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The central limit theorem for complex Riesz-Raikov sums

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Abstract

For complex Riesz-Raikov sums, the central limit theorem is proved. As a byproduct, metric discrepancy results are proved for complex geometric progressions.

Résumé

Le théorème limite central pour des sommes de Riesz-Raikov complexes Nous démontrons un théorème limite central pour les sommes de Riesz-Raikov complexes. En application de nos méthodes, nous établissons aussi des résultats de discrépance pour les progressions géométriques complexes.

Let $\mathbf{I} = \mathbf{Z}\bigl[\sqrt{-1}\,\bigr]$, $\mathbf{K} = \mathbf{Q}\bigl[\sqrt{-1}\,\bigr]$, and $D = \{z \in \mathbf{C} \mid \Re z, \Im z \in [\,0,1)\}$. We denote by μ the Lebesgue measure on $\mathbf{C}$. Let f be a locally square integrable real valued function on $\mathbf{C}$ satisfying

$$f(z+\sqrt{-1}\,) = f(z+1) = f(z), \qquad \int_D f(z)\,\mu(dz) = 0, \qquad \int_D f^2(z)\,\mu(dz) < \infty. \tag{1}$$

We denote the Fourier series of f by $\sum_{n\in\mathbf{I}^\times} \widehat{f}(n)\exp(2\pi\sqrt{-1}\,\Re(\overline{n}z))$. For a positive integer d, we put $f_d(z) = \sum_{n\in\mathbf{I}\colon |n|<d} \widehat{f}(n)\exp(2\pi\sqrt{-1}\,\Re(\overline{n}z))$ and $R(f,d) = \|f - f_d\|_2$. We assume the condition

$$R(f,d) = O\bigl((\log d)^{-1-\varepsilon}\bigr) \quad \text{for some} \quad \varepsilon > 0. \tag{2}$$

It is known that a function f of bounded variation in the sense of Hardy-Krause satisfies the condition $\bigl|\widehat{f}(n)\bigr| = O\bigl((|\Re n| \vee 1)^{-1}(|\Im n| \vee 1)^{-1}\bigr)$ (Cf. Zaremba [16]), which implies $R(f,d) = O(d^{-1})$ and (2).

Theorem 1 *Assume that a real valued function f on $\mathbf{C}$ satisfies (1) and (2), and that $\theta \in \mathbf{C}$ satisfies $|\theta| > 1$. Regarding $\sum f(\theta^k z)$ as a random variable on the probability space $(D, \mathcal{B}_D, \mu)$, we have*

$$\frac{1}{\sqrt{N}} \sum_{k=1}^{N} f(\theta^k z) \xrightarrow{\mathcal{D}} N(0, \sigma^2(\theta, f)). \tag{3}$$

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Here the limiting variance $\sigma^2(\theta, f)$ is given by

$$\sigma^2(\theta, f) = \int_D f^2(z)\,\mu(dz), \quad \text{or} \quad \sigma^2(\theta, f) = \int_D f^2(z)\,\mu(dz) + 2\sum_{k=1}^{\infty}\int_D f(p^k z) f(q^k z)\,\mu(dz), \tag{4}$$

according as $\theta^r \notin \mathbf{K}$ $(r = 1, 2, \dots)$ holds or not. In the second case p and $q \in \mathbf{I}$ are relatively irreducible and satisfy $\theta^r = p/q$, where r is the minimal positive integer satisfying $\theta^r \in \mathbf{K}$. In this case $\sigma^2(\theta, f) = 0$ holds if and only if there exists a function g satisfying (1) and

$$f(z) = g(pz) - g(qz). \tag{5}$$

For real Riesz-Raikov sums $\sum f(\theta^k x)$ with $\theta > 1$ and $x \in [\,0, 1)$, the results were proved by Kac [8], Petit [14], and [4]. Complex case with $\theta \in \mathbf{I}$ was studied by Leonov [10] and Conze-Le Borge-Roger [3].

