,\infty)$ for some $\beta\in\mathbb{R}$ and $L:(0,\infty)\to(0,\infty)$ *varying slowly at* 0 (i.e. such that $\lim_{t\downarrow 0}L(\alpha t)/L(t)=1$ for any $\alpha\in(0,\infty)$). Then (2.12) yields $c_{2.3}\leq t^{\beta+d_{\mathrm{s}}/2}L(t)\leq c_{2.4}$ for $t\in(0,1]$, which and [**9**, Section VIII.8, Lemma 2] imply $\beta=-d_{\mathrm{s}}/2$. It follows that $t^{d_{\mathrm{s}}/2}p_t(x,x)=L(t)$ varies slowly at 0, which contradicts (3.16). Thus $p_t(x,x)$ does not vary regularly at 0.
(2) Let $G:\mathbb{R}\to\mathbb{R}$ be T-periodic with $T\in(0,\infty)$. Let $\varepsilon\in(0,\infty)$, let $\delta_0\in(0,1\wedge D^2]$ be such that $c_{3.6}\exp\bigl(-c_{3.7}(D^2/\delta_0)^{\frac{1}{d_{\mathrm{w}}-1}}\bigr)<\varepsilon/3$, and let $\delta\in(0,\delta_0]$ and $n_2\geq n_0$ be as in Lemma 3.9 for ε,δ_0 and $\alpha=e^{-T}$. By $x\in K\setminus N$ we can choose $n\geq n_2$ so that $\gamma_{w_nv_n}^2\leq e^{-T}$ and $x\in K\setminus N_{x_n}$, and then $\lim_{k\to\infty}\pi(\sigma^{m_k}(\omega))=x_n$ for some strictly increasing $\{m_k\}_{k\in\mathbb{N}}\subset\mathbb{N}$. Let $k_1\in\mathbb{N}$ be as in Lemma 3.11 for these $x,\omega,n,\{m_k\}_{k\in\mathbb{N}},\varepsilon,\delta$ and $\alpha=\gamma_{w_nv_n}^2$. Since G_n is $\log(\gamma_{w_nv_n}^{-2})$-periodic, $\min_{\mathbb{R}}G_n=G_n(-\log t_{n,1})$ for some $t_{n,1}\in[\gamma_{w_nv_n}^2\delta,\delta]$, and then there exist $t_{n,0}\in[e^{-T}\delta,\delta]$ and $l_n\in\mathbb{N}\cup\{0\}$ such that $-\log t_{n,1}=-\log t_{n,0}+l_nT$. Now it follows from $G=G(\cdot+l_nT)$, (3.13) with $\alpha=\gamma_{w_nv_n}^2$, (3.11) with $\alpha=e^{-T}$ and $G_n(-\log t_{n,1})=$

$\min_{\mathbb{R}} G_n$ that for any $k \geq k_1$,

$$\begin{aligned}&\sum_{j\in\{0,1\}}(-1)^j\Bigl(\bigl(\gamma_{[\omega]_{m_k}}^2 t_{n,j}\bigr)^{d_{\mathrm{s}}/2} p_{\gamma_{[\omega]_{m_k}}^2 t_{n,j}}(x,x)-G\bigl(-\log(\gamma_{[\omega]_{m_k}}^2 t_{n,j})\bigr)\Bigr)\\&=\bigl(\gamma_{[\omega]_{m_k}}^2 t_{n,0}\bigr)^{d_{\mathrm{s}}/2} p_{\gamma_{[\omega]_{m_k}}^2 t_{n,0}}(x,x)-\bigl(\gamma_{[\omega]_{m_k}}^2 t_{n,1}\bigr)^{d_{\mathrm{s}}/2} p_{\gamma_{[\omega]_{m_k}}^2 t_{n,1}}(x,x)\\&\geq G_n(-\log t_{n,0})-G_n(-\log t_{n,1})-2\varepsilon\geq M_{y,z}-3\varepsilon,\end{aligned}$$

which implies

$$\max_{j\in\{0,1\}}\Bigl|\bigl(\gamma_{[\omega]_{m_k}}^2 t_{n,j}\bigr)^{d_{\mathrm{s}}/2} p_{\gamma_{[\omega]_{m_k}}^2 t_{n,j}}(x,x)-G\bigl(-\log(\gamma_{[\omega]_{m_k}}^2 t_{n,j})\bigr)\Bigr|\geq\frac{M_{y,z}}{2}-\frac{3}{2}\varepsilon. \tag{3.17}$$

Letting $k\to\infty$ and then $\varepsilon\downarrow 0$ in (3.17) shows (2.15). □

Proof of Theorems 2.17 and 2.18. Setting $N_{\mathrm{RV}} := N_{\mathrm{P}} := N$ with $N \in \mathcal{B}(K)$ as in Lemma 3.12, we conclude Theorems 2.17 and 2.18 from Lemmas 3.10 and 3.12. □

4. Post-critically finite self-similar fractals

In this and the next sections, we apply Theorems 2.17 and 2.18 to concrete examples. First in this section, we consider the case of post-critically finite self-similar fractals, and the next section treats the case of generalized Sierpiński carpets.

Throughout this section, we assume that $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is a post-critically finite self-similar structure with K connected and $\#K \geq 2$; see [**23**, Theorem 1.6.2] for a simple equivalent condition for K to be connected. In particular, $2 \leq \#V_0 < \infty$ and V_* is countably infinite and dense in K, so that $K \neq \overline{V_0} = V_0$ and $V_{**} = V_*$.

4.1. Harmonic structures and resulting self-similar Dirichlet spaces. First in this subsection, we briefly describe the construction of a homogeneously scaled self-similar Dirichlet space over K; see [**23**, Chapter 3] for details. Let $D = (D_{xy})_{x,y\in V_0}$ be a real symmetric matrix of size $\#V_0$ (which we also regard as a linear operator on $\mathbb{R}^{V_0}$) such that

(D1) $\{u \in \mathbb{R}^{V_0} \mid Du = 0\} = \mathbb{R}\mathbf{1}_{V_0}$,
(D2) $D_{xy} \geq 0$ for any $x, y \in V_0$ with $x \neq y$.

We define $\mathcal{E}^{(0)}(u,v) := -\sum_{x,y\in V_0} D_{xy}u(y)v(x)$ for $u, v \in \mathbb{R}^{V_0}$, so that $(\mathcal{E}^{(0)}, \mathbb{R}^{V_0})$ is a Dirichlet form on $L^2(V_0, \#)$. Furthermore let $\mathbf{r} = (r_i)_{i\in S} \in (0,\infty)^S$ and define

$$\mathcal{E}^{(m)}(u,v) := \sum_{w\in W_m}\frac{1}{r_w}\mathcal{E}^{(0)}(u\circ F_w|_{V_0}, v\circ F_w|_{V_0}),\qquad u,v\in\mathbb{R}^{V_m} \tag{4.1}$$

for each $m \in \mathbb{N}$, where $r_w := r_{w_1}r_{w_2}\cdots r_{w_m}$ for $w = w_1w_2\ldots w_m \in W_m$ ($r_\emptyset := 1$).

Definition 4.1. The pair $(D,\mathbf{r})$ with D and $\mathbf{r}$ as above is called a *harmonic structure on* $\mathcal{L}$ if and only if $\mathcal{E}^{(0)}(u,u) = \inf_{v\in\mathbb{R}^{V_1},\, v|_{V_0}=u}\mathcal{E}^{(1)}(v,v)$ for any $u \in \mathbb{R}^{V_0}$; note that then $\mathcal{E}^{(m)}(u,u) = \min_{v\in\mathbb{R}^{V_{m+1}},\, v|_{V_m}=u}\mathcal{E}^{(m+1)}(v,v)$ for any $m \in \mathbb{N}\cup\{0\}$ and any $u \in \mathbb{R}^{V_m}$. If $\mathbf{r} \in (0,1)^S$ in addition, then $(D,\mathbf{r})$ is called *regular*.

In the rest of this section, we assume that $(D,\mathbf{r})$ is a *regular* harmonic structure on $\mathcal{L}$. Let $d_H \in (0,\infty)$ be such that $\sum_{i\in S} r_i^{d_H} = 1$, set $\mu_i := r_i^{d_H}$ for $i \in S$ and let μ be the self-similar measure on $\mathcal{L}$ with weight $(\mu_i)_{i\in S}$. We set $d_{\mathrm{s}} := 2d_H/(d_H+1)$, so that $r_i = \mu_i^{2/d_\mathrm{s}-1}$ for each $i \in S$. In this case, $\{\mathcal{E}^{(m)}(u|_{V_m}, u|_{V_m})\}_{m\in\mathbb{N}\cup\{0\}}$ is non-decreasing and hence has the limit in $[0,\infty]$ for any $u \in C(K)$. Then we define

$$\begin{aligned}\mathcal{F} &:= \{u \in C(K) \mid \lim_{m\to\infty} \mathcal{E}^{(m)}(u|_{V_m}, u|_{V_m}) < \infty\},\\ \mathcal{E}(u,v) &:= \lim_{m\to\infty} \mathcal{E}^{(m)}(u|_{V_m}, v|_{V_m}) \in \mathbb{R}, \qquad u, v \in \mathcal{F},\end{aligned} \tag{4.2}$$

so that $(\mathcal{E},\mathcal{F})$ is easily seen to satisfy the conditions (SSDF1), (SSDF2) and (SSDF3) of Definition 2.7. By [**23**, Theorem 3.3.4], $(\mathcal{E},\mathcal{F})$ is a resistance form on K whose resistance metric $R : K \times K \to [0,\infty)$ is compatible with the original topology of K, and then [**26**, Corollary 6.4 and Theorem 9.4] imply that $(\mathcal{E},\mathcal{F})$ is a non-zero symmetric regular Dirichlet form on $L^2(K,\mu)$; see [**23**, Definition 2.3.1] or [**26**, Definition 3.1] for the definition of resistance forms and their resistance metrics. Thus $(\mathcal{L},\mu,\mathcal{E},\mathcal{F})$ is a homogeneously scaled self-similar Dirichlet space with weight $(\mu_i)_{i\in S}$ and spectral dimension d_s. Note that $d_\mathrm{s} \in (0,2)$ in this case.

Remark 4.2. As described in [**23**, Sections 3.1–3.3], even for a *non-regular* harmonic structure $(D, \mathbf{r} = (r_i)_{i\in S})$ on $\mathcal{L}$, in a similar way as above we can still construct a resistance form $(\mathcal{E},\mathcal{F})$ on (a certain proper Borel subset of) K which satisfies (suitable modifications of) (SSDF1), (SSDF2) and (SSDF3). Such $(D,\mathbf{r})$, however, does not give rise to a *homogeneously scaled* self-similar Dirichlet space since $r_i < 1$ for some $i \in S$ by [**23**, Proposition 3.1.8] and $r_j \geq 1$ for some $j \in S$ by the non-regularity of $(D,\mathbf{r})$. This is why we have assumed from the beginning that our harmonic structure $(D,\mathbf{r})$ on $\mathcal{L}$ is regular.

