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A MULTI-GROUP SIR EPIDEMIC MODEL WITH AGE STRUCTURE

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Abstract. This paper provides the first detailed analysis of a multi-group SIR epidemic model with age structure, which is given by a nonlinear system of $3n$ partial differential equations. The basic reproduction number $\mathcal{R}_0$ is obtained as the spectral radius of the next generation operator, and it is shown that if $\mathcal{R}_0 < 1$, then the disease-free equilibrium is globally asymptotically stable, while if $\mathcal{R}_0 > 1$, then an endemic equilibrium exists. The global asymptotic stability of the endemic equilibrium is also shown under additional assumptions such that the transmission coefficient is independent from the age of infective individuals and the mortality and removal rates are constant. To our knowledge, this is the first paper which applies the method of Lyapunov functional and graph theory to a multi-dimensional PDE system.

1. **Introduction.** It is well-known that the spread of infectious diseases can be modeled by differential equations. The SIR model is one of the most popular epidemic models, in which total host population is divided into three classes called *susceptible* (S), *infective* (I) and *removed* (R). For general information of mathematical epidemiology, the reader may refer to Diekmann *et al.* [6].

The age-structured SIR models have been studied by many authors (see [1, 4, 7, 10, 11, 15, 16, 18, 28, 31, 35, 40, 41, 42, 46]). A typical age-structured SIR model

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is given by the following nonlinear system of partial differential equations:

$$\begin{cases} \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right) S(t,a) = -\lambda(t,a) S(t,a) - \mu(a) S(t,a), \\ \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right) I(t,a) = \lambda(t,a) S(t,a) - \{\mu(a)+\gamma(a)\} I(t,a), \\ \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right) R(t,a) = \gamma(a) I(t,a) - \mu(a) R(t,a), \quad t \geq 0, \quad a \geq 0, \\ \qquad S(t,0) = \displaystyle\int_0^{+\infty} f(\sigma) P(t,\sigma)\, \mathrm{d}\sigma, \quad I(t,0) = R(t,0) = 0, \\ \qquad S(0,a) = S_0(a), \quad I(0,a) = I_0(a), \quad R(0,a) = R_0(a), \end{cases} \tag{1.1}$$

where $S(t,a)$, $I(t,a)$ and $R(t,a)$ denote the densities of susceptible, infective and removed individuals of age a at time t, respectively. $\mu(a)$ denotes the age-specific per capita mortality rate, $\gamma(a)$ denotes the age-specific per capita removal rate and $f(a)$ denotes the age-specific per capita fertility rate. The density of total host population of age a at time t is given by $P(t,a) := S(t,a) + I(t,a) + R(t,a)$. The force of infection to a susceptible individual of age a at time t is given by

$$\lambda(t,a) = \int_0^{+\infty} \beta(a,\sigma) I(t,\sigma)\, \mathrm{d}\sigma,$$

where $\beta(a,\sigma)$ denotes the disease transmission coefficient between a susceptible individual of age a and an infective individual of age σ. For model (1.1), the basic reproduction number $\mathcal{R}_0$ is obtained as the spectral radius of a linear operator called *the next generation operator* (see Diekmann *et al.* [6]). The global asymptotic stability of the disease-free equilibrium of (1.1) was proved for $\mathcal{R}_0 < 1$ in Inaba [18]. However, the global asymptotic stability of an endemic equilibrium of (1.1) for $\mathcal{R}_0 > 1$ has been an open problem for a long time and even the possibility of unstable endemic equilibria has been shown for $\mathcal{R}_0 > 1$ (see Thieme [40], Andreasen [1], Cha *et al.* [4], Franceschetti *et al.* [9]). On the other hand, in Melnik and Korobeinikov [35] the global asymptotic stability of the endemic equilibrium for systems in which only S has the age structure was proved by constructing a Lyapunov functional.

Under the assumption that host population is not homogeneous but heterogeneous, we can divide each of the classes (S, I and R) into several homogeneous groups according to the heterogeneity (e.g., sex, position) of each individual. Such heterogeneity in the host population can result from different contact modes such as those among children and adults for childhood diseases (e.g., measles and mumps), or different behaviors such as the numbers of sexual partners for some sexually transmitted infections (e.g., Herpes, Condyloma acuminatum). Such models are typically called *multi-group epidemic models* (see van den Driessche and Watmough [43]). The multi-group SIR epidemic models as ODEs have been studied by many authors (see [2, 12, 14, 45, 25, 28, 39]). For a multi-group SIR epidemic model, Guo *et al.* [12] proved the complete global asymptotic stability of each equilibrium by using a graph-theoretic approach. By using the results or ideas of Guo *et al.* [12], the papers [39, 45, 28] investigated uniqueness and global stability of the endemic equilibrium for several classes of multigroup models, and some open problems were resolved.

In this paper, we are concerned with an SIR epidemic model in quite a general form, containing both of the age structure and the multi-group structure. This model is suitable for a disease which exhibits the age dependency and the heterogeneous state dependency. An SIR epidemic model we shall consider in this paper is formulated by the following nonlinear system of first-order partial differential equations:

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) S_j(t,a) = -\lambda_j(t,a) S_j(t,a) - \mu_j(a) S_j(t,a), \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) I_j(t,a) = \lambda_j(t,a) S_j(t,a) - \{\mu_j(a) + \gamma_j(a)\} I_j(t,a), \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) R_j(t,a) = \gamma_j(a) I_j(t,a) - \mu_j(a) R_j(t,a), \quad t \geq 0, \ a \geq 0, \\ S_j(t,0) = \sum_{k=1}^{n} \int_0^{+\infty} f_{jk}(\sigma) P_k(t,\sigma) \mathrm{d}\sigma, \ I_j(t,0) = R_j(t,0) = 0, \\ S_j(0,a) = S_{j,0}(a), \ I_j(0,a) = I_{j,0}(a), \ R_j(0,a) = R_{j,0}(a), \quad j = 1, 2, \ldots, n, \end{cases} \tag{1.2}$$

where $S_j(t,a)$, $I_j(t,a)$ and $R_j(t,a)$ denote the densities of susceptible, infective and removed individuals of age a at time t in group j, respectively. $\mu_j(a)$ and $\gamma_j(a)$ denote the age-specific per capita mortality and removal rates for individuals in group j, respectively. $f_{jk}(a)$ denotes the age-specific per capita fertility rate at which an individual of age a in group k gives birth to an offspring in group j. $P_j(t,a) := S_j(t,a) + I_j(t,a) + R_j(t,a)$ denotes the density of total host population of age a at time t in group j, and

$$\lambda_j(t,a) = \sum_{k=1}^{n} \int_0^{+\infty} \beta_{jk}(a,\sigma) I_k(t,\sigma) \mathrm{d}\sigma, \quad j = 1, 2, \ldots, n$$

denotes the force of infection to a susceptible individual of age a at time t in group j, where $\beta_{jk}(a,\sigma)$ denotes the disease transmission coefficient between a susceptible individual of age a in group j and an infective individual of age σ in group k.

To our knowledge, there are very few studies on epidemic models with both the age-structure and multi-group structure. In Feng *et al.* [8], the global behavior of an SIS epidemic model with both of these structures was studied. In Li *et al.* [30], an age-structured epidemic with two-group structure including super-infection was studied. An SIR epidemic model similar to (1.2) was studied in Kuniya [28], where the main focus was put on a corresponding ODE model derived by using a discretization approach as in Tudor [41] and Hethcote [15]. Therefore, a detailed mathematical analysis for the PDE model (1.2) has been an open problem. By virtue of both of the age-structure and multi-group structure, model (1.2) is thought to be realistic for sexually transmitted diseases, in which the age of sexual activity and the heterogeneity as sex play important roles. In this viewpoint, the detailed mathematical analysis for model (1.2) is an important task not only for mathematical but also biological purposes. In this paper, we shall investigate the existence, uniqueness and asymptotic stability of each equilibrium of (1.2) by using a functional analytic approach as in Webb [46] and Inaba [18] and a Lyapunov functional approach as in Magal *et al.* [31], McCluskey [33] and Melnik and Korobeinikov [35].

The organization of this paper is as follows. In Section 2, we prepare some preliminaries. In Section 3, using an abstract formulation, we show the mathematical well-posedness of the problem. In Section 4, we derive the basic reproduction number $\mathcal{R}_0$ for model (1.2) as the spectral radius of the next generation operator. In Section 5, we investigate the existence of each equilibrium. It is shown that the basic reproduction number $\mathcal{R}_0$ plays the role of a threshold value for the existence of a nontrivial endemic equilibrium. In Section 6, we investigate the uniqueness of the endemic equilibrium for two cases of separable mixing and general mixing. In Section 7, we prove the global asymptotic stability of the disease-free equilibrium for $\mathcal{R}_0 < 1$. In Section 8, under the assumptions that the disease transmission coefficients $\beta_{jk}(a,\sigma)$ are independent of the age of infective individuals and the mortality and removed rates are constant, we prove the global asymptotic stability of an endemic equilibrium for $\mathcal{R}_0 > 1$. In Section 9, we perform numerical simulation to verify the validity of our main theorem.

2. **Preliminaries.** We make the following assumption on each coefficient of system (1.2):

Assumption 2.1. *(i) For each $j \in \{1,2,\dots,n\}$, $\mu_j, \gamma_j \in L^\infty_+(0,+\infty)$. Moreover, there exists a positive constant $\underline{\mu} > 0$ such that $\mu_j(a) > \underline{\mu}$ for all $a \geq 0$.*

(ii) For each $j,k \in \{1,2,\dots,n\}$, $\beta_{jk} \in L^\infty_+\left((0,+\infty)^2\right)$ and $f_{jk} \in L^1_+(0,+\infty) \cap L^\infty_+(0,+\infty)$. Moreover, there exists a positive constant $a_\dagger > 0$ such that $f_{jk}(a) = 0$ for all $a > a_\dagger$.

From (i) and (ii) of Assumption 2.1, we see that for each $j,k \in \{1,2,\dots,n\}$, there exist positive constants $\gamma_j^+ > 0$ and $\beta_{jk}^+ > 0$ such that

$$\operatorname*{ess.sup}_{a\in[0,+\infty)} \gamma_j(a) \leq \gamma_j^+ < +\infty, \qquad \operatorname*{ess.sup}_{(a,\sigma)\in([0,+\infty))^2} \beta_{jk}(a,\sigma) \leq \beta_{jk}^+ < +\infty. \tag{2.1}$$

By adding the first three equations of (1.2), we obtain the following Lotka-McKendrick-von Foerster system:

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) P_j(t,a) = -\mu_j(a) P_j(t,a), \\ P_j(t,0) = \displaystyle\sum_{k=1}^n \int_0^{a_\dagger} f_{jk}(\sigma) P_k(t,\sigma)\, \mathrm{d}\sigma, \\ P_j(0,a) = P_{j,0}(a), \quad j = 1,2,\dots,n, \quad t \geq 0, \quad a \geq 0. \end{cases} \tag{2.2}$$

Note that from (ii) of Assumption 2.1, the interval of integration in the second equation of (2.2) is bounded above by $a_\dagger$. Let

$$Q(a) := \operatorname{diag}(\mu_1(a), \dots, \mu_n(a)) \tag{2.3}$$

and $M(a) := (f_{jk}(a))_{1\leq j,k\leq n}$. We make the following assumption:

Assumption 2.2. *The net reproduction rate matrix*

$$\int_0^{a_\dagger} M(a)\, \mathrm{e}^{-\int_0^a Q(\sigma)\mathrm{d}\sigma} \mathrm{d}a$$

is indecomposable (irreducible). The Frobenius root of this matrix is 1.

Under Assumption 2.2, it follows from Inaba [17, Propositions 3.2 and 3.3] that the system (2.2) has a nonnegative demographic steady state $P_j^* \in L^1_+(0,+\infty)\backslash\{0\}$, $j = 1,2,\dots,n$, which is globally stable. In what follows, we assume that the demographic steady state has been already reached, that is, $P_j(t,\cdot) \equiv P_j^*(\cdot)$, $j = 1,2,\dots,n$. Thus, system (1.2) can be rewritten as follows:

$$\begin{cases} \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right) S_j(t,a) = -\lambda_j(t,a) S_j(t,a) - \mu_j(a) S_j(t,a), \\ \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right) I_j(t,a) = \lambda_j(t,a) S_j(t,a) - \{\mu_j(a)+\gamma_j(a)\} I_j(t,a), \\ \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right) R_j(t,a) = \gamma_j(a) I_j(t,a) - \mu_j(a) R_j(t,a), \quad t \geq 0,\ a \geq 0, \\ \qquad \lambda_j(t,a) = \displaystyle\sum_{k=1}^n \int_0^{+\infty} \beta_{jk}(a,\sigma) I_k(t,\sigma)\,\mathrm{d}\sigma, \\ S_j(t,0) = b_j,\ I_j(t,0) = R_j(t,0) = 0, \\ S_j(0,a) = S_{j,0}(a),\ I_j(0,a) = I_{j,0}(a),\ R_j(0,a) = R_{j,0}(a), \quad j = 1,2,\dots,n, \end{cases} \tag{2.4}$$

where

$$b_j := \sum_{k=1}^n \int_0^{+\infty} f_{jk}(\sigma) P_k^*(\sigma)\,\mathrm{d}\sigma, \quad j = 1,2,\dots,n \tag{2.5}$$

are positive constants.

3. Well-posedness of the problem.

To define the state space for (2.4), we prepare the function spaces

$$X := L^1(0,+\infty;\mathbb{R}^n), \quad Y := X \times X, \quad Z := X \times X \times X$$

with norm

$$\begin{aligned} &\|\varphi\|_X := \int_0^{+\infty} \|\varphi(a)\|\,\mathrm{d}a = \int_0^{+\infty} \sum_{j=1}^n |\varphi_j(a)|\,\mathrm{d}a, \quad \varphi = (\varphi_1,\dots,\varphi_n)^{\mathrm{T}} \in X, \\ &\|\psi\|_Y := \|\psi_1\|_X + \|\psi_2\|_X, \quad \|\psi\|_Z := \|\psi_1\|_X + \|\psi_2\|_X + \|\psi_3\|_X, \\ &\psi_i = (\psi_{i,1},\dots,\psi_{i,n})^{\mathrm{T}} \in X,\ i = 1,2,3, \end{aligned}$$

respectively, where T denotes the transpose of a vector. Let us denote their positive cones by X_+, Y_+ and Z_+, respectively. Under the assumption that the system has reached the demographic steady state, we define the following state space:

$$\Omega := \left\{ \begin{pmatrix} S \\ I \\ R \end{pmatrix} \in Z_+ :\ S + I + R = P^* \right\},$$

where $S := (S_1,\dots,S_n)^{\mathrm{T}} \in X$, $I := (I_1,\dots,I_n)^{\mathrm{T}} \in X$, $R := (R_1,\dots,R_n)^{\mathrm{T}} \in X$ and $P^* := (P_1^*,\dots,P_n^*)^{\mathrm{T}} \in X$. In what follows, we assume that the initial condition of (2.4) is taken from Ω.

To show the existence and uniqueness of a global classical solution of problem (2.4) in Ω, it is sufficient to consider the following reduced $2n$-dimensional system:

$$\begin{cases} \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right) I_j(t,a) = \lambda_j(t,a)\left\{P_j^*(a) - I_j(t,a) - R_j(t,a)\right\} \\ \qquad\qquad\qquad\qquad - \left\{\mu_j(a)+\gamma_j(a)\right\} I_j(t,a), \\ \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right) R_j(t,a) = \gamma_j(a) I_j(t,a) - \mu_j(a) R_j(t,a), \quad t \geq 0, \ a \geq 0, \\ \qquad \lambda_j(t,a) = \displaystyle\sum_{k=1}^{n} \int_0^{+\infty} \beta_{jk}(a,\sigma) I_k(t,\sigma)\,\mathrm{d}\sigma, \\ \qquad I_j(t,0) = R_j(t,0) = 0, \\ \qquad I_j(0,a) = I_{j,0}(a), \ R_j(0,a) = R_{j,0}(a), \quad j = 1,2,\ldots,n, \end{cases} \tag{3.1}$$

Note that S_j $(j=1,2,\ldots,n)$ is given by $S_j = P_j^* - I_j - R_j$ in Ω. Let us define the state space for (3.1) by

$$\tilde{\Omega} := \left\{ \begin{pmatrix} I \\ R \end{pmatrix} \in Y_+ : \ \mathbf{0} \leq I + R \leq P^* \right\}. \tag{3.2}$$

Let us define a linear operator $A : D(A) \subset Y \to Y$ by

$$\begin{cases} A\psi(a) := \begin{pmatrix} -\dfrac{\mathrm{d}}{\mathrm{d}a}\psi_1(a) - Q(a)\psi_1(a) \\ -\dfrac{\mathrm{d}}{\mathrm{d}a}\psi_2(a) - Q(a)\psi_2(a) \end{pmatrix}, \\ D(A) := \left\{ \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in Y : \ \psi_1 \text{ and } \psi_2 \text{ are absolutely continuous,} \right. \\ \qquad\qquad \left. \psi_1', \psi_2' \in L^1(0,+\infty;\mathbb{R}^n), \ \psi(0) = \mathbf{0} \right\}. \end{cases} \tag{3.3}$$

Furthermore, let us define a nonlinear operator $F : \tilde{\Omega} \subset Y \to Y$ by

$$F(\psi)(a) := \begin{pmatrix} (\Lambda\psi_1)(a)\left\{P^*(a) - \psi_1(a) - \psi_2(a)\right\} - \Gamma(a)\psi_1(a) \\ \Gamma(a)\psi_1(a) \end{pmatrix}, \quad \psi \in \tilde{\Omega}, \tag{3.4}$$

where Λ is a linear operator on X defined by

$$(\Lambda\varphi)(a) := \operatorname*{diag}_{1\leq j\leq n}\left(\sum_{k=1}^{n}\int_0^{+\infty} \beta_{jk}(a,\sigma)\varphi_k(\sigma)\,\mathrm{d}\sigma\right), \quad \varphi \in X \tag{3.5}$$

and $\Gamma(a)$ is a diagonal matrix defined by

$$\Gamma(a) := \mathrm{diag}\left(\gamma_1(a),\ldots,\gamma_n(a)\right). \tag{3.6}$$

Then, setting $u := (I,R)^{\mathrm{T}} \in Y$, we rewrite problem (3.1) as the following abstract Cauchy problem in $\tilde{\Omega}$:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t) + F(u(t)), \\ u(0) = u_0 \in \tilde{\Omega}, \end{cases} \tag{3.7}$$

where $u_0 = (I_{1,0}, \ldots, I_{n,0}, R_{1,0}, \ldots, R_{n,0})^{\mathrm{T}} \in Y$. We easily see that operator A is an infinitesimal generator of a C_0-semigroup $\left\{\mathrm{e}^{tA}\right\}_{t\geq 0} : Y \to Y$ defined by

$$\left(\mathrm{e}^{tA}\psi\right)(a) = \begin{cases} \mathbf{0}, & t-a>0, \\ \mathrm{e}^{-\int_{a-t}^{a} Q(\sigma)\mathrm{d}\sigma}\psi(a-t), & a-t>0. \end{cases} \tag{3.8}$$

From (3.8), the space $\tilde{\Omega}$ is positively invariant under the semiflow defined by e^{tA}, that is, $\mathrm{e}^{tA}\left(\tilde{\Omega}\right) \subset \tilde{\Omega}$ for all $t \geq 0$.