We can also derive metric discrepancy results for complex geometric progressions $\{\theta^k z\}$ with $|\theta| > 1$. Relating results for real θ are proved in [5]. Denote $\langle z \rangle = \Re z - \lfloor \Re z \rfloor + (\Im z - \lfloor \Im z \rfloor)\sqrt{-1} \in D$. For $0 \le a < a' < 1$ and $0 \le b < b' < 1$, denote $\widetilde{\mathbf{1}}_{a,a',b,b'}(z) = \mathbf{1}_{[a,a')+\sqrt{-1}\,[b,b')}(\langle z \rangle) - (a' - a)(b' - b)$ and

$$D_N\{z_k\} = \sup_{a,a',b,b':0\le a<a'<1, 0\le b<b'<1} \frac{1}{N}\Big|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{a,a',b,b'}(z_k)\Big|.$$

Theorem 2 *For any $\theta \in \mathbf{C}$ with $|\theta| > 1$, we have the law of the iterated logarithm*

$$\varlimsup_{N\to\infty} \frac{N D_N\{\theta^k z\}}{\sqrt{2N \log\log N}} = \Sigma_\theta = \sup_{a,a',b,b':0\le a<a'<1, 0\le b<b'<1} \sigma(\theta, \widetilde{\mathbf{1}}_{a,a',b,b'}) \quad \mu\text{-a.e. } z. \tag{6}$$

Especially when $\theta^r \notin \mathbf{K}$ for all $r \in \mathbf{N}$, we have $\Sigma_\theta = \frac{1}{2}$.

1. Preliminary

Put $\mathbf{I}^\times = \mathbf{I} \setminus \{0\}$ and $\mathbf{I}^+ = \{p \in \mathbf{I} \mid \Im p > 0\} \cup \{p \in \mathbf{I} \mid \Im p = 0, \Re p > 0\}$. Let μ_0 be the probability measure on $\mathbf{C}$ given by $\mu_0(dz) = \frac{1}{\pi^2}\big(\frac{\sin \Re z}{\Re z}\big)^2\big(\frac{\sin \Im z}{\Im z}\big)^2 \mu(dz)$. There exists a $C_0 > 0$ such that

$$\mathbf{1}_D\,\mu(dz) \le (C_0/2)\mu_0(dz) \qquad (z \in \mathbf{C}) \tag{7}$$

Note that $\int_{\mathbf{C}} \exp\big(2\pi\sqrt{-1}\,\Re(\overline{w}z)\big)\,\mu_0(dz) = \big(1 - \pi|\Re w|\big)^+\big(1 - \pi|\Im w|\big)^+$, and it equals to zero if $|w| \ge 1/2$. Hence we have

$$\Big|\int_{\mathbf{C}} \Big(\sum_{k=1}^{N} f(\theta^k z)\Big)^2 \mu_0(dz)\Big| \le \sum_{k,l=1}^{N} \sum_{n,m\in\mathbf{I}^\times} \big|\widehat{f}(n)\widehat{f}(m)\big|\mathbf{1}(|\overline{n}\theta^k + \overline{m}\theta^l| < 1/2).$$

For $\zeta \in \mathbf{C}$, denote by $\varphi(\zeta)$ the $n \in \mathbf{I}$ such that $|\zeta + \overline{n}| < 1/2$ if it exists, and put $\varphi(\zeta) = \infty$ if such n does not exist. Put $\widehat{f}(\infty) = 0$. Suppose that $l \le k$ and $|\overline{n}\theta^k + \overline{m}\theta^l| < 1/2$ for some n and $m \in \mathbf{I}^\times$. Then $|\overline{n}\theta^{k-l} + \overline{m}| < 1/2$, and hence $m = \varphi(\overline{n}\theta^{k-l})$. Hence (7) implies $\big|\int_{\mathbf{D}}\big(\sum_{k=1}^{N} f(\theta^k z)\big)^2 \mu(dz)\big| \le C_0 \sum_{k=1}^{N}\sum_{l=1}^{k}\sum_{n\in\mathbf{I}^\times}\big|\widehat{f}(n)\widehat{f}(\varphi(\overline{n}\theta^{k-l}))\big|$. If $n \in \mathbf{I}_0^h = \{n \in \mathbf{I}^\times \mid |\theta|^h \le |n| < |\theta|^{h+1}\}$ then $|\theta|^{h+k-l} \le |\overline{n}\theta^{k-l}|$ and $|\theta|^{h+k-l-1} \le |\varphi(\overline{n}\theta^{k-l})|$. By $\sum_{n\in\mathbf{I}_0^\times} = \sum_{h=0}^{\infty}\sum_{n\in\mathbf{I}_0^h}$, we see that

$$\sum_{n\in\mathbf{I}^\times} \big|\widehat{f}(n)\widehat{f}(\varphi(\overline{n}\theta^{k-l}))\big| \le \sum_h \Big(\sum_{n\in\mathbf{I}_0^h} \big|\widehat{f}(n)\big|^2\Big)^{1/2}\Big(\sum_{n\in\mathbf{I}_0^h} \big|\widehat{f}(\varphi(\overline{n}\theta^{k-l}))\big|^2\Big)^{1/2} \le \sum_h R(f, |\theta|^h) R(f, |\theta|^{h+k-l-1})$$

