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Wang, Jinliang

Zhang, Ran

Kuniya, Toshikazu

(Citation)
IMA Journal of Applied Mathematics, 81(2):321-343

(Issue Date)
2016-04

(Resource Type)
journal article

(Version)
Accepted Manuscript

IMA Journal of Applied Mathematics (2015) Page 1 of 21
doi:10.1093/imamat/xxx000

The dynamics of an SVIR epidemiological model with infection age

JINLIANG WANG AND RAN ZHANG
School of Mathematical Science, Heilongjiang University, Harbin 150080, P.R. China
jinliangwang@hlju.edu.cn ranzhang90@aliyun.com

AND

TOSHIKAZU KUNIYA*
Graduate School of System Informatics, Kobe University, 1-1 Rokkodai-cho, Nada-ku, Kobe 657-8501, Japan
*Corresponding author: tkuniya@port.kobe-u.ac.jp

[Received on 21 September 2015]

In this paper, we study the global dynamics of an SVIR epidemiological model with infection age structure. Biologically, we assume that effective contacts between vaccinated individuals and infectious individuals are less than that between susceptible individuals and infectious individuals. Using Lyapunov functions, we show that the global stability of each equilibrium is completely determined by the basic reproduction number $\Re_0$: if $\Re_0 \leqslant 1$ then the disease-free equilibrium is globally asymptotically stable; while if $\Re_0 > 1$, then there exists a unique endemic equilibrium which is globally asymptotically stable.

Keywords: SVIR epidemiological model, infection age structure, basic reproduction number, Lyapunov function, uniform persistence, global asymptotic stability

2000 Math Subject Classification: 34K30, 35K57, 35Q80, 92D25

1. Introduction

Since Kermack & McKendrick (1927), a lot of differential equations have been studied as models for the spread of infectious diseases. As one of the most basic models, the SIR epidemiological model has been studied by many authors (see Diekmann & Heesterbeek (2000) and references therein), in which the total population is divided by susceptible S, infectious I and recovered R. Main interests of them are the local and global stability of equilibria (Korobeinikov, 2004), bifurcation (Franceschetti *et al.*, 2012), nonlinear incidence rates (Beretta *et al.*, 2001; Li & Jin, 2005; Li *et al.*, 1999) and so on.

It is needless to say that vaccination is an effective strategy against infectious diseases (Gabbuti *et al.*, 2007). Considering a continuous vaccination strategy, Liu *et al.* (2008) formulated the following system of ordinary differential equations:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \mu - \beta S(t)I(t) - (\mu+\alpha)S(t), \\ \dfrac{\mathrm{d}V(t)}{\mathrm{d}t} = \alpha S(t) - \beta_1 V(t)I(t) - (\mu+\gamma_1)V(t), \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta S(t)I(t) + \beta_1 V(t)I(t) - \gamma I(t) - \mu I(t), \\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = \gamma_1 V(t) + \gamma I(t) - \mu R(t), \end{cases} \tag{1.1}$$

where $S(t), V(t), I(t)$ and $R(t)$ denote the susceptible, vaccinated, infectious and recovered populations

Coefficients	Interpretation
μ	The recruitment rate and natural death rate;
β	The rate of disease transmission between susceptible and infectious individuals;
β_1	The rate of disease transmission between vaccinated and infected individuals;
γ	The recovery rate;
α	The vaccination rate;
γ_1	The rate at which a vaccinated individual obtains immunity.

Table 1. The biological interpretation of coefficients of model (1.1)

at time t, respectively. All coefficients are assumed to be positive and the biological interpretation of them is listed in Table 1. In Liu *et al.* (2008), it was shown that the global dynamics of model (1.1) is completely determined by the basic reproduction number. That is, if the number is less than unity, then the disease-free equilibrium is globally asymptotically stable, while if the number is greater than unity, then a positive endemic equilibrium exists and it is globally asymptotically stable. It was observed in Liu *et al.* (2008) that vaccination has an effect of decreasing the basic reproduction number.

On the other hand, the infection age (time elapsed since the infection) is also known as an important factor for modeling infectious diseases (Chen *et al.*, 2014). It has been observed that the infection age can cause qualitative changes in the dynamics of solutions (Thieme & Castillo-Chavez, 1993). Recently, Röst & Wu (2008) formulated an SEIR model with infection age structure. They rewrote the model as a differential equation with infinite delay and proved the asymptotic smoothness, persistence, local stability of disease-free and endemic equilibria in terms of the basic reproduction number. However, the global stability of the endemic equilibrium was not proved. After them, by constructing an appropriate Lyapunov function, McCluskey (2009) proved that the endemic equilibrium is globally asymptotically stable whenever it exists. In Wang *et al.* (2011), the model was extended to an SVEIR epidemiological model by introducing the vaccinated population and it was shown that the global stability property as in the previous SEIR model also holds for the model. In the study, it was observed that both of the time for which vaccinated individuals to obtain immunity and the possibility for them to be infected before acquiring immunity can lead to the overevaluation of the effect of vaccination.

In the models studied in Röst & Wu (2008) and Wang *et al.* (2011), the removal rate from infectious class was assumed to be constant and not a function of infection age. By virtue of this assumption, the analysis can be simplified to that for the differential equations with infinite delay. However, for more realistic modeling, the fact that the removal (recovery) rate can depend on the infection age of each individual should not be neglected. This fact leads to a formulation of partial differential equation formulation (Webb, 1985). There have already been several studies on vaccination-age structured models. For example, Iannelli *et al.* (2005) considered a two-strain epidemiological model with super-infection and vaccination-age. Duan *et al.* (2014a) investigated the global stability of an vaccination-age structured SVIR model. Motivated by these works, in this paper, we investigate the dynamics of an SVIR epidemiological model with infection age. As a main result of this paper, we succeed in proving that the global stability of each equilibrium is completely determined by the basic reproduction number $\Re_0$: if $\Re_0 \leqslant 1$, then the disease-free equilibrium is globally asymptotically stable, while if $\Re_0 > 1$, then there exists a unique endemic equilibrium and it is globally asymptotically stable. For the proof, we first show the relative compactness of the orbit generated by the model in order to make use of the well-known invariance principle. In addition, we show the uniform persistence of the system to make the Lyapunov function well-defined.

This paper is organized as follows: In Section 2, we formulate a new SVIR model with infection age. In Section 3, the preliminary results, including positiveness, boundness of the solutions, existence of equilibria of the model and the basic reproduction number $\Re_0$ are obtained. In Section 4, the asymptotic smoothness of the semi-flow generated by system is shown. In Section 5, the existence of global compact attractor is proved. In Section 6, the uniform persistence of the system is proved by reformulating it as a Volterra integral equation. In Section 7, by constructing suitable Lyapunov function, the basic reproduction number $\Re_0$ is proved to be a threshold parameter which determines the global asymptotic stability of each equilibrium. In the last section, we give some brief summaries and discussions on our results.

2. Model formulation

In this section, we formulate an SVIR epidemiological model with infection age. We divide the total population into susceptible, vaccinated, infectious and recovered populations. The number of each of them at time t is denoted by $S(t)$, $V(t)$, $I(t)$ and $R(t)$, respectively. The infectious class is structured by age of infection (i.e., time since entry into class $I(t)$). The density of individuals with infection-age a at time t is given by $i(t,a)$ with $I(t)=\int_0^\infty i(t,a)\mathrm{d}a$.

Following the line of Liu *et al.* (2008); Wang *et al.* (2011), we also assume that before obtaining immunity the vaccinees still have the possibility of infection while contacting with infected individuals. Then we arrive at the following system mixed by ordinary differential equations and a partial differential equations:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \beta S(t)\displaystyle\int_0^\infty p(a)i(t,a)\mathrm{d}a - (d+\alpha)S(t), \\ \dfrac{\mathrm{d}V(t)}{\mathrm{d}t} = \alpha S(t) - \beta_1 V(t)\displaystyle\int_0^\infty p(a)i(t,a)\mathrm{d}a - (d+\gamma_1)V(t), \\ \dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -\delta(a)i(t,a), \\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = \gamma_1 V(t) + \displaystyle\int_0^\infty \xi(a)i(t,a)\mathrm{d}a - dR(t), \end{cases} \tag{2.1}$$

subject to the boundary and initial conditions

$$\begin{cases} i(t,0) = (\beta S(t) + \beta_1 V(t))\displaystyle\int_0^\infty p(a)i(t,a)\mathrm{d}a. \\ S(0) = \varphi_S \geqslant 0, V(0) = \varphi_V \geqslant 0, i(0,a) = \varphi_i(a) \in L^1_+(0,\infty) \text{ and } R(0) = \varphi_R \geqslant 0, \end{cases} \tag{2.2}$$

where $L^1_+(0,\infty)$ is the space of functions on $(0,\infty)$ that are nonnegative and Lebesgue integrable. Λ and d are the recruitment rate and natural death rate of the population, respectively. $p(a)$ is the kernel of disease transmission with infectious individuals whose infection age is a. $\delta(a)$ is the age-specific removal (including death and recovery) rate of infected individuals. The biological interpretation of the other coefficients of model (2.1) are the same as those listed in Table 1. In this paper, it is biologically assumed that $\beta_1 \leqslant \beta$, which is based on the fact that the vaccinated individuals are thought to have partial immunity and can be recognized as the transmission characters of the disease, and hence the effective contacts with infectious individuals may decrease compared to those of susceptibles (Liu *et al.*, 2008).