Lemma 3.1. *The operator $F : \tilde{\Omega} \to Y$ defined by (3.4) is Lipschitz continuous and there exists a positive constant $\alpha > 0$ such that*

$$(I + \alpha F)\left(\tilde{\Omega}\right) \subset \tilde{\Omega}, \tag{3.9}$$

where I denotes the identity operator.

Proof. Since the Lipschitz continuity of F is obvious, we proceed to prove (3.9).

From (2.1), we see that for $(I, R)^{\mathrm{T}} \in \tilde{\Omega}$, the following inequality holds:

$$\lambda_j(t,a) \leq \sum_{k=1}^{n} \beta_{jk}^{+} \int_0^{+\infty} P_k^*(\sigma)\,\mathrm{d}\sigma =: \lambda_j^{+} < +\infty, \quad j = 1, 2, \ldots, n. \tag{3.10}$$

For $\psi = (\psi_1, \psi_2)^{\mathrm{T}} \in \tilde{\Omega}$, let

$$\begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} := (I + \alpha F) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

Then we have

$$\begin{aligned} \tilde{\psi}_1 + \tilde{\psi}_2 &= \psi_1 + \psi_2 + \alpha\Lambda\psi_1 (P^* - \psi_1 - \psi_2) \\ &\leq \psi_1 + \psi_2 + \alpha\lambda^{+} (P^* - \psi_1 - \psi_2), \end{aligned}$$

where $\lambda^{+} := \max_{j\in\{1,2,\ldots,n\}} \lambda_j^{+}$. Hence, if we choose α such that $\alpha < 1/\lambda^{+}$, then $\tilde{\psi}_1 + \tilde{\psi}_2 \leq P^*$ follows. Furthermore,

$$\begin{aligned} \tilde{\psi}_1 &= \psi_1 + \alpha\left\{\Lambda\psi_1 (P^* - \psi_1 - \psi_2) - \Gamma(a)\psi_1\right\} \\ &= \left\{I - \alpha\Gamma(a)\right\}\psi_1 + \alpha\Lambda\psi_1 (P^* - \psi_1 - \psi_2) \\ &\geq \left(1 - \alpha\gamma^{+}\right)\psi_1 + \alpha\Lambda\psi_1 (P^* - \psi_1 - \psi_2), \end{aligned}$$

where $\gamma^{+} := \max_{j\in\{1,2,\ldots,n\}} \gamma_j^{+}$. Hence, if we choose α such that $\alpha < 1/\gamma^{+}$, then $\tilde{\psi}_1 \geq \mathbf{0}$ follows. Consequently, we can choose $\alpha < \min(1/\lambda^{+}, 1/\gamma^{+})$ such that $\left(\tilde{\psi}_1, \tilde{\psi}_2\right)^{\mathrm{T}} \in \tilde{\Omega}$ holds, which completes the proof. □

Let $\alpha > 0$ be a constant such that (3.9) holds. Using this α, as in Busenberg *et al.* [3], we rewrite the abstract Cauchy problem (3.7) as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}u(t) = \left(A - \dfrac{1}{\alpha}\right)u(t) + \dfrac{1}{\alpha}(I + \alpha F)u(t), \\ u(0) = u_0 \in \tilde{\Omega} \end{cases} \tag{3.11}$$

and investigate the existence of solution of problem (3.11). The mild solution of (3.11) is given by the solution of integral equation

$$u(t) = \mathrm{e}^{-\frac{1}{\alpha}t}\mathrm{e}^{tA}u_0 + \frac{1}{\alpha}\int_0^t \mathrm{e}^{-\frac{1}{\alpha}(t-s)}\mathrm{e}^{(t-s)A}(I + \alpha F)u(s)\,\mathrm{d}s.$$

As in Busenberg *et al.* [3], we construct an iterative sequence $\{u^n : n \in \mathbb{N} \cup \{0\}\}$ such that

$$\begin{aligned} u^0(t) &:= u_0, \\ u^{n+1}(t) &:= \mathrm{e}^{-\frac{1}{\alpha}t}\mathrm{e}^{tA}u_0 + \frac{1}{\alpha}\int_0^t \mathrm{e}^{-\frac{1}{\alpha}(t-s)}\mathrm{e}^{(t-s)A}(I+\alpha F)u^n(s)\,\mathrm{d}s, \quad n \in \mathbb{N}\cup\{0\}. \end{aligned}$$

From Lemma 3.1, and by using the facts that u^{n+1} is a linear convex combination of $\mathrm{e}^{tA}u^0 \in \tilde{\Omega}$ and $\mathrm{e}^{(t-s)A}(I+\alpha F)u^n \in \tilde{\Omega}$, we conclude that $u^{n+1} \in \tilde{\Omega}$ if $u^n \in \tilde{\Omega}$. Thus, from the Lipschitz continuity of the operator F, u^n converges to the mild solution $u \in \tilde{\Omega}$ of (3.11) uniformly as $n \to +\infty$.

It is easy to see that if the mild solution $u \in \tilde{\Omega}$ of (3.11) belongs to $D(A)$, then it is differentiable and hence it is a classical solution of problem (3.7). In conclusion, we have the following proposition:

Proposition 3.1. *Let $\tilde{\Omega}$ be defined by (3.2). The abstract Cauchy problem (3.11) has a unique mild solution $U(t)u_0$, where $\{U(t)\}_{t\geq 0}$ denotes a semiflow on Y. The set $\tilde{\Omega}$ is positively invariant with respect to the semiflow $U(t)$ and if $u_0 \in D(A)$, then $U(t)u_0$ gives a classical solution of (3.11).*

This proposition implies the mathematical well-posedness of the original problem (2.4) for any $u_0 \in \tilde{\Omega} \cap D(A)$.

4. **The basic reproduction number $\mathcal{R}_0$.** In this section, we define the *basic reproduction number* $\mathcal{R}_0$, which is epidemiologically the expected number of secondary cases produced by a typical infective individual during its entire period of infectiousness in a fully susceptible population. The basic reproduction number is the most important idea characterizing the renewal process in structured populations, so it has been developing as a central dogma in mathematical epidemiology since an epoch-making paper by Diekmann, Heesterbeek and Metz at 1990 ([5]). For recent developments of $\mathcal{R}_0$, the reader may refer to Diekmann *et al.* [6] and Inaba [22].

Note that $(P^*, \mathbf{0}, \mathbf{0})$ is the disease-free equilibrium of system (2.4). Linearizing system (2.4) around the disease-free equilibrium $(P^*, \mathbf{0}, \mathbf{0})$, we obtain the following linear system in the disease invasion phase:

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right)\tilde{I}_j(t,a) = \tilde{\lambda}_j(t,a)P_j^*(a) - \{\mu_j(a) + \gamma_j(a)\}\tilde{I}_j(t,a), \\ \qquad \tilde{\lambda}_j(t,a) = \displaystyle\sum_{k=1}^n \int_0^{+\infty} \beta_{jk}(a,\sigma)\tilde{I}_k(t,\sigma)\,\mathrm{d}\sigma, \quad \tilde{I}_j(t,0) = 0, \\ \qquad j = 1,2,\ldots,n, \quad t \geq 0, \quad a \geq 0, \end{cases} \tag{4.1}$$

where $\tilde{I}_j$ $(j = 1,2,\ldots,n)$ denotes the perturbation of the density of infective individuals from the disease-free steady state $I_j \equiv 0$ $(j = 1,2,\ldots,n)$.

Let $B : D(B) \subset X \to X$ be a linear operator defined by

$$\begin{cases} B\varphi(a) := -\dfrac{\mathrm{d}}{\mathrm{d}a}\varphi(a) - \{Q(a) + \Gamma(a)\}\varphi(a), \\ D(B) := \{\varphi \in X : \ \varphi \text{ is absolutely continuous,} \\ \qquad\qquad \varphi' \in L^1(0,+\infty;\mathbb{R}^n), \ \varphi(0) = \mathbf{0}\}, \end{cases} \tag{4.2}$$

where $Q(a)$ and $\Gamma(a)$ are diagonal matrices defined by (2.3) and (3.6), respectively. Then, the linear system (4.1) is rewritten as the following abstract Cauchy problem in X:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} I(t) = BI(t) + PI(t), \\ I(0) = I_0 \in X, \end{cases} \tag{4.3}$$

where $I = (I_1, \dots, I_n)^{\mathrm{T}} \in X$, $I_0 = (I_{1,0}, \dots, I_{n,0})^{\mathrm{T}} \in X$ and P is a bounded linear operator on X defined by

$$P\varphi(a) := \begin{pmatrix} P_1^*(a) \sum_{k=1}^n \int_0^{+\infty} \beta_{1k}(a,\sigma)\varphi_k(\sigma)\,\mathrm{d}\sigma \\ \vdots \\ P_n^*(a) \sum_{k=1}^n \int_0^{+\infty} \beta_{nk}(a,\sigma)\varphi_k(\sigma)\,\mathrm{d}\sigma \end{pmatrix}, \quad \varphi \in X. \tag{4.4}$$

Let $V(t) = \mathrm{e}^{tB}$ be the C_0-semigroup generated by the generator B. Using the variation of constants formula, we have

$$I(t) = V(t)I_0 + \int_0^t V(t-s)PI(s)\,\mathrm{d}s. \tag{4.5}$$

Applying P from the left hand side to (4.5), we obtain an abstract renewal equation:

$$v(t) = g(t) + \int_0^t \Psi(s)v(t-s)ds, \tag{4.6}$$

where $v(t) := PI(t)$ denotes the density of newly infecteds in the linear invasion phase, $g(t) := PV(t)I_0$ and $\Psi(s) := PV(s)$.

Then the *next generation operator* is defined by

$$\mathcal{K} = \int_0^\infty \Psi(s)ds = P(-B)^{-1}, \tag{4.7}$$

where we have used a relation that

$$(z-B)^{-1} = \int_0^\infty e^{-zs}V(s)ds,$$

for $z \in \rho(B)$ ($\rho(B)$ denotes the resolvent set of B) and $0 \in \rho(B)$.

Then the next generation operator is calculated as follows:

$$(\mathcal{K}\varphi)(a) := ((\mathcal{K}_1\varphi)(a), \dots, (\mathcal{K}_n\varphi)(a))^{\mathrm{T}}, \tag{4.8}$$

where

$$(\mathcal{K}_j\varphi)(a) := P_j^*(a) \sum_{k=1}^n \int_0^{+\infty} \int_\rho^{+\infty} \beta_{jk}(a,\sigma)\,\mathrm{e}^{-\int_\rho^\sigma \{\mu_k(\eta)+\gamma_k(\eta)\}\mathrm{d}\eta}\mathrm{d}\sigma\ \varphi_k(\rho)\,\mathrm{d}\rho. \tag{4.9}$$

According to Diekmann, Heesterbeek and Metz [5], we define the basic reproduction number $\mathcal{R}_0$ by the spectral radius of the next generation operator $\mathcal{K}$:

$$\mathcal{R}_0 := r(\mathcal{K}), \tag{4.10}$$

where $r(A)$ denotes the spectral radius of a bounded linear operator A.

In fact, we can show that $\mathcal{R}_0$ gives the asymptotic per generation growth factor of infected population in the linear invasion phase, the Malthusian parameter (asymptotic growth rate of infecteds) is positive $\mathcal{R}_0 > 1$, while it is negative if $\mathcal{R}_0 < 1$ (see Inaba [22]).

5. **Existence of equilibria.** Instead of system (2.4), we next focus on the following reduced $2n$-dimensional system:

$$\begin{cases} \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right)S_j\left(t,a\right)=-\lambda_j\left(t,a\right)S_j\left(t,a\right)-\mu_j\left(a\right)S_j\left(t,a\right), \\ \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right)I_j\left(t,a\right)=\lambda_j\left(t,a\right)S_j\left(t,a\right)-\left\{\mu_j\left(a\right)+\gamma_j\left(a\right)\right\}I_j\left(t,a\right), \\ \qquad \lambda_j\left(t,a\right)=\displaystyle\sum_{k=1}^{n}\int_0^{+\infty}\beta_{jk}\left(a,\sigma\right)I_k\left(t,\sigma\right)\mathrm{d}\sigma, \quad t\geq 0,\ a\geq 0, \\ \qquad S_j\left(t,0\right)=b_j,\ I_j\left(t,0\right)=0, \\ \qquad S_j\left(0,a\right)=S_{j,0}\left(a\right),\ I_j\left(0,a\right)=I_{j,0}\left(a\right),\quad j=1,2,\ldots,n. \end{cases} \tag{5.1}$$

Note that R_j $(j=1,2,\ldots,n)$ is obtained as $R_j=P_j^*-S_j-I_j$. In this section, we investigate the existence of equilibria of system (5.1).

Denote $S^*:=\left(S_1^*,\ldots,S_n^*\right)^{\mathrm{T}}\in X$ and $I^*:=\left(I_1^*,\ldots,I_n^*\right)^{\mathrm{T}}\in X$ the equilibria of system (5.1). They must satisfy the following equations:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}a}S_j^*\left(a\right)=-\lambda_j^*\left(a\right)S_j^*\left(a\right)-\mu_j\left(a\right)S_j^*\left(a\right), \\ \dfrac{\mathrm{d}}{\mathrm{d}a}I_j^*\left(a\right)=\lambda_j^*\left(a\right)S_j^*\left(a\right)-\left\{\mu_j\left(a\right)+\gamma_j\left(a\right)\right\}I_j^*\left(a\right), \quad a\geq 0, \\ \lambda_j^*\left(a\right)=\displaystyle\sum_{k=1}^{n}\int_0^{+\infty}\beta_{jk}\left(a,\sigma\right)I_k^*\left(\sigma\right)\mathrm{d}\sigma, \\ S_j^*\left(0\right)=b_j,\quad I_j^*\left(0\right)=0,\quad j=1,2,\ldots,n. \end{cases} \tag{5.2}$$

It is obvious that $S_j^*\left(a\right)\equiv P_j^*\left(a\right),\ I_j^*\left(a\right)\equiv 0,\ j=1,2,\ldots,n$ is a solution set of (5.2) and hence, it is the disease-free equilibrium of system (5.1). From the variation of constants formula, we have

$$\begin{aligned} S_j^*\left(a\right)= &\ b_j\mathrm{e}^{-\int_0^a\left\{\lambda_j^*(\sigma)+\mu_j(\sigma)\right\}\mathrm{d}\sigma}, \\ = &\ P_j^*\left(a\right)\mathrm{e}^{-\int_0^a\lambda_j^*(\sigma)\mathrm{d}\sigma},\quad j=1,2,\ldots,n \end{aligned} \tag{5.3}$$

and

$$I_j^*\left(a\right)=\int_0^a\tilde{\lambda}_j^*\left(\sigma\right)\mathrm{e}^{-\int_\sigma^a\{\mu_j(\rho)+\gamma_j(\rho)\}\mathrm{d}\rho}\mathrm{d}\sigma,\quad j=1,2,\ldots,n, \tag{5.4}$$

where

$$\tilde{\lambda}_j^*(a):=\lambda_j^*\left(a\right)S_j^*\left(a\right),\quad j=1,2,\cdots,n. \tag{5.5}$$

Substituting (5.4) into the third equation of (5.2), we have

$$\lambda_j^*(a)=\sum_{k=1}^{n}\int_0^{+\infty}\beta_{jk}\left(a,\sigma\right)\int_0^{\sigma}\tilde{\lambda}_k^*\left(\rho\right)\mathrm{e}^{-\int_\rho^\sigma\{\mu_k(\eta)+\gamma_k(\eta)\}\mathrm{d}\eta}\mathrm{d}\rho\ \mathrm{d}\sigma,\quad j=1,2,\cdots,n. \tag{5.6}$$