, since $n \mapsto \varphi(\overline{n}\theta^{k-l})$ is injective on $\mathbf{I}_0^h \cap \{n \in \mathbf{I}^\times \mid \varphi(\overline{n}\theta^{k-l}) \neq \infty\}$. Hence we have

$$\left|\int_{\mathbf{D}} \Big(\sum_{k=1}^N f(\theta^k z)\Big)^2 \mu(dz)\right| \le C_0 \sum_{k=1}^N \sum_{l=1}^k \sum_{h=0}^\infty R(f,|\theta|^h) R(f,|\theta|^{h+k-l-1}) \le C_0 N \Big(\sum_{h=0}^\infty R(f,|\theta|^{h-1})\Big)^2. \quad (8)$$

By $(|\Re z| \vee 1)(|\Im z| \vee 1) \ge |z|/\sqrt{2}$, for any $w \in \mathbf{C}$ we have $\left|\int_{D+w} \exp\big(2\pi\sqrt{-1}\,\Re(\overline{n}\lambda z)\big)\,\mu(dz)\right| \le 4/\big(\Re(n\overline{\lambda}) \vee 1\big)\big(\Im(n\overline{\lambda}) \vee 1\big) \le 4\sqrt{2}/|n\lambda|$. Hence we have the following two lemmas for a trigonometric polynomial $h(z) = \sum_{n \in \mathbf{I}^\times : |n| \le d} c_n \exp(2\pi\sqrt{-1}\,\Re(\overline{n}z))$.

Lemma 1.1 *For $|\lambda| \ge 1$ and $w \in \mathbf{C}$, $\left|\int_{D+w} h(\lambda z)\,\mu(dz)\right| \le (4\sqrt{2}/|\lambda|) \sum_{n \in \mathbf{I}^\times : |n| \le d} |c_n|/|n| = O\big(1/|\lambda|\big)$.*

Lemma 1.2 *Suppose that a sequence $\{\lambda_k\}$ of complex numbers satisfies the modulus Hadamard'gap condition $|\lambda_{k+1}/\lambda_k| \ge Q > 1$. Then there exists a constant C_Q depending only on Q such that*

$$\int_D \Big(\max_{l=1}^N \sum_{k=M+1}^{M+l} h(\lambda_k z)\Big)^4 \mu(dz) \le C_Q \Big(\sum_{n \in \mathbf{I}^\times : |n| \le d} |c_n|\Big)^4 N^2$$

Proof: By applying the triangle inequality for L^4-norm, we see that it is sufficient to prove for $h(z) = \cos(2\pi\Re(\overline{j}z) + \gamma_j)$. By dividing into subsequences, we see that it is sufficient to prove under the condition $Q \ge 3$. In this case, $\{\cos(2\pi\Re(\overline{j}\lambda_k z) + \gamma_j)\}_k$ forms a multiplicative system under the measure μ_0, and the above inequality with respect to measure μ_0 is already proved as a combination of the main lemma of Komlós-Révész [9] and the Erdős-Stečkin result [13]. Thanks to (7), we can conclude the proof. □

Lemma 1.3 *There exists a positive constant $C_{\theta,d}$ such that $|\overline{n}\theta^k + \overline{m}\theta^l| \ge C_{\theta,d}|\theta^{k\wedge l}|$ for any k, $l \in \mathbf{N}$, n, $m \in \mathbf{I}^\times$ with $|n|$, $|m| \le d$ and $\overline{n}\theta^k + \overline{m}\theta^l \neq 0$.*

Proof: Assume $k \ge l$. Because of $\lim_{j\to\infty} |\overline{n}\theta^j + \overline{m}| = \infty$, $D_{m,n} = \inf\{|\overline{n}\theta^j + \overline{m}| \mid j \ge 0, |\overline{n}\theta^j + \overline{m}| \neq 0\}$ is positive. Hence $|\overline{n}\theta^k + \overline{m}\theta^l| \ge D_{m,n}|\theta^l|$ provided that the left hand side is not zero. □