Let $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ be the scale on Σ associated with $(\mathcal{L},\mu,\mathcal{E},\mathcal{F})$ and set $d_\mathrm{w} := d_H + 1$. Then $(\mathcal{L},\mathcal{S})$ is locally finite by [**23**, Lemma 4.2.3] and [**25**, Lemma 1.3.6], and by [**23**, Proof of Lemma 4.2.4] there exists $c_R \in (0,\infty)$ such that $R(x,y) \geq c_R s^{2/d_\mathrm{w}}$ for any $s \in (0,1]$, any $w, v \in \Lambda_s$ with $K_w \cap K_v = \emptyset$ and any $(x,y) \in K_w \times K_v$, which and [**23**, Lemma 3.3.5] easily imply that $R^{d_\mathrm{w}/2}$ is adapted to $\mathcal{S}$. Finally, (CHK) holds by [**26**, Theorem 10.4] (or by [**23**, Section 5.1]) and so does (CUHK) with $\rho := R^{d_\mathrm{w}/2}$ by [**26**, Theorem 15.10] (see also [**21**, Lemma 2.5]).

Thus in order to apply Theorems 2.17 and 2.18 to the present case, it suffices to verify (2.13) and (2.14). In the rest of this section, we give a few criteria for (2.13) and (2.14) and apply them to concrete examples. In Subsection 4.2 we treat the case where $(\mathcal{L},(D,\mathbf{r}),\mu)$ possesses certain good symmetry, including the case of affine nested fractals, and Subsection 4.3 presents alternative criteria for (2.13) and (2.14) which are applicable for some cases with weaker (or even without) symmetry.

The following definitions play central roles in the rest of this section.

Definition 4.3. (1) We define the *symmetry group* $\mathcal{G}$ *of* $(\mathcal{L},(D,\mathbf{r}),\mu)$ by

$$\mathcal{G} := \left\{ g \;\middle|\; \begin{array}{l} g \text{ is a homeomorphism from } K \text{ to itself, } g(V_0) = V_0,\ \mu\circ g = \mu, \\ u\circ g, u\circ g^{-1} \in \mathcal{F} \text{ and } \mathcal{E}(u\circ g, u\circ g) = \mathcal{E}(u,u) \text{ for any } u \in \mathcal{F} \end{array} \right\}, \tag{4.3}$$

which clearly forms a subgroup of the group of homeomorphisms of K.
(2) For each $x \in V_*$, we define

$$m_x := \min\{m \in \mathbb{N}\cup\{0\} \mid x \in V_m\} \quad \text{and} \quad n_x := \#\{w \in W_{m_x} \mid x \in K_w\}. \tag{4.4}$$

4.2. Cases with good symmetry and affine nested fractals. Assuming certain good symmetry of $(\mathcal{L},(D,\mathbf{r}),\mu)$, we have the following criterion for (2.13) and (2.14), which is an immediate consequence of [**21**, Remark 6.4].

PROPOSITION 4.4. *Let $q \in V_0$ and suppose that $\{g(q) \mid g \in \mathcal{G}\} = V_0$ and that $r_i = r$ for any $i \in S$ for some $r \in (0,1)$. Then for each $x \in V_*$, $n_x = \#\{w \in W_m \mid x \in K_w\}$ for any $m \geq m_x$ and $\lim_{t\downarrow 0} p_t(x,x)/p_t(q,q) = n_x^{-1}$. In particular, if*

$$n_y \neq n_z \quad \text{for some } y,z \in V_* \setminus V_0 \tag{4.5}$$

in addition, then the conditions (2.13) *and* (2.14) *are satisfied.*

Next we recall the definition of affine nested fractals and apply Proposition 4.4 to them. Throughout the rest of this subsection, we assume the following:

$$\begin{array}{l} d \in \mathbb{N},\ K \text{ is a compact subset of } \mathbb{R}^d, \text{ and for each } i \in S,\ F_i = f_i|_K \\ \text{for some contractive similitude } f_i \text{ on } \mathbb{R}^d \text{ with contraction ratio } \alpha_i. \end{array} \tag{4.6}$$

Recall that $f : \mathbb{R}^d \to \mathbb{R}^d$ is called a *contractive similitude on* $\mathbb{R}^d$ if and only if there exist $\alpha \in (0,1)$, $U \in O(d)$ and $b \in \mathbb{R}^d$ such that $f(x) = \alpha Ux + b$ for any $x \in \mathbb{R}^d$. Then such α is called the *contraction ratio of* f. According to [**23**, Theorem 1.2.3], any finite family of contractive similitudes on $\mathbb{R}^d$ actually gives rise to a self-similar structure satisfying (4.6) by taking the associated self-similar set.

NOTATION. For $x, y \in \mathbb{R}^d$ with $x \neq y$, let $g_{xy} : \mathbb{R}^d \to \mathbb{R}^d$ denote the reflection in the hyperplane $H_{xy} := \{z \in \mathbb{R}^d \mid |z-x| = |z-y|\}$.

DEFINITION 4.5. (1) A homeomorphism $g : K \to K$ is called a *symmetry of* $\mathcal{L}$ if and only if, for any $m \in \mathbb{N} \cup \{0\}$, there exists an injective map $g^{(m)} : W_m \to W_m$ such that $g(F_w(V_0)) = F_{g^{(m)}(w)}(V_0)$ for any $w \in W_m$.
(2) We set $\mathcal{G}_{\mathrm{s}} := \{g \mid g \text{ is a symmetry of } \mathcal{L},\ g = f|_K \text{ for some isometry } f \text{ of } \mathbb{R}^d\}$.
(3) $\mathcal{L}$ is called an *affine nested fractal* if and only if it is post-critically finite, K is connected and $g_{xy}|_K \in \mathcal{G}_{\mathrm{s}}$ for any $x, y \in V_0$ with $x \neq y$. An affine nested fractal $\mathcal{L}$ is called a *nested fractal* if and only if $\alpha_i = \alpha$ for any $i \in S$ for some $\alpha \in (0,1)$.
(4) A real matrix $A = (A_{xy})_{x,y\in V_0}$ is called $\mathcal{G}_{\mathrm{s}}$*-invariant* if and only if $A_{xy} = A_{g(x)g(y)}$ for any $x, y \in V_0$ and any $g \in \mathcal{G}_{\mathrm{s}}$. Also $\mathbf{a} = (a_i)_{i\in S} \in (0,\infty)^S$ is called $\mathcal{G}_{\mathrm{s}}$*-invariant* if and only if $a_i = a_j$ for any $i, j \in S$ satisfying $g(F_i(V_0)) = F_j(V_0)$ for some $g \in \mathcal{G}_{\mathrm{s}}$.

By [**23**, Proof of Proposition 3.8.9], if $\mathcal{L}$ is an affine nested fractal, then $A = (A_{xy})_{x,y\in V_0}$ is $\mathcal{G}_{\mathrm{s}}$-invariant if and only if $A_{xy} = A_{x'y'}$ whenever $|x-y| = |x'-y'|$.

Now we can conclude the following theorem for affine nested fractals.

THEOREM 4.6. *Assume that $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is an affine nested fractal, that D is $\mathcal{G}_{\mathrm{s}}$-invariant and that $r_i = r$ for any $i \in S$ for some $r \in (0,1)$. Further assume that*

$$\#(F_i(V_0) \cap F_j(V_0)) \leq 1 \quad \text{for any } i,j \in S \text{ with } i \neq j. \tag{4.7}$$

If $\mathcal{L}$ satisfies (4.5)*, then the conclusions of Theorems* 2.17 *and* 2.18 *hold true.*

PROOF. We have $\mathcal{G}_{\mathrm{s}} \subset \mathcal{G}$ by [**21**, Proof of Theorem 4.5] and [**23**, Corollary 3.8.21]. Since $g_{xy}|_K \in \mathcal{G}_{\mathrm{s}}$ and $g_{xy}(x) = y$ for any $x, y \in V_0$ with $x \neq y$, Proposition 4.4 is applicable and hence so are Theorems 2.17 and 2.18. □

REMARK 4.7. (1) *If $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is an affine nested fractal satisfying* (4.7)*, then a harmonic structure $(D, \mathbf{r})$ on $\mathcal{L}$ as in Theorem* 4.6 *exists and is unique (up to constant multiples of D).* Here the existence part is essentially due to Lindstrøm [**32**]; see [**23**, Section 3.8] and references therein for further details. Also see [**17, 33, 34, 35**] for more recent results on existence of harmonic structures.

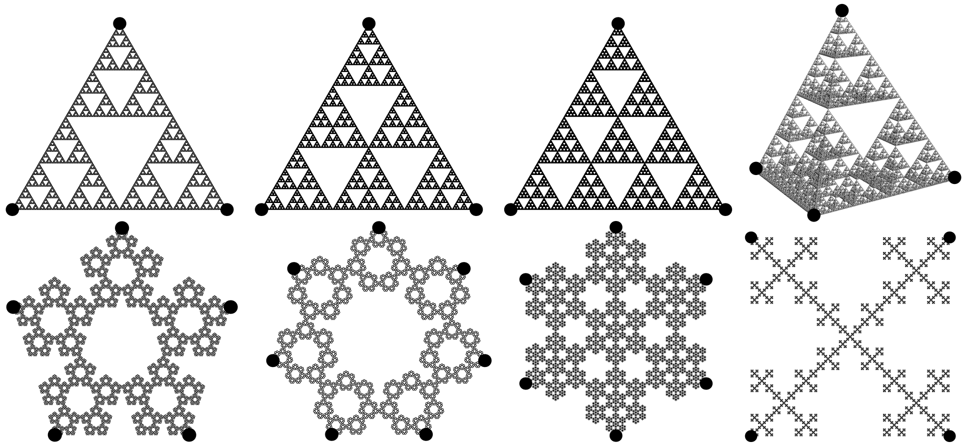

FIGURE 2. Examples of nested fractals. From the *upper left*, two-dimensional level-l Sierpiński gasket ($l = 2, 3, 4$), three-dimensional standard (level-2) Sierpiński gasket, pentagasket (5-polygasket), heptagasket (7-polygasket), snowflake and the Vicsek set. In each fractal, the set V_0 of its boundary points is marked by solid circles.

(2) For the same reason as [**21**, Theorem 4.5] (see [**21**, Remark 4.6-(2)]), it is unclear whether the (technical) assumption (4.7) can be removed from Theorem 4.6.

At the last of this subsection, we provide some examples of nested fractals.

EXAMPLE 4.8 (Sierpiński gaskets). Let $d, l \in \mathbb{N} \setminus \{1\}$, let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be the d-dimensional level-l Sierpiński gasket as in [**21**, Example 5.1] and let $(D, \mathbf{r})$ be the harmonic structure on $\mathcal{L}$ described there. Then clearly $\mathcal{L}$ is a nested fractal and the conditions of Theorem 4.6 except (4.5) are satisfied. Moreover, it is easy to see that (4.5) is satisfied if and only if $l \geq 3$ (see Figure 2). Thus by Theorem 4.6, if $l \geq 3$ then the conclusions of Theorems 2.17 and 2.18 are valid.