Since the variable $R(t)$ does not appear in the first three differential equations of (2.1), we restrict our attention to the following reduced system:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \beta S(t)\displaystyle\int_0^\infty p(a)i(t,a)\mathrm{d}a - (d+\alpha)S(t), \\ \dfrac{\mathrm{d}V(t)}{\mathrm{d}t} = \alpha S(t) - \beta_1 V(t)\displaystyle\int_0^\infty p(a)i(t,a)\mathrm{d}a - (d+\gamma_1)V(t), \\ \dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -\delta(a)i(t,a), \end{cases} \tag{2.3}$$

subject to the boundary conditions and initial conditions

$$\begin{cases} i(t,0) = (\beta S(t) + \beta_1 V(t))\displaystyle\int_0^\infty p(a)i(t,a)\mathrm{d}a. \\ S(0) = \varphi_S \geqslant 0, V(0) = \varphi_V \geqslant 0, \text{ and } i(0,a) = \varphi_i(a) \in L_+^1(0,\infty). \end{cases} \tag{2.4}$$

We make the following assumptions on the coefficients of system (2.3):

Assumption 2.1 (i) $p(a), \delta(a) \in L_+^\infty(0,+\infty)$, with essential upper bounds $\bar{p}, \bar{\delta}$, respectively.

(ii) $p(a), \delta(a)$ are Lipschitz continuous on $\mathbb{R}_+$ with Lipschitz coefficients M_p, M_δ, respectively.

(iii) There exist a_p, a_δ such that p, δ are positive in a neighborhood of a_p, a_δ, respectively.

(iv) $\delta(a) \geqslant d$ for all $a > 0$.

Before the analysis, we introduce some additional notations. Following Webb (1985), we define the phase space for system (2.3) by $\mathscr{Y} = \mathbb{R}_+ \times \mathbb{R}_+ \times L_+^1(0,+\infty)$, equipped with the norm

$$\|(x,y,\varphi)\|_{\mathscr{Y}} = |x| + |y| + \int_0^\infty |\varphi(a)|\mathrm{d}a.$$

Biologically, this norm is interpreted as the total population size.

Let us denote the solution of system (2.3) by $X(t) := (S(t), V(t), i(t,\cdot))$. The initial condition in (2.4) is denoted by

$$X_0 := (S(0), V(0), i(0,\cdot)) = (\varphi_S, \varphi_V, \varphi_i(\cdot)) \in \mathscr{Y}. \tag{2.5}$$

It follows from the standard theory of functional differential equation Hale (1971) that system (2.3) with initial condition (2.5) has a unique nonnegative solution $X(t) \in \mathscr{Y}$ for all $t \geqslant 0$. The continuous semi-flow $\Phi : \mathbb{R}_+ \times \mathscr{Y} \to \mathscr{Y}$ defined by system (2.3) takes the following form

$$\Phi(t, X_0) := X(t),\ t \geqslant 0.$$

For convenience, throughout the paper, we shall often use the symbol $\Phi_t(X_0)$ instead of $\Phi(t,X_0)$.

3. Preliminaries

For $a \geqslant 0$, let

$$\Omega(a) = \mathrm{e}^{-\int_0^a \delta(\tau)\mathrm{d}\tau}. \tag{3.1}$$

Biologically, it is the survival rate at which an individual to stay in the infectious class after a period of time $a \geqslant 0$.

It follows from (i) and (iv) of Assumption 2.1 that

$$0 \leqslant \Omega(a) \leqslant e^{-da} \tag{3.2}$$

for each $a \geqslant 0$. Additionally, the equation

$$\Omega'(a) = -\delta(a)\Omega(a)$$

hold for almost all $a \geqslant 0$. Let

$$A = \int_0^\infty p(a)\Omega(a)\mathrm{d}a. \tag{3.3}$$

It follows from (i) and (iii) of Assumption 2.1 and equation (3.2) that A is positive and finite.

Consider

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) i(t,a) = -\delta(a) i(t,a).$$

We integrate it along the characteristic line $t - a = const.$ and obtain

$$i(t,a) = \begin{cases} (\beta S(t-a) + \beta_1 V(t-a))P(t-a)\Omega(a), & \text{for} \quad 0 \leqslant a \leqslant t; \\ \varphi_i(a-t)\dfrac{\Omega(a)}{\Omega(a-t)}, & \text{for} \quad 0 \leqslant t \leqslant a, \end{cases} \tag{3.4}$$

where

$$P(t) = \int_0^\infty p(a) i(t,a)\mathrm{d}a. \tag{3.5}$$

By virtue of equation (3.4) and Assumption 2.1, we have the following result on the boundedness of the solutions of system (2.3):

PROPOSITION 3.1 For system (2.3) with initial conditions (2.5), the following statements hold:

(i) $\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi_t(X_0)\| \leqslant \Lambda - d\|\Phi_t(X_0)\|$ for all $t \geqslant 0$;

(ii) $\|\Phi_t(X_0)\| \leqslant \max\left\{\frac{\Lambda}{d}, \frac{\Lambda}{d} + e^{-dt}\left(\|X_0\| - \frac{\Lambda}{d}\right)\right\} \leqslant \max\left\{\frac{\Lambda}{d}, \|X_0\|\right\}$ for $t \geqslant 0$;

(iii) $\limsup_{t\to\infty}\|\Phi_t(X_0)\| \leqslant \frac{\Lambda}{d}$;

(iv) Φ is point dissipative; that is there is a bounded set that attracts all points in $\mathscr{Y}$.

Proof. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi_t(X_0)\| = \frac{\mathrm{d}S(t)}{\mathrm{d}t} + \frac{\mathrm{d}V(t)}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty i(t,a)\mathrm{d}a. \tag{3.6}$$

From equation (3.4), we have

$$\int_0^\infty i(t,a)\mathrm{d}a = \int_0^t (\beta S(t-a) + \beta_1 V(t-a))P(t-a)\Omega(a)\mathrm{d}a + \int_t^\infty \varphi_i(a-t)\frac{\Omega(a)}{\Omega(a-t)}\mathrm{d}a.$$

Taking change of variable $a = t - \sigma$ and $a = t + \tau$ in the first and second integral, respectively and differentiating by t yield

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty i(t,a)\mathrm{d}a &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (\beta S(\sigma)P(\sigma) + \beta_1 V(\sigma)P(\sigma))\Omega(t-\sigma)\mathrm{d}\sigma + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \varphi_i(\tau)\frac{\Omega(t+\tau)}{\Omega(\tau)}\mathrm{d}\tau \\ &= (\beta S(t)P(t) + \beta_1 V(t)P(t))\Omega(0) + \int_0^t (\beta S(\sigma)P(\sigma) + \beta_1 V(\sigma)P(\sigma))\Omega'(t-\sigma)\mathrm{d}\sigma \\ &\quad + \int_0^\infty \varphi_i(\tau)\frac{\Omega'(t+\tau)}{\Omega(\tau)}\mathrm{d}\tau. \end{aligned}$$

Noting that

$$\Omega(0) = 1 \ \text{ and } \ \Omega'(a) = -\delta(a)\Omega(a),$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty i(t,a)\mathrm{d}a = \beta S(t)P(t) + \beta_1 V(t)P(t) - \int_0^\infty \delta(a)i(t,a)\mathrm{d}a. \tag{3.7}$$

Then equation (3.6) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi_t(X_0)\| = \Lambda &- \beta S(t)\int_0^\infty p(a)i(t,a)\mathrm{d}a - (d+\alpha)S(t)\\ &+ \alpha S(t) - \beta_1 V(t)\int_0^\infty p(a)i(t,a)\mathrm{d}a - (d+\gamma_1)V(t)\\ &+ \beta S(t)P(t) + \beta_1 V(t)P(t) - \int_0^\infty \delta(a)i(t,a)\mathrm{d}a\\ \leqslant \Lambda &- dS(t) - dV(t) - \int_0^\infty \delta(a)i(t,a)\mathrm{d}a\end{aligned}$$

It follows from (iv) of Assumption 2.1 that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi_t(X_0)\| \ \leqslant \ \Lambda - dS(t) - dV(t) - d\int_0^\infty i(t,a)\mathrm{d}a \ = \ \Lambda - d\|\Phi_t(X_0)\|.$$

This proves the first statement. It follows from the variation of constants formula that for $t \geqslant 0$

$$\|\Phi_t(X_0)\| \leqslant \frac{\Lambda}{d} - e^{-dt}\left(\frac{\Lambda}{d} - \|X_0\|\right). \tag{3.8}$$

Hence the second, third and fourth statements directly follow. This completes the proof. □

Proposition 3.1 implies that for $X_0 \in \mathscr{Y}$ and $\|X_0\| \leqslant K$ for some $K \geqslant \frac{\Lambda}{d}$, the number of population classes $S(t)$, $V(t)$ and $\int_0^\infty i(t,a)\mathrm{d}a$ will eventually remain bounded. It follows from the (i) of Assumption 2.1 and Proposition 3.1 that $P(t) \leqslant \bar{p}K$. Furthermore, Proposition 3.1 implies that the state space

$$\Omega := \left\{X_0 \in \mathscr{Y} : \ \|\Phi_t(X_0)\| \leqslant \frac{\Lambda}{d}\right\}$$

is positively invariant for semi-flow Φ of system (2.3) and attracts all points in $\mathscr{Y}$.