We denote $\int_{\mathbf{C}} g(z)\,\mu_R(dz) = \lim_{T\to\infty} \frac{1}{4T^2} \int_{[-T,T]+\sqrt{-1}[-T,T]} g(z)\,\mu(dz)$ if the limit exists. Now we verify (9) and (10) below. Since they are clear when $\theta^r \notin \mathbf{K}$ ($r = 1, 2, \dots$) we consider the other case. Note that $\left|\int_D f(p^k z) f(q^k z)\,\mu(dz)\right| = \left|\sum_{l \in \mathbf{I}^\times} \widehat{f}(l(\overline{q})^k)\widehat{f}(-l(\overline{p})^k)\right| \le R(f,|q|^k)R(f,|p|^k)$, which is summable in k. Hence,

$$\sigma^2(\theta, f_d) \to \sigma^2(\theta, f) \quad (d \to \infty). \quad (9)$$

For a trigonometric polynomial h satisfying (1) and $\lambda \in \mathbf{C}^\times$, we have $\int_{\mathbf{C}} h(\Theta\lambda z)h(\lambda z)\,\mu_R(dz) = 0$ if $\Theta \notin \mathbf{K}$, and $\int_{\mathbf{C}} h((P/Q)\lambda z)h(\lambda z)\,\mu_R(dz) = \int_D h(Pz)h(Qz)\,\mu(dz)$ if P, $Q \in \mathbf{I}^\times$. Hence we have

$$\int_{\mathbf{C}} \Big(\sum_{k=M+1}^{M+N} h(\theta^k z)\Big)^2 \mu_R(dz) = N\int_D h(z)^2\,\mu(dz) + 2\sum_{l=1}^\infty (N - lr)^+ \int_D h(p^l z)h(q^l z)\,\mu(dz).$$

By noting that $\int_D h(p^l z)h(q^l z)\,\mu(dz) = 0$ for large l and that $0 \le N - (N - lr)^+ \le lr$, we have

$$\int_{\mathbf{C}} \Big(\sum_{k=M+1}^{M+N} h(\theta^k z)\Big)^2 \mu_R(dz) = N\sigma^2(\theta, h) + O(1). \quad (10)$$

2. Coboundary

When $\theta^r = p/q \in \mathbf{I}$, we may assume $q = 1$. Put $\Pi(p,q) = \{w \in \mathbf{I} \mid \overline{p} \nmid w,\ \overline{q} \nmid w\}$ when $q \neq 1$, $\Pi(p,1) = \{w \in \mathbf{I} \mid \overline{p} \nmid w\}$, and $\Pi(p,q)^+ = \Pi(p,q) \cap \mathbf{I}^+$. We have $\sigma^2(\theta,f) = \sum_{n \in \mathbf{I}^\times} |\widehat{f}(n)|^2 + 2\sum_{k=1}^\infty \sum_{n \in \mathbf{I}^\times} \sum_{m \in \mathbf{I}^\times} \widehat{f}(n)\widehat{f}(m)\mathbf{1}(\overline{n}p^k + \overline{m}q^k = 0)$, and by the derivation of (9) we see that this series is absolutely convergent. If we write $n = \overline{p}^s\overline{q}^t w$ and $m = \overline{p}^u\overline{q}^v w'$ by using s, t, u, $v = 0, 1, 2, \dots$, and w, $w' \in \Pi(p,q)$, we have

$$\sigma^2(\theta,f) = 2\sum_{w\in\Pi(p,q)^+}\sum_{l=0}^{\infty}\Big|\sum_{s=0}^{l}\widehat{f}(\overline{p}^s\overline{q}^{l-s}w)\Big|^2 \quad (q\neq 1),\quad \sigma^2(\theta,f) = 2\sum_{w\in\Pi(p,1)^+}\Big|\sum_{s=0}^{\infty}\widehat{f}(\overline{p}^s w)\Big|^2 \quad (q=1)$$