Set the following linear operator on X corresponding to the right-hand side of this equation:

$$\mathcal{H}(\varphi)(a):=\left(\mathcal{H}_1(\varphi)(a),\mathcal{H}_2(\varphi)(a),\cdots,\mathcal{H}_n(\varphi)(a)\right)^{\mathrm{T}},\quad \varphi\in X, \tag{5.7}$$

where

$$\mathcal{H}_j(\varphi)(a) := \sum_{k=1}^{n} \int_0^{+\infty} \beta_{jk}(a,\sigma) \int_0^{\sigma} \varphi_k(\rho)\, \mathrm{e}^{-\int_\rho^\sigma \{\mu_k(\eta)+\gamma_k(\eta)\}\mathrm{d}\eta} \mathrm{d}\rho\, \mathrm{d}\sigma,\ j=1,2,\cdots,n.$$

Then, $\lambda_j^* = \mathcal{H}_j(\tilde{\lambda}^*)$, $j=1,2,\cdots,n$. From (5.3) and (5.5), we have

$$\tilde{\lambda}_j^*(a) = \lambda_j^*(a)\, P_j^*(a)\, \mathrm{e}^{-\int_0^a \lambda_j^*(\sigma)\mathrm{d}\sigma} = P_j^*(a)\, \mathcal{H}_j(\tilde{\lambda}^*)(a)\mathrm{e}^{-\int_0^a \mathcal{H}_j(\tilde{\lambda}^*)(\sigma)\mathrm{d}\sigma},\ j=1,2,\cdots,n.$$

Hence, defining a nonlinear operator

$$\Phi(\varphi)(a) := \left(\Phi_1(\varphi)(a), \Phi_2(\varphi)(a), \ldots, \Phi_n(\varphi)(a)\right)^{\mathrm{T}} \in X, \quad \varphi \in X, \tag{5.8}$$

where

$$\Phi_j(\varphi)(a) := P_j^*(a)\, \mathcal{H}_j(\varphi)(a)\mathrm{e}^{-\int_0^a \mathcal{H}_j(\varphi)(\sigma)\mathrm{d}\sigma}, \quad j=1,2,\ldots,n, \quad \varphi\in X,$$

we can show the existence of the endemic equilibrium by finding a positive nontrivial fixed-point of operator Φ since it implies the existence of positive $\tilde{\lambda}^*$. Noticing that $\mathcal{H}$ is linear and $\mathcal{H}(0)=0$, we see that the Fréchet derivative of Φ at $\varphi=0$ is

$$\Phi'[0](\varphi)(a) = \left(P_1^*(a)\mathcal{H}_1(\varphi)(a), P_2^*(a)\mathcal{H}_2(\varphi)(a), \cdots, P_n^*(a)\mathcal{H}_n(\varphi)(a)\right)^{\mathrm{T}}, \quad \varphi \in X.$$

Noticing that

$$\begin{aligned}
&P_j^*(a)\mathcal{H}_j(\varphi)(a)\\
=&P_j^*(a)\sum_{k=1}^{n}\int_0^{+\infty}\beta_{jk}(a,\sigma)\int_0^{\sigma}\varphi_k(\rho)\,\mathrm{e}^{-\int_\rho^\sigma\{\mu_k(\eta)+\gamma_k(\eta)\}\mathrm{d}\eta}\mathrm{d}\rho\ \mathrm{d}\sigma\\
=&P_j^*(a)\sum_{k=1}^{n}\int_0^{+\infty}\int_\rho^{+\infty}\beta_{jk}(a,\sigma)\,\mathrm{e}^{-\int_\rho^\sigma\{\mu_k(\eta)+\gamma_k(\eta)\}\mathrm{d}\eta}\mathrm{d}\sigma\ \varphi_k(\rho)\,\mathrm{d}\rho,\ j=1,2,\cdots,n,
\end{aligned}$$

we see that $\Phi'[0]$ is equal to the next generation operator $\mathcal{K}$ defined by (4.7). In what follows, we shall show that the basic reproduction number $\mathcal{R}_0 = r(\mathcal{K})$ determines the existence of the positive fixed point of operator Φ.

Before the proof of our main result in this section, we adopt the following assumption:

Assumption 5.1. *(i) For each $j,k\in\{1,2,\ldots,n\}$, the domain of β_{jk} is extended such that $\beta_{jk}(a,\sigma)=0$ for $a,\sigma\in(-\infty,0)$;*

(ii) For each $j,k\in\{1,2,\ldots,n\}$, the following holds uniformly for $\sigma\geq 0$:

$$\lim_{h\to 0}\int_0^{+\infty}\left|\beta_{jk}(a+h,\sigma)-\beta_{jk}(a,\sigma)\right|\mathrm{d}a = 0;$$

(iii) For each $j,k\in\{1,2,\cdots,n\}$, there exist $\epsilon_0>0$ and $\tilde{a}\in(0,+\infty)$ such that

$$\beta_{jk}(a,\sigma)\geq\epsilon_0 \quad \text{for almost all } (a,\sigma)\in[0,+\infty)\times[\tilde{a},+\infty).$$

Under Assumption 5.1, we first prove the following lemma:

Lemma 5.1. *Let $\mathcal{K}$ be defined by (4.7).*

(i) $\mathcal{K}$ is compact.

(ii) $\mathcal{K}$ is nonsupporting.

Proof. Let B_0 be an arbitrary bounded subset of X. Note that there exists a positive constant $C_0 > 0$ such that $\|\varphi\|_X \le C_0$ for all $\varphi \in B_0$.

First we show that $\lim_{h\to 0} \|(\mathcal{K}\varphi)(a+h) - (\mathcal{K}\varphi)(a)\|_X = 0$ uniformly for $\varphi \in B_0$. In fact, we have

$$
\begin{aligned}
\|(\mathcal{K}\varphi)(a+h) - (\mathcal{K}\varphi)(a)\|_X &= \sum_{j=1}^{n} \int_0^{+\infty} |(\mathcal{K}_j\varphi)(a+h) - (\mathcal{K}_j\varphi)(a)|\,\mathrm{d}a \\
\le \sum_{j,k=1}^{n} \int_0^{+\infty} \int_\rho^{+\infty} \int_0^{+\infty} & \left|P_j^*(a+h)\beta_{jk}(a+h,\sigma) - P_j^*(a)\beta_{jk}(a,\sigma)\right| \mathrm{d}a \\
& \times \mathrm{e}^{-\int_\rho^\sigma \{\mu_k(\eta)+\gamma_k(\eta)\}\mathrm{d}\eta} \mathrm{d}\sigma\, |\varphi_k(\rho)|\,\mathrm{d}\rho.
\end{aligned} \tag{5.9}
$$

Since $P_j^*(a) = b_j \exp\left(-\int_0^a \mu_j(\sigma)\mathrm{d}\sigma\right),\ j = 1, 2, \cdots, n$ is continuous, monotone non-increasing and $\lim_{a\to+\infty} P_j^*(a) = 0$, Assumption 5.1 implies that for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$
\int_0^{+\infty} \left|P_j^*(a+h)\beta_{jk}(a+h,\sigma) - P_j^*(a)\beta_{jk}(a,\sigma)\right| \mathrm{d}a \le \epsilon \tag{5.10}
$$

holds whenever $|h| < \delta$. Hence, from (5.10) and Assumption 2.1 (i), the inequality (5.9) can be further evaluated as

$$
\begin{aligned}
\|(\mathcal{K}\varphi)(a+h) - (\mathcal{K}\varphi)(a)\|_X &\le \epsilon \sum_{j,k=1}^{n} \int_0^{+\infty} \int_\rho^{+\infty} \mathrm{e}^{-\underline{\mu}(\sigma-\rho)}\mathrm{d}\sigma\, |\varphi_k(\rho)|\,\mathrm{d}\rho \\
&= \frac{\epsilon}{\underline{\mu}} \|\varphi\|_X = \frac{\epsilon}{\underline{\mu}} C_0,
\end{aligned}
$$

which implies the uniform convergence $\lim_{h\to 0} \|(\mathcal{K}\varphi)(a+h) - (\mathcal{K}\varphi)(a)\|_X = 0$ with respect to $\varphi \in B_0$.

Next we show that $\lim_{A\to+\infty} \sum_{j=1}^n \int_A^{+\infty} |(\mathcal{K}_j\varphi)(a)|\,\mathrm{d}a = 0$ uniformly for $\varphi \in B_0$. In fact, we have

$$
\begin{aligned}
& \int_A^{+\infty} |(\mathcal{K}_j\varphi)(a)|\,\mathrm{d}a \\
= & \int_A^{+\infty} \left| P_j^*(a) \sum_{k=1}^{n} \int_0^{+\infty} \int_\rho^{+\infty} \beta_{jk}(a,\sigma)\, \mathrm{e}^{-\int_\rho^\sigma \{\mu_k(\eta)+\gamma_k(\eta)\}\mathrm{d}\eta} \mathrm{d}\sigma\ \varphi_k(\rho)\,\mathrm{d}\rho \right| \mathrm{d}a \\
\le & \ b_j\beta_j^+ \int_A^{+\infty} \left| \mathrm{e}^{-\int_0^a \mu_j(\sigma)\mathrm{d}\sigma} \sum_{k=1}^{n} \int_0^{+\infty} \int_\rho^{+\infty} \mathrm{e}^{-\underline{\mu}(\sigma-\rho)}\mathrm{d}\sigma \varphi_k(\rho)\mathrm{d}\rho \right| \mathrm{d}a \\
\le & \ b_j\beta_j^+ \frac{1}{\underline{\mu}} \int_A^{+\infty} \mathrm{e}^{-\underline{\mu}a}\mathrm{d}a\, \|\varphi\|_X = \frac{b_j\beta_j^+}{\underline{\mu}^2} \mathrm{e}^{-\underline{\mu}A} C_0.
\end{aligned}
$$

Since the right-hand side of this inequality converses to zero as $A \to +\infty$ uniformly for $\varphi \in B_0$, the assertion is true. Consequently, from the well-known compactness criterion for L^1-space (see, e.g., Yosida [47, p.275]), we conclude that operator $\mathcal{K}$ is compact.

Next we prove (ii). Let X^* be the dual space of X, that is, the set of all linear functionals on X. Let $\langle f, \varphi\rangle$ denote the value of $f \in X^*$ at $\varphi \in X$. Let X_+^* be the dual cone of X, that is, the subset of X^* consisting of all positive linear functionals on X. We write $f \in X_+^*$ if $f \in X^*$ and $\langle f, \varphi\rangle \ge 0$ for all $\varphi \in X_+$.

For each $j, k \in \{1, 2, \cdots, n\}$, let

$$\tilde{\beta}_{jk}(a) = \begin{cases} \epsilon_0, & a \in [\tilde{a}, +\infty)\,; \\ 0, & \text{otherwise.} \end{cases}$$

Then, it follows from Assumption 5.1 (iii) that $\beta_{jk}(a, \sigma) \geq \tilde{\beta}_{jk}(\sigma)$. Let $\tilde{f} \in X_+^*$ be a positive linear functional defined by

$$\left\langle \tilde{f}, \varphi \right\rangle := \sum_{k=1}^{n} \int_0^{+\infty} \int_\rho^{+\infty} \tilde{\beta}_{jk}\left(\sigma\right) \mathrm{e}^{-\int_\rho^\sigma \gamma_k(\eta)\mathrm{d}\eta} \mathrm{d}\sigma \ \varphi_k\left(\rho\right) \mathrm{d}\rho.$$

Note that $\tilde{f}$ is strictly positive (i.e., $\left\langle \tilde{f}, \varphi \right\rangle > 0$ for all $\varphi \in X_+ \setminus \{\mathbf{0}\}$). From (4.7), we have

$$\mathcal{K}\varphi \geq \left\langle \tilde{f}, \varphi \right\rangle P^*.$$

Then, for any positive integer $n \in \mathbb{N}$, we have

$$\mathcal{K}^n\varphi \geq \left\langle \tilde{f}, \varphi \right\rangle \left\langle \tilde{f}, P^* \right\rangle^{n-1} P^*$$

and hence, for every pair $\varphi \in X_+ \setminus \{\mathbf{0}\}$ and $f \in X_+^* \setminus \{0\}$, it holds that $\langle f, \mathcal{K}^n\varphi \rangle > 0$. This implies that $\mathcal{K}$ is nonsupporting (see Marek [32] and Sawashima [37] for a precise definition of a nonsupporting operator). □

Next we prove the following lemma.

Lemma 5.2. *Let Φ be defined by (5.8).*

(i) Φ is compact.

(ii) $\Phi\left(X_+\right)$ is bounded. That is, there exists a positive constant $\tilde{M}_0 > 0$ such that for all $\varphi \in X_+$, $\left\|\Phi\left(\varphi\right)\right\|_X \leq \tilde{M}_0$ holds.

Proof. Let us define a nonlinear operator $\tilde{\Phi}$ corresponding to Φ by

$$\tilde{\Phi}(\varphi)(a) := \left(\tilde{\Phi}_1(\varphi)(a), \tilde{\Phi}_2(\varphi)(a), \cdots, \tilde{\Phi}_n(\varphi)(a)\right)^{\mathrm{T}} \in X, \quad \varphi \in X,$$

where

$$\tilde{\Phi}_j\left(\varphi\right)\left(a\right) := P_j^*\left(a\right) \varphi_j(a) \mathrm{e}^{-\int_0^a \varphi_j(\sigma)\mathrm{d}\sigma}, \quad j = 1, 2, \ldots, n, \quad \varphi \in X.$$

Then, Φ is given by the composition $\tilde{\Phi} \circ \mathcal{H}$. To show (ii), it suffices to show that $\tilde{\Phi}(X_+)$ is bounded since $\mathcal{H}(X_+) \subset X_+$. In fact,

$$\begin{aligned} \left\|\tilde{\Phi}(\varphi)\right\|_X &= \sum_{j=1}^{n} \int_0^{+\infty} P_j^*(a)\varphi_j(a)\mathrm{e}^{-\int_0^a \varphi_j(\sigma)\mathrm{d}\sigma}\mathrm{d}a \\ &\leq \sum_{j=1}^{n} b_j \int_0^{+\infty} \varphi_j(a)\mathrm{e}^{-\int_0^a \varphi_j(\sigma)\mathrm{d}\sigma}\mathrm{d}a \\ &= \sum_{j=1}^{n} b_j \left(1 - \mathrm{e}^{-\int_0^{+\infty} \varphi_j(\sigma)\mathrm{d}\sigma}\right) \leq \sum_{j=1}^{n} b_j, \end{aligned}$$

which implies that $\tilde{\Phi}(X_+)$ is bounded and hence, $\Phi(X_+)$ is bounded.

To show (i), it suffices to show that $\mathcal{H}$ is compact since the composition of a bounded operator and a compact operator is also compact. In fact, under Assumption 5.1 (ii), we can show the compactness of $\mathcal{H}$ in a similar way as in the proof of Lemma 5.1 (i). □

In order to show the existence of a nontrivial fixed-point of Φ, we shall apply the Krasnoselskii fixed-point theorem (see Krasnoselskii [26, Theorem 4.11]). Before the proof, we recall some basic facts on the theory of linear positive operators. Let E be a real or complex Banach space. A closed subset $C \subset E$ is called *cone* (or *positive cone*) if (i) $C + C \subset C$, (ii) $\lambda \geq 0 \Rightarrow \lambda C \subset C$, (iii) $C \cap (-C) = \{0\}$ and (iv) $C \neq \{0\}$. The cone C is called *total* if the set $\{\phi - \psi : \phi, \psi \in C\}$ is dense in E. The cone C is called *solid* if its interior C^0 is nonempty. Combining the Krein-Rutman theorem (see Krein and Rutman [27]) and Sawashima's theorem (see Sawashima [37]), we can obtain the following corollary (see Inaba [21, Corollary 7.6]):

Corollary 5.1. *Let C be a cone of a Banach space E and T be a bounded linear operator from E to itself. Suppose that C is total, the spectral radius $r(T) > 0$ of operator T is positive and T is compact and nonsupporting with respect to C. Then, the following holds:*

(i) $r(T) \in P_\sigma(T) \setminus \{0\}$, where $P_\sigma(\cdot)$ denotes the point spectrum of an operator. Moreover, $r(T)$ is a simple pole of the resolvent $R(\lambda, T) := (\lambda - T)^{-1}$.
(ii) The eigenspace corresponding to $r(T)$ is one-dimensional and its eigenvector $\psi \in C$ is a quasi-interior point, that is, $\langle f, \psi \rangle > 0$ for all $f \in C^* \setminus \{0\}$. Moreover, any eigenvector of T in C is proportional to ψ.
(iii) The adjoint eigenspace corresponding to $r(T)$ is one-dimensional and its eigenfunctional $f \in C^* \setminus \{0\}$ is strictly positive.
(iv) $r(T)$ is a dominant point of the spectrum $\sigma(T)$, that is, $|\mu| < r(T)$ holds for all $\mu \in \sigma(T) \setminus \{r(T)\}$.
(v) $B_1 := \lim_{n \to +\infty} r(T)^{-n} T^n$ converges in the topology of the operator norm and it is a strictly nonsupporting operator given by

$$B_1 = \frac{1}{2\pi i} \int_{\Gamma_0} R(\lambda, T) \, \mathrm{d}\lambda,$$

where Γ_0 denotes the positively oriented circle centered at $r(T)$ such that no points of the spectrum $\sigma(T)$ except $r(T)$ lie on and inside Γ_0.