For the system (2.3), there always exists the disease-free equilibrium $P_0 = (S_0, V_0, i_0(a))$, where $S_0 = \frac{\Lambda}{d+\alpha}$, $V_0 = \frac{\alpha S_0}{d+\gamma_1}$, and $i_0(a) = 0$. Moreover, the endemic equilibrium $P^* = (S^*, V^*, i^*(a))$ whose elements are non-zero is obtained as the solution of the following algebraic equations:

$$\begin{cases} \Lambda = (d+\alpha)S^* + \beta S^* \int_0^\infty p(a)i^*(a)\mathrm{d}a,\\ \alpha S^* = (d+\gamma_1)V^* + \beta_1 V^* \int_0^\infty p(a)i^*(a)\mathrm{d}a,\\ \dfrac{\mathrm{d}i^*(a)}{\mathrm{d}a} = -\delta(a)i^*(a),\\ i^*(0) = (\beta S^* + \beta_1 V^*)\int_0^\infty p(a)i^*(a)\mathrm{d}a. \end{cases} \tag{3.9}$$

From the third equation, we obtain $i^*(a) = i^*(0)\Omega(a)$. Putting it into the fourth equation gives

$$1 = (\beta S^* + \beta_1 V^*)\int_0^\infty p(a)\Omega(a)\mathrm{d}a. \tag{3.10}$$

In order to obtain the endemic equilibrium, we next define the key threshold parameter. According to Driessche & Wagmough (2002), the expected number of secondary cases produced by a typical infectious individual during its entire period of infectiousness is defined as *basic reproduction number*, $\Re_0$, which is given by

$$\Re_0 = (\beta S_0 + \beta_1 V_0)G. \tag{3.11}$$

where S_0, V_0 have been defined above and $G = \int_0^\infty p(a)\Omega(a)\mathrm{d}a$.

Denote $I^* = \int_0^\infty p(a)i^*(a)\mathrm{d}a$. From the first and second equations of (3.9), we have

$$S^* = \frac{\Lambda}{d+\alpha+\beta I^*}, \quad V^* = \frac{\alpha\Lambda}{(d+\alpha+\beta I^*)(d+\gamma_1+\beta_1 I^*)}.$$

Plugging it into (3.10) yields

$$H(I^*) = a_0(I^*)^2 + a_1 I^* + a_2 = 0, \tag{3.12}$$

where $a_0 = \beta\beta_1$, $a_1 = \beta(d+\gamma_1) + \beta_1(d+\alpha) - \beta\beta_1\Lambda G$, $a_2 = (d+\alpha_1)(d+\alpha)(1-\Re_0)$.

Since $a_0 > 0$, it has $G(\pm\infty) = +\infty$. When $\Re_0 \leqslant 1$, we know that $H(0) \geqslant 0$ and

$$H'(I^*) = 2a_0 I^* + a_1 = 2a_0 I^* + \beta\beta_1(d+\alpha)\left\{\frac{d+\gamma_1}{(d+\alpha)\beta_1} + \frac{1}{\beta} - \frac{\Lambda G}{d+\alpha}\right\}.$$

Moreover, $\Re_0 \leqslant 1$ is equivalent to

$$\frac{\Lambda G}{d+\alpha} \leqslant \frac{d+\gamma_1}{(d+\alpha)\beta_1+\varepsilon}, \quad \text{where} \quad \varepsilon = (\beta-\beta_1)d + \beta\gamma_1,$$

which implies that $\frac{\Lambda G}{d+\alpha} < \frac{d+\gamma_1}{(d+\alpha)\beta_1}$. Therefore, $H'(I^*) > 0$ for any $I^* \geqslant 0$ when $\Re_0 \leqslant 1$. In this case, it is obvious that equation (3.12) has not positive root.

On the other hand, when $\Re_0 > 1$, it has that $H(0) = a_2 < 0$. From the properties of the quadratic function $H(I^*)$, equation (3.12) has a unique positive real root I^*. Consequently, we have the following Proposition for system (2.3).

PROPOSITION 3.2 System (2.3) has an endemic equilibrium $P^*(S^*, V^*, i^*(a))$ if and only if $\Re_0 > 1$.

4. Asymptotic smoothness

In this section, we will prove that the semi-flow Φ is asymptotically smooth. To this end, we first prove that $P(t)$ is Lipschitz continuous on $\mathbb{R}_+$. Second, we introduce the result in Smith & Thieme (2011), which was used to prove the asymptotic smoothness of semi-flow. Finally, we prove our main theorem by using their result.

PROPOSITION 4.1 The function $P(t)$ defined by (3.5) is Lipschitz continuous on $\mathbb{R}_+$.

Proof. Let $K \geqslant \max\{\frac{\Lambda}{d}, \|X_0\|\}$. It follows from Proposition 3.1 that $\|X(t)\| \leqslant K$ for all $t \geqslant 0$. Let $h > 0$, then

$$\begin{aligned} P(t+h) - P(t) &= \int_0^\infty p(a)i(t+h,a)\mathrm{d}a - \int_0^\infty p(a)i(t,a)\mathrm{d}a \\ &= \int_0^h p(a)i(t+h,a)\mathrm{d}a + \int_h^\infty p(a)i(t+h,a)\mathrm{d}a - \int_0^\infty p(a)i(t,a)\mathrm{d}a \\ &= \int_0^h p(a)i(t+h-a,0)\Omega(a)\mathrm{d}a + \int_h^\infty p(a)i(t+h,a)\mathrm{d}a - \int_0^\infty p(a)i(t,a)\mathrm{d}a. \end{aligned} \tag{4.1}$$

From the Assumption 2.1, we have the following facts hold, $p(a) \leqslant \bar{p}$, $i(t+h-a,0) \leqslant \bar{\beta}\bar{p}K^2$. Hence $\int_0^h p(a)i(t+h-a,0)\Omega(a)\mathrm{d}a$ can be bounded by $\bar{p}^2\bar{\beta}K^2h$, where $\bar{\beta} = \beta + \beta_1$. Taking the substitution $\sigma = a-h$ for the second integral of (4.1) gives

$$P(t+h)-P(t) \leqslant \bar{p}^2\bar{\beta}K^2h + \int_0^\infty p(\sigma+h)i(t+h,\sigma+h)\mathrm{d}\sigma - \int_0^\infty p(a)i(t,a)\mathrm{d}a.$$

Note that

$$i(t+h,\sigma+h) = i(t,\sigma)\frac{\Omega(\sigma+h)}{\Omega(\sigma)}.$$

Hence (4.1) becomes

$$\begin{aligned} P(t+h)-P(t) &\leqslant \bar{p}^2\bar{\beta}K^2h + \int_0^\infty \left(p(a+h)\frac{\Omega(a+h)}{\Omega(a)} - p(a)\right)i(t,a)\mathrm{d}a \\ &= \bar{p}^2\bar{\beta}K^2h + \int_0^\infty \left(p(a+h)e^{-\int_a^{a+h}\delta(\tau)\mathrm{d}\tau} - p(a)\right)i(t,a)\mathrm{d}a \\ &= \bar{p}^2\bar{\beta}K^2h + \int_0^\infty p(a+h)\left(e^{-\int_a^{a+h}\delta(\tau)\mathrm{d}\tau} - 1\right)i(t,a)\mathrm{d}a + \int_0^\infty (p(a+h)-p(a))i(t,a)\mathrm{d}a. \end{aligned} \tag{4.2}$$

It follows from (i) of Assumption 2.1 that $0 \geqslant -\int_a^{a+h}\delta(\tau)\mathrm{d}\tau \geqslant -\bar{\delta}h$. Hence, we have $1 \geqslant e^{-\int_a^{a+h}\delta(\tau)\mathrm{d}\tau} \geqslant e^{-\bar{\delta}h} \geqslant 1-\bar{\delta}h$. It follows that

$$\int_0^\infty p(a+h)\left(e^{-\int_a^{a+h}\delta(\tau)\mathrm{d}\tau} - 1\right)i(t,a)\mathrm{d}a \leqslant \bar{p}\bar{\delta}Kh.$$

By using (ii) of Assumption 2.1, we have $\int_0^\infty |p(a+h)-p(a)|\, i(t,a)\mathrm{d}a \leqslant M_phK$. Therefore

$$|P(t+h)-P(t)| \leqslant \bar{p}^2\bar{\beta}K^2h + \bar{p}\bar{\delta}Kh + M_pKh. \tag{4.3}$$

That is, $P(t)$ is Lipschitz continuous with coefficient $M_P = (\bar{p}^2\bar{\beta}K + \bar{p}\bar{\delta} + M_k)K$. □

The following result is well-known and useful for the proof of our main theorem.