If $\sigma^2(\theta,f)=0$ and $q\neq 1$, then $\sum_{s=0}^{l}\widehat{f}(\overline{p}^s\overline{q}^{l-s}w)=0$ for $l=0, 1, 2, \dots$ and $w\in\Pi(p,q)^+$, and hence for $w\in\Pi(p,q)$. Put $\widehat{g}(\overline{p}^s\overline{q}^{l-s}w) = -\sum_{j=0}^{s}\widehat{f}(\overline{p}^j\overline{q}^{l+1-j}w) = \sum_{j=0}^{\infty}\widehat{f}(\overline{p}^{s+1+j}\overline{q}^{l-s-j}w)$, where we use convention $\widehat{f}(n)=0$ for $n\notin\mathbf{I}$. Thanks to Schwarz inequality, we have

$$\Big(\sum_{w\in\Pi(p,q)}\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}\Big|\sum_{j=0}^{\infty}\widehat{f}(\overline{p}^{s+1+j}\overline{q}^{l-s-j}w)\Big|^2\Big)^{1/2} \le \sum_{j=0}^{\infty}\Big(\sum_{w\in\Pi(p,q)}\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}\big|\widehat{f}(\overline{p}^{s+1+j}\overline{q}^{l-s-j}w)\big|^2\Big)^{1/2},$$

which implies $\big(\sum_{w\in\Pi(p,q)}\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}\big|\widehat{g}(\overline{p}^s\overline{q}^{l-s}w)\big|^2\big)^{1/2} \le \sum_{j=0}^{\infty}R(f,|p|^j) < \infty$. Hence $\{\widehat{g}(n)\}_{n\in\mathbf{I}}\in\ell^2$ and $g(z)=\sum_{n\in\mathbf{I}^\times}\widehat{g}(n)\exp\big(2\pi\sqrt{-1}\,\Re(\overline{n}z)\big)\in L^2$. We can verify (5) by comparing the Fourier coefficients.

If $\sigma^2(\theta,f)=0$ and $q=1$, then $\sum_{s=0}^{\infty}\widehat{f}(\overline{p}^s w)=0$ for $w\in\Pi(p,1)$. By putting $\widehat{g}(\overline{p}^s w) = -\sum_{j=0}^{s}\widehat{f}(\overline{p}^j w) = \sum_{j=0}^{\infty}\widehat{f}(\overline{p}^{s+1+j}w)$, we see $\{\widehat{g}(n)\}\in\ell^2$ and $g(z)=\sum_{n\in\mathbf{I}}\widehat{g}(n)\exp\big(2\pi\sqrt{-1}\,\Re(\overline{n}z)\big)\in L^2$.

3. Almost sure invariance principle

We follow the method of Aistleitner [1], which originated with Berkes [2] and Philipp [15].

Proposition 3 *Let h be a trigonometric polynomial satisfying (1). By enlarging the probability space, we can define a standard gaussian i.i.d. $\{Z_i\}$ such that $\sum_{k=1}^{N}h(\theta^k z) = \sum_{k\le N\sigma^2(\theta,h)}Z_k + O(N^{124/250})$ a.s.*

Proof: If $\sigma^2(\theta,h)=0$, then h can be expressed as (5) and the sum is a telescoping sum. Thus we have $\sum_{k=1}^{N}h(\theta^k z)=O(1)$ and the above conclusion is clear. We assume $\sigma^2(\theta,h)>0$.

Divide $\mathbf{N}$ into consecutive blocks Δ_1', Δ_1, Δ_2', Δ_2, $\dots$ with $^\#\Delta_i' = \lfloor 1+9\log_{|\theta|}i\rfloor$ and $^\#\Delta_i = i$. Put $i^- = \min\Delta_i$ and $i^+ = \max\Delta_i$. We see $9\log_{|\theta|}i \le {^\#\Delta_i'} = i^- - (i-1)^+ - 1$ and $|\theta^{i^-}/\theta^{(i-1)^+}| > i^9$.

Denote $\rho(i) = \lceil\log_2 i^4|\theta^{i^+}|\rceil$. Let $\mathcal{F}_i$ be a σ-field on D generated by intervals $J_{i,j,j'} = \{z\in D \mid j2^{-\rho(i)} \le \Re z < (j+1)2^{-\rho(i)},\ j'2^{-\rho(i)} \le \Im z < (j'+1)2^{-\rho(i)}\}$ (j, $j' = 0, \dots, 2^{\rho(i)}-1$). Note $i^4|\theta^{i^+}| \le 2^{\rho(i)} \le 2i^4|\theta^{i^+}|$ and put $T_i(z) = \sum_{k\in\Delta_i}h(\theta^k z)$, $Y_i = E(T_i \mid \mathcal{F}_i) - E(T_i \mid \mathcal{F}_{i-1})$. Clearly $\{Y_i,\mathcal{F}_i\}$ forms a martingale difference sequence, i.e., $E(Y_i \mid \mathcal{F}_{i-1}) = 0$. We show

$$\|Y_i - T_i\|_\infty = O