As a consequence of the Krasnoselskii fixed-point theorem, we can obtain the following corollary (see also Inaba [18, Proposition 4.6] and Kawachi [24, Theorem 11]):

Corollary 5.2. *Let E be a real Banach space and E_+ be its positive cone. Let Ψ be a positive nonlinear operator on E. Suppose that*

(i) Ψ is compact, $\Psi(E_+)$ is bounded and $\Psi(0) = 0$, where $0 \in E_+$ is the origin of the Banach space E.
(ii) Ψ has the strong Fréchet derivative $T := \Psi'[0] : E \to E$ at the origin.
(iii) T has a positive eigenvector $v_0 \in E_+ \setminus \{0\}$ corresponding to an eigenvalue $\lambda_0 > 1$, and it has no eigenvector corresponding to eigenvalue 1.

Then, Ψ has at least one nontrivial fixed-point in $E_+ \setminus \{0\}$.

From Lemmas 5.1-5.2 and Corollaries 5.1-5.2, we prove the following main proposition of this section:

Proposition 5.1. *Let $\mathcal{R}_0$ and Φ be defined by (4.10) and (5.8), respectively. If $\mathcal{R}_0 > 1$, then Φ has at least one nontrivial fixed-point in $X_+ \setminus \{\mathbf{0}\}$.*

Proof. For nonlinear operator Φ and real Banach space X, Corollary 5.2 (i) follows from Lemma 5.2. Corollary 5.2 (ii) immediately follows with the strong Fréchet derivative $\mathcal{K} = \Phi'[\mathbf{0}]$. Since the compactness and nonsupporting property of $\mathcal{K}$ follows from Lemma 5.1, Corollary 5.1 can be applied to show that $r(\mathcal{K}) (= \mathcal{R}_0 > 1)$ is the only positive eigenvalue of $\mathcal{K}$ corresponding to a positive eigenvector and there is no eigenvector of $\mathcal{K}$ corresponding to eigenvalue 1. Therefore, Corollary 5.2 (iii) follows. Consequently, Φ has at least one nontrivial fixed-point in $X_+ \setminus \{\mathbf{0}\}$. $\square$

As mentioned above, the existence of a nontrivial fixed-point of operator Φ implies the existence of a nontrivial endemic equilibrium (S^*, I^*) of system (5.1). In conclusion, from Proposition 5.1, the following proposition is established:

Proposition 5.2. *Let $\mathcal{R}_0$ be defined by (4.10). If $\mathcal{R}_0 > 1$, then system (5.1) has at least one nontrivial endemic equilibrium (S^*, I^*), where $S^* = (S_1^*, \dots, S_n^*)^{\mathrm{T}} \in X_+ \setminus \{\mathbf{0}\}$ and $I^* = (I_1^*, \dots, I_n^*)^{\mathrm{T}} \in X_+ \setminus \{\mathbf{0}\}$.*

6. **Uniqueness of each equilibrium.** It is clear that the disease-free equilibrium $(P^*, \mathbf{0})$ of system (5.1) is unique. In this section, we investigate the uniqueness of endemic equilibrium (S^*, I^*) of system (5.1). Under the situation that such an endemic equilibrium exists, we see from (5.3), (5.5) and (5.6) that each component of $\lambda^* \in X_+ \setminus \{\mathbf{0}\}$ should satisfy the following integral equation.

$$\begin{aligned}
&\lambda_j^*(a)\\
=\;& \sum_{k=1}^{n} \int_0^{+\infty} \beta_{jk}(a,\sigma) \int_0^{\sigma} \lambda_k^*(\rho) P_k^*(\rho) \mathrm{e}^{-\int_0^{\rho} \lambda_k^*(\eta)\mathrm{d}\eta} \mathrm{e}^{-\int_{\rho}^{\sigma} \{\mu_k(\eta)+\gamma_k(\eta)\}\mathrm{d}\eta} \mathrm{d}\rho \, \mathrm{d}\sigma\\
=\;& \sum_{k=1}^{n} \int_0^{+\infty} \beta_{jk}(a,\sigma) P_k^*(\sigma) \int_0^{\sigma} \lambda_k^*(\rho) \mathrm{e}^{-\int_0^{\rho} \lambda_k^*(\eta)\mathrm{d}\eta} \mathrm{e}^{-\int_{\rho}^{\sigma} \gamma_k(\eta)\mathrm{d}\eta} \mathrm{d}\rho \, \mathrm{d}\sigma\\
=\;& \sum_{k=1}^{n} \int_0^{+\infty} \int_{\rho}^{+\infty} \beta_{jk}(a,\sigma) P_k^*(\sigma) \mathrm{e}^{-\int_{\rho}^{\sigma} \gamma_k(\eta)\mathrm{d}\eta} \mathrm{d}\sigma \lambda_k^*(\rho) \mathrm{e}^{-\int_0^{\rho} \lambda_k^*(\eta)\mathrm{d}\eta} \mathrm{d}\rho,\\
=\;& \sum_{k=1}^{n} \int_0^{+\infty} \phi_{jk}(a,\rho) \lambda_k^*(\rho) \mathrm{e}^{-\int_0^{\rho} \lambda_k^*(\eta)\mathrm{d}\eta} \mathrm{d}\rho, \quad j = 1, 2, \cdots, n, \qquad (6.1)
\end{aligned}$$

where

$$\phi_{jk}(a,\rho) := \int_{\rho}^{+\infty} \beta_{jk}(a,\sigma) P_k^*(\sigma) \mathrm{e}^{-\int_{\rho}^{\sigma} \gamma_k(\eta)\mathrm{d}\eta} \mathrm{d}\sigma, \quad j,k = 1, 2, \cdots, n. \qquad (6.2)$$

Hence, we define a nonlinear operator

$$\hat{\Phi}(\varphi)(a) := \left(\hat{\Phi}_1(\varphi)(a), \hat{\Phi}_2(\varphi)(a), \cdots, \hat{\Phi}_n(\varphi)(a)\right)^{\mathrm{T}} \in X, \quad \varphi \in X, \qquad (6.3)$$

where

$$\hat{\Phi}_j(\varphi)(a) := \sum_{k=1}^{n} \int_0^{+\infty} \phi_{jk}(a,\rho) \varphi_k(\rho) \mathrm{e}^{-\int_0^{\rho} \varphi_k(\eta)\mathrm{d}\eta} \mathrm{d}\rho, \quad j = 1, 2, \cdots, n,$$

and consider the situation where $\hat{\Phi}$ has a nontrivial fixed point $\lambda^* = \hat{\Phi}(\lambda^*)$. We shall investigate its uniqueness.

6.1. **Case of separable mixing.** First we consider the case of separable mixing (see Diekmann *et al.* [6]). That is, we assume the following:

Assumption 6.1. *For each $j, k \in \{1, 2, \ldots, n\}$, there exist positive bounded functions $\beta_j^1, \beta_k^2 \in L_+^\infty(0, +\infty)$ such that $\beta_{jk}(a, \sigma) = \beta_j^1(a)\,\beta_k^2(\sigma)$.*

Under assumption 6.1, each component of $\hat{\Phi}(\varphi) = \left(\hat{\Phi}_1(\varphi), \ldots, \hat{\Phi}_n(\varphi)\right)^{\mathrm{T}}$ is written as

$$\hat{\Phi}_j(\varphi)(a) = \beta_j^1(a) \sum_{k=1}^n \int_0^{+\infty} \int_\rho^{+\infty} \beta_k^2(\sigma)\, P_k^*(\sigma)\, \mathrm{e}^{-\int_\rho^\sigma \gamma_k(\eta)\mathrm{d}\eta} \mathrm{d}\sigma\; \varphi_k(\rho)\, \mathrm{e}^{-\int_0^\rho \varphi_k(\eta)\mathrm{d}\eta} \mathrm{d}\rho, \quad (6.4)$$

where $j = 1, 2, \ldots, n$. Thus, the solution of the fixed-point problem $\lambda^* = \hat{\Phi}(\lambda^*)$ is given by

$$\lambda^* = \left(c\beta_1^1, c\beta_2^1, \ldots, c\beta_n^1\right)^{\mathrm{T}}, \quad (6.5)$$

where $c > 0$ is a positive constant. Substituting (6.5) into (6.4) and rearranging the equation, we arrive at the following characteristic equation for unknown c:

$$1 = \sum_{k=1}^n \int_0^{+\infty} \int_\rho^{+\infty} \beta_k^2(\sigma)\, P_k^*(\sigma)\, \mathrm{e}^{-\int_\rho^\sigma \gamma_k(\eta)\mathrm{d}\eta} \mathrm{d}\sigma\; \beta_k^1(\rho)\, \mathrm{e}^{-c\int_0^\rho \beta_k^1(\eta)\mathrm{d}\eta} \mathrm{d}\rho.$$

Since the right-hand side of this equation is strictly monotone decreasing with respect to c, it has only one positive root $c > 0$. This implies the uniqueness of λ^* and therefore, we obtain the following proposition:

Proposition 6.1. *Let $\mathcal{R}_0$ be defined by (4.10). Under Assumption 6.1, if $\mathcal{R}_0 > 1$, then system (5.1) has a unique endemic equilibrium $(S^*, I^*) \in Y_+ \setminus \{\mathbf{0}\}$.*

6.2. **Case of general mixing.** Next we consider the case of general mixing. In this section, we make the following assumption:

Assumption 6.2. *For each $j, k \in \{1, 2, \ldots, n\}$,*

$$\beta_{jk}(a, \rho)\, P_k^*(\rho) - \gamma_k(\rho)\, \phi_{jk}(a, \rho) \geq 0 \quad \forall (a, \rho) \in [0, +\infty) \times [0, +\infty), \quad (6.6)$$

where ϕ_{jk} is defined by (6.2).

We have

$$\begin{aligned} &\beta_{jk}(a, \rho)\, P_k^*(\rho) - \gamma_k(\rho)\, \phi_{jk}(a, \rho) \\ &= \beta_{jk}(a, \rho)\, P_k^*(\rho) - \gamma_k(\rho) \int_\rho^{+\infty} \beta_{jk}(a, \sigma)\, P_k^*(\sigma)\, \mathrm{e}^{-\int_\rho^\sigma \gamma_k(\eta)\mathrm{d}\eta} \mathrm{d}\sigma \\ &= \int_\rho^{+\infty} \left\{\gamma_k(\sigma)\, \beta_{jk}(a, \rho)\, P_k^*(\rho) - \gamma_k(\rho)\, \beta_{jk}(a, \sigma)\, P_k^*(\sigma)\right\} \mathrm{e}^{-\int_\rho^\sigma \gamma_k(\eta)\mathrm{d}\eta} \mathrm{d}\sigma \\ &\quad + \beta_{jk}(a, \rho)\, P_k^*(\rho)\, \mathrm{e}^{-\int_\rho^{+\infty} \gamma_k(\eta)\mathrm{d}\eta}, \end{aligned}$$

and noting additionally, when $\gamma_j(a)$, $j = 1, 2, \ldots, n$ is monotone nondecreasing with respect to a and $\beta_{jk}(a, \sigma)\, P_k^*(\sigma)$, $j, k = 1, 2, \ldots, n$ is monotone nonincreasing with respect to σ, Assumption 6.2 is immediately satisfied.

Under Assumption 6.2, we first prove the following lemma:

Lemma 6.1. *Let $\hat{\Phi}$ be defined by (6.3). Under Assumption 6.2, $\hat{\Phi}$ is monotone nondecreasing on X.*

Proof. For $\varphi \in X$ and $j \in \{1, 2, \ldots, n\}$, we have

$$\begin{aligned} \hat{\Phi}_j(\varphi)(a) &= \sum_{k=1}^{n} \int_0^{+\infty} \phi_{jk}(a,\rho)\,\varphi_k(\rho)\,\mathrm{e}^{-\int_0^\rho \varphi_k(\eta)\mathrm{d}\eta}\mathrm{d}\rho \\ &= \sum_{k=1}^{n} \int_0^{+\infty} \phi_{jk}(a,\rho)\,\frac{\mathrm{d}}{\mathrm{d}\rho}\left(-\mathrm{e}^{-\int_0^\rho \varphi_k(\eta)\mathrm{d}\eta}\right)\mathrm{d}\rho \\ &= \sum_{k=1}^{n} \left[\phi_{jk}(a,\rho)\left(-\mathrm{e}^{-\int_0^\rho \varphi_k(\eta)\mathrm{d}\eta}\right)\right]_{\rho=0}^{\rho=+\infty} \\ &\quad + \sum_{k=1}^{n} \int_0^{+\infty} \frac{\mathrm{d}}{\mathrm{d}\rho}\phi_{jk}(a,\rho)\ \mathrm{e}^{-\int_0^\rho \varphi_k(\eta)\mathrm{d}\eta}\mathrm{d}\rho. \end{aligned} \tag{6.7}$$

Observe that

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\phi_{jk}(a,\rho) = -\beta_{jk}(a,\rho)\,P_k^*(\rho) + \gamma_k(\rho)\,\phi_{jk}(a,\rho)\,, \quad j,k = 1,2,\ldots,n.$$

Using this to simplify (6.7) gives

$$\hat{\Phi}_j(\varphi)(a) = \sum_{k=1}^{n} \phi_{jk}(a,0) - \sum_{k=1}^{n} \int_0^{+\infty} \left\{\beta_{jk}(a,\rho)\,P_k^*(\rho) - \gamma_k(\rho)\,\phi_{jk}(a,\rho)\right\}\mathrm{e}^{-\int_0^\rho \varphi_k(\eta)\mathrm{d}\eta}\mathrm{d}\rho.$$

Hence, from the inequality (6.6), it follows that $\hat{\Phi}_j(\varphi) \geq \hat{\Phi}_j(\tilde{\varphi})\,,\ j = 1,2,\ldots,n$ if $\varphi \geq \tilde{\varphi},\ \ \varphi, \tilde{\varphi} \in X$. This implies that $\hat{\Phi}$ is monotone nondecreasing on X. □

In what follows, we will prove the uniqueness of an endemic equilibrium (S^*, I^*) of system (5.1) by using Lemma 6.1. For each $j \in \{1, 2, \cdots, n\}$ and $\varphi \in X$, let

$$\hat{\Phi}_j^+(\varphi) := \sum_{k=1}^{n} \beta_{jk}^+ \int_0^{+\infty} \int_\rho^{+\infty} P_k^*(\sigma)\,\mathrm{e}^{-\int_\rho^\sigma \gamma_k(\eta)\mathrm{d}\eta}\mathrm{d}\sigma\ \varphi_k(\rho)\,\mathrm{e}^{-\int_0^\rho \varphi_k(\eta)\mathrm{d}\eta}\mathrm{d}\rho$$

and

$$\hat{\Phi}_j^-(\varphi) := \sum_{k=1}^{n} \int_0^{+\infty} \int_\rho^{+\infty} \tilde{\beta}_{jk}(\sigma) P_k^*(\sigma)\,\mathrm{e}^{-\int_\rho^\sigma \gamma_k(\eta)\mathrm{d}\eta}\mathrm{d}\sigma\ \varphi_k(\rho)\,\mathrm{e}^{-\int_0^\rho \varphi_k(\eta)\mathrm{d}\eta}\mathrm{d}\rho.$$

Note that $\mathbf{0} < \hat{\Phi}^-(\varphi)\,\mathbf{e} \leq \hat{\Phi}(\varphi) \leq \hat{\Phi}^+(\varphi)\,\mathbf{e}$ holds for all $\varphi \in X_+$, where $\mathbf{e} = (1, 1, \cdots 1)^{\mathrm{T}} \in \mathbb{R}^n$ and

$$\hat{\Phi}^+(\varphi) := \mathrm{diag}\ \left(\hat{\Phi}_1^+(\varphi), \cdots, \hat{\Phi}_n^+(\varphi)\right) \text{ and } \hat{\Phi}^-(\varphi) := \mathrm{diag}\ \left(\hat{\Phi}_1^-(\varphi), \cdots, \hat{\Phi}_n^-(\varphi)\right)^{\mathrm{T}}.$$

Under these settings, we prove the following proposition.

Proposition 6.2. *Let $\hat{\Phi}$ be defined by (6.3). Under Assumption 6.2, $\hat{\Phi}$ has at most one nontrivial fixed-point in $X_+ \setminus \{\mathbf{0}\}$.*

Proof. Let $\varphi^* := (\varphi_1^*, \ldots, \varphi_n^*)^{\mathrm{T}} \in X_+ \setminus \{\mathbf{0}\}$ and $\tilde{\varphi}^* := (\tilde{\varphi}_1^*, \ldots, \tilde{\varphi}_n^*)^{\mathrm{T}} \in X_+ \setminus \{\mathbf{0}\}$ be two fixed-points of $\hat{\Phi}$. We have

$$\begin{aligned} \varphi^* \ = \ \hat{\Phi}(\varphi^*) \ &\geq \ \hat{\Phi}^-(\varphi^*)\,\mathbf{e} \ = \ \hat{\Phi}^-(\varphi^*)\left\{\hat{\Phi}^+(\tilde{\varphi}^*)\right\}^{-1}\hat{\Phi}^+(\tilde{\varphi}^*)\,\mathbf{e} \\ &\geq \ \hat{\Phi}^-(\varphi^*)\left\{\hat{\Phi}^+(\tilde{\varphi}^*)\right\}^{-1}\hat{\Phi}(\tilde{\varphi}^*) \ = \ \hat{\Phi}^-(\varphi^*)\left\{\hat{\Phi}^+(\tilde{\varphi}^*)\right\}^{-1}\tilde{\varphi}^*. \end{aligned}$$

Since all of the diagonal entries of matrix $\hat{\Phi}^{-}(\varphi^*)\left\{\hat{\Phi}^{+}(\tilde{\varphi}^*)\right\}^{-1}$ are positive for $\tilde{\varphi}^* \in X_+\setminus\{0\}$, we see that there exists a positive constant $q := \sup\{r \geq 0 :\ \varphi^* \geq r\tilde{\varphi}^*\} > 0$.