PROPOSITION 4.2 (McCluskey, 2012) Let $D \subseteq \mathbb{R}$. For $j = 1,2$, suppose $f_j : D \to \mathbb{R}$ is a bounded Lipschitz continuous function with bound K_j and Lipschitz coefficient M_j. Then the product f_1f_2 is also Lipschitz continuous with coefficient $K_1M_2 + K_2M_1$.

THEOREM 4.3 (Hale & Waltman, 1989) The semi-flow $\Phi : \mathbb{R}_+ \times \mathscr{Y} \to \mathscr{Y}$ is asymptotically smooth if there are maps $\Theta, \Psi : \mathbb{R}_+ \times \mathscr{Y} \to \mathscr{Y}$ s.t. $\Phi(t,X) = \Theta(t,X) + \Psi(t,X)$, and the following hold for any bounded closed set C that is forward invariant under Φ:

(i) $\lim_{t\to\infty} \mathrm{diam}\Theta(t,C) = 0$;

(ii) there exists $t_C \geqslant 0$ such that $\Psi(t,C)$ has compact closure for each $t \geqslant t_C$.

In order to verify (ii) of Theorem 4.3, we need the following theorem:

THEOREM 4.4 (Smith & Thieme, 2011) A set $C \in L^1_+(\mathbb{R}_{\geqslant 0})$ has compact closure if and only if the following conditions hold:

(i) $\sup_{f\in C}\int_0^\infty f(a)\mathrm{d}a < \infty$;

(ii) $\lim_{r\to\infty}\int_r^\infty f(a)\mathrm{d}a \to 0$ uniformly in $f \in C$;

(iii) $\lim_{h\to 0^+}\int_0^\infty |f(a+h)-f(a)|\mathrm{d}a \to 0$ uniformly in $f \in C$;

(iv) $\lim_{h\to 0^+}\int_0^h f(a)\mathrm{d}a \to 0$ uniformly in $f \in C$.

Now we are in a position to prove our main theorem in this section. A similar argument can be found in McCluskey (2012).

THEOREM 4.5 The semi-flow Φ generated by (2.3) is asymptotically smooth.

Proof. According to Theorems 4.3 and 4.4, we prove that each forward invariant bounded closed set under Φ is attracted by a nonempty compact set. To this end, let $C \subset \mathscr{Y}$ be bounded by an upper bounded $K > \frac{\Lambda}{d}$. Hence we consider the solution $\Phi(t,X_0) = (S(t),V(t),i(t,\cdot))$ with $X_0 \in C$, where $i(t,\cdot)$ is defined in (3.4). It suffices to show that conditions in Theorems 4.3 and 4.4 can be fulfilled for system (2.3).

Proof of (i) of Theorem 4.3. For $t \geqslant 0$, let $\Psi(t,X_0) = (S(t),V(t),\tilde{i}(t,\cdot))$ and $\Theta(t,X_0) = (0,0,\tilde{\varphi}_i(t,\cdot))$, where

$$\tilde{i}(t,a) = \begin{cases} (\beta S(t-a)P(t-a) + \beta_1 V(t-a)P(t-a))\Omega(a) & \text{for} \quad 0 \leqslant a \leqslant t; \\ 0 & \text{for} \quad t < a, \end{cases}$$

and

$$\tilde{\varphi}_i(t,a) = \begin{cases} 0 & \text{for} \quad 0 \leqslant a \leqslant t; \\ \varphi_i(a-t)\dfrac{\Omega(a)}{\Omega(a-t)} & \text{for} \quad t < a. \end{cases}$$

Then it is easy to see that $\Phi = \Theta + \Psi$. Denote by $\|\cdot\|_1$ the standard norm on L^1. Thus,

$$\begin{aligned} \|\tilde{\varphi}_i(t,\cdot)\|_1 &= \int_0^\infty |\tilde{\varphi}_i(t,a)|\mathrm{d}a \\ &= \int_t^\infty \varphi_i(a-t)\frac{\Omega(a)}{\Omega(a-t)}\mathrm{d}a \\ &= \int_0^\infty \varphi_i(\sigma)\frac{\Omega(\sigma+t)}{\Omega(\sigma)}\mathrm{d}\sigma \\ &= \int_0^\infty \varphi_i(\sigma)e^{-\int_\sigma^{\sigma+t}\delta(\tau)\mathrm{d}\tau}\mathrm{d}\sigma \\ &\leqslant e^{-dt}\int_0^\infty \varphi_i(\sigma)\mathrm{d}\sigma \\ &\leqslant Ke^{-dt}. \end{aligned}$$

This shows that $\|\tilde{\varphi}_i(t,\cdot)\|_1 \to 0$ as $t \to \infty$, which implies that $\Theta(t,X_0)$ approaches $\mathbf{0} \in \mathscr{Y}$ with uniform exponential speed. This completes the proof of (i) of Theorem 4.3.

Proof of (ii) of Theorem 4.3. It follows from Proposition 3.1 that $S(t)$ and $V(t)$ remain bounded in the compact set $[0,K]$. We only have to show that $\tilde{i}(t,a)$ remains in a pre-compact subset of L^1_+ that is independent of X_0. It can be really realized by verifying conditions (i)-(iv) of Theorem 4.4.

From Proposition 3.1 and Equation (3.2),

$$0 \leqslant \tilde{i}(t,a) = \begin{cases} (\beta S(t-a) + \beta_1 V(t-a))P(t-a)\Omega(a) & \text{for} \quad 0 \leqslant a \leqslant t; \\ 0 & \text{for} \quad t < a, \end{cases}$$

can be evaluated by

$$\tilde{i}(t,a) \leqslant (\beta+\beta_1)\bar{p}K^2 e^{-da}.$$

It is easy to see that the conditions (i), (ii) and (iv) of Theorem 4.4 follow directly. Now, we verify that condition (iii) holds for system (2.3). Let $h \in (0,t)$, then

$$\begin{aligned}
&\int_0^\infty |\tilde{i}(t,a+h)-\tilde{i}(t,a)|\mathrm{d}a = \\
&\int_0^{t-h} |(\beta S(t-a-h)+\beta_1 V(t-a-h))P(t-a-h)\Omega(a+h) \\
&-(\beta S(t-a)+\beta_1 V(t-a))P(t-a)\Omega(a)|\mathrm{d}a \\
&+\int_{t-h}^{t} |0-(\beta S(t-a)+\beta_1 V(t-a))P(t-a)\Omega(a)|\mathrm{d}a \\
\leqslant &\int_0^{t-h} |(\beta S(t-a-h)+\beta_1 V(t-a-h))P(t-a-h)\Omega(a+h) \\
&-(\beta S(t-a)+\beta_1 V(t-a))P(t-a)\Omega(a)|\mathrm{d}a + \bar{\beta}\bar{p}K^2 h \\
\leqslant &\bar{\beta}\bar{p}K^2 h + \Delta + \Xi,
\end{aligned} \tag{4.4}$$

where

$$\Delta = \int_0^{t-h} (\beta S(t-a-h)+\beta_1 V(t-a-h))P(t-a-h)|\Omega(a+h)-\Omega(a)|\mathrm{d}a,$$

and

$$\begin{aligned}
\Xi = &\int_0^{t-h} |(\beta S(t-a-h)P(t-a-h)+\beta_1 V(t-a-h)P(t-a-h)) \\
&-(\beta S(t-a)P(t-a)+\beta_1 V(t-a)P(t-a))|\Omega(a)\mathrm{d}a \\
\leqslant &\int_0^{t-h} |(\beta S(t-a-h)P(t-a-h)-(\beta S(t-a)P(t-a))|\Omega(a)\mathrm{d}a \\
&+\int_0^{t-h} |(\beta_1 V(t-a-h)P(t-a-h)-\beta_1 V(t-a)P(t-a))|\Omega(a)\mathrm{d}a.
\end{aligned}$$

Recall that

$$\begin{aligned}
\int_0^{t-h} |\Omega(a+h)-\Omega(a)|\mathrm{d}a &= \int_0^{t-h} (\Omega(a)-\Omega(a+h))\mathrm{d}a \\
&= \int_0^{t-h} \Omega(a)\mathrm{d}a - \int_h^t \Omega(a)\mathrm{d}a \\
&= \int_0^{t-h} \Omega(a)\mathrm{d}a - \int_h^{t-h} \Omega(a)\mathrm{d}a - \int_{t-h}^{t} \Omega(a)\mathrm{d}a \\
&= \int_0^{h} \Omega(a)\mathrm{d}a - \int_{t-h}^{t} \Omega(a)\mathrm{d}a \\
&\leqslant h.
\end{aligned}$$