On the other hand, *it is unclear whether* (2.13) *and* (2.14) *hold when* $l = 2$, as already remarked at the end of the introduction.

EXAMPLE 4.9 (Polygaskets). Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be the (N, l)-polygasket with $N, l \in \mathbb{N}$, $N \geq 4$, $l < N/2$ in [**21**, Example 5.5] and let $(D, \mathbf{r})$ be the harmonic structure on $\mathcal{L}$ described there. Then we easily see that the conditions of Proposition 4.4 including (4.5) are satisfied and hence the conclusions of Theorems 2.17 and 2.18 hold in this case.

Note that this example includes the case of the N-polygasket with $N \in \mathbb{N}$, $N \geq 5$, $N/4 \notin \mathbb{N}$ in [**21**, Example 5.3] (see Figure 2), which is the $(N, \lceil N/4 \rceil)$-polygasket with $\lceil N/4 \rceil := \min\{n \in \mathbb{N} \mid n \geq N/4\}$ and is realized in $\mathbb{R}^2$ as a nested fractal.

4.3. Cases possibly without good symmetry. We follow the framework of Subsection 4.1 throughout this subsection. Recall that Proposition 4.4 above is based on the assumption of good symmetry of $(\mathcal{L}, (D, \mathbf{r}), \mu)$. On the other hand, even under weaker assumptions on symmetry of $(\mathcal{L}, (D, \mathbf{r}), \mu)$, we can still verify (2.13) or (2.14) in some cases, as follows. Recall that $K^I = K \setminus \overline{V_0} = K \setminus V_0$.

PROPOSITION 4.10. *Let $y, z \in K^I$ and let Λ_y, Λ_z be partitions of Σ. Define $\Gamma_y := \{w \in \Lambda_y \mid y \in K_w\}$ and $\Gamma_z := \{w \in \Lambda_z \mid z \in K_w\}$, let $w_y \in \Gamma_y$ and assume $y \in F_{w_y}(V_0)$ if $\Gamma_y = \{w_y\}$. Let $\varphi : \Gamma_y \to \Gamma_z$ and suppose that for each $w \in \Gamma_y$, $r_{\varphi(w)}/r_w = r_{\varphi(w_y)}/r_{w_y}$ and there exists $g_w \in \mathcal{G}$ such that $g_w\bigl(F_w^{-1}(y)\bigr) = F_{\varphi(w)}^{-1}(z)$. Set $\tilde{y} := F_{\varphi(w_y)}(y)$ and $\tilde{z} := F_{w_y}(z)$* (note that $\tilde{y}, \tilde{z} \in K^I$).
(1) *If φ is injective and not surjective, then* $\limsup_{t\downarrow 0} p_t(\tilde{y},\tilde{y})/p_t(\tilde{z},\tilde{z}) > 1$.
(2) *If $n \in \mathbb{N}$ and $\#\varphi^{-1}(v) = n$ for any $v \in \Gamma_z$, then* $\lim_{t\downarrow 0} p_t(\tilde{y},\tilde{y})/p_t(\tilde{z},\tilde{z}) = n^{-1}$.

We need the following definition and lemma for the proof of Proposition 4.10.

DEFINITION 4.11. Let U be a non-empty open subset of K.
(1) Let $\lambda \in (0,\infty)$ and set $\mathcal{E}_\lambda(u,v) := \mathcal{E}(u,v) + \int_K uv\,d\mu$ for $u, v \in \mathcal{F}$. We define

$$\operatorname{cap}^U_\lambda(B) := \inf\{\mathcal{E}_\lambda(u,u) \mid u \in \mathcal{F}_U,\ u \geq 1\ \mu\text{-a.e. on } B\},\ B \subset U \text{ open in } U, \tag{4.8}$$

$$\operatorname{Cap}^U_\lambda(A) := \inf\bigl\{\operatorname{cap}^U_\lambda(B) \bigm| B \subset U \text{ open in } U,\ A \subset B\bigr\},\ A \subset U, \tag{4.9}$$

so that $\operatorname{Cap}^U_\lambda$ is an extension of $\operatorname{cap}^U_\lambda$. We call $\operatorname{Cap}^U_\lambda$ the λ-*order capacity on* U.
(2) We define the *Dirichlet resolvent kernel* $u^U = u^U_\lambda(x,y)$ *on* U by

$$u^U_\lambda(x,y) := \int_0^\infty e^{-\lambda t} p^U_t(x,y)\,dt, \qquad (\lambda,x,y) \in (0,\infty)\times U \times U, \tag{4.10}$$

where $p^U = p^U_t(x,y)$ is the Dirichlet heat kernel on U introduced in Lemma 3.2.

By (2.11), $p^U \leq p$ and $d_{\mathrm{s}} \in (0,2)$, $u^U_\lambda(x,y) \leq u^K_\lambda(x,y) \leq c_{4.1}\lambda^{d_{\mathrm{s}}/2-1}$ for any $(\lambda,x,y) \in (0,\infty)\times U\times U$ for some $c_{4.1} \in (0,\infty)$, and $u^U : (0,\infty)\times U \times U \to [0,\infty)$ is continuous by the continuity of p^U. Note that $\operatorname{Cap}^U_\lambda(\{x\}) \in (0,\infty)$ for any $(\lambda, x) \in (0,\infty)\times U$ by [**26**, (3.1)] and hence $\mathcal{F}_U = \{u \in \mathcal{F} \mid u|_{K\setminus U} = 0\}$ by [**11**, Corollary 2.3.1]. Then since $p^U = p^U_t(x,y)$ is the transition density of a $\mu|_U$-symmetric diffusion on U whose Dirichlet form on $L^2(U,\mu|_U)$ is $(\mathcal{E}^U, \mathcal{F}_U)$ by [**26**, Theorem 10.4] (or by [**19**, Lemma 7.11-(2)]), we easily see from [**11**, (2.2.11), (2.2.13) and Exercise 4.2.2] that $\operatorname{Cap}^U_\lambda(\{x\}) = 1/u^U_\lambda(x,x)$ for any $(\lambda, x) \in (0,\infty)\times U$.

LEMMA 4.12. *Let $x \in K$, Λ be a partition of Σ and set $\Gamma := \{w \in \Lambda \mid x \in K_w\}$. Assume $x \in F_w(V_0)$ if $\#\Gamma = 1$ and $\Gamma = \{w\}$, so that $x \in F_w(V_0)$ for any $w \in \Gamma$. Set $U := \{x\} \cup \bigcup_{w\in\Gamma} K^I_w$ and $U^q := \{q\} \cup K^I$ for $q \in V_0$. Then for any $\lambda \in (0,\infty)$,*

$$\bigl(u^U_\lambda(x,x)\bigr)^{-1} = \sum_{w\in\Gamma} \frac{\gamma_w^{d_{\mathrm{s}}-2}}{u^{U^{F_w^{-1}(x)}}_{\gamma_w^2\lambda}\bigl(F_w^{-1}(x), F_w^{-1}(x)\bigr)}. \tag{4.11}$$

PROOF. Note that U and U^q, $q \in V_0$, are open subsets of K by [**23**, Proposition 1.3.6] and $\#V_0 < \infty$. If $u \in \mathcal{F}_U$ and $u(x) = 1$ then $u \circ F_w \in \mathcal{F}_{U^{F_w^{-1}(x)}}$ and $u \circ F_w\bigl(F_w^{-1}(x)\bigr) = 1$ for any $w \in \Gamma$, and conversely if $u_w \in \mathcal{F}_{U^{F_w^{-1}(x)}}$ and $u_w\bigl(F_w^{-1}(x)\bigr) = 1$ for each $w \in \Gamma$, then $u \in C(K)$ defined by $u|_{K_w} := u_w \circ F_w^{-1}$ for $w \in \Gamma$ and $u|_{K\setminus U} := 0$ belongs to $\mathcal{F}_U$ by (4.2) and satisfies $u(x) = 1$. Therefore using [**11**, Theorem 2.1.5-(i)] and (2.4), we see that for any $\lambda \in (0,\infty)$,

$$\begin{aligned}\bigl(u_\lambda^U(x,x)\bigr)^{-1} &= \mathrm{Cap}_\lambda^U(\{x\}) = \inf\{\mathcal{E}_\lambda(u,u) \mid u \in \mathcal{F}_U,\, u(x)=1\}\\ &= \inf\biggl\{\sum_{w\in\Gamma}\frac{1}{r_w}\mathcal{E}_{\gamma_w^2\lambda}(u\circ F_w, u\circ F_w)\biggm| u\in\mathcal{F}_U,\, u(x)=1\biggr\}\\ &= \sum_{w\in\Gamma}\gamma_w^{d_\mathrm{s}-2}\inf\Bigl\{\mathcal{E}_{\gamma_w^2\lambda}(u_w,u_w)\Bigm| u_w\in\mathcal{F}_{U^{F_w^{-1}(x)}},\, u_w\bigl(F_w^{-1}(x)\bigr)=1\Bigr\}\\ &= \sum_{w\in\Gamma}\gamma_w^{d_\mathrm{s}-2}\,\mathrm{Cap}_{\gamma_w^2\lambda}^{U^{F_w^{-1}(x)}}\bigl(F_w^{-1}(x)\bigr) = \sum_{w\in\Gamma}\frac{\gamma_w^{d_\mathrm{s}-2}}{u_{\gamma_w^2\lambda}^{U^{F_w^{-1}(x)}}\bigl(F_w^{-1}(x),F_w^{-1}(x)\bigr)},\end{aligned}$$

proving (4.11). □

Proof of Proposition 4.10. Set $\Gamma_{\tilde y} := \{\varphi(w_y)w \mid w \in \Gamma_y\}$, $\Gamma_{\tilde z} := \{w_y w \mid w \in \Gamma_z\}$ and define $\tilde\varphi : \Gamma_{\tilde y} \to \Gamma_{\tilde z}$ by $\tilde\varphi(\varphi(w_y)w) := w_y\varphi(w)$ for $w \in \Gamma_y$, so that $r_w = r_{\tilde\varphi(w)}$ for any $w \in \Gamma_{\tilde y}$. Also set $U^{\tilde y} := \{\tilde y\} \cup \bigcup_{w\in\Gamma_{\tilde y}} K_w^I$ and $U^{\tilde z} := \{\tilde z\} \cup \bigcup_{w\in\Gamma_{\tilde z}} K_w^I$. By $\tilde y \in K_{\varphi(w_y)}^I$ and $\tilde z \in K_{w_y}^I$ we can choose partitions $\Lambda_{\tilde y}, \Lambda_{\tilde z}$ of Σ so that $\Gamma_{\tilde y} = \{w \in \Lambda_{\tilde y} \mid \tilde y \in K_w\}$ and $\Gamma_{\tilde z} = \{w \in \Lambda_{\tilde z} \mid \tilde z \in K_w\}$, and in the situations of (1) and (2) we have $\tilde y \in F_w(V_0)$ for any $w \in \Gamma_{\tilde y}$ and $\tilde z \in F_w(V_0)$ for any $w \in \Gamma_{\tilde z}$. Note that $u_\lambda^{U^q}(a,b) = u_\lambda^{U^{g(q)}}(g(a),g(b))$ for $g \in \mathcal{G}$, $q \in V_0$ and $(\lambda,a,b) \in (0,\infty)\times U^q \times U^q$, where $U^q := \{q\} \cup K^I$ for $q \in V_0$. Therefore recalling that $g_w\bigl(F_{\varphi(w_y)w}^{-1}(\tilde y)\bigr) = F_{w_y\varphi(w)}^{-1}(\tilde z)$ for each $w \in \Gamma_y$, we see from Lemma 4.12 that for any $\lambda \in (0,\infty)$,