Now we suppose that $0 < q < 1$. For $\varphi \in X_+$ and $j \in \{1, 2, \ldots, n\}$, we have

$$\begin{aligned}
&\hat{\Phi}_j(q\varphi)\\
= \ & q\hat{\Phi}_j(\varphi) \ + \ q\sum_{k=1}^{n}\int_0^{+\infty}\phi_{jk}(a,\rho)\,\varphi_k(\rho)\,\mathrm{e}^{-\int_0^\rho \varphi_k(\eta)\mathrm{d}\eta}\left(\mathrm{e}^{(1-q)\int_0^\rho \varphi_k(\eta)\mathrm{d}\eta}-1\right)\mathrm{d}\rho\\
\geq \ & q\hat{\Phi}_j(\varphi) \ + \ \eta_j(\varphi, q)
\end{aligned}\tag{6.8}$$

where

$$\begin{aligned}
\eta_j(\varphi,q) := q\ \sum_{k=1}^{n}\int_0^{+\infty}&\int_\rho^{+\infty}\tilde{\beta}_{jk}(\sigma)\,P_k^*(\sigma)\,\mathrm{e}^{-\int_\rho^\sigma \gamma_k(\eta)\mathrm{d}\eta}\mathrm{d}\sigma\\
&\times\varphi_k(\rho)\,\mathrm{e}^{-\int_0^\rho \varphi_k(\eta)\mathrm{d}\eta}\left(\mathrm{e}^{(1-q)\int_0^\rho \varphi_k(\eta)\mathrm{d}\eta}-1\right)\mathrm{d}\rho.
\end{aligned}$$

Note that $\eta_j(\varphi,q)\,,\ j = 1, 2, \cdots, n$ is positive for all $\varphi \in X_+\setminus\{\mathbf{0}\}$. From (6.8), we have

$$\hat{\Phi}(q\varphi) \geq q\hat{\Phi}(\varphi) + \eta(\varphi,q)\,\mathbf{e},\tag{6.9}$$

where $\eta = \operatorname{diag}(\eta_1, \eta_2, \cdots, \eta_n)$. From the monotonicity of $\hat{\Phi}$ (see Lemma 6.1) and the inequality (6.9), we have

$$\begin{aligned}
\varphi^* \ = \ \hat{\Phi}(\varphi^*) \ &\geq \ \hat{\Phi}(q\tilde{\varphi}^*)\\
&\geq \ q\hat{\Phi}(\tilde{\varphi}^*) + \eta(\tilde{\varphi}^*, q)\,\mathbf{e}\\
&= \ q\tilde{\varphi}^* + \eta(\tilde{\varphi}^*, q)\left\{\hat{\Phi}^{+}(\tilde{\varphi}^*)\right\}^{-1}\hat{\Phi}^{+}(\tilde{\varphi}^*)\,\mathbf{e}\\
&\geq \ q\tilde{\varphi}^* + \eta(\tilde{\varphi}^*, q)\left\{\hat{\Phi}^{+}(\tilde{\varphi}^*)\right\}^{-1}\hat{\Phi}(\tilde{\varphi}^*)\\
&= \ q\tilde{\varphi}^* + \eta(\tilde{\varphi}^*, q)\left\{\hat{\Phi}^{+}(\tilde{\varphi}^*)\right\}^{-1}\tilde{\varphi}^*.
\end{aligned}\tag{6.10}$$

Since all of the diagonal entries of matrix $\eta(\tilde{\varphi}^*, q)\left\{\hat{\Phi}^{+}(\tilde{\varphi}^*)\right\}^{-1}$ are positive, we see from (6.10) that there exists an $\eta > 0$ such that $\varphi^* \geq (q+\eta)\,\tilde{\varphi}^*$. This contradicts with the definition of q as the supremum of set $\{r \geq 0 :\ \varphi^* \geq r\tilde{\varphi}^*\}$. Therefore, we have that $q \geq 1$ and $\varphi^* \geq q\tilde{\varphi}^* \geq \tilde{\varphi}^*$. Exchanging the role of φ^* and $\tilde{\varphi}^*$, we can prove $\varphi^* \leq \tilde{\varphi}^*$ in a similar way and hence, $\varphi^* = \tilde{\varphi}^*$. □

Consequently, from Proposition 6.2, we obtain the following proposition:

Proposition 6.3. *Let $\mathcal{R}_0$ be defined by (4.10). Under Assumption 6.2, if $\mathcal{R}_0 > 1$, then system (5.1) has a unique nontrivial endemic equilibrium $(S^*, I^*) \in Y_+\setminus\{\mathbf{0}\}$.*

7. Global stability of the disease-free equilibrium.

In this section, we investigate the global asymptotic stability of the disease-free equilibrium $(P^*, \mathbf{0}) \in Y_+$ of system (5.1).

Using the generator B introduced in Section 4, the second equation of system (5.1) can be rewritten as the following abstract Cauchy problem in X:

$$\left\{\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}I(t) = BI(t) + (\Lambda I(t))\,S(t)\,,\\
&I(0) = I_0 \in X,
\end{aligned}\right.\tag{7.1}$$

where $S=(S_1,\ldots,S_n)^{\mathrm{T}}\in X$, $I=(I_1,\ldots,I_n)^{\mathrm{T}}\in X$, $I_0=(I_{1,0},\ldots,I_{n,0})^{\mathrm{T}}\in X$ and Λ is a linear operator on X defined by (3.5). If we regard $S(t)$ as a given function, we can regard (7.1) as a linear non-autonomous equation for $I(t)$. Since $S(t)\le P^*$, we have

$$\begin{aligned} I(t) &= \mathrm{e}^{tB}I_0+\int_0^t \mathrm{e}^{(t-s)B}\left(\Lambda I(s)\right)S(s)\,\mathrm{d}s \\ &\le \mathrm{e}^{tB}I_0+\int_0^t \mathrm{e}^{(t-s)B}PI(s)\,\mathrm{d}s. \end{aligned} \tag{7.2}$$

Thus, for the C_0-semigroup $\{W(t)\}_{t\ge0}$ generated by the perturbed operator $B+P$, we have from (7.2) that

$$\mathbf{0} \;\le\; I(t) \;\le\; W(t)\,I_0 \quad \forall t\ge0. \tag{7.3}$$

We define the growth bound of $W(t)$ by $\omega_0(B+P):=\lim_{t\to+\infty}t^{-1}\ln\|W(t)\|$ and the α-growth bound of $W(t)$ by $\omega_1(B+P):=\lim_{t\to+\infty}t^{-1}\ln\alpha[W(t)]$, where $\alpha[A]$ denotes the measure of noncompactness of a bounded linear operator A.

From Proposition 4.13 in Webb [46], we have

$$\omega_0(B+P)=\max\left\{\omega_1(B+P),\ \sup_{z\in\sigma(B+P)\setminus E_\sigma(B+P)}\Re z\right\}, \tag{7.4}$$

where $\sigma(A)$ and $E_\sigma(A)$ denote the spectrum and the essential spectrum of a closed linear operator A, respectively. To prove the global asymptotic stability of the disease-free equilibrium, it is sufficient to show that the growth bound $\omega_0(B+P)$ is negative. To this end, we first prove the following lemma:

Lemma 7.1. *Let B and P be defined by (4.2) and (4.4), respectively. The following holds:*

$$\omega_1(B+P)=\omega_1(B)\le-\underline{\mu}, \tag{7.5}$$

where $\underline{\mu}>0$ is a positive constant defined in (i) of Assumption 2.1.

Proof. Observe that

$$\left(V(t)\varphi\right)(a) \;=\; \begin{cases} 0, & t-a>0,\\ \mathrm{e}^{-\int_{a-t}^{a}\{Q(\sigma)+\Gamma(\sigma)\}\mathrm{d}\sigma}\varphi(a-t), & a-t>0. \end{cases}$$

As in the proof of Theorem 4.6 in Webb [46], we have

$$\alpha\left([V(t)]\right)\le\|V(t)\|\le\mathrm{e}^{-\underline{\mu}t}. \tag{7.6}$$

Moreover, we can prove the compactness of operator P based on the assumption 5.1 and the fact that $P_j^*\in L^1\cap L^\infty$, so it follows that $\omega_1(B+P)=\omega_1(B)$ (see Webb [46, Proposition 4.14]). Hence we obtain (7.5). □

For $z\in\mathbb{C}$ with $\Re z>-\underline{\mu}$, the following operator is well defined on X:

$$\left(U_z\varphi\right)(a):=\begin{pmatrix} \sum_{k=1}^n\int_0^{+\infty}\phi_{1k}(z,a,\rho)\,\varphi_k(\rho)\,\mathrm{d}\rho \\ \vdots \\ \sum_{k=1}^n\int_0^{+\infty}\phi_{nk}(z,a,\rho)\,\varphi_k(\rho)\,\mathrm{d}\rho \end{pmatrix}, \tag{7.7}$$

with

$$\phi_{jk}(z,a,\rho):=P_j^*(a)\int_\rho^{+\infty}\beta_{jk}(a,\sigma)\,\mathrm{e}^{-\int_\rho^\sigma\{z+\mu_k(\eta)+\gamma_k(\eta)\}\mathrm{d}\eta}\mathrm{d}\sigma,\quad j,k=1,2,\ldots,n.$$

Note that $U_0=\mathcal{K}$, where $\mathcal{K}$ is the next generation operator defined by (4.8).

Lemma 7.2. *Let B, P and U_z be defined by (4.2), (4.4) and (7.7), respectively. Let*

$$\Sigma := \left\{ z \in \mathbb{C} : 1 \in P_\sigma \left(U_z \right), \Re z > -\underline{\mu} \right\}, \tag{7.8}$$

where $P_\sigma(A)$ denotes the point spectrum of a closed linear operator A. Then it follows that

$$\Sigma = P_\sigma \left(B + P \right) \cap \left\{ z \in \mathbb{C} : \Re z > -\underline{\mu} \right\}.$$

Proof. First note that the compactness of operator U_z can be proved as in the proof of (i) of Lemma 5.1. Next let us prove that if $z \in \Sigma$, then $z \in P_\sigma \left(B + P \right) \cap \left\{ z \in \mathbb{C} : \Re z > -\underline{\mu} \right\}$. Suppose that there exists an eigenvector

$$\psi_z := \left(\psi_{z,1}, \ldots, \psi_{z,n} \right)^{\mathrm{T}} \in X$$

corresponding to the eigenvalue 1, that is, $\psi_z = U_z \psi_z$. Let us define

$$\tilde{\psi}_j \left(a \right) := \int_0^a \mathrm{e}^{-\int_\sigma^a \{ z + \mu_j(\rho) + \gamma_j(\rho) \} \mathrm{d}\rho} \psi_{z,j} \left(\sigma \right) \mathrm{d}\sigma, \quad j = 1, 2, \ldots, n,$$

with $\tilde{\psi} := \left(\tilde{\psi}_1, \ldots, \tilde{\psi}_n \right)^{\mathrm{T}} \in X$. Then, we have

$$\left(B\tilde{\psi} \right)_j \left(a \right) = -\psi_{z,j} \left(a \right) + z \tilde{\psi}_j \left(a \right), \quad j = 1, 2, \ldots, n$$

and

$$\begin{aligned} \left(P\tilde{\psi} \right)_j \left(a \right) &= P_j^* \left(a \right) \sum_{k=1}^n \int_0^{+\infty} \beta_{jk} \left(a, \sigma \right) \int_0^\sigma \mathrm{e}^{-\int_\rho^\sigma \{ z + \mu_k(\eta) + \gamma_k(\eta) \} \mathrm{d}\eta} \psi_{z,j} \left(\rho \right) \mathrm{d}\rho \mathrm{d}\sigma \\ &= \left(U_z \psi_z \right)_j \left(a \right) \\ &= \psi_{z,j} \left(a \right), \quad j = 1, 2, \ldots, n. \end{aligned}$$

Thus, $\left(B + P \right) \tilde{\psi} = z \tilde{\psi}$ and hence, $\tilde{\psi}$ is an eigenvector of operator $B + P$ corresponding to the eigenvalue $z \in \mathbb{C}$ with $\Re z > -\underline{\mu}$. Finally suppose that $z \in P_\sigma \left(B + P \right)$ and $\Re z > -\underline{\mu}$. Then there exists $\tilde{\varphi} \in D\left(B \right) \subset X$ such that

$$\left(z - \left(B + P \right) \right) \tilde{\varphi} = 0.$$

Then, we have

$$\frac{\mathrm{d}}{\mathrm{d}a} \tilde{\varphi} \left(a \right) = -z \tilde{\varphi} \left(a \right) - \left\{ Q \left(a \right) + \Gamma \left(a \right) \right\} \tilde{\varphi} \left(a \right) + P \tilde{\varphi} \left(a \right),$$

and hence, integration yields

$$\tilde{\varphi}_j \left(a \right) = \int_0^a \mathrm{e}^{-\int_\sigma^a \{ z + \mu_j(\rho) + \gamma_j(\rho) \} \mathrm{d}\rho} \left(P \tilde{\varphi} \right)_j \left(\sigma \right) \mathrm{d}\sigma, \quad j = 1, 2, \ldots, n. \tag{7.9}$$

From (7.9),

$$\begin{aligned} P\tilde{\varphi} \left(a \right) &= \begin{pmatrix} P_1^* \left(a \right) \sum_{k=1}^n \int_0^{+\infty} \beta_{1k} \left(a, \sigma \right) \int_0^\sigma \mathrm{e}^{-\int_\rho^\sigma \{ z + \mu_k(\eta) + \gamma_k(\eta) \} \mathrm{d}\eta} \\ \times \left(P \tilde{\varphi} \left(\rho \right) \right)_k \mathrm{d}\rho \mathrm{d}\sigma \\ \vdots \\ P_n^* \left(a \right) \sum_{k=1}^n \int_0^{+\infty} \beta_{nk} \left(a, \sigma \right) \int_0^\sigma \mathrm{e}^{-\int_\rho^\sigma \{ z + \mu_k(\eta) + \gamma_k(\eta) \} \mathrm{d}\eta} \\ \times \left(P \tilde{\varphi} \left(\rho \right) \right)_k \mathrm{d}\rho \mathrm{d}\sigma \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=1}^n \int_0^{+\infty} \phi_{1k} \left(z, a, \rho \right) \left(P \tilde{\varphi} \left(\rho \right) \right)_k \mathrm{d}\rho \\ \vdots \\ \sum_{k=1}^n \int_0^{+\infty} \phi_{nk} \left(z, a, \rho \right) \left(P \tilde{\varphi} \left(\rho \right) \right)_k \mathrm{d}\rho \end{pmatrix} = U_z P \tilde{\varphi}, \end{aligned}$$

which implies that $P\tilde{\varphi}$ is an eigenfunction associated with eigenvalue unity of U_z, that is, $z \in \Sigma$. □

Lemma 7.3. *The following sign relation holds:*

$$\mathrm{sign}(\mathcal{R}_0 - 1) = \mathrm{sign}(\omega_0(B+P)). \tag{7.10}$$

Proof. For $z > -\underline{\mu}$, positive operator U_z is nonsupporting, which can be proved as in (ii) of Lemma 5.1. Then, since the kernel $\phi_{jk}(z, a, \rho)$ of U_z is monotone decreasing with respect to $z > -\underline{\mu}$, so $r(U_z)$ is also monotone decreasing with respect to $z > -\underline{\mu}$ and $\lim_{z\to\infty} r(U_z) = 0$. Suppose that $r(U_0) = r(\mathcal{K}) = \mathcal{R}_0 \geq 1$. Then $r(U_z) = 1$ has a unique nonnegative root z_0 such that $\mathrm{sign}(\mathcal{R}_0 - 1) = \mathrm{sign}(z_0)$, and z_0 is the dominant normal eigenvalue of $B+P$. In fact, from $\sup_{z\in E_\sigma(B+P)} \Re z \leq \omega_1(B+P) < -\underline{\mu}$ (see Webb [46, Proposition 4.13]), we know that there exist only normal eigenvalues in the half plane $\Re z > -\underline{\mu}$ and the dominant property of z_0 is proved as Lemma 5.6 of Inaba [18]. From (7.4) and (7.5), we obtain $z_0 = \omega_0(B+P)$. Therefore $\omega_0(B+P) > 0$ if $\mathcal{R}_0 > 1$ and $\omega_0(B+P) = 0$ if $\mathcal{R}_0 = 1$. On the other hand, if $\mathcal{R}_0 < 1$, there is no normal eigenvalue with nonnegative real part. Since there exist at most finite number of normal eigenvalue in the half plane $\Re z > -\underline{\mu}$, it follows from (7.4) that $\omega_0(B+P) < 0$. □

Proposition 7.1. *If $\mathcal{R}_0 < 1$, then the disease-free equilibrium $(P^*, \mathbf{0}) \in Y_+$ of system (5.1) is globally asymptotically stable.*

Proof. If $\mathcal{R}_0 < 1$, we have $\omega_0(B+P) < 0$, so $\lim_{t\to\infty} W(t)I_0 = 0$. From (7.3), we conclude the global asymptotic stability of the disease-free equilibrium $(P^*, \mathbf{0}) \in Y_+$. □

8. **Global stability of an endemic equilibrium.** Finally, in this section, we investigate the global asymptotic stability of endemic equilibrium $(S^*, I^*) \in Y_+$ of system (5.1). We make the following additional assumption.

Assumption 8.1. *(i) For each $j, k \in \{1, 2, \ldots, n\}$, $\beta_{jk}(a, \sigma)$ is independent of the age σ and state k of infective individuals, that is, $\beta_{jk}(a, \sigma) \equiv \beta_j(a)$.*
(ii) For each $j \in \{1, 2, \ldots, n\}$, $\mu_j(a)$ and $\gamma_j(a)$ are positive constants, that is, $\mu_j(a) \equiv \mu_j$ and $\gamma_j(a) \equiv \gamma_j$.