It follows that

$$\Delta \leqslant \bar{\beta}\bar{p}K^2 h.$$

For Ξ, combining Proposition 3.1 with the expression for $\frac{\mathrm{dS(t)}}{\mathrm{d}t}$, we obtain that $|\frac{\mathrm{dS(t)}}{\mathrm{d}t}| \leqslant \Lambda + (d+\alpha)K + \beta \bar{p} K^2 := B_S$, and therefore $S(\cdot)$ is Lipschitz on $[0,\infty)$ with coefficient B_S. Similarly, $V(\cdot)$ is Lipschitz on $[0,\infty)$ with coefficient $B_V = (d+\alpha+\gamma_1)K + \beta_1 \bar{p} K^2$. By Proposition 4.1, $P(t)$ is Lipschitz continuous with coefficient $M_P = (\bar{p}^2\bar{\beta}K + \bar{p}\bar{\delta} + M_k)K$. Thus, Proposition 4.2 indicates that $S(\cdot)P(\cdot)$ and $V(\cdot)P(\cdot)$ is Lipschitz on $[0,\infty)$ with coefficient $M_{SP} = KM_P + \bar{p}KB_S$ and $M_{VP} = KM_P + \bar{p}KB_V$. Let $M = \beta M_{SP} + \beta_1 M_{VP}$. Obviously,

$$\Xi \leqslant Mh \int_0^{t-h} e^{-da}\mathrm{d}a \leqslant \frac{Mh}{d}.$$

Hence

$$\int_0^\infty |i(t,a+h) - i(t,a)|\mathrm{d}a \leqslant \left(2\bar{\beta}\bar{p}K^2 + \frac{Mh}{d}\right)h.$$

This completes the verification of condition (iii) of Theorem 4.4. It is concluded that $i(y,\cdot)$ remains in a pre-compact subset C_K^i of $L_+^1(0,+\infty)$. Thus, $\Phi(t,C) \subseteq [0,K]\times[0,K]\times C_K^i$, which has compact closure in $\mathscr{Y}$, which completes the proof of (ii) of Theorem 4.3. □

5. Attractor

Following the line of McCluskey (2012); Smith & Thieme (2011), we first give the necessary definitions needed in this section.

DEFINITION 5.1 *Total trajectory.* A total trajectory of Φ is a function $X : \mathbb{R} \to \mathscr{Y}$ such that $\Phi_s(X(t)) = X(t+s)$ for all $t \in \mathbb{R}$ and all $s \geqslant 0$.

DEFINITION 5.2 *Compact attractor.* A non-empty compact set $\tilde{A}$ is a compact attractor of a class $\mathscr{C}$ of set if $\tilde{A}$ is invariant and $\mathrm{d}\left(\Phi_t(C),\tilde{A}\right) \to 0$ for each $C \in \mathscr{C}$.

For each $X_0 \in \tilde{A}$, there exists a total trajectory X such that $X(0) = X_0$ and $X(t) \in \tilde{A}$ for all $t \in \mathbb{R}$.

It is noted that total trajectory $i(t,a) = i(t-a,0)\Omega(a)$ for all $t \in \mathbb{R}$ and $a \in \mathbb{R}_+$ has nice properties, for example, Lipschitz continuous, see Proposition 4.1.

Based on the point dissipativeness (see Propositions 3.1) and asymptotic smoothness (see Theorem 4.5) of Φ, we establish the following main theorem in this section, which follows from Theorem 2.33 of Smith & Thieme (2011).

THEOREM 5.3 The semi-flow Φ has a global attractor $\mathscr{A}$ contained in $\mathscr{Y}$, which attracts the bound sets of $\mathscr{Y}$.

6. Uniform persistence for $\Re_0 > 1$

In this section, we show the uniform persistence of system (2.3). For the simplicity of notation, denote $i(t,0)$ by $\hat{i}(t)$. Thus (3.4) can be rewritten as

$$i(t,a) = \begin{cases} \hat{i}(t-a)\Omega(a), & t \geqslant a \geqslant 0; \\ i_0(a-t)\frac{\Omega(a)}{\Omega(a-t)}, & a \geqslant t \geqslant 0. \end{cases} \tag{6.1}$$

Thus, boundary condition in (2.4) becomes

$$\hat{i}(t) = (\beta S(t) + \beta_1 V(t))\left\{\int_0^t p(a)\Omega(a)\hat{i}(t-a)\mathrm{d}a + \int_t^\infty p(a)\frac{\Omega(a)}{\Omega(a-t)}i_0(a-t)\mathrm{d}a\right\}. \tag{6.2}$$

We first have the following result.

LEMMA 6.1 If $\Re_0 > 1$, then there exists a positive constant $\varepsilon > 0$ such that

$$\limsup_{t\to\infty} \hat{i}(t) > \varepsilon. \tag{6.3}$$

Proof. Since $\Re_0 > 1$, there exists a sufficiently small $\varepsilon > 0$ such that

$$\left(\beta\frac{\Lambda-\varepsilon}{d+\alpha} + \beta_1\frac{\alpha\frac{\Lambda-\varepsilon}{d+\alpha}-\varepsilon}{d+\gamma_1}\right)\int_0^\infty p(a)\Omega(a)\mathrm{d}a > 1. \tag{6.4}$$

For such ε, we show that (6.3) holds true for all t. To this end, by way of contradiction, suppose that there exists a sufficiently large $T > 0$ such that

$$\hat{i}(t) \;=\; (\beta S(t) + \beta_1 V(t))\int_0^\infty p(a)i(t,a)\mathrm{d}a \;\leqslant\; \varepsilon \quad \text{for all } t \geqslant T, \tag{6.5}$$

then it follows from the first equation of (2.3) that

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} \geqslant \Lambda - \varepsilon - (d+\alpha)S(t) \quad \text{for all } t \geqslant T.$$

Hence, from comparison theorem, we can easily get

$$S(t) \geqslant \frac{\Lambda-\varepsilon}{d+\alpha} \quad \text{for all } t \geqslant T. \tag{6.6}$$

Furthermore, from the second equation of (2.3), we have

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \geqslant \alpha\frac{\Lambda-\varepsilon}{d+\alpha} - \varepsilon - (d+\gamma_1)V(t) \quad \text{for all } t \geqslant T.$$

It follows that

$$V(t) \geqslant \frac{\alpha\frac{\Lambda-\varepsilon}{d+\alpha}-\varepsilon}{d+\gamma_1} \quad \text{for all } t \geqslant T. \tag{6.7}$$

Combining (6.2), (6.6) and (6.7), we have

$$\hat{i}(t) \geqslant \left(\beta\frac{\Lambda-\varepsilon}{d+\alpha} + \beta_1\frac{\alpha\frac{\Lambda-\varepsilon}{d+\alpha}-\varepsilon}{d+\gamma_1}\right)\int_0^t p(a)\Omega(a)\hat{i}(t-a)\mathrm{d}a \quad \text{for all } t \geqslant T. \tag{6.8}$$

Now, without loss of generality, we can perform the time-shift of system (2.3) with respect to T. That is, replacing the initial condition of system (2.3) by $X_1 := \Phi(T, X_0)$, we can consider that (6.5) and (6.8) hold for $t \geqslant 0$. Note that for any $\lambda > 0$, the Laplace transform $\mathscr{L}[\hat{i}](\lambda)$ of $\hat{i}$ is bounded:

$$\mathscr{L}[\hat{i}](\lambda) := \int_0^\infty e^{-\lambda t}\hat{i}(t)\mathrm{d}t \leqslant \varepsilon\int_0^\infty e^{-\lambda t}\mathrm{d}t = \frac{\varepsilon}{\lambda} < +\infty.$$

Hence, for any $\lambda > 0$, taking the Laplace transform of both sides of (6.8), we have

$$\mathscr{L}[\hat{i}](\lambda) \quad \geqslant \quad \left(\beta\frac{\Lambda-\varepsilon}{d+\alpha} + \beta_1\frac{\alpha\frac{\Lambda-\varepsilon}{d+\alpha}-\varepsilon}{d+\gamma_1}\right)\int_0^\infty p(a)\Omega(a)e^{-\lambda a}\mathrm{d}a\,\mathscr{L}[\hat{i}](\lambda).$$

Dividing both sides by $\mathscr{L}[\hat{i}](\lambda)$, we have the following inequality.

$$1 \geqslant \left(\beta\frac{\Lambda-\varepsilon}{d+\alpha}+\beta_1\frac{\alpha\frac{\Lambda-\varepsilon}{d+\alpha}-\varepsilon}{d+\gamma_1}\right)\int_0^\infty p(a)\Omega(a)e^{-\lambda a}\mathrm{d}a. \tag{6.9}$$

Note that this inequality holds for any $\lambda > 0$. Therefore, from the continuity of the right-hand side of (6.9) with respect to λ and the boundedness of integral $\int_0^\infty p(a)\Omega(a)\mathrm{d}a$, we obtain

$$1 \quad \geqslant \quad \left(\beta\frac{\Lambda-\varepsilon}{d+\alpha}+\beta_1\frac{\alpha\frac{\Lambda-\varepsilon}{d+\alpha}-\varepsilon}{d+\gamma_1}\right)\int_0^\infty p(a)\Omega(a)\mathrm{d}a,$$

which contradicts to (6.4). This completes the proof. □

Next, in order to apply a technique used in Chapter 9 of Smith & Thieme (2011) (see also Section 8 of McCluskey (2012)), we consider a total Φ-trajectory of system (2.1) in space $\mathscr{Y}$. Let $\phi:\mathbb{R}\to\mathscr{Y}$ be a total Φ-trajectory such that $\phi(r) := (S(r),V(r),i(r,\cdot))$, $r\in\mathbb{R}$. Then, it follows that $\phi(r+t) = \Phi(t,\phi(r))$, $t \geqslant 0$, $r\in\mathbb{R}$ and

$$i(r,a) \quad = \quad i(r-a,0)\,\Omega(a) \;=\; \hat{i}(r-a)\Omega(a), \quad r\in\mathbb{R}, \quad a,b\geqslant 0.$$