$$\begin{aligned}(4.12)\quad &\bigl(u_\lambda^{U^{\tilde y}}(\tilde y,\tilde y)\bigr)^{-1}\\ &= \sum_{w\in\Gamma_{\tilde y}}\frac{\gamma_w^{d_\mathrm{s}-2}}{u_{\gamma_w^2\lambda}^{U^{F_w^{-1}(\tilde y)}}\bigl(F_w^{-1}(\tilde y),F_w^{-1}(\tilde y)\bigr)} = \sum_{w\in\Gamma_{\tilde y}}\frac{\gamma_{\tilde\varphi(w)}^{d_\mathrm{s}-2}}{u_{\gamma_{\tilde\varphi(w)}^2\lambda}^{U^{F_{\tilde\varphi(w)}^{-1}(\tilde z)}}\bigl(F_{\tilde\varphi(w)}^{-1}(\tilde z),F_{\tilde\varphi(w)}^{-1}(\tilde z)\bigr)}\\ &= \begin{cases}\bigl(u_\lambda^{U^{\tilde z}}(\tilde z,\tilde z)\bigr)^{-1} - \sum_{w\in\Gamma_{\tilde z}\setminus\tilde\varphi(\Gamma_{\tilde y})}\dfrac{\gamma_w^{d_\mathrm{s}-2}}{u_{\gamma_w^2\lambda}^{U^{F_w^{-1}(\tilde z)}}\bigl(F_w^{-1}(\tilde z),F_w^{-1}(\tilde z)\bigr)} & \text{for (1)},\\ n\bigl(u_\lambda^{U^{\tilde z}}(\tilde z,\tilde z)\bigr)^{-1} & \text{for (2)}.\end{cases}\end{aligned}$$

(1) By Proposition 2.16, there exist $c_{4.2}, c_{4.3} \in (0,\infty)$ such that for any $x \in K$,

$$(4.13)\qquad c_{4.2} \le \lambda^{1-d_\mathrm{s}/2}u_\lambda^K(x,x) \le c_{4.3}, \qquad \lambda \in [1,\infty).$$

We easily see from Lemma 3.3, (2.11) and (4.13) that $\lim_{\lambda\to\infty} u_\lambda^U(x,x)/u_\lambda^K(x,x) = 1$ for any non-empty open subset U of K and any $x \in U$. It follows from this fact, (4.12) and (4.13) that $\liminf_{\lambda\to\infty} u_\lambda^K(\tilde y,\tilde y)/u_\lambda^K(\tilde z,\tilde z) \ge 1 + c_{4.2}c_{4.3}^{-1}\#(\Gamma_z\setminus\varphi(\Gamma_y)) > 1$, which immediately implies $\limsup_{t\downarrow 0} p_t(\tilde y,\tilde y)/p_t(\tilde z,\tilde z) > 1$.

(2) (4.12) implies that $p_t^{U^{\tilde z}}(\tilde z,\tilde z) = np_t^{U^{\tilde y}}(\tilde y,\tilde y)$ for any $t \in (0,\infty)$, from which the assertion is immediate since $\lim_{t\downarrow 0} p_t^U(x,x)/p_t(x,x) = 1$ for any non-empty open subset U of K and any $x \in U$ by Lemma 3.3 and (2.11). □

Remark 4.13. As shown in the previous proof, in the situation of Proposition 4.10-(1) it actually holds that $\liminf_{\lambda\to\infty} u_\lambda^K(\tilde y,\tilde y)/u_\lambda^K(\tilde z,\tilde z) > 1$. Unfortunately, however, here we cannot conclude from this fact that $\liminf_{t\downarrow 0} p_t(\tilde y,\tilde y)/p_t(\tilde z,\tilde z) > 1$, for $p_{(\cdot)}(\tilde y,\tilde y)$ and $p_{(\cdot)}(\tilde z,\tilde z)$ may not vary regularly at 0 and hence Tauberian theorems for the Laplace transform may not be applicable to them.

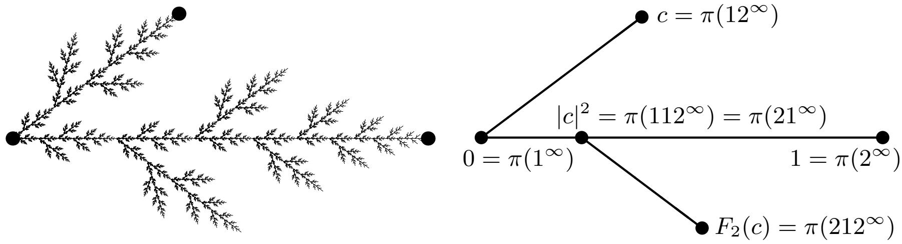

FIGURE 3. Hata's tree-like set ($c = 0.4 + 0.3\sqrt{-1}$) and the set V_1

Now a simple application of Proposition 4.10-(1) yields the following theorem. Recall (4.4) for the definition of n_x for $x \in V_*$.

THEOREM 4.14. *If $n_x = 1$ for some $x \in V_* \setminus V_0$, then the conclusions of Theorem 2.17 are valid for **any** regular harmonic structure $(D, \mathbf{r})$ on $\mathcal{L}$, where the set N_{RV} can be chosen independently of $(D, \mathbf{r})$.*

PROOF. By $x \in V_{m_x}$ there exist $q \in V_0$ and $w_x \in W_{m_x}$ such that $x = F_{w_x}(q)$, and $\{w \in W_{m_x} \mid x \in K_w\} = \{w_x\}$ by $n_x = 1$. On the other hand, by $V_0 = \pi(\mathcal{P})$ and (2.1) we can choose $\omega \in \mathcal{P}$ and $v \in W_* \setminus \{\emptyset\}$ so that $q = \pi(\omega)$ and $\sigma_v(\omega) \in \mathcal{C}$. Then taking $\tau \in W_*$ such that $K_\tau \subset K^I$, which is possible by [**23**, Proposition 1.3.6], we see that $z := F_{\tau v}(q) = F_\tau(\pi(\sigma_v(\omega))) \in K^I$ and that $\#\{w \in W_{|\tau v|} \mid z \in K_w\} \geq 2$. Now for any regular harmonic structure $(D, \mathbf{r})$ on $\mathcal{L}$, Proposition 4.10-(1) easily yields $\limsup_{t \downarrow 0} p_t(F_{\tau v}(x), F_{\tau v}(x))/p_t(F_{w_x}(z), F_{w_x}(z)) > 1$, and hence Theorem 2.17 applies with N_{RV} determined solely by $F_{\tau v}(x)$, $F_{w_x}(z)$ and π. $\square$

At the last of this section, we apply Proposition 4.10 and Theorem 4.14 to some examples.

EXAMPLE 4.15. Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be any one of the (N, l)-polygasket with $N, l \in \mathbb{N}$, $N \geq 4$, $l < N/2$ in [**21**, Example 5.5], the snowflake and the Vicsek set (see Figure 2). Then the assumption of Theorem 4.14 is clearly satisfied and hence the conclusions of Theorem 2.17 hold for *any* regular harmonic structure on $\mathcal{L}$.

EXAMPLE 4.16 (Hata's tree-like set). Following [**23**, Example 1.2.9], let $c \in \mathbb{C} \setminus \mathbb{R}$ satisfy $|c|, |1 - c| \in (0, 1)$, set $S := \{1, 2\}$ and define $f_i : \mathbb{C} \to \mathbb{C}$ for $i \in S$ by $f_1(z) := c\overline{z}$ and $f_2(z) := (1 - |c|^2)\overline{z} + |c|^2$. Let K be the *self-similar set associated with* $\{f_i\}_{i \in S}$, i.e. the unique non-empty compact subset of $\mathbb{C} \cong \mathbb{R}^2$ that satisfies $K = \bigcup_{i \in S} f_i(K)$, and set $F_i := f_i|_K$ for $i \in S$. Then $\mathcal{L} := (K, S, \{F_i\}_{i \in S})$ is a self-similar structure with K connected, $\mathcal{P} = \{12^\infty, 1^\infty, 2^\infty\}$ and $V_0 = \{c, 0, 1\}$. Also $F_2(c) \in V_1 \setminus V_0$ and $n_{F_2(c)} = 1$. $\mathcal{L}$ is called *Hata's tree-like set* (see Figure 3).

Let $r \in (0, 1)$, set $\mathbf{r} = (r_i)_{i \in S} := (r, 1 - r^2)$ and let $D = (D_{xy})_{x, y \in V_0}$ be the real symmetric matrix given by $D_{c0} = -D_{cc} := 1/r$, $D_{01} = -D_{11} := 1$, $D_{c1} := 0$ and $D_{00} := -1 - 1/r$. Then $(D, \mathbf{r})$ is a regular harmonic structure on $\mathcal{L}$ and, except for constant multiples of D, any harmonic structure on $\mathcal{L}$ is of this form. Now Theorem 4.14 applies again and hence the conclusions of Theorem 2.17 are valid in this case. Note that this case is beyond the reach of the author's preceding result [**21**, Theorem 3.4], since $\mathcal{G} = \{\mathrm{id}_K\}$ by virtue of the following proposition.

Proposition 4.17. *Let $\mathcal{L}=(K,S,\{F_i\}_{i\in S})$ and $(D,\mathbf{r})$ be as in Example 4.16 and let $R:K\times K\to[0,\infty)$ be the resistance metric of the resistance form $(\mathcal{E},\mathcal{F})$ on K resulting from $(D,\mathbf{r})$ by (4.2). If $g:K\to K$ is surjective and satisfies $g(V_0)=V_0$ and $R(g(x),g(y))=R(x,y)$ for any $x,y\in K$, then $g=\mathrm{id}_K$.*

Proof. Since $r^{-1}R(F_1(x),F_1(y))=(1-r^2)^{-1}R(F_2(x),F_2(y))=R(x,y)$ for any $x,y\in K$ by $K_1\cap K_2=\{|c|^2\}$, (4.2) and (2.4), an induction in m easily implies that $\sup_{x\in K\setminus K_{2^m}}R(0,x)<1$ for $m\in\mathbb{N}$, so that $1=\max_{x\in K}R(0,x)$ is attained only by $x=1$. Then since $R(F_{21}(x),F_{21}(y))=r(1-r^2)R(x,y)$ for any $x,y\in K$ by (4.2) and (2.4) again, $r(1-r^2)=\max_{x\in K_{21}}R(|c|^2,x)$ is attained only by $x=F_2(c)$.