Note that (i) of Assumption 8.1 is a special case of separable mixing discussed in Section 6.1. Therefore, if $\mathcal{R}_0 > 1$ and Assumption 5.1 holds, then it follows from Proposition 6.1 that system (5.1) has a unique endemic equilibrium (S^*, I^*). Now Assumptions 2.1 (ii), 5.1 (iii) and 6.1 imply that for each $j \in \{1, 2, \cdots, n\}$ there exist $\beta_j^+, \epsilon_0 \in (0, +\infty)$ such that

$$\epsilon_0 \;\leq\; \beta_j(a) \;\leq\; \beta_j^+ \quad \text{for almost all } a \in [0, +\infty)\,. \tag{8.1}$$

Under Assumption 8.1, system (5.1) can be rewritten as follows:

$$\begin{cases} \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right) S_j(t,a) = -\lambda_j(t,a) S_j(t,a) - \mu_j S_j(t,a), \\ \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right) I_j(t,a) = \lambda_j(t,a) S_j(t,a) - (\mu_j+\gamma_j) I_j(t,a), \\ \qquad S_j(t,0) = b_j, \quad I_j(t,0) = 0, \\ \qquad \lambda_j(t,a) = \beta_j(a) \displaystyle\sum_{k=1}^{n} \int_0^{+\infty} I_k(t,\sigma)\,\mathrm{d}\sigma, \\ \qquad\quad j = 1,2,\ldots,n, \quad t \geq 0, \quad a \geq 0. \end{cases} \tag{8.2}$$

Let $J_j(t) := \int_0^{+\infty} I_j(t,a)\,\mathrm{d}a$ and $r_j := \mu_j + \gamma_j$ for $j = 1,2,\ldots,n$. Then, integrating the second equation of (8.2) with respect to a from 0 to $+\infty$, we arrive at the following system:

$$\begin{cases} \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right) S_j(t,a) = -\left\{\beta_j(a) \displaystyle\sum_{k=1}^{n} J_k(t) + \mu_j\right\} S_j(t,a), \\ \qquad \dfrac{\mathrm{d}}{\mathrm{d}t} J_j(t) = \displaystyle\int_0^{+\infty} \beta_j(a) \sum_{k=1}^{n} J_k(t) S_j(t,a)\,\mathrm{d}a - r_j J_j(t), \\ \qquad S_j(t,0) = b_j, \quad j = 1,2,\ldots,n, \quad t \geq 0, \quad a \geq 0. \end{cases} \tag{8.3}$$

Note that $I_j(t,0) \equiv I_j(t,+\infty) \equiv 0$ for all $j \in \{1,2,\ldots,n\}$. In what follows, we investigate the global asymptotic stability of the unique endemic equilibrium (S^*, J^*) of system (8.3), where $J^* := (J_1^*, \ldots, J_n^*)^{\mathrm{T}}$ and

$$J_j^* := \int_0^{+\infty} I_j^*(a)\,\mathrm{d}a, \quad j = 1,2,\ldots,n.$$

The components of (S^*, J^*) must satisfy

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}a} S_j^*(a) = -\left\{\beta_j(a) \displaystyle\sum_{k=1}^{n} J_k^* + \mu_j\right\} S_j^*(a), \\ r_j J_j^* = \displaystyle\int_0^{+\infty} \beta_j(a) \sum_{k=1}^{n} J_k^* S_j^*(a)\,\mathrm{d}a, \\ S_j^*(0) = b_j, \quad j = 1,2,\ldots,n, \quad t \geq 0, \quad a \geq 0. \end{cases} \tag{8.4}$$

Note that from the boundary conditions for (8.3) and (8.4), we have the equality $S_j(t,0) = S_j^*(0) = b_j$, $j = 1,2,\ldots,n$.

For the proof of the global asymptotic stability of the endemic equilibrium (S^*, J^*), we need the invariance principle (see Walker [44, Theorem 4.2 of Chapter IV]) for Lyapunov functionals in infinite-dimensional spaces. To use the principle, we have to show the relative compactness of the positive orbit defined by system (8.3). Now let us define a function space for system (8.3) by $\mathcal{Y} := X \times \mathbb{R}^n$ and its norm by

$$\left\|(\varphi,\psi)^{\mathrm{T}}\right\|_{\mathcal{Y}} := \|\varphi\|_X + \|\psi\| = \sum_{j=1}^{n} \int_0^{+\infty} |\varphi_j(a)|\,\mathrm{d}a + \sum_{j=1}^{n} |\psi_j|$$

with positive cone $\mathcal{Y}_+$. Since the existence and uniqueness of solution for system (8.3) are assured by Proposition 3.1, we can obtain a C_0-semigroup $\{\mathcal{U}(t)\}_{t\geq 0}$:

$\mathcal{Y}_+ \to \mathcal{Y}_+$ defined by system (8.3) as

$$\mathcal{U}(t)\, x_0 := (S(t,\cdot), J(t))^{\mathrm{T}}, \tag{8.5}$$

where $x_0 := (S_0, J_0)^{\mathrm{T}} \in \mathcal{Y}_+$, $S_0 = (S_{0,1}, \dots, S_{0,n})^{\mathrm{T}}$, $J_0 = (J_{0,1}, \dots, J_{0,n})^{\mathrm{T}}$ is the initial condition of system (8.3). Let us define a closed subset $\mathcal{C} \subset \mathcal{Y}$ of $\mathcal{Y}$ by

$$\mathcal{C} := \left\{ (\varphi, \psi)^{\mathrm{T}} \in \mathcal{Y}_+ : \left\| (\varphi, \psi)^{\mathrm{T}} \right\|_{\mathcal{Y}} \le \sum_{j=1}^{n} \frac{b_j}{\mu_j} \right\}. \tag{8.6}$$

Then, it is easy to see that $\mathcal{C}$ is positively invariant for $\{\mathcal{U}(t)\}_{t\ge 0}$. In fact, integrating the first differential equation in system (8.3) with respect to a from 0 to $+\infty$ and adding each equation, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^{+\infty} S_j(t,a)\,\mathrm{d}a + J_j(t) \right) &= b_j - \mu_j \int_0^{+\infty} S_j(t,a)\,\mathrm{d}a - r_j J_j(t) \\ &\le b_j - \mu_j \left(\int_0^{+\infty} S_j(t,a)\,\mathrm{d}a + J_j(t) \right). \end{aligned}$$

Hence, for $x_0 \in \mathcal{C}$,

$$0 \le \int_0^{+\infty} S_j(t,a)\,\mathrm{d}a + J_j(t) \le \frac{b_j}{\mu_j}, \quad \forall t \ge 0 \tag{8.7}$$

and this implies that $\mathcal{U}(t)\,\mathcal{C} \subset \mathcal{C}$ for all $t \ge 0$.

Now we define

$$\tilde{\varphi}_j(t,a) := \begin{cases} 0, & t-a>0, \\ S_j(t,a), & a-t>0, \end{cases} \quad j = 1, 2, \dots, n$$

and

$$\tilde{S}_j(t,a) := S_j(t,a) - \tilde{\varphi}_j(t,a), \quad j = 1, 2, \dots, n$$

and divide $\{\mathcal{U}(t)\}_{t\ge 0}$ into two families of maps

$$\mathcal{V}(t)\, x_0 := (\tilde{\varphi}(t,\cdot), \mathbf{0})^{\mathrm{T}}, \tag{8.8}$$

and

$$\mathcal{W}(t)\, x_0 := \left(\tilde{S}(t,\cdot), J(t) \right)^{\mathrm{T}}, \tag{8.9}$$

where $\tilde{\varphi} = (\tilde{\varphi}_1, \cdots, \tilde{\varphi}_n)^{\mathrm{T}}$, $\tilde{S} = \left(\tilde{S}_1, \cdots, \tilde{S}_n\right)^{\mathrm{T}}$ and $J = (J_1, \dots, J_n)^{\mathrm{T}}$. Thus, $\mathcal{U}(t)\, x_0 = \mathcal{V}(t)\, x_0 + \mathcal{W}(t)\, x_0$ holds for all $t \ge 0$. We first prove the following lemma:

Lemma 8.1. *Let $\mathcal{V}(t)$ and $\mathcal{C}$ defined by (8.8) and (8.6), respectively. There exists a function $\delta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $r > 0$,*

$$\lim_{t \to +\infty} \delta(t,r) = 0 \tag{8.10}$$

and

$$\|\mathcal{V}(t)\, x_0\|_{\mathcal{Y}} \le \delta(t,r) \quad \forall x_0 \in \mathcal{C}, \quad \|x_0\|_{\mathcal{Y}} \le r, \quad t \ge 0. \tag{8.11}$$

Proof. Integrating the equation of S_j in system (8.3) along the characteristic line $t - a =$ constant, we have

$$\tilde{\varphi}_j(t,a) := \begin{cases} 0, & t-a>0, \\ S_{j,0}(a-t)\, \mathrm{e}^{-\int_0^t \left\{ \beta_j(a-t+\sigma) \sum_{k=1}^n J_k(\sigma) + \mu_j \right\} \mathrm{d}\sigma}, & a-t>0 \end{cases}$$

for $j=1,2,\dots,n$. Hence, for $x_0 \in \mathcal{C}$ satisfying $\|x_0\|_{\mathcal{Y}} \le r$,

$$\begin{aligned}\|\mathcal{V}(t)\,x_0\|_{\mathcal{Y}} &= \|\tilde{\varphi}(t,\cdot)\|_X + \|\mathbf{0}\| = \sum_{j=1}^{n}\int_0^{+\infty} |\tilde{\varphi}_j(t,a)|\,\mathrm{d}a \\ &= \sum_{j=1}^{n}\int_t^{+\infty} S_{j,0}(a-t)\,\mathrm{e}^{-\int_0^t\left\{\beta_j(a-t+\sigma)\sum_{k=1}^n J_k(\sigma)+\mu_j\right\}\mathrm{d}\sigma}\mathrm{d}a \\ &\le \mathrm{e}^{-\underline{\mu}t}\sum_{j=1}^{n}\int_0^{+\infty} S_{j,0}(a)\,\mathrm{d}a \\ &= \mathrm{e}^{-\underline{\mu}t}\,\|S_0\|_X \le \mathrm{e}^{-\underline{\mu}t}\,r\end{aligned}$$

and this implies that (8.10) and (8.11) hold with $\delta(t,r) := \mathrm{e}^{-\underline{\mu}t}\,r$. □

Next we prove the following lemma.

Lemma 8.2. *Let $\mathcal{W}(t)$ and $\mathcal{C}$ be defined by (8.9) and (8.6), respectively. For all $t \ge 0$, $\mathcal{W}(t)$ maps any bounded subsets of $\mathcal{C}$ into sets with compact closure in $\mathcal{Y}$.*

Proof. From (8.7) we see that for x_0 in any bounded subset of $\mathcal{C}$, $J(t)$ remains in the compact set $\left\{\psi \in \mathbb{R}_+^n : 0 \le \psi_j \le \frac{b_j}{\mu_j},\ j=1,\dots,n\right\}$ for all $t \ge 0$. Therefore, it suffices to show that $\tilde{S}$ remains in a pre-compact subset of X which is independent of x_0. Now

$$\tilde{S}_j(t,a) := \begin{cases} S_j(t,a), & t-a>0, \\ 0, & a-t>0, \end{cases} \quad j=1,2,\dots,n$$

and hence, integrating the equation of S_j in system (8.3) along the characteristic line $t-a=$ constant gives

$$\tilde{S}_j(t,a) := \begin{cases} b_j\mathrm{e}^{-\int_0^a\left\{\beta_j(\sigma)\sum_{k=1}^n J_k(t-a+\sigma)+\mu_j\right\}\mathrm{d}\sigma}, & t-a>0, \\ 0, & a-t>0, \end{cases} \quad j=1,2,\dots,n. \tag{8.12}$$

To complete the proof, we have to show the following (see Smith and Thieme [38, Theorem B.2]).

(i) The supremum of $\displaystyle\sum_{j=1}^{n}\int_0^{+\infty}\tilde{S}_j(t,a)\,\mathrm{d}a$ for all initial data $x_0 \in \mathcal{C}$ is finite.

(ii) $\displaystyle\lim_{h\to+\infty}\sum_{j=1}^{n}\int_h^{+\infty}\tilde{S}_j(t,a)\,\mathrm{d}a = 0$ uniformly in $x_0 \in \mathcal{C}$.

(iii) $\displaystyle\lim_{h\to+0}\sum_{j=1}^{n}\int_0^{+\infty}\left|\tilde{S}_j(t,a+h)-\tilde{S}_j(t,a)\right|\mathrm{d}a = 0$ uniformly in $x_0 \in \mathcal{C}$.

(iv) $\displaystyle\lim_{h\to+0}\sum_{j=1}^{n}\int_0^{h}\tilde{S}_j(t,a)\,\mathrm{d}a = 0$ uniformly in $x_0 \in \mathcal{C}$.

Now, from (8.12),

$$0 \le \tilde{S}_j(t,a) \le b_j\mathrm{e}^{-\underline{\mu}a}, \quad j=1,2,\dots,n$$

and hence, (i), (ii) and (iv) immediately follow.

We show (iii). Since we shall consider $h \to +0$, we can assume that $h \in (0,t)$ without loss of generality. Then, from (8.12), we have

$$\begin{aligned}
&\sum_{j=1}^{n} \int_0^{+\infty} \left| \tilde{S}_j(t,a+h) - \tilde{S}_j(t,a) \right| \mathrm{d}a \\
&= \sum_{j=1}^{n} \int_{t-h}^{t} \left| 0 - \tilde{S}_j(t,a) \right| \mathrm{d}a + \sum_{j=1}^{n} \int_0^{t-h} \left| \tilde{S}_j(t,a+h) - \tilde{S}_j(t,a) \right| \mathrm{d}a \\
&\le \sum_{j=1}^{n} b_j h + \sum_{j=1}^{n} b_j \int_0^{t-h} \left| \mathrm{e}^{-\int_0^{a+h} \left\{ \beta_j(\sigma) \sum_{k=1}^{n} J_k(t-a-h+\sigma) + \mu_j \right\} \mathrm{d}\sigma} \right. \\
&\qquad\qquad \left. - \mathrm{e}^{-\int_0^{a} \left\{ \beta_j(\sigma) \sum_{k=1}^{n} J_k(t-a+\sigma) + \mu_j \right\} \mathrm{d}\sigma} \right| \mathrm{d}a \\
&\le \sum_{j=1}^{n} b_j h + \sum_{j=1}^{n} b_j \int_0^{t-h} \left| \int_0^{a+h} \left\{ \beta_j(\sigma) \sum_{k=1}^{n} J_k(t-a-h+\sigma) + \mu_j \right\} \mathrm{d}\sigma \right. \\
&\qquad\qquad \left. - \int_0^{a} \left\{ \beta_j(\sigma) \sum_{k=1}^{n} J_k(t-a+\sigma) + \mu_j \right\} \mathrm{d}\sigma \right| \mathrm{d}a,
\end{aligned}$$

where we use the relation $|\mathrm{e}^{-x} - \mathrm{e}^{-y}| \le |x-y|$. Thus,

$$\sum_{j=1}^{n} \int_0^{+\infty} \left| \tilde{S}_j(t,a+h) - \tilde{S}_j(t,a) \right| \mathrm{d}a \le \sum_{j=1}^{n} b_j h \left\{ 1 + \mu_j (t-h) \right\} + \mathcal{A} + \mathcal{B}, \quad (8.13)$$

where

$$\begin{aligned}
\mathcal{A} := \sum_{j=1}^{n} b_j \int_0^{t-h} &\left| \int_0^{a+h} \beta_j(\sigma) \sum_{k=1}^{n} J_k(t-a-h+\sigma) \mathrm{d}\sigma \right. \\
&\left. - \int_0^{a+h} \beta_j(\sigma) \sum_{k=1}^{n} J_k(t-a+\sigma) \mathrm{d}\sigma \right| \mathrm{d}a
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B} := \sum_{j=1}^{n} b_j \int_0^{t-h} &\left| \int_0^{a+h} \beta_j(\sigma) \sum_{k=1}^{n} J_k(t-a+\sigma) \mathrm{d}\sigma \right. \\
&\left. - \int_0^{a} \beta_j(\sigma) \sum_{k=1}^{n} J_k(t-a+\sigma) \mathrm{d}\sigma \right| \mathrm{d}a.
\end{aligned}$$

Now, we have

$$\begin{aligned}
\mathcal{A} &\le \sum_{j=1}^{n} b_j \int_0^{t-h} \int_0^{a+h} \beta_j(\sigma) \left| \sum_{k=1}^{n} J_k(t-a-h+\sigma) - \sum_{k=1}^{n} J_k(t-a+\sigma) \right| \mathrm{d}\sigma \mathrm{d}a \\
&\le \sum_{j=1}^{n} b_j \beta_j^{+} \int_0^{t-h} \int_0^{a+h} \left\| J(t-a-h+\sigma) - J(t-a+\sigma) \right\| \mathrm{d}\sigma \mathrm{d}a \qquad (8.14)
\end{aligned}$$

where β_j^{+} $(j = 1, 2, \ldots, n)$ is a positive upper bound of $\beta_j(a)$ whose existence is assured by Assumption 2.1. Note that it follows from the second equation of (8.3) that