Hence, (2.3) becomes

$$\begin{cases} \dfrac{\mathrm{d}S(r)}{\mathrm{d}r} = \Lambda-(d+\alpha)S(r)-\beta S(r)\displaystyle\int_0^\infty p(a)\Omega(a)\hat{i}(r-a)\mathrm{d}a, \\ \dfrac{\mathrm{d}V(r)}{\mathrm{d}r} = \alpha S(r)-(d+\gamma_1)V(r)-\beta_1 V(r)\displaystyle\int_0^\infty p(a)\Omega(a)\hat{i}(r-a)\mathrm{d}a, \\ \hat{i}(r) = (\beta S(r)+\beta_1 V(r))\displaystyle\int_0^\infty p(a)\Omega(a)\hat{i}(r-a)\mathrm{d}a, \quad r\in\mathbb{R}. \end{cases}$$

We prove the following lemma.

LEMMA 6.2 For total Φ-trajectory ϕ in $\mathscr{Y}$, $S(r)$ and $V(r)$ are strictly positive on $\mathbb{R}$ and $\hat{i}(r)=0$ for all $r\geqslant 0$ if $\hat{i}(r)=0$ for all $r\leqslant 0$.

Proof. Suppose that $S(r^*)=0$ for a number $r^*\in\mathbb{R}$ and show a contradiction. In this case, it follows from the first equation of (6.10) that $\mathrm{d}S(r^*)/\mathrm{d}r = \Lambda > 0$. This implies that $S(r^*-\eta) < 0$ for sufficiently small $\eta > 0$ and it contradicts to the fact that the total Φ-trajectory ϕ remains in $\mathscr{Y}$. Consequently, $S(r)$ is strictly positive on $\mathbb{R}$. Similarly, we can prove $V(r)$ is strictly positive on $\mathbb{R}$.

By changing the variables, we can rewrite the third equation of (6.10) as follows.

$$\hat{i}(r) \quad = \quad (\beta S(r)+\beta_1 V(r))\int_{-\infty}^r p(r-a)\Omega(r-a)\hat{i}(a)\mathrm{d}a.$$

Hence, if $\hat{i}(r)=0$ for all $r\leqslant 0$, then we have

$$\hat{i}(r) \leqslant (\beta S_0+\beta_1 V_0)\bar{p}\int_0^r \hat{i}(a)\mathrm{d}a, \quad r\geqslant 0.$$

This is a Gronwall-like inequality and it follows that $\hat{i}(r)=0$ for all $r\geqslant 0$. In fact, let

$$\hat{I}(r) := \int_0^r \hat{i}(a)\mathrm{d}a, \quad r\geqslant 0.$$

Then,

$$\begin{aligned}\frac{\mathrm{d}\hat{I}(r)}{\mathrm{d}r} &= \hat{i}(r)\\ &\leqslant (\beta S_0+\beta_1 V_0)\bar{p}\int_0^r \hat{i}(a)\mathrm{d}a\\ &\leqslant (\beta S_0+\beta_1 V_0)\bar{p}\hat{I}(r), \quad r\geqslant 0\end{aligned}$$

and hence,

$$\hat{I}(r) \leqslant \hat{I}(0)e^{(\beta S_0+\beta_1 V_0)\bar{p}r} = 0, \quad r\geqslant 0.$$

This implies that $\hat{i}(r)=0$ for all $r\geqslant 0$. □

We prove the following lemma on the total Φ-trajectory ϕ:

LEMMA 6.3 For total Φ-trajectory ϕ in $\mathscr{Y}$, $\hat{i}(r)$ is strictly positive or identically zero on $\mathbb{R}$.

Proof. From the second statement of Lemma 6.2, by performing appropriate shifts, we see that $\hat{i}(r)=0$ for all $r\geqslant r^*$ if $\hat{i}(r)=0$ for all $r\leqslant r^*$, where $r^*\in\mathbb{R}$ is arbitrary. This implies that either $\hat{i}(r)$ is identically zero on $\mathbb{R}$ or there exists a decreasing sequence $\{r_j\}_{j=1}^{\infty}$ such that $r_j\to-\infty$ as $j\to\infty$ and $\hat{i}(r_j)>0$. In the latter case, letting $\hat{i}_j(t):=\hat{i}(t+r_j)$, $t\geqslant 0$, we have from the third equation of (6.10) that

$$\hat{i}_j(t)\geqslant(\beta\underline{S}+\beta_1\underline{V})\int_0^t p(a)\Omega(a)\hat{i}_j(t-a)\mathrm{d}a+\hat{j}_j(t),\ t\geqslant 0,$$

where $\underline{S}:=\inf_{r\in\mathbb{R}}S(r)>0$, $\underline{V}:=\inf_{r\in\mathbb{R}}V(r)>0$ and

$$\hat{j}_j(t):= (\beta S(t+r_j)+\beta_1 V(t+r_j))\int_t^{\infty} p(a)\Omega(a)\hat{i}_j(t-a)\mathrm{d}a,\ t\geqslant 0.$$

Then, since $\hat{j}_j(0)=\hat{i}(r_j)>0$ and $\hat{j}_j(r)$ is continuous at 0, it follows from Corollary B.6 of Smith & Thieme (2011) that there exists a number $r^*>0$, which depends only on $(\beta\underline{S}+\beta_1\underline{V})\,p(a)\Omega(a)$, such that $\hat{i}_j(t)>0$ for all $t>r^*$. From the definition of $\hat{i}_j$, this implies that $\hat{i}(t)>0$ for all $t>r^*+r_j$. Since $r_j\to-\infty$ as $j\to\infty$, we obtain that $\hat{i}(r)>0$ for all $r\in\mathbb{R}$ by letting $j\to\infty$. Consequently, $\hat{i}(r)$ is strictly positive on $\mathbb{R}$. □

Now, let us define a function $\rho:\mathscr{Y}\to\mathbb{R}_+$ on $\mathscr{Y}$ by

$$\rho\,(x,y,\varphi):=(\beta x+\beta_1 y)\int_0^{\infty} p(a)\varphi(a)\mathrm{d}a, \quad (x,y,\varphi)\in\mathscr{Y}.$$

Then, it follows from the previous argument that

$$\rho\,(\Phi_t(X_0))=\hat{i}(t).$$

Then, Lemma 6.1 implies the uniform weak ρ-persistence of semi-flow Φ for $\Re_0>1$. Moreover, from the Theorem 5.3 and Lemmas 6.2, 6.3 and the Lipschitz continuity of $\hat{i}$ (which immediately follows from Proposition 4.1), we can apply Theorem 5.2 of Smith & Thieme (2011) to conclude that the uniform weak ρ-persistence of semi-flow Φ implies the uniform (strong) ρ-persistence. In conclusion, we obtain the following theorem.

THEOREM 6.1 If $\Re_0>1$, then semi-flow Φ is uniformly (strongly) ρ-persistent.

The uniform persistence of system (2.1) for $\Re_0 > 1$ immediately follows from Theorem 6.1. In fact, it follows from (6.1) that

$$\|i(t,\cdot)\|_{L^1} \geqslant \int_0^t \hat{i}(t-a)\Omega(a)\mathrm{d}a$$

and hence, from a variation of the Lebesgue-Fatou lemma ((Smith, 2003, Section B.2)), we have

$$\liminf_{t\to\infty} \|i(t,\cdot)\|_{L^1} \geqslant \hat{i}^{\infty} \int_0^{\infty} \Omega(a)\mathrm{d}a,$$

where $\hat{i}^{\infty} := \liminf_{t\to\infty} \hat{i}(t)$. Under Theorem 6.1, there exists a positive constant $\varepsilon > 0$ such that $\hat{i}^{\infty} > \varepsilon$ if $\Re_0 > 1$ and hence, the persistence of $i(t,a)$ with respect to $\|\cdot\|_{L^1}$ follows. By a similar argument, we can prove that $S(t)$ and $V(t)$ are also persistent with respect to $|\cdot|$. Consequently, we have the following theorem.