Let $\mathcal{G}_R$ be the collection of surjections $g:K\to K$ satisfying $g(V_0)=V_0$ and $R(g(x),g(y))=R(x,y)$ for any $x,y\in K$, and let $g\in\mathcal{G}_R$. We first show $g|_{V_1}=\mathrm{id}_{V_1}$ and $g(K_i)=K_i$, $i\in S$. It follows from $R(c,0)<R(0,1)<R(c,1)$ and $g(V_0)=V_0$ that $g|_{V_0}=\mathrm{id}_{V_0}$. Define $\gamma:[0,2]\to K$ by $\gamma(t):=(1-t)c$ for $t\in[0,1]$ and $\gamma(t):=t-1$ for $t\in[1,2]$. We easily see that $R(c,\gamma(t))=R(c,\gamma(s))+R(\gamma(s),\gamma(t))$ for any $s,t\in[0,2]$ with $s\leq t$, so that $R(c,\gamma(\cdot))$ is strictly increasing. By $K_1\cap K_2=\{|c|^2\}$, a continuous path $g\circ\gamma:[0,2]\to K$ from $c\in K_1$ to $1\in K_2$ has to admit $t\in(0,2)$ such that $g\circ\gamma(t)=|c|^2$. Then $R(c,|c|^2)=R(c,g\circ\gamma(t))=R(c,\gamma(t))$ and hence $t=1+|c|^2$ by the strict monotonicity of $R(c,\gamma(\cdot))$. Thus $g(|c|^2)=|c|^2$, and in particular g defines a homeomorphism $g|_{K\setminus\{|c|^2\}}:K\setminus\{|c|^2\}\to K\setminus\{|c|^2\}$. Set $U:=\bigcup_{m\in\mathbb{N}\cup\{0\}}K_{1^{2m}2}$. Then since $g(c)=c\in K_1\setminus\{|c|^2\}$, $g(1)=1\in F_2(U)$ and $K\setminus\{|c|^2\}$ consists of three connected components $K_1\setminus\{|c|^2\}$, $F_2(U)$ and $F_{21}(U)$, it follows that $g(K_1\setminus\{|c|^2\})=K_1\setminus\{|c|^2\}$, $g(F_2(U))=F_2(U)$ and $g(F_{21}(U))=F_{21}(U)$. Thus $g(K_1)=K_1$ and $g(K_2)=K_2$. Moreover, $\max_{x\in K_{21}}R(|c|^2,x)$ is attained by $g(F_2(c))\in K_{21}$ and hence $g(F_2(c))=F_2(c)$.

Now let $m\in\mathbb{N}$ and assume that $g|_{V_m}=\mathrm{id}_{V_m}$ for any $g\in\mathcal{G}_R$. Then for $g\in\mathcal{G}_R$ and $i\in S$, by $g|_{V_1}=\mathrm{id}_{V_1}$ and $g(K_i)=K_i$ we have $g_i:=F_i^{-1}\circ g\circ F_i\in\mathcal{G}_R$, hence $g_i|_{V_m}=\mathrm{id}_{V_m}$ and therefore $g|_{V_{m+1}}=\mathrm{id}_{V_{m+1}}$. Thus $g|_{V_*}=\mathrm{id}_{V_*}$ for any $g\in\mathcal{G}_R$ by induction in m, which proves $\mathcal{G}_R=\{\mathrm{id}_K\}$ since V_* is dense in K. □

Example 4.18. Following [**23**, Example 4.4.9], let $S:=\{1,2,3,4\}$ and define $f_i:\mathbb{C}\to\mathbb{C}$ for $i\in S$ by $f_1(z):=\frac{1}{2}(z+1)$, $f_2(z):=\frac{1}{2}(z-1)$, $f_3(z):=\frac{\sqrt{-1}}{4}(z+1)$ and $f_4(z):=\frac{\sqrt{-1}}{4}(z-1)$. Let K be the self-similar set associated with $\{f_i\}_{i\in S}$ and set $F_i:=f_i|_K$, $i\in S$. Then $\mathcal{L}:=(K,S,\{F_i\}_{i\in S})$ is a self-similar structure with K connected, $\mathcal{P}=\{1^\infty,2^\infty\}$ and $V_0=\{-1,1\}$. Defining $g,h:\mathbb{C}\to\mathbb{C}$ by $g(z):=-\overline{z}$ and $h(z):=\overline{z}$, we easily see that $\mathcal{G}_{\mathrm{s}}=\{\mathrm{id}_K,g|_K,h|_K,gh|_K\}$, and thus $\mathcal{L}$ is an affine nested fractal. Moreover, $F_3(1)=\sqrt{-1}/2\in V_1\setminus V_0$ and $n_{F_3(1)}=1$.

Set $D=(D_{xy})_{x,y\in V_0}:=\begin{pmatrix}-1&1\\1&-1\end{pmatrix}$, which is $\mathcal{G}_{\mathrm{s}}$-invariant, and let $\mathbf{r}=(r_i)_{i\in S}\in(0,1)^S$ be such that $r_1+r_2=1$. Then $(D,\mathbf{r})$ is clearly a regular harmonic structure on $\mathcal{L}$ and the conclusions of Theorem 2.17 hold by Theorem 4.14.

Next assume $r_1=r_2=\frac{1}{2}$ and that $r_3=r_4=\left(\frac{1}{2}\right)^m$ for some $m\in\mathbb{N}$, so that $\mathbf{r}$ is $\mathcal{G}_{\mathrm{s}}$-invariant and hence $g|_K\in\mathcal{G}_{\mathrm{s}}\subset\mathcal{G}$ by [**21**, Proof of Theorem 4.5] and [**23**, Corollary 3.8.21]. Set $y:=0$, $z:=\frac{\sqrt{-1}}{2}$, $\Lambda_y:=\{iw\mid i\in\{1,2\},\,w\in W_m\}\cup\{ij\mid i\in\{3,4\},\,j\in S\}$, $\Lambda_z:=S$ and let Γ_y,Γ_z be as in Proposition 4.10. Then $\Gamma_y=\{12^m,21^m,32,41\}$, $\Gamma_z=\{3\}$ and $r_w=\left(\frac{1}{2}\right)^{m+1}$ for any $w\in\Gamma_y$, from which and $g|_K\in\mathcal{G}$ we can easily verify the assumptions of Proposition 4.10-(2) with $\varphi(w):=3$, $w\in\Gamma_y$. Thus (2.14) is satisfied and hence the conclusion of Theorem 2.18 is valid in this case.

5. Sierpiński carpets

In this last section, we apply Theorems 2.17 and 2.18 to the canonical heat kernel on generalized Sierpiński carpets, which are among the most typical examples of *infinitely ramified* self-similar fractals and have been intensively studied e.g. in [**1, 2, 3, 4, 5, 6, 31, 25, 16, 19, 18**].

We fix the following setting throughout this section.

FRAMEWORK 5.1. Let $d, l \in \mathbb{N}$, $d \geq 2$, $l \geq 2$ and set $Q_0 := [0,1]^d$. Let $S \subset \{0, 1, \dots, l-1\}^d$ be non-empty, and for each $i \in S$ define $f_i : \mathbb{R}^d \to \mathbb{R}^d$ by $f_i(x) := l^{-1}i + l^{-1}x$. Set $Q_1 := \bigcup_{i \in S} f_i(Q_0)$, which satisfies $Q_1 \subset Q_0$. Let K be the self-similar set associated with $\{f_i\}_{i \in S}$, i.e. the unique non-empty compact subset of $\mathbb{R}^d$ that satisfies $K = \bigcup_{i \in S} f_i(K)$, and set $F_i := f_i|_K$ for $i \in S$, so that $\mathrm{GSC}(d, l, S) := (K, S, \{F_i\}_{i \in S})$ is a self-similar structure. Also let $\rho : K \times K \to [0, \infty)$ be the Euclidean metric on K given by $\rho(x, y) := |x - y|$, set $d_{\mathrm{f}} := \log_l \#S$ and let μ be the self-similar measure on $\mathcal{L}$ with weight $(1/\#S)_{i \in S}$.

Recall that d_{f} is the Hausdorff dimension of (K, ρ) and that μ is a constant multiple of the d_{f}-dimensional Hausdorff measure on (K, ρ); see e.g. [**23**, Theorem 1.5.7 and Proposition 1.5.8].

The following definition is essentially due to M. T. Barlow and R. F. Bass [**5**].

DEFINITION 5.2 (Generalized Sierpiński carpets). $\mathrm{GSC}(d, l, S)$ is called a *generalized Sierpiński carpet* if and only if S satisfies the following four conditions:

(GSC1) (Symmetry) $f(Q_1) = Q_1$ for any isometry f of $\mathbb{R}^d$ with $f(Q_0) = Q_0$.

(GSC2) (Connectedness) Q_1 is connected.

(GSC3) (Non-diagonality) $\mathrm{int}_{\mathbb{R}^d}\bigl(Q_1 \cap \prod_{k=1}^d [(i_k - \varepsilon_k)l^{-1}, (i_k + 1)l^{-1}]\bigr)$ is either empty or connected for any $(i_k)_{k=1}^d \in \mathbb{Z}^d$ and any $(\varepsilon_k)_{k=1}^d \in \{0,1\}^d$.

(GSC4) (Borders included) $\{(x_1, 0, \dots, 0) \in \mathbb{R}^d \mid x_1 \in [0,1]\} \subset Q_1$.

As special cases of Definition 5.2, $\mathrm{GSC}(2, 3, S_{\mathrm{SC}})$ and $\mathrm{GSC}(3, 3, S_{\mathrm{MS}})$ are called the *Sierpiński carpet* and the *Menger sponge*, respectively, where $S_{\mathrm{SC}} := \{0,1,2\}^2 \setminus \{(1,1)\}$ and $S_{\mathrm{MS}} := \bigl\{(i_1, i_2, i_3) \in \{0,1,2\}^3 \bigm| \sum_{k=1}^3 \mathbf{1}_{\{1\}}(i_k) \leq 1\bigr\}$ (see Figure 4).

We remark that there are several equivalent ways of stating the non-diagonality condition, as in the following proposition.

PROPOSITION 5.3 ([**20**, §2]). *Set* $|x|_1 := \sum_{k=1}^d |x_k|$ *for* $x = (x_k)_{k=1}^d \in \mathbb{R}^d$. *Then* (GSC3) *is equivalent to any one of the following three conditions:*

$(\mathrm{ND})_{\mathbb{N}}$ $\mathrm{int}_{\mathbb{R}^d}\bigl(Q_1 \cap \prod_{k=1}^d [(i_k - 1)l^{-m}, (i_k + 1)l^{-m}]\bigr)$ *is either empty or connected for any* $m \in \mathbb{N}$ *and any* $(i_k)_{k=1}^d \in \{1, \dots, l^m - 1\}^d$.