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}J(t)\right\| \;\leq\; \sum_{j=1}^{n}\left\{\beta_j^+\frac{b_j}{\mu_j}\sum_{k=1}^{n}\frac{b_k}{\mu_k}+r_j\frac{b_j}{\mu_j}\right\} \;=:\; M_J$$

and therefore, $J(t)$ is Lipschitz continuous on $\mathbb{R}_+$ with Lipschitz coefficient M_J. Hence, from (8.14), we obtain

$$\mathcal{A} \;\leq\; \sum_{j=1}^{n} b_j\beta_j^+ M_J h\int_0^{t-h}(a+h)\mathrm{d}a \;=\; \sum_{j=1}^{n} b_j\beta_j^+ M_J h\left\{\frac{1}{2}(t-h)^2+h(t-h)\right\}. \tag{8.15}$$

Moreover, we have

$$\begin{aligned}\mathcal{B} &= \sum_{j=1}^{n} b_j\int_0^{t-h}\left|\int_a^{a+h}\beta_j(\sigma)\sum_{k=1}^{n}J_k(t-a+\sigma)\,\mathrm{d}\sigma\right|\mathrm{d}a\\ &\leq \sum_{j=1}^{n} b_j\beta_j^+\int_0^{t-h}\int_a^{a+h}\|J(t-a+\sigma)\|\,\mathrm{d}\sigma\mathrm{d}a.\end{aligned}$$

Since $\|J(t)\| \leq \sum_{j=1}^{n} b_j/\mu_j$ for all $t\geq 0$, we obtain

$$\mathcal{B} \;\leq\; \sum_{j=1}^{n} b_j\beta_j^+\sum_{k=1}^{n}\frac{b_k}{\mu_k}h(t-h). \tag{8.16}$$

Consequently, it follows from (8.13), (8.15) and (8.16) that

$$\begin{aligned}&\sum_{j=1}^{n}\int_0^{+\infty}\left|\tilde{S}_j(t,a+h)-\tilde{S}_j(t,a)\right|\mathrm{d}a\\ &\leq \sum_{j=1}^{n} b_j h\{1+\mu_j(t-h)\}\\ &\quad+\sum_{j=1}^{n} b_j\beta_j^+ M_J h\left\{\frac{1}{2}(t-h)^2+h(t-h)\right\}+\sum_{j=1}^{n} b_j\beta_j^+\sum_{k=1}^{n}\frac{b_k}{\mu_k}h(t-h)\\ &\to 0 \text{ as } h\to +0 \text{ uniformly in } x_0\in\mathcal{C},\end{aligned}$$

and hence, (iii) is shown and the proof is complete. □

Using Lemmas 8.1 and 8.2, we prove the following proposition on the relative compactness of the positive orbit $\{\mathcal{U}(t)x_0 : t\geq 0\}$:

Proposition 8.1. *Under Assumption 8.1, let $\mathcal{U}(t)$ and $\mathcal{C}$ be defined by (8.5) and (8.6), respectively. For $x_0\in\mathcal{C}$, $\{\mathcal{U}(t)x_0 : t\geq 0\}$ has compact closure in $\mathcal{Y}$.*

Proof. Since

$$\sup_{t\geq 0}\|\mathcal{U}(t)x_0\|_{\mathcal{Y}} \leq \sum_{j=1}^{n}\frac{b_j}{\mu_j}$$

for any $x_0\in\mathcal{C}$, it follows from Lemmas 8.1-8.2 and Proposition 3.13 in Webb [46] that $\{\mathcal{U}(t)x_0 : t\geq 0\}$ has compact closure in $\mathcal{Y}$. □

Since the relative compactness of the solution orbit is proved in Proposition 8.1, we can use the invariance principle stated in Walker [44, Theorem 4.2 in Chapter IV] together with a Lyapunov functional in order to prove the global asymptotic stability of endemic equilibrium (S^*, J^*). Thus, following the graph-theoretic approach of Guo *et al.* [12], we construct a Lyapunov functional. Set constants

$$\tilde{\beta}_{jk} := \int_0^{+\infty} \beta_j(a) S_j^*(a) \,\mathrm{d}a \; J_k^*, \quad j,k = 1,2,\dots,n$$

and a Laplacian matrix

$$\tilde{B} := \begin{pmatrix} \sum_{k\neq 1} \tilde{\beta}_{1k} & -\tilde{\beta}_{21} & \cdots & -\tilde{\beta}_{n1} \\ -\tilde{\beta}_{12} & \sum_{k\neq 2} \tilde{\beta}_{2k} & & -\tilde{\beta}_{n2} \\ \vdots & & \ddots & \vdots \\ -\tilde{\beta}_{1n} & -\tilde{\beta}_{2n} & \cdots & \sum_{k\neq n} \tilde{\beta}_{nk} \end{pmatrix}.$$

Then, from Guo *et al.* [12, Lemma 2.1], we see that the solution space of a linear system

$$\tilde{B}\kappa = \mathbf{0}, \quad \kappa = (\kappa_1, \dots, \kappa_n)^{\mathrm{T}}. \tag{8.17}$$

has dimension 1 and one of its basis is given by

$$\kappa = (\kappa_1, \dots, \kappa_n)^{\mathrm{T}} = (c_{11}, \dots, c_{nn})^{\mathrm{T}},$$

where $c_{jj} > 0$ $j = 1,2,\dots,n$ denotes the cofactor of the j-th diagonal entry of matrix $\tilde{B}$. Note that for such κ it follows

$$\sum_{k=1}^{n} \tilde{\beta}_{jk}\kappa_j = \sum_{k=1}^{n} \tilde{\beta}_{kj}\kappa_k. \tag{8.18}$$

Using this κ, we consider the following Lyapunov functional:

$$L(S,J) := \sum_{j=1}^{n} \kappa_j \left\{ \int_0^{+\infty} S_j^*(a)\, g\left(\frac{S_j(t,a)}{S_j^*(a)}\right) \mathrm{d}a + J_j^* g\left(\frac{J_j(t)}{J_j^*}\right) \right\}, \tag{8.19}$$

where $g(w) := w - 1 - \ln w$. To check that this function L is well-defined, it suffices to show that the integration in the right-hand side is bounded. We make the following assumption:

Assumption 8.2. *For each $j \in \{1,2,\cdots,n\}$, $S_{j,0} \in X_+$, $J_j(0) > 0$. In addition,*

$$\int_0^{+\infty} |\ln S_{j,0}(a)|\, \mathrm{e}^{-\mu_j a} \mathrm{d}a < +\infty. \tag{8.20}$$

In fact, since $S_j(\cdot, a)$ has a structure compatible with $b_j \mathrm{e}^{-\mu_j a}$ and $|\ln(b_j \mathrm{e}^{-\mu_j a})|\, \mathrm{e}^{-\mu_j a}$ is integrable, (8.20) is thought to be a natural assumption. Under Assumption 8.2, we have the following lemma on the boundedness of the Lyapunov functional $L(S,J)$:

Lemma 8.3. *Let $\mathcal{R}_0$ and $L(S,J)$ be defined by (4.10) and (8.19), respectively. If $\mathcal{R}_0 > 1$ and Assumption 8.2 holds, then $L(S,J)$ is well-defined.*

Proof. For $t-a>0$,

$$
\begin{aligned}
& \left| S_j^*(a)\ln\frac{S_j(t,a)}{S_j^*(a)} \right| \\
= \; & \left| S_j^*(a)\ln S_j(t,a) - S_j^*(a)\ln S_j^*(a) \right| \\
\le \; & \left| b_j \mathrm{e}^{-\int_0^a \left\{ \beta_j(\sigma)\sum_{k=1}^n J_k^* + \mu_j \right\} \mathrm{d}\sigma} \ln\left(b_j \mathrm{e}^{-\int_0^a \left\{ \beta_j(\sigma)\sum_{k=1}^n J_k(t-a+\sigma)+\mu_j \right\}\mathrm{d}\sigma} \right) \right| \\
& + \left| b_j \mathrm{e}^{-\int_0^a \left\{ \beta_j(\sigma)\sum_{k=1}^n J_k^* + \mu_j \right\} \mathrm{d}\sigma} \ln\left(b_j \mathrm{e}^{-\int_0^a \left\{ \beta_j(\sigma)\sum_{k=1}^n J_k^*+\mu_j \right\}\mathrm{d}\sigma} \right) \right| \\
\le \; & b_j \left| \ln b_j \right| \mathrm{e}^{-\mu_j a} + b_j \left(\beta_j^+ \sum_{k=1}^n \frac{b_k}{\mu_k} + \mu_j \right) a \mathrm{e}^{-\mu_j a} \\
& + b_j \left| \ln b_j \right| \mathrm{e}^{-\mu_j a} + b_j \left(\beta_j^+ \sum_{k=1}^n \frac{b_k}{\mu_k} + \mu_j \right) a \mathrm{e}^{-\mu_j a}, \quad j = 1,2,\cdots,n. \qquad (8.21)
\end{aligned}
$$

For $a-t\ge 0$,

$$
\begin{aligned}
& \left| S_j^*(a)\ln\frac{S_j(t,a)}{S_j^*(a)} \right| \\
= \; & \left| S_j^*(a)\ln S_j(t,a) - S_j^*(a)\ln S_j^*(a) \right| \\
\le \; & \left| b_j \mathrm{e}^{-\int_0^a \left\{ \beta_j(\sigma)\sum_{k=1}^n J_k^* + \mu_j \right\} \mathrm{d}\sigma} \ln\left(S_{j,0}(a-t) \mathrm{e}^{-\int_{a-t}^a \left\{ \beta_j(\sigma)\sum_{k=1}^n J_k(t-a+\sigma)+\mu_j \right\}\mathrm{d}\sigma} \right) \right| \\
& + \left| b_j \mathrm{e}^{-\int_0^a \left\{ \beta_j(\sigma)\sum_{k=1}^n J_k^* + \mu_j \right\} \mathrm{d}\sigma} \ln\left(b_j \mathrm{e}^{-\int_0^a \left\{ \beta_j(\sigma)\sum_{k=1}^n J_k^*+\mu_j \right\}\mathrm{d}\sigma} \right) \right| \\
\le \; & b_j \left| \ln S_{j,0}(a-t) \right| \mathrm{e}^{-\mu_j a} + b_j \left(\beta_j^+ \sum_{k=1}^n \frac{b_k}{\mu_k} + \mu_j \right) t \mathrm{e}^{-\mu_j a} \\
& + b_j \left| \ln b_j \right| \mathrm{e}^{-\mu_j a} + b_j \left(\beta_j^+ \sum_{k=1}^n \frac{b_k}{\mu_k} + \mu_j \right) a \mathrm{e}^{-\mu_j a} \\
\le \; & b_j \left| \ln S_{j,0}(a-t) \right| \mathrm{e}^{-\mu_j a} + b_j \left(\beta_j^+ \sum_{k=1}^n \frac{b_k}{\mu_k} + \mu_j \right) a \mathrm{e}^{-\mu_j a} \\
& + b_j \left| \ln b_j \right| \mathrm{e}^{-\mu_j a} + b_j \left(\beta_j^+ \sum_{k=1}^n \frac{b_k}{\mu_k} + \mu_j \right) a \mathrm{e}^{-\mu_j a}, \quad j = 1,2,\cdots,n. \qquad (8.22)
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\int_0^{+\infty} \left| S_j^*(a) \ln \frac{S_j(t,a)}{S_j^*(a)} \right| \mathrm{d}a \\
\leq\ & b_j \left|\ln b_j\right| \int_0^t \mathrm{e}^{-\mu_j a} \mathrm{d}a + b_j \int_t^{+\infty} \left|\ln S_{j,0}(a-t)\right| \mathrm{e}^{-\mu_j a} \mathrm{d}a \\
& + b_j \left|\ln b_j\right| \int_0^{+\infty} \mathrm{e}^{-\mu_j a} \mathrm{d}a + 2b_j \left(\beta_j^+ \sum_{k=1}^n \frac{b_k}{\mu_k} + \mu_j \right) \int_0^{+\infty} a \mathrm{e}^{-\mu_j a} \mathrm{d}a \\
=\ & \frac{b_j \left|\ln b_j\right|}{\mu_j} \left(1 - \mathrm{e}^{-\mu_j t}\right) + b_j \int_0^{+\infty} \left|\ln S_{j,0}(a)\right| \mathrm{e}^{-\mu_j (a+t)} \mathrm{d}a \\
& + \frac{b_j \left|\ln b_j\right|}{\mu_j} + \frac{2b_j}{\mu_j^2} \left(\beta_j^+ \sum_{k=1}^n \frac{b_k}{\mu_k} + \mu_j \right) \\
\leq\ & \frac{2b_j \left|\ln b_j\right|}{\mu_j} + b_j \int_0^{+\infty} \left|\ln S_{j,0}(a)\right| \mathrm{e}^{-\mu_j a} \mathrm{d}a + \frac{2b_j}{\mu_j^2} \left(\beta_j^+ \sum_{k=1}^n \frac{b_k}{\mu_k} + \mu_j \right).
\end{aligned}
$$

Under Assumption 8.2, the second term in the right-hand side is finite. Thus, $S_j^*(a) \ln \left(S_j(t.a)/S_j^*(a)\right)$, $j = 1, 2, \cdots, n$ is integrable and hence, the Lyapunov functional $L(S, J)$ is well-defined. □

Under these preparations, we prove the following proposition:

Proposition 8.2. *Let $\mathcal{R}_0$ be defined by (4.10). Under Assumption 8.1, if $\mathcal{R}_0 > 1$, then system (8.3) has a unique globally asymptotically stable endemic equilibrium (S^*, J^*) which attracts any initial values satisfying Assumption 8.2.*

Proof. The uniqueness of (S^*, J^*) follows from Proposition 6.1. To prove the global stability of (S^*, J^*), we use the Lyapunov functional $L\,(S, J)$ defined by (8.19). The differentiation of L along the trajectories of system (8.3) is

$$
\begin{aligned}
L'\,(S, J) &= \sum_{j=1}^n \kappa_j \left\{ \int_0^{+\infty} \left(1 - \frac{S_j^*\,(a)}{S_j\,(t,a)}\right) \frac{\partial}{\partial t} S_j\,(t,a)\, \mathrm{d}a + \left(1 - \frac{J_j^*}{J_j\,(t)}\right) \frac{\mathrm{d}}{\mathrm{d}t} J_j\,(t) \right\} \\
&= \sum_{j=1}^n \kappa_j \Bigg[- \int_0^{+\infty} \left(1 - \frac{S_j^*\,(a)}{S_j\,(t,a)}\right) \frac{\partial}{\partial a} S_j\,(t,a)\, \mathrm{d}a \\
&\qquad - \int_0^{+\infty} \left(1 - \frac{S_j^*\,(a)}{S_j\,(t,a)}\right) \left\{ \beta_j\,(a) \sum_{k=1}^n J_k\,(t) + \mu_j \right\} S_j\,(t,a)\, \mathrm{d}a \\
&\qquad + \sum_{k=1}^n \int_0^{+\infty} \beta_j\,(a)\, S_j\,(t,a)\, \mathrm{d}a\ J_k\,(t) - r_j J_j\,(t) \\
&\qquad - \sum_{k=1}^n \int_0^{+\infty} \beta_j\,(a)\, S_j\,(t,a)\, \mathrm{d}a\ J_k\,(t) \frac{J_j^*}{J_j\,(t)} + r_j J_j^* \Bigg].
\end{aligned}
\tag{8.23}
$$

Note that it follows from (8.18) and the second equation of (8.4) that

$$\begin{aligned}
\sum_{j=1}^{n}\kappa_j\sum_{k=1}^{n}\int_0^{+\infty}\beta_j(a)\,S_j^*(a)\,\mathrm{d}aJ_k(t) &= \sum_{j=1}^{n}\sum_{k=1}^{n}\kappa_j\tilde{\beta}_{jk}\frac{J_k(t)}{J_k^*}\\
&= \sum_{j=1}^{n}\sum_{k=1}^{n}\kappa_k\tilde{\beta}_{kj}\frac{J_j(t)}{J_j^*}\\
&= \sum_{j=1}^{n}\frac{J_j(t)}{J_j^*}\sum_{k=1}^{n}\kappa_k\tilde{\beta}_{kj}\\
&= \sum_{j=1}^{n}\frac{J_j(t)}{J_j^*}\sum_{k=1}^{n}\kappa_j\tilde{\beta}_{jk}\\
&= \sum_{j=1}^{n}\kappa_j\frac{J_j(t)}{J_j^*}\sum_{k=1}^{n}\tilde{\beta}_{jk} \;=\; \sum_{j=1}^{n}\kappa_j r_j J_j(t).
\end{aligned} \tag{8.24}$$

Then, substituting (8.24) into (8.23), we have

$$\begin{aligned}
L'(S,J) \;=\; \sum_{j=1}^{n}\kappa_j\Bigg[&-\int_0^{+\infty}\left(1-\frac{S_j^*(a)}{S_j(t,a)}\right)\frac{\partial}{\partial a}S_j(t,a)\,\mathrm{d}a\\
&-\int_0^{+\infty}\left(1-\frac{S_j^*(a)}{S_j(t,a)}\right)\mu_j S_j(t,a)\,\mathrm{d}a\\
&-\sum_{k=1}^{n}\int_0^{+\infty}\beta_j(a)\,S_j(t,a)\,\mathrm{d}a\;J_k(t)\,\frac{J_j^*}{J_j(t)}+r_jJ_j^*\Bigg]
\end{aligned} \tag{8.25}$$