THEOREM 6.2 If $\Re_0 > 1$, the semiflow $\{\Phi(t)\}_{t\geqslant 0}$ generated by (2.3) is uniformly persistent in $\mathscr{Y}$, that is, there exists a constant $\varepsilon > 0$ such that for each $X_0 \in \mathscr{Y}$,

$$\liminf_{t\to+\infty} S(t) \geqslant \varepsilon, \;\; \liminf_{t\to+\infty} V(t) \geqslant \varepsilon, \;\; \liminf_{t\to+\infty} \|i(t,\cdot)\|_{L^1} \geqslant \varepsilon.$$

7. Global asymptotic stability

It is important to analyze the stability of these equilibria, as it will indicate whether the disease will die out eventually, or it will persist for all time. In particular, the global stability of the epidemiological model with infection age becomes much more interesting from the realistic views to theoretical views Chen *et al.* (2014). In this section , we will establish the global stability of disease-free equilibrium and endemic equilibrium. This is achieved by using the Lyapunov functional approach.

THEOREM 7.1 If $\Re_0 \leqslant 1$, then the disease-free equilibrium P_0 is globally asymptotically stable.

Proof. Let

$$\alpha(a) = \int_a^{\infty} p(\sigma) e^{-\int_a^{\sigma} \delta(s)ds} \mathrm{d}\sigma. \tag{7.1}$$

Then, it follows that

$$\alpha(0) = \frac{\Re_0}{\beta S_0 + \beta_1 V_0} \tag{7.2}$$

and

$$\alpha'(a) = \frac{\mathrm{d}\alpha(a)}{\mathrm{d}a} = \alpha(a)\delta(a) - p(a). \tag{7.3}$$

Using the fact that $h(z) = z - 1 - \ln z \geqslant h(1) = 0$ for all $z > 0$, we have $S_0 h(\frac{S(t)}{S_0}) \geqslant 0$, $V_0 h(\frac{V(t)}{V_0}) \geqslant 0$. We constructive the Lyapunov function as follows,

$$U_1 = \; S_0 h\left(\frac{S(t)}{S_0}\right) + V_0 h\left(\frac{V(t)}{V_0}\right) + (\beta S_0 + \beta_1 V_0) \int_0^{\infty} \alpha(a) i(t,a) \mathrm{d}a. \tag{7.4}$$

The Lyapunov function U_1 is nonnegative and defined with respect to the disease free equilibrium $P_0 = (S_0, V_0, i_0(a))$, which is a global minimum. Differentiating U_1 along the solution of system (2.3), we obtain

$$\frac{\mathrm{d}U_1(t)}{\mathrm{d}t} = \left(1 - \frac{S_0}{S}\right)\frac{\mathrm{d}S(t)}{\mathrm{d}t} + \left(1 - \frac{V_0}{V}\right)\frac{\mathrm{d}V(t)}{\mathrm{d}t} + (\beta S_0 + \beta_1 V_0)\int_0^{\infty} \alpha(a) \frac{\partial i(t,a)}{\partial t} \mathrm{d}a. \tag{7.5}$$

By using $\Lambda = (d+\alpha)S_0$ and $\alpha S_0 = (d+\gamma_1)V_0$, we have

$$\begin{aligned}\frac{\mathrm{d}U_1(t)}{\mathrm{d}t} = (d+\alpha)S_0 - \beta S(t)\int_0^\infty p(a)i(t,a)\mathrm{d}a - (d+\alpha)S \\ + \alpha S(t) - \beta_1 V(t)\int_0^\infty p(a)i(t,a)\mathrm{d}a - (d+\gamma_1)V(t) \\ - \frac{(d+\alpha)S_0^2}{S(t)} + \beta S_0\int_0^\infty p(a)i(t,a)\mathrm{d}a + (d+\alpha)S_0 \\ - \frac{\alpha S(t)V_0}{V} + \beta_1 V_0\int_0^\infty p(a)i(t,a)\mathrm{d}a + (d+\gamma_1)V_0 \\ - (\beta S_0 + \beta_1 V_0)\int_0^\infty \alpha(a)\left[\frac{\partial i(t,a)}{\partial a} + \delta(a)i(t,a)\right]\mathrm{d}a.\end{aligned} \tag{7.6}$$

Collecting terms in (7.6), yields

$$\begin{aligned}\frac{\mathrm{d}U_1(t)}{\mathrm{d}t} = dS_0\left[2 - \frac{S(t)}{S_0} - \frac{S_0}{S(t)}\right] + \alpha S_0\left[3 - \frac{V(t)}{V_0} - \frac{S_0}{S(t)} - \frac{S(t)V_0}{S_0V(t)}\right] \\ - (\beta S(t) + \beta_1 V(t))\int_0^\infty p(a)i(t,a)\mathrm{d}a + (\beta S_0 + \beta_1 V_0)\int_0^\infty p(a)i(t,a)\mathrm{d}a \\ - (\beta S_0 + \beta_1 V_0)\int_0^\infty \alpha(a)\left[\frac{\partial i(t,a)}{\partial a} + \delta(a)i(t,a)\right]\mathrm{d}a.\end{aligned} \tag{7.7}$$

Here using integration by parts, we have

$$\begin{aligned}\int_0^\infty \alpha(a)\frac{\partial i(t,a)}{\partial a}\mathrm{d}a = \int_0^\infty \alpha(a)di(t,a) \\ = \alpha(a)i(t,a)\mid_0^\infty - \int_0^\infty \alpha'(a)i(t,a)\mathrm{d}a \\ = \alpha(a)i(t,a)\mid_\infty - \alpha(0)(\beta S(t) + \beta_1 V(t))\int_0^\infty p(a)i(t,a)\mathrm{d}a \\ - \int_0^\infty (\alpha(a)\delta(a) - p(a))i(t,a)\mathrm{d}a.\end{aligned} \tag{7.8}$$

Combining (7.7) and (7.8), we obtain

$$\begin{aligned}\frac{\mathrm{d}U_1(t)}{\mathrm{d}t} = dS_0\left[2 - \frac{S(t)}{S_0} - \frac{S_0}{S(t)}\right] + \alpha S_0\left[3 - \frac{V(t)}{V_0} - \frac{S_0}{S(t)} - \frac{S(t)V_0}{S_0V(t)}\right] \\ - (\beta S_0 + \beta_1 V_0)\alpha(a)i(t,a)\mid_\infty \\ + (\Re_0 - 1)(\beta S(t) + \beta_1 V(t))\int_0^\infty p(a)i(t,a)\mathrm{d}a.\end{aligned} \tag{7.9}$$

Due to the fact that the arithmetic mean is greater than or equal to the geometric mean, (7.9) becomes

$$\begin{aligned}\frac{\mathrm{d}U_1(t)}{\mathrm{d}t} = dS_0\left[2 - \frac{S(t)}{S_0} - \frac{S_0}{S(t)}\right] + \alpha S_0\left[3 - \frac{V(t)}{V_0} - \frac{S_0}{S(t)} - \frac{S(t)V_0}{S_0V(t)}\right] \\ + (\Re_0 - 1)(\beta S(t) + \beta_1 V(t))\int_0^\infty p(a)i(t,a)\mathrm{d}a \\ \leqslant 0, \quad \text{if } \Re_0 \leqslant 1.\end{aligned} \tag{7.10}$$

The above equalities strictly hold only at $S(t)=S_0$ and $V(t)=V_0$. Therefore, $\Re_0 \leqslant 1$ ensures that $\frac{\mathrm{d}U_1(t)}{\mathrm{d}t} \leqslant 0$ holds. Every solution of system (2.3) tends to $\mathscr{M}_0$, where $\mathscr{M}_0$ is the largest invariant subset in $\{\frac{\mathrm{d}U_1(t)}{\mathrm{d}t}=0\}$. Note that when $\Re_0 \leqslant 1$, the equality holds only if $S(t)=S_0$ and $V(t)=V_0$. Hence we have that $\mathscr{M}_0=\{P_0\}$, and it follows that P^* is globally asymptotically stable. This completes the proof. □

The following lemma will be used in the proof of global attractivity of endemic equilibrium if the basic reproduction number is greater than one. Since it directly comes from boundary conditions (2.4) and equilibrium equations (3.9), we omit the proof of it.

LEMMA 7.1 Let $P^*=(S^*,V^*,i^*(a))$ be the endemic equilibrium of system (2.3), then the following equalities hold true:

(i)

$$\begin{aligned}&\beta S^*\int_0^\infty \frac{S(t)i(t,a)}{S^*i^*(a)}p(a)i^*(a)\mathrm{d}a+\beta_1V^*\int_0^\infty\frac{V(t)i(t,a)}{V^*i^*(a)}p(a)i^*(a)\mathrm{d}a\\&=(\beta S^*+\beta_1V^*)\int_0^\infty p(a)i^*(a)\frac{i(t,0)}{i^*(0)}\mathrm{d}a;\end{aligned}\tag{7.11}$$

(ii)

$$\begin{aligned}&\beta S^*\int_0^\infty\left(1-\frac{S(t)i(t,a)i^*(0)}{S^*i^*(a)i(t,0)}\right)p(a)i^*(a)\mathrm{d}a\\&=-\beta_1V^*\int_0^\infty\left(1-\frac{V(t)i(t,a)i^*(0)}{V^*i^*(a)i(t,0)}\right)p(a)i^*(a)\mathrm{d}a.\end{aligned}\tag{7.12}$$

Now we are in the position to state the main result of this section.

THEOREM 7.2 If $\Re_0>1$, then the endemic equilibrium P^* is globally asymptotically stable.