$(\mathrm{ND})_2$ *The case of* $m = 2$ *of* $(\mathrm{ND})_{\mathbb{N}}$ *holds.*

(NDF) *For any* $i, j \in S$ *with* $f_i(Q_0) \cap f_j(Q_0) \neq \emptyset$ *there exists* $\{n(k)\}_{k=0}^{|i-j|_1} \subset S$ *such that* $n(0) = i$, $n(|i-j|_1) = j$ *and* $|n(k) - n(k+1)|_1 = 1$ *for any* $k \in \{0, \dots, |i-j|_1 - 1\}$.

REMARK 5.4. Only the case of $m = 1$ of $(\mathrm{ND})_{\mathbb{N}}$ was assumed in the original definition of generalized Sierpiński carpets in [**5**, Section 2], but Barlow, Bass, Kumagai and Teplyaev [**6**] have recently realized that it is too weak for [**5**, Proof of Theorem 3.19] and has to be replaced by $(\mathrm{ND})_{\mathbb{N}}$ (or equivalently, by (GSC3)).

Now in view of (NDF) in Proposition 5.3, (GSC2) and (GSC3) together imply that $\mathrm{int}_{\mathbb{R}^d} Q_1$ is connected, so that Definition 5.2 turns out to be equivalent to the definition of generalized Sierpiński carpets in [**6**, Subsection 2.2].

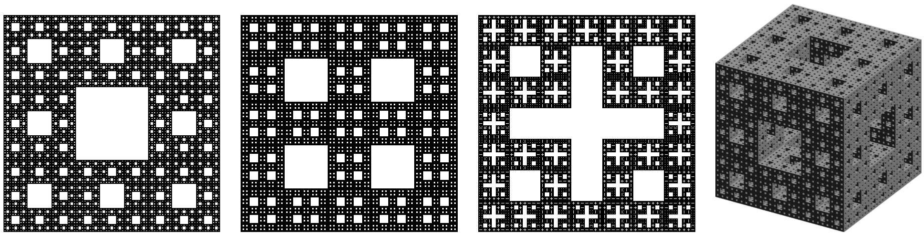

FIGURE 4. Sierpiński carpet, some other generalized Sierpiński carpets with $d = 2$ and Menger sponge

In the rest of this section, we assume that $\mathcal{L} := \mathrm{GSC}(d, l, S) = (K, S, \{F_i\}_{i\in S})$ is a generalized Sierpiński carpet. Then we easily see the following proposition.

PROPOSITION 5.5. *Set* $S_{k,\varepsilon} := \{(i_n)_{n=1}^d \in S \mid i_k = (l-1)\varepsilon\}$ *for* $k \in \{1, 2, \dots, d\}$ *and* $\varepsilon \in \{0, 1\}$. *Then* $\mathcal{P} = \bigcup_{k=1}^d (S_{k,0}^{\mathbb{N}} \cup S_{k,1}^{\mathbb{N}})$, $\overline{V_0} = V_0 = K \setminus (0, 1)^d$ *and* $V_{**} = V_*$.

Analysis on generalized Sierpiński carpets was initiated by M. T. Barlow and R. F. Bass in [**1**]: they obtained a non-degenerate μ-symmetric diffusion X on K in the case of $d = 2$ by taking a certain scaling limit of (a suitable subsequence of) the reflecting Brownian motions $X^{(m)}$ on $Q_m := \bigcup_{w\in W_m} f_w(Q_0)$, where $f_w := f_{w_1} \circ \cdots \circ f_{w_m}$ ($f_\emptyset := \mathrm{id}_{\mathbb{R}^d}$) for $w = w_1 \dots w_m \in W_*$. Then they studied the diffusion X intensively in a series of papers [**2, 3, 4**] and extended their results to the case of $d \geq 3$ in [**5**]. On the other hand, Kusuoka and Zhou [**31**] also obtained a non-degenerate diffusion on K in the case of $d = 2$ by constructing a (homogeneously scaled self-similar) Dirichlet form on $L^2(K, \mu)$ via a discrete approximation of K. It had been a long-standing problem to prove that the constructions in [**1, 5**] and in [**31**] give rise to the same diffusion on K, until Barlow, Bass, Kumagai and Teplyaev [**6**] finally solved it by proving the uniqueness of a non-zero conservative symmetric regular Dirichlet form on $L^2(K, \mu)$ possessing certain local symmetry properties. The following is a summary of the main results of [**6**].

DEFINITION 5.6. (1) We define

$$\mathcal{G}_0 := \{f|_K \mid f \text{ is an isometry of } \mathbb{R}^d \text{ with } f(Q_0) = Q_0\}, \tag{5.1}$$

which forms a subgroup of the group of homeomorphisms of K by virtue of (GSC1).
(2) Define $\psi : \mathbb{R}^d \to Q_0$ by $\psi\big((x_k)_{k=1}^d\big) := \big(\min_{n\in\mathbb{Z}} |x_k - 2n|\big)_{k=1}^d$. For each $w \in W_*$, we set $q^w := F_w(0)$ and define the *folding map* $\varphi_w : K \to K_w$ *into* K_w by

$$\varphi_w(x) := q^w + l^{-|w|}\psi\big(l^{|w|}(x - q^w)\big), \tag{5.2}$$

so that $\varphi_w|_{K_w} = \mathrm{id}_{K_w}$ and $\varphi_w \circ \varphi_v = \varphi_w$ for any $w, v \in W_*$ with $|w| = |v|$.
(3) For $u \in L^2(K, \mu)$ and $\delta \in (0, \infty)$, we define

$$J_\delta(u) := \delta^{-d_{\mathrm{f}}} \int_K \int_{B_\delta(x,\rho)} (u(x) - u(y))^2 d\mu(y) d\mu(x). \tag{5.3}$$

Note that $\mu \circ g = \mu$ for any $g \in \mathcal{G}_0$. We set $\mu|_A := \mu|_{\mathcal{B}(A)}$ for $A \in \mathcal{B}(K)$. For each $w \in W_*$, if $u : K_w \to [-\infty, \infty]$ is Borel measurable then $\int_K |u \circ \varphi_w| d\mu = (\#S)^{|w|} \int_{K_w} |u| d\mu$, so that $\varphi_w^* u := u \circ \varphi_w$ defines a bounded linear operator $\varphi_w^* : L^2(K_w, \mu|_{K_w}) \to L^2(K, \mu)$, which is called the *unfolding operator from* K_w.

THEOREM 5.7 ([**6**, Theorem 1.2 and Subsection 4.7]). (1) *There exists a unique (up to constant multiples of $\mathcal{E}$) non-zero conservative symmetric regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,\mu)$ satisfying the following conditions:*

(BBKT1) $u\circ\varphi_w\in\mathcal{F}$ *for any* $u\in\mathcal{F}$ *and any* $w\in W_*$.

(BBKT2) *For any* $m\in\mathbb{N}$ *and any* $u\in\mathcal{F}$,

$$\mathcal{E}(u,u)=\frac{1}{(\#S)^m}\sum_{w\in W_m}\mathcal{E}(u\circ\varphi_w,u\circ\varphi_w). \tag{5.4}$$

(BBKT3) *Let* $w,v\in W_*$, $|w|=|v|$ *and* $g\in\mathcal{G}_0$. *If* $u\in L^2(K_v,\mu|_{K_v})$ *and* $u\circ\varphi_v\in\mathcal{F}$, *then* $u^g_{w,v}:=u\circ F_v\circ g\circ F_w^{-1}\circ\varphi_w\in\mathcal{F}$ *and* $\mathcal{E}(u^g_{w,v},u^g_{w,v})=\mathcal{E}(u\circ\varphi_v,u\circ\varphi_v)$.

(2) $(K,\mu,\mathcal{E},\mathcal{F})$ *satisfies* (CHK) *and there exist* $d_{\mathrm{w}}\in[2,\infty)$ *and* $c_{5.1},c_{5.2}\in(0,\infty)$ *such that, with* $d_{\mathrm{s}}:=2d_{\mathrm{f}}/d_{\mathrm{w}}$, *for any* $(t,x,y)\in(0,1]\times K\times K$,

$$\frac{c_{5.1}}{t^{d_{\mathrm{s}}/2}}\exp\biggl(-\Bigl(\frac{\rho(x,y)^{d_{\mathrm{w}}}}{c_{5.1}t}\Bigr)^{\frac{1}{d_{\mathrm{w}}-1}}\biggr)\leq p_t(x,y)\leq\frac{c_{5.2}}{t^{d_{\mathrm{s}}/2}}\exp\biggl(-\Bigl(\frac{\rho(x,y)^{d_{\mathrm{w}}}}{c_{5.2}t}\Bigr)^{\frac{1}{d_{\mathrm{w}}-1}}\biggr). \tag{5.5}$$

(3) $\mathcal{F}=\{u\in L^2(K,\mu)\mid\limsup_{\delta\downarrow0}\delta^{-d_{\mathrm{w}}}J_\delta(u)<\infty\}$, *and there exist* $c_{5.3},c_{5.4}\in(0,\infty)$ *such that for any* $u\in\mathcal{F}$,

$$c_{5.3}\mathcal{E}(u,u)\leq\limsup_{\delta\downarrow0}\delta^{-d_{\mathrm{w}}}J_\delta(u)\leq\sup_{\delta\in(0,\infty)}\delta^{-d_{\mathrm{w}}}J_\delta(u)\leq c_{5.4}\mathcal{E}(u,u). \tag{5.6}$$

REMARK 5.8. *The strict inequality* $d_{\mathrm{w}}>2$ *holds if* $\#S<l^d$. In the case of $d=2$, this estimate follows from [**3**, Proof of Proposition 5.2] (see also [**4**, (2.5)]), whereas for $d\geq3$ this fact is only stated in [**5**, Remarks 5.4-1.] without proof.

In fact, by virtue of [**18**, Proof of Proposition 5.1], we can also deduce from Theorem 5.7-(1),(3) the following simpler characterization of $(\mathcal{E},\mathcal{F})$ although it is more restrictive than that in Theorem 5.7-(1).

PROPOSITION 5.9. $(\mathcal{E},\mathcal{F})$ *is the unique (up to constant multiples of* $\mathcal{E}$*) non-zero conservative symmetric regular Dirichlet form on* $L^2(K,\mu)$ *possessing the following properties:*

(GSCDF1) *If* $u\in\mathcal{F}\cap C(K)$ *and* $g\in\mathcal{G}_0$ *then* $u\circ g\in\mathcal{F}$ *and* $\mathcal{E}(u\circ g,u\circ g)=\mathcal{E}(u,u)$.

(GSCDF2) $\mathcal{F}\cap C(K)=\{u\in C(K)\mid u\circ F_i\in\mathcal{F}$ *for any* $i\in S\}$.

(GSCDF3) *There exists* $r\in(0,\infty)$ *such that for any* $u\in\mathcal{F}\cap C(K)$,

$$\mathcal{E}(u,u)=\sum_{i\in S}\frac{1}{r}\mathcal{E}(u\circ F_i,u\circ F_i). \tag{5.7}$$

Moreover, $d_{\mathrm{w}}=\log_l(\#S/r)$ *and* $d_{\mathrm{s}}=2\log_{\#S/r}\#S$.

We need the following lemma, which easily follows by a direct calculation.

LEMMA 5.10. *Let* $w,v,\tau\in W_*$, $|w|=|v|$, $\varepsilon^{w,v}=(\varepsilon^{w,v}_k)_{k=1}^d:=\psi\bigl(l^{|w|}(q^v-q^w)\bigr)$ *and define* $f_{w,v}:\mathbb{R}^d\to\mathbb{R}^d$ *by* $f_{w,v}(x):=\bigl(\varepsilon^{w,v}_k+(1-2\varepsilon^{w,v}_k)x_k\bigr)_{k=1}^d$ *for* $x=(x_k)_{k=1}^d\in\mathbb{R}^d$, *so that* $g_{w,v}:=f_{w,v}|_K\in\mathcal{G}_0$. *Then* $\varphi_{w\tau}\circ F_v=F_w\circ\varphi_\tau\circ g_{w,v}$.

PROOF OF PROPOSITION 5.9. We first prove that $(\mathcal{E},\mathcal{F})$ as in Theorem 5.7-(1) possesses the stated properties. (GSCDF1) is immediate from (BBKT3) with $w=v=\emptyset$. We easily see from Theorem 5.7-(3) that $u\circ F_w\in\mathcal{F}$ for any $w\in W_*$ and any $u\in\mathcal{F}$, and [**18**, Proof of Proposition 5.1] shows that $u\in\mathcal{F}$ whenever

$u \in C(K)$ and $u \circ F_i \in \mathcal{F}$ for any $i \in S$, proving (GSCDF2). (Note that [**18**, Proof of Proposition 5.1] for $f \in C(K)$ is based only on Theorem 5.7-(3) and (NDF).)

(GSCDF3) is stated in [**6**, Theorem 1.2] without explicit proof. In fact, it can be directly deduced from Theorem 5.7-(1),(3), as follows. Noting that $u \circ F_i \in \mathcal{F}$ for $i \in S$ and $u \in \mathcal{F}$, define $\mathcal{RE} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ by $(\mathcal{RE})(u, v) := \sum_{i \in S} \mathcal{E}(u \circ F_i, v \circ F_i)$. By (5.6) there exists $c_{5.5} \in (0, \infty)$ such that $(\mathcal{RE})(u, u) \leq c_{5.5}\mathcal{E}(u, u)$ for any $u \in \mathcal{F}$, and we can easily verify (BBKTk), $k = 1, 2, 3$ for $(\mathcal{RE}, \mathcal{F})$ from those for $(\mathcal{E}, \mathcal{F})$ and Lemma 5.10. It follows that $(\mathcal{E} + \mathcal{RE}, \mathcal{F})$ is a non-zero conservative symmetric regular Dirichlet form on $L^2(K, \mu)$ satisfying (BBKTk), $k = 1, 2, 3$, and hence $\mathcal{E} + \mathcal{RE} = \theta\mathcal{E}$ for some $\theta \in (0, \infty)$ by Theorem 5.7-(1). Since $(\mathcal{E}, \mathcal{F})$ is non-zero, $\lambda := \theta - 1 \in [0, \infty)$ and $\mathcal{RE} = \lambda\mathcal{E}$. Furthermore take $u \in \mathcal{F} \cap C(K) \setminus \{0\}$ such that $\mathrm{supp}_K[u] \subset K^I$. Then $\mathcal{E}(u, u) > 0$ by Theorem 5.7-(3) and $V_0 \neq \emptyset$. For any $w \in W_*$, $(F_w)_* u \in \mathcal{F}$ by (GSCDF2), $\mathcal{E}\bigl((F_w)_* u, (F_w)_* u\bigr) = \lambda^{|w|}\mathcal{E}(u, u)$ by $\mathcal{RE} = \lambda\mathcal{E}$, and we easily see from $\mathcal{E}(u, u) > 0$ and (5.6) that $(\lambda\#S/l^{d_{\mathrm{w}}})^{|w|} \in [c_{5.3}/c_{5.4}, c_{5.4}/c_{5.3}]$. Letting $|w| \to \infty$ yields $\lambda = l^{d_{\mathrm{w}}}/\#S > 0$, proving (GSCDF3), $d_{\mathrm{w}} = \log_l(\#S/r)$ and $d_{\mathrm{s}} = 2\log_{\#S/r}\#S$ with $r := \lambda^{-1}$.

Next for the proof of the uniqueness, suppose that $(\mathcal{E}', \mathcal{F}')$ is a non-zero conservative symmetric regular Dirichlet form on $L^2(K, \mu)$ with the stated properties. The regularity of $(\mathcal{E}', \mathcal{F}')$ easily implies that $u \circ F_i \in \mathcal{F}'$ for any $i \in S$ and any $u \in \mathcal{F}'$ and that (GSCDF1) and (GSCDF3) with $\mathcal{F}'$ in place of $\mathcal{F}' \cap C(K)$ are valid. Furthermore we see from Lemma 5.10 with $\tau = \emptyset$ and the assumed properties of $(\mathcal{E}', \mathcal{F}')$ that for $w \in W_*$ and $u \in L^2(K_w, \mu|_{K_w})$, $u \circ \varphi_w \in \mathcal{F}'$ if and only if $u \circ F_w \in \mathcal{F}'$, and if $u \circ \varphi_w \in \mathcal{F}'$ then $\mathcal{E}'(u \circ \varphi_w, u \circ \varphi_w) = (\#S/r)^{|w|}\mathcal{E}'(u \circ F_w, u \circ F_w)$. Now it is immediate from these facts and (GSCDF1) that $(\mathcal{E}', \mathcal{F}')$ satisfies (BBKTk), $k = 1, 2, 3$, and hence $(\mathcal{E}', \mathcal{F}') = (\theta\mathcal{E}, \mathcal{F})$ for some $\theta \in (0, \infty)$ by Theorem 5.7-(1). □

It follows from Proposition 5.9 that $(\mathcal{L}, \mu, \mathcal{E}, \mathcal{F})$ is a homogeneously scaled self-similar Dirichlet space with weight $(1/\#S)_{i \in S}$ and spectral dimension d_{s}. Moreover for its associated scale $\mathcal{S} = \{\Lambda_s\}_{s \in (0,1]}$ on Σ, we easily see that $\#\Lambda^1_{s,x} \leq 4^d$ for any $(s, x) \in (0, 1] \times K$ and that $\rho^{d_{\mathrm{w}}/2}$ is a $(2/d_{\mathrm{w}})$-qdistance on K adapted to $\mathcal{S}$, so that $(\mathcal{L}, \mu, \mathcal{E}, \mathcal{F})$ satisfies (CUHK) by Theorem 5.7-(2).

Finally, we verify that Theorems 2.17 and 2.18 are applicable to $(\mathcal{L}, \mu, \mathcal{E}, \mathcal{F})$ if $\#S < l^d$. Recall that $\mathcal{L} = \mathrm{GSC}(d, l, S) = (K, S, \{F_i\}_{i \in S})$ is a generalized Sierpiński carpet, that μ is the self-similar measure on $\mathcal{L}$ with weight $(1/\#S)_{i \in S}$ and that $(\mathcal{E}, \mathcal{F})$ is the Dirichlet form on $L^2(K, \mu)$ as in Theorem 5.7 and Proposition 5.9.

Theorem 5.11. *If $\#S < l^d$, then the conclusions of Theorems 2.17 and 2.18 hold true for the continuous heat kernel $p = p_t(x, y)$ of $(K, \mu, \mathcal{E}, \mathcal{F})$.*

Since $(\mathcal{L}, \mu, \mathcal{E}, \mathcal{F})$ satisfies (CUHK), for the proof of Theorem 5.11 it suffices to verify (2.14), which is an easy consequence of the following Proposition. Recall that $p^U = p^U_t(x, y)$ denotes the Dirichlet heat kernel on U introduced in Lemma 3.2 for a non-empty open subset U of K. Note that for any $w, v \in W_*$ with $|w| = |v|$, $\varphi_v|_{K_w} = F_v \circ g_{v,w} \circ F_w^{-1}$ by Lemma 5.10 and hence $\varphi_v|_{K_w} : K_w \to K_v$ is a surjective isometry with respect to the metric ρ.

Proposition 5.12. *Let $\varepsilon = (\varepsilon_k)_{k=1}^d \in \{0, 1\}^d$, $m \in \mathbb{N} \cup \{0\}$, $i = (i_k)_{k=1}^d \in l^{-m}\mathbb{Z}^d$, set $R^{i,\varepsilon}_m := \prod_{k=1}^d [i_k - \varepsilon_k l^{-m}, i_k + l^{-m}]$, $W^{i,\varepsilon}_m := \{w \in W_m \mid K_w \subset R^{i,\varepsilon}_m\}$ and suppose $W^{i,\varepsilon}_m \neq \emptyset$. Also set $U^{i,\varepsilon}_m := K \cap \mathrm{int}_{\mathbb{R}^d} R^{i,\varepsilon}_m$, $U^\varepsilon := K \cap \prod_{k=1}^d (-\varepsilon_k, 1)$, $V^\varepsilon_0 := U^\varepsilon \cap \prod_{k=1}^d [0, 1 - \varepsilon_k]$, let $\tau \in W^{i,\varepsilon}_m$ and let $g_\tau \in \mathcal{G}_0$ be such that $F_\tau \circ g_\tau(U^\varepsilon) =$*

$K_\tau \cap U^{i,\varepsilon}_m$. *Then with* $\gamma := \sqrt{r/\#S}$, *for any* $(t,x,y) \in (0,\infty) \times U^\varepsilon \times U^\varepsilon$,

$$p^{U^\varepsilon}_t(x,y) = \gamma^{md_s} \sum_{w \in W^{i,\varepsilon}_m} p^{U^{i,\varepsilon}_m}_{\gamma^{2m}t}\bigl(F_\tau \circ g_\tau(x), (\varphi_\tau|_{K_w})^{-1} \circ F_\tau \circ g_\tau(y)\bigr). \tag{5.8}$$

In particular, for any $t \in (0,\infty)$ *and any* $(x,y) \in (U^\varepsilon \times V^\varepsilon_0) \cup (V^\varepsilon_0 \times U^\varepsilon)$,

$$p^{U^\varepsilon}_t(x,y) = (\#W^{i,\varepsilon}_m)\gamma^{md_s} p^{U^{i,\varepsilon}_m}_{\gamma^{2m}t}\bigl(F_\tau \circ g_\tau(x), F_\tau \circ g_\tau(y)\bigr). \tag{5.9}$$

Note that $V^\varepsilon_0 = K \cap \bigl((0,1)^{\{k \in \{1,\dots,d\} | \varepsilon_k = 0\}} \times \{0\}^{\{k \in \{1,\dots,d\} | \varepsilon_k = 1\}}\bigr) \neq \emptyset$; indeed, $\bigl(l^{-1}(1-\varepsilon_k)\bigr)^d_{k=1} \in V^\varepsilon_0$ by (GSC1) and (GSC4).

Proof. Recalling Definition 3.1, throughout this proof we regard $\mathcal{F}_{U^{i,\varepsilon}_m}$ and $\mathcal{F}_{U^\varepsilon}$ as linear subspaces of $L^2\bigl(U^{i,\varepsilon}_m, \mu|_{U^{i,\varepsilon}_m}\bigr)$ and $L^2(U^\varepsilon, \mu|_{U^\varepsilon})$, respectively, in the natural manner. For $w \in W^{i,\varepsilon}_m$, noting that $\varphi_w(U^{i,\varepsilon}_m) = K_w \cap U^{i,\varepsilon}_m$ and $\mu|_{U^{i,\varepsilon}_m} \circ \bigl(\varphi_w|_{U^{i,\varepsilon}_m}\bigr)^{-1} = (\#W^{i,\varepsilon}_m)\mu|_{K_w \cap U^{i,\varepsilon}_m}$, we set $\varphi^\star_w u := u \circ \varphi_w|_{U^{i,\varepsilon}_m}$ for $u : U^{i,\varepsilon}_m \to [-\infty,\infty]$, so that it defines a bounded linear operator $\varphi^\star_w : L^2\bigl(U^{i,\varepsilon}_m, \mu|_{U^{i,\varepsilon}_m}\bigr) \to L^2\bigl(U^{i,\varepsilon}_m, \mu|_{U^{i,\varepsilon}_m}\bigr)$. Then define $\Theta := \Theta^{i,\varepsilon}_m := (\#W^{i,\varepsilon}_m)^{-1} \sum_{w \in W^{i,\varepsilon}_m} \varphi^\star_w$. We have $\Theta^2 = \Theta$ by $\varphi_w \circ \varphi_{w'} = \varphi_w$, $w, w' \in W_m$. We claim that Θ is self-adjoint on $L^2\bigl(U^{i,\varepsilon}_m, \mu|_{U^{i,\varepsilon}_m}\bigr)$ and that

$$\Theta\bigl(\mathcal{F}_{U^{i,\varepsilon}_m}\bigr) \subset \mathcal{F}_{U^{i,\varepsilon}_m} \quad \text{and} \quad \mathcal{E}(\Theta u, v) = \mathcal{E}(u, \Theta v), \quad u, v \in \mathcal{F}_{U^{i,\varepsilon}_m}. \tag{5.10}$$

Indeed, let $w \in W^{i,\varepsilon}_m$ and let $u \in \mathcal{F} \cap C(K)$ satisfy $\mathrm{supp}_K[u] \subset U^{i,\varepsilon}_m$. Then we have $(u \circ \varphi_w)\mathbf{1}_{U^{i,\varepsilon}_m} \in C(K)$, $\mathrm{supp}_K\bigl[(u \circ \varphi_w)\mathbf{1}_{U^{i,\varepsilon}_m}\bigr] \subset U^{i,\varepsilon}_m$, $\bigl((u \circ \varphi_w)\mathbf{1}_{U^{i,\varepsilon}_m}\bigr) \circ F_{w'} = 0$ for $w' \in W_m \setminus W^{i,\varepsilon}_m$ and $\bigl((u \circ \varphi_w)\mathbf{1}_{U^{i,\varepsilon}_m}\bigr) \circ F_{w'} = u \circ F_w \circ g_{w,w'} \in \mathcal{F}$ for $w' \in W^{i,\varepsilon}_m$ by Lemma 5.10, so that $(u \circ \varphi_w)\mathbf{1}_{U^{i,\varepsilon}_m} \in \mathcal{F} \cap C(K)$ by (GSCDF2) and

$$\mathcal{E}\bigl((u \circ \varphi_w)\mathbf{1}_{U^{i,\varepsilon}_m}, (u \circ \varphi_w)\mathbf{1}_{U^{i,\varepsilon}_m}\bigr) = (\#W^{i,\varepsilon}_m)r^{-m}\mathcal{E}(u \circ F_w, u \circ F_w) \leq (\#W^{i,\varepsilon}_m)\mathcal{E}(u,u).$$

These facts together with the regularity of $(\mathcal{E},\mathcal{F})$ easily implies $\varphi^\star_w\bigl(\mathcal{F}_{U^{i,\varepsilon}_m}\bigr) \subset \mathcal{F}_{U^{i,\varepsilon}_m}$ and hence $\Theta\bigl(\mathcal{F}_{U^{i,\varepsilon}_m}\bigr) \subset \mathcal{F}_{U^{i,\varepsilon}_m}$. Moreover for $u, v \in \mathcal{F}_{U^{i,\varepsilon}_m}$, by Lemma 5.10,

$$(\#W^{i,\varepsilon}_m)\mathcal{E}(\Theta u, v) = \sum_{w \in W^{i,\varepsilon}_m} \mathcal{E}(\varphi^\star_w u, v) = \sum_{w,w' \in W^{i,\varepsilon}_m} \frac{1}{r^m}\mathcal{E}(u \circ F_w \circ g_{w,w'}, v \circ F_{w'}),$$

which is seen to be equal to $(\#W^{i,\varepsilon}_m)\mathcal{E}(u, \Theta v)$ by the same calculation in the converse direction and $g^{-1}_{w,w'} = g_{w,w'} = g_{w',w}$. Thus (5.10) follows, and a similar calculation also shows that Θ is self-adjoint on $L^2\bigl(U^{i,\varepsilon}_m, \mu|_{U^{i,\varepsilon}_m}\bigr)$. As a consequence, we can easily verify that $T^{U^{i,\varepsilon}_m}_t \Theta = \Theta T^{U^{i,\varepsilon}_m}_t$ for any $t \in (0,\infty)$, in exactly the same way as [**6**, Proof of Proposition 2.21, (b) $\Rightarrow$ (c)].

Next we set $\iota^{i,\varepsilon}_m u := u \circ (g^{-1}_\tau \circ F^{-1}_\tau \circ \varphi_\tau)|_{U^{i,\varepsilon}_m}$ for $u : U^\varepsilon \to [-\infty,\infty]$ and $\kappa^{i,\varepsilon}_m u := u \circ (F_\tau \circ g_\tau)|_{U^\varepsilon}$ for $u : U^{i,\varepsilon}_m \to [-\infty,\infty]$, so that they define bounded linear operators $\iota := \iota^{i,\varepsilon}_m : L^2(U^\varepsilon, \mu|_{U^\varepsilon}) \to L^2\bigl(U^{i,\varepsilon}_m, \mu|_{U^{i,\varepsilon}_m}\bigr)$ and $\kappa := \kappa^{i,\varepsilon}_m : L^2\bigl(U^{i,\varepsilon}_m, \mu|_{U^{i,\varepsilon}_m}\bigr) \to L^2(U^\varepsilon, \mu|_{U^\varepsilon})$. Clearly $\kappa\iota = \mathrm{id}_{L^2(U^\varepsilon, \mu|_{U^\varepsilon})}$ and hence ι is injective. Similarly to the proof of (5.10), we easily see $\kappa\bigl(\mathcal{F}_{U^{i,\varepsilon}_m}\bigr) \subset \mathcal{F}_{U^\varepsilon}$, $\iota(\mathcal{F}_{U^\varepsilon}) \subset \mathcal{F}_{U^{i,\varepsilon}_m}$, hence

$$\iota^{-1}\bigl(\mathcal{F}_{U^{i,\varepsilon}_m}\bigr) = \mathcal{F}_{U^\varepsilon}, \quad \text{and} \quad \mathcal{E}(\iota u, \iota u) = (\#W^{i,\varepsilon}_m)r^{-m}\mathcal{E}(u,u), \quad u \in \mathcal{F}_{U^\varepsilon}. \tag{5.11}$$

On the other hand, it follows by $\varphi_w \circ \varphi_{w'} = \varphi_w$, $w, w' \in W_m$, that $\Theta\iota = \iota$ and $\iota\kappa\Theta = \Theta$, which and the last assertion of the previous paragraph imply that for any $t \in (0,\infty)$, $T^{U^{i,\varepsilon}_m}_t \iota = T^{U^{i,\varepsilon}_m}_t \Theta\iota = \Theta T^{U^{i,\varepsilon}_m}_t \iota = \iota\kappa\Theta T^{U^{i,\varepsilon}_m}_t \iota$ and hence $T^{U^{i,\varepsilon}_m}_t \iota\bigl(L^2(U^\varepsilon, \mu|_{U^\varepsilon})\bigr) \subset$

$\iota\bigl(L^2(U^\varepsilon,\mu|_{U^\varepsilon})\bigr)$. Therefore $\bigl\{\iota^{-1}T^{U^{i,\varepsilon}_m}_{\gamma^{2m}t}\iota\bigr\}_{t\in(0,\infty)}$ is a well-defined symmetric strongly continuous contraction semigroup on $L^2(U^\varepsilon,\mu|_{U^\varepsilon})$, and then in view of [**11**, Lemma 1.3.4-(i)], (5.11) means that its associated closed symmetric form is $(\mathcal{E}^{U^\varepsilon},\mathcal{F}_{U^\varepsilon})$. Thus $T^{U^\varepsilon}_t=\iota^{-1}T^{U^{i,\varepsilon}_m}_{\gamma^{2m}t}\iota$ for any $t\in(0,\infty)$, which together with the uniqueness of p^{U^ε} immediately yields (5.8). Since $F_\tau\circ g_\tau(V^\varepsilon_0)\subset\bigcap_{w\in W^{i,\varepsilon}_m}K_w$, (5.9) follows from (5.8) and the symmetry of $p^{U^\varepsilon}_t(x,y)$ and $p^{U^{i,\varepsilon}_m}_t(x,y)$ in x,y. □

Proof of Theorem 5.11. We follow the notation of Proposition 5.12 in this proof. Let $\varepsilon:=(1,0,\dots,0)\in\{0,1\}^d$ and $\mathbf{0}:=(0,\dots,0)\in\mathbb{Z}^d$, so that $W^{l^{-1}\varepsilon,\varepsilon}_1=\{\mathbf{0},\varepsilon\}$ by (GSC4). By $\#S<l^d$ and (GSC1), $i-\varepsilon\in\{0,1,\dots,l-1\}^d\setminus S$ for some $i\in S$, and then $W^{i,\varepsilon}_1=\{i\}$. Now for $x\in V^\varepsilon_0$, $F_\varepsilon(x),F_i(x)\in K^I$, and (5.9) implies that $2p^{U^{l^{-1}\varepsilon,\varepsilon}_1}_t\bigl(F_\varepsilon(x),F_\varepsilon(x)\bigr)=p^{U^{i,\varepsilon}_1}_t\bigl(F_i(x),F_i(x)\bigr)$ for any $t\in(0,\infty)$, from which it follows that $\lim_{t\downarrow 0}p_t\bigl(F_i(x),F_i(x)\bigr)/p_t\bigl(F_\varepsilon(x),F_\varepsilon(x)\bigr)=2$ by virtue of Lemma 3.3 and (5.5). Thus (2.14) holds and hence Theorems 2.17 and 2.18 apply. □

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