Furthermore, it follows from

$$\frac{\partial}{\partial a}g\left(\frac{S_j(t,a)}{S_j^*(a)}\right)=\left(1-\frac{S_j^*(a)}{S_j(t,a)}\right)\frac{1}{S_j^*(a)}\left\{\frac{\partial}{\partial a}S_j(t,a)-\frac{S_j(t,a)}{S_j^*(a)}\frac{\mathrm{d}}{\mathrm{d}a}S_j^*(a)\right\},$$

and the first equation of (8.4) that we have

$$\begin{aligned}
&S_j^*(a)\,\frac{\partial}{\partial a}g\left(\frac{S_j(t,a)}{S_j^*(a)}\right)=\\
&\qquad\left(1-\frac{S_j^*(a)}{S_j(t,a)}\right)\left[\frac{\partial}{\partial a}S_j(t,a)+\left\{\beta_j(a)\sum_{k=1}^{n}J_k^*+\mu_j\right\}S_j(t,a)\right].
\end{aligned} \tag{8.26}$$

From (8.25), (8.26) and the second equation in (8.4), using integration by parts, we have

$$\begin{aligned} L'(S,J) = \sum_{j=1}^{n} \kappa_j \Bigg[&- \int_0^{+\infty} S_j^*(a) \frac{\partial}{\partial a} g\left(\frac{S_j(t,a)}{S_j^*(a)}\right) \mathrm{d}a \\ &+ \int_0^{+\infty} \left(1 - \frac{S_j^*(a)}{S_j(t,a)}\right) \beta_j(a) \sum_{k=1}^{n} J_k^* \, S_j(t,a) \, \mathrm{d}a \\ &- \sum_{k=1}^{n} \int_0^{+\infty} \beta_j(a) S_j(t,a) \, \mathrm{d}a \; J_k(t) \frac{J_j^*}{J_j(t)} + r_j J_j^* \Bigg] \\ = \sum_{j=1}^{n} \kappa_j \Bigg[&- \left[S_j^*(a) g\left(\frac{S_j(t,a)}{S_j^*(a)}\right)\right]_{a=0}^{a=+\infty} + \int_0^{+\infty} \frac{\partial}{\partial a} S_j^*(a) g\left(\frac{S_j(t,a)}{S_j^*(a)}\right) \mathrm{d}a \\ &+ \sum_{k=1}^{n} \int_0^{+\infty} \beta_j(a) S_j^*(a) J_k^* \left(\frac{S_j(t,a)}{S_j^*(a)} - 1\right) \mathrm{d}a \\ &- \sum_{k=1}^{n} \int_0^{+\infty} \beta_j(a) S_j^*(a) J_k^* \left(\frac{J_j^* J_k(t) S_j(t,a)}{J_j(t) J_k^* S_j^*(a)} - 1\right) \mathrm{d}a \Bigg]. \end{aligned} \tag{8.27}$$

Since

$$\left[S_j^*(a) g\left(\frac{S_j(t,a)}{S_j^*(a)}\right)\right]_{a=0}^{a=+\infty} = 0 \tag{8.28}$$

(note that $\lim_{a\to+\infty} S_j^*(a) = 0$ since the survival rate $\ell(a) := \mathrm{e}^{-\mu_j a}$ converges to zero as $a \to +\infty$) and

$$\begin{aligned} \sum_{j=1}^{n} \kappa_j \sum_{k=1}^{n} \int_0^{+\infty} & \beta_j(a) S_j^*(a) J_k^* \\ & \times \left(-\ln \frac{S_j(t,a)}{S_j^*(a)} + \ln \frac{J_j^* J_k(t) S_j(t,a)}{J_j(t) J_k^* S_j^*(a)} + \ln \frac{J_j(t)}{J_j^*} - \ln \frac{J_k(t)}{J_k^*} \right) \mathrm{d}a = 0, \end{aligned} \tag{8.29}$$

combining (8.27)-(8.29) gives

$$
\begin{aligned}
L'(S,J) = \sum_{j=1}^{n} \kappa_j \Bigg[& \int_0^{+\infty} \frac{\partial}{\partial a} S_j^*(a)\, g\left(\frac{S_j(t,a)}{S_j^*(a)}\right) \mathrm{d}a \\
& + \sum_{k=1}^{n} \int_0^{+\infty} \beta_j(a) S_j^*(a) J_k^* g\left(\frac{S_j(t,a)}{S_j^*(a)}\right) \mathrm{d}a \\
& - \sum_{k=1}^{n} \int_0^{+\infty} \beta_j(a) S_j^*(a) J_k^* g\left(\frac{J_j^* J_k(t) S_j(t,a)}{J_j(t) J_k^* S_j^*(a)}\right) \mathrm{d}a \\
& + \sum_{k=1}^{n} \int_0^{+\infty} \beta_j(a) S_j^*(a) J_k^* \left(\ln \frac{J_j(t)}{J_j^*} - \ln \frac{J_k(t)}{J_k^*}\right) \mathrm{d}a \\
= \sum_{j=1}^{n} \kappa_j \Bigg[& - \int_0^{+\infty} \mu_j S_j^*(a)\, g\left(\frac{S_j(t,a)}{S_j^*(a)}\right) \mathrm{d}a \\
& - \sum_{k=1}^{n} \int_0^{+\infty} \beta_j(a) S_j^*(a) J_k^* g\left(\frac{J_j^* J_k(t) S_j(t,a)}{J_j(t) J_k^* S_j^*(a)}\right) \mathrm{d}a \\
& + \sum_{k=1}^{n} \int_0^{+\infty} \beta_j(a) S_j^*(a) J_k^* \left(\ln \frac{J_j(t)}{J_j^*} - \ln \frac{J_k(t)}{J_k^*}\right) \mathrm{d}a \Bigg]. \qquad (8.30)
\end{aligned}
$$

Since the first two terms in the right-hand side of (8.30) are nonpositive, it is sufficient to show the nonpositivity of the last term. In fact,

$$
\begin{aligned}
& \sum_{j=1}^{n} \kappa_j \sum_{k=1}^{n} \int_0^{+\infty} \beta_j(a) S_j^*(a) J_k^* \left(\ln \frac{J_j(t)}{J_j^*} - \ln \frac{J_k(t)}{J_k^*}\right) \mathrm{d}a \\
& \qquad = \sum_{j=1}^{n} \sum_{k=1}^{n} \kappa_j \tilde{\beta}_{jk} \left(\ln \frac{J_j(t)}{J_j^*} - \ln \frac{J_k(t)}{J_k^*}\right)
\end{aligned}
$$

and hence, using the graph-theoretic approach as in Guo *et al.* [12], we can show that this term equals zero. Therefore, $L'(S,J) \leq 0$ and it is easy to show that the equality holds if and only if $(S,J) = (S^*, J^*)$. Note that this implies that the largest positive invariant subset of set $\{(S,J) \in \mathcal{Y} : L'(S,J) = 0\}$ in $\mathcal{Y}$ is the singleton $\{(S^*, J^*)\}$. Since the relative compactness of the solution orbit is proved in Proposition 8.1, we can apply the invariance principle stated in Walker [44, Theorem 4.2 in Chapter IV] and conclude that the endemic equilibrium (S^*, J^*) is globally asymptotically stable. The proof is complete. □

Compiling Propositions 5.2, 6.1, 6.3, 7.1 and 8.2, we establish the following theorem.

Theorem 8.1. *Let $\mathcal{R}_0$ be the basic reproduction number defined by (4.10).*

- *(i) If $\mathcal{R}_0 < 1$, then the disease-free equilibrium $(P^*, \mathbf{0}) \in Y_+$ of system (5.1) is globally asymptotically stable.*
- *(ii) If $\mathcal{R}_0 > 1$, then system (5.1) has at least one endemic equilibrium $(S^*, I^*) \in Y_+ \setminus \{\mathbf{0}\}$.*
- *(iii) If $\mathcal{R}_0 > 1$ and either of Assumptions 6.1 or 6.2 holds, then the endemic equilibrium $(S^*, I^*) \in Y_+ \setminus \{\mathbf{0}\}$ of system (5.1) is unique.*

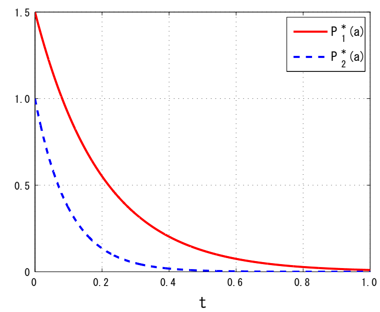


FIGURE 1. Demographic steady states $P_1^*(a)$ and $P_2^*(a)$ for parameter (9.1)

(iv) If $\mathcal{R}_0 > 1$ and Assumption 8.1 holds, then system (5.1) has a unique globally asymptotically stable endemic equilibrium $(S^*, I^*) \in Y_+ \setminus \{\mathbf{0}\}$ which attracts any initial values satisfying Assumption 8.2.

9. **Numerical examples.** In this section, we verify the validity of Theorem 8.1 by performing numerical simulation. Since we can not consider the infinitely large age interval $a \in [0, +\infty)$ in numerical simulation, we set the maximum age a_M and normalize it as $a_M = 1$ for simplicity. Let us consider two-group case ($n = 2$) for sexually transmitted diseases, in which $j = 1$ and $j = 2$ imply female and male populations, respectively. Let us fix the following demographic parameters:

$$b_1 = 1.5, \quad b_2 = 1, \quad \mu_1(a) \equiv \mu_1 = 5, \quad \mu_2(a) \equiv \mu_2 = 10. \tag{9.1}$$

In this case, the demographic steady states $P_j^*(a) = b_j \exp(-\mu_j a)$, $j = 1, 2$ are almost zero at $a = a_M = 1$ (see Figure 1). Therefore, we can expect that our simulation would be a good approximation for the case of infinite age interval $[0, +\infty)$ in which $P_j^*(a) \to 0$, $j = 1, 2$ as $a \to +\infty$.

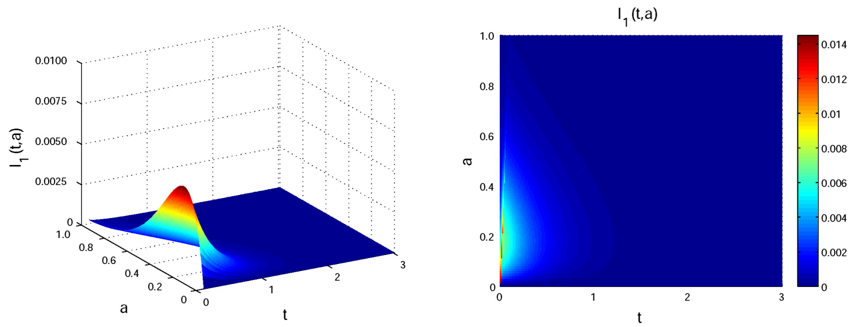
Let us fix the following epidemic parameters

$$\gamma_1 = 0.25, \quad \gamma_2 = 0.5, \quad \beta_{21}(a,\sigma) = \beta_{22}(a,\sigma) = \beta_2(a) = 5\left(1 + 0.9\cos\frac{3\pi a}{2}\right)$$

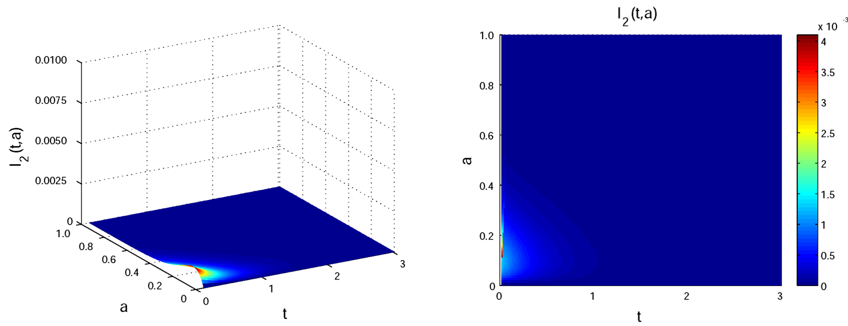
and vary $\beta_{11}(a,\sigma) = \beta_{12}(a,\sigma) = \beta_1(a)$. Note that Assumption 8.1 holds in this case. Therefore, from Theorem 8.1, we can expect that the basic reproduction number $\mathcal{R}_0$ would play the role of the complete threshold value for the global asymptotic stability of each equilibrium. From (4.9), the component $\mathcal{K}_j\varphi$ in the vector of the next generation operator (4.8) can be approximated by the following one.

$$\begin{aligned}
(\mathcal{K}_j\varphi)(a) &= P_j^*(a)\beta_j(a)\sum_{k=1}^{2}\int_0^{a_M}\int_\rho^{a_M} \mathrm{e}^{-(\mu_k+\gamma_k)(\sigma-\rho)}\mathrm{d}\sigma\ \varphi_k(\rho)\,\mathrm{d}\rho \\
&\approx P_j^*(a)\beta_j(a)\sum_{k=1}^{2}\frac{1}{\mu_k+\gamma_k}\int_0^{a_M}\varphi_k(\rho)\,\mathrm{d}\rho,
\end{aligned}$$

where we used the approximation $\mathrm{e}^{-(\mu_j+\gamma_j)a_M} \approx 0$, $j = 1, 2$. Hence, the basic reproduction number $\mathcal{R}_0$ can be calculated as the following value, which is the



(a) Time variation of $I_1(t,a)$ (b) Time variation of $I_1(t,a)$



(c) Time variation of $I_2(t,a)$ (d) Time variation of $I_2(t,a)$

FIGURE 2. Time variation of $I_1(t,a)$ and $I_2(t,a)$ for $\beta_1(a) = 10.5\,(1+0.9\cos 3\pi a/2)$ (in this case, $\mathcal{R}_0 \approx 0.9627 < 1$)

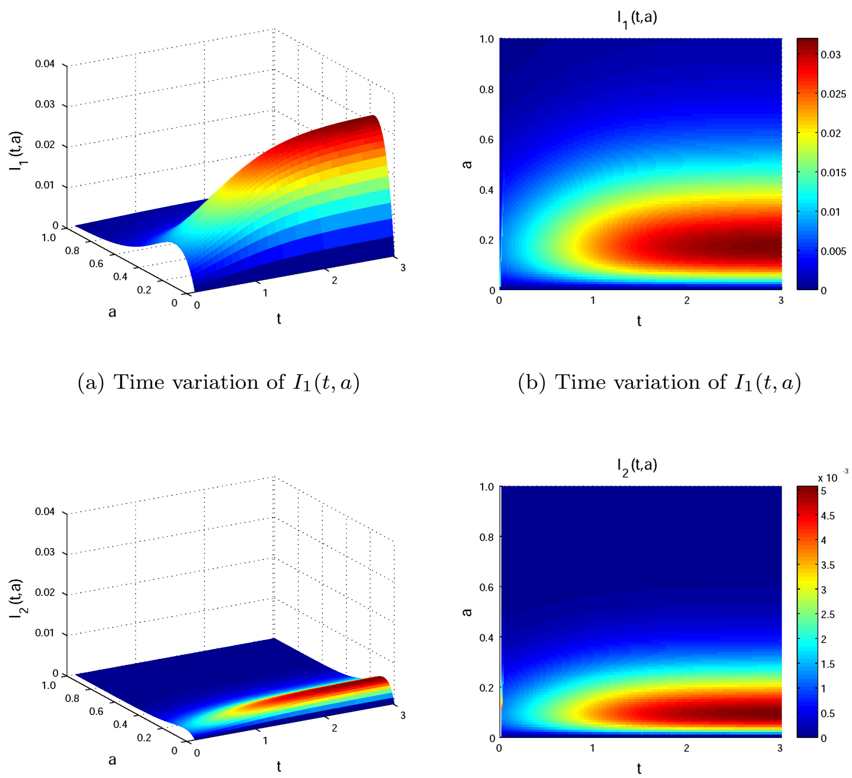
eigenvalue associated with the eigenvector $(P_1^*(a)\beta_1(a), P_2^*(a)\beta_2(a))$:

$$\mathcal{R}_0 = \sum_{k=1}^{2} \frac{1}{\mu_k + \gamma_k} \int_0^{a_M} P_k^*(\rho)\,\beta_k(\rho)\,\mathrm{d}\rho$$

For $\beta_1(a) = 10.5\,(1+0.9\cos 3\pi a/2)$, we have $\mathcal{R}_0 \approx 0.9627 < 1$. Hence, from Theorem 8.1 (i), we see that the disease-free equilibrium is globally asymptotically stable. In fact, in Figure 2, both of the infective individuals converge to zero.

On the other hand, for $\beta_1(a) = 11.5\,(1+0.9\cos 3\pi a/2)$, we have $\mathcal{R}_0 \approx 1.0465 > 1$. Hence, from Theorem 8.1 (iv), we see that the endemic equilibrium is globally asymptotically stable. In fact, in Figure 3, both of the infective individuals converge to the positive distributions.

10. **Discussion.** In this paper, we have formulated a multi-group SIR epidemic model (1.2) with age structure. For the reduced system (5.1), we have shown that the basic reproduction number $\mathcal{R}_0$ defined by (4.10) plays the role of a threshold value for the existence, uniqueness and asymptotic stability of each equilibrium.



(a) Time variation of $I_1(t,a)$

(b) Time variation of $I_1(t,a)$

(c) Time variation of $I_2(t,a)$

(d) Time variation of $I_2(t,a)$

FIGURE 3. Time variation of $I_1(t,a)$ and $I_2(t,a)$ for $\beta_1(a) = 11.5\,(1 + 0.9\cos 3\pi a/2)$ (in this case, $\mathcal{R}_0 \approx 1.0465 > 1$)

In this paper, we have mainly focused on the stability of each equilibrium. However, in other references, some irregular cases such as unstable endemic equilibrium (see Andreasen [1], Cha *et al.* [4] and Thieme [40]) or multiple endemic equilibria (see Franceschetti *et al.* [9]) when $\mathcal{R}_0 > 1$ have been observed for age-structured SIR epidemic models. Investigating such nontrivial properties of our model (1.2) has been left as an open problem.

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