Proof. Define a Lyapunov function

$$U_2(t)=S^*h\left(\frac{S(t)}{S^*}\right)+V^*h\left(\frac{V(t)}{V^*}\right)+(\beta S^*+\beta_1V^*)\int_0^\infty\alpha^*(a)h\left(\frac{i(t,a)}{i^*(a)}\right)\mathrm{d}a,\tag{7.13}$$

where $\alpha^*(a)=\int_a^\infty p(\sigma)\mathrm{d}\sigma\in L_+^1$. Such type of Lyapunov function has also been used for infection-age epidemiological models Chen *et al.* (2014); Magal *et al.* (2010); McCluskey (2012).

From Lemma 9.4 of McCluskey (2012), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\beta S^*+\beta_1V^*)\int_0^\infty\alpha(a)h\left(\frac{i(t,a)}{i^*(a)}\right)\mathrm{d}a\\&=(\beta S^*+\beta_1V^*)\int_0^\infty\left[\frac{i(t,0)}{i^*(0)}-\frac{i(t,a)}{i^*(a)}+\ln\frac{i(t,a)}{i^*(a)}-\ln\frac{i(t,0)}{i^*(0)}\right]\mathrm{d}a.\end{aligned}$$

Combining the derivatives of $S(t)$, $V(t)$ and $i(t,\cdot)$ yields

$$\begin{aligned}\frac{\mathrm{d}U_2(t)}{\mathrm{d}t} &= (1-\frac{S^*}{S})\left[\Lambda-(d+\alpha)S(t)-\beta S(t)\int_0^\infty p(a)i(t,a)\mathrm{d}a\right]\\ &+(1-\frac{V^*}{V})\left[\alpha S(t)-\beta_1 V(t)\int_0^\infty p(a)i(t,a)\mathrm{d}a-(d+\gamma_1)V(t)\right]\\ &+(\beta S^*+\beta_1 V^*)\int_0^\infty\left[\frac{i(t,0)}{i^*(0)}-\frac{i(t,a)}{i^*(a)}+\ln\frac{i(t,a)}{i^*(a)}-\ln\frac{i(t,0)}{i^*(0)}\right]\mathrm{d}a. \qquad (7.14)\end{aligned}$$

By using $\Lambda=(d+\alpha)S^*+\beta S^*\int_0^\infty p(a)i^*(a)\mathrm{d}a$ and $\alpha S^*=\beta_1 V^*\int_0^\infty p(a)i^*(a)\mathrm{d}a+(d+\gamma_1)V^*$, (7.14) becomes

$$\begin{aligned}\frac{\mathrm{d}U_2(t)}{\mathrm{d}t} &= dS^*\left(2-\frac{S(t)}{S^*}-\frac{S^*}{S(t)}\right)+(d+\gamma_1)V^*\left(3-\frac{V(t)}{V^*}-\frac{S^*}{S(t)}-\frac{S(t)V^*}{S^*V(t)}\right)\\ &+\beta S^*\int_0^\infty\left(1-\frac{S(t)i(t,a)}{S^*i^*(a)}-\frac{S^*}{S(t)}+\frac{i(t,a)}{i^*(a)}\right)p(a)i^*(a)\mathrm{d}a\\ &+\beta_1 V^*\int_0^\infty\left(2-\frac{V(t)i(t,a)}{V^*i^*(a)}-\frac{S^*}{S(t)}+\frac{i(t,a)}{i^*(a)}-\frac{S(t)V^*}{S^*V(t)}\right)p(a)i^*(a)\mathrm{d}a\\ &+(\beta S^*+\beta_1 V^*)\int_0^\infty\left[\frac{i(t,0)}{i^*(0)}-\frac{i(t,a)}{i^*(a)}+\ln\frac{i(t,a)}{i^*(a)}-\ln\frac{i(t,0)}{i^*(0)}\right]\mathrm{d}a. \qquad (7.15)\end{aligned}$$

With the aid of Lemma 7.1, we can easily obtain

$$\begin{aligned}\frac{\mathrm{d}U_2(t)}{\mathrm{d}t} &= dS^*\left(2-\frac{S(t)}{S^*}-\frac{S^*}{S(t)}\right)+(d+\gamma_1)V^*\left(3-\frac{V(t)}{V^*}-\frac{S^*}{S(t)}-\frac{S(t)V^*}{S^*V(t)}\right)\\ &-\beta S^*\int_0^\infty h\left(\frac{S^*}{S(t)}\right)\mathrm{d}a-\beta V^*\int_0^\infty h\left(\frac{S^*}{S(t)}\right)\mathrm{d}a\\ &-\beta V^*\int_0^\infty h\left(\frac{S(t)V^*}{S^*V(t)}\right)\mathrm{d}a-\beta S^*\int_0^\infty h\left(\frac{S^*i(t,a)}{S(t)i^*(a)}\right)\mathrm{d}a\\ &-\beta V^*\int_0^\infty h\left(\frac{V^*i(t,a)}{V(t)i^*(a)}\right)\mathrm{d}a. \qquad (7.16)\end{aligned}$$

Let Γ be the largest invariant set of $\{(S(t),V(t),i(t,a)):\frac{\mathrm{d}U_2(t)}{\mathrm{d}t}=0\}$. We show that $\Gamma=\{P^*\}$. Obviously, $\{P^*\}\subseteq\Gamma$. Now, $\frac{\mathrm{d}U_2(t)}{\mathrm{d}t}=0$ if and only if

$$S(t)=S^*,\ \ V(t)=V^*, \text{and}\ \ i(t,a)=i^*(a),\ a\in(0,+\infty).$$

This proves that $\{P^*\}\supseteq\Gamma$. Therefore, $\Gamma=\{P^*\}$ and it follows that P^* is globally asymptotically stable. This completes the proof. □

REMARK 7.1 Recently, this kind of Lyapunov function are extensively used for the analysis of differential equations with discrete or distributed delay (see, e.g., McCluskey (2009, 2010a,b,c), Korobeinikov (2004), Wang *et al.* (2011)), virus infection models (see, e.g., Liu & Wang (2010), Li & Shu (2010) and Huang *et al.* (2010)) and human infection models (see, e.g. Huang *et al.* (2010), Magal *et al.* (2010) and McCluskey (2012)).

8. Discussion and conclusion

In this paper, we have investigated the global behavior of the SVIR epidemiological model (2.1) with infection age structure. By constructing suitable Lyapunov functions U_1 and U_2, we have succeeded in showing that the basic reproduction number $\Re_0$ is the complete threshold value for the global asymptotic stability of each equilibrium: if $\Re_0 \leqslant 1$, then the disease-free equilibrium is globally asymptotically stable, while if $\Re_0 > 1$, then an endemic equilibrium uniquely exists and it is globally asymptotically stable.

If we assume that $p(a) = p$ and $\delta(a) = \delta$ and integrate the third equation of model (2.3) with respect to age a, then it can be rewritten as the following system of ordinary differential equations:

$$\begin{cases} \frac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \beta p S(t) I(t) - (d+\alpha) S(t), \\ \frac{\mathrm{d}V(t)}{\mathrm{d}t} = \alpha S(t) - \beta_1 p V(t) I(t) - (d+\gamma_1) V(t), \\ \frac{\mathrm{d}I(t)}{\mathrm{d}t} = (\beta p S(t) + \beta_1 p V(t)) I(t) - \delta I(t), \end{cases} \tag{8.1}$$

where $I(t) = \int_0^\infty i(t,a)\mathrm{d}a$. In Liu *et al.* (2008), it was shown that the global behavior of this system is completely determined by $\Re_0|_{(8.1)} = (\beta S_0 + \beta_1 V_0)p/\delta$. From this perspective, we can conclude that our result is an extension of the previous result to more general age-structured model (2.3). We can also reduce the model (2.3) to a time-delay model, for which we refer the reader to McCluskey (2012).

Functional analysis approach used in this paper to prove the asymptotic smoothness and persistence comes from the book Smith & Thieme (2011); Webb (1985) and recent work McCluskey (2012). It should be pointed that the same dynamics scenario can also been obtained from reformulating (2.3) as a non-densely defined Cauchy problem. For this approach, we refer the reader to the recent works: Chen *et al.* (2014); Duan *et al.* (2014a,b); Magal *et al.* (2010); Magal & McCluskey (2013).

Acknowledgments

The authors would like to thank the anonymous referees and the editor for their helpful suggestions and comments which led to the improvement of our original manuscript. J. Wang was supported by National Natural Science Foundation of China (nos. 11401182, 11471089), Science and Technology Innovation Team in Higher Education Institutions of Heilongjiang Province (No. 2014TD005), Project funded by China Postdoctoral Science Foundation (No. 2014M552295), and Project funded by Chongqing Postdoctoral Foundation (No. Xm2014024). T. Kuniya was supported by Grant-in-Aid for Young Scientists (B) of Japan Society for the Promotion of Science (No. 15K17585) and the program of the Japan Initiative for Global Research Network on Infectious Diseases (J-GRID); from Japan Agency for Medical Research and Development, AMED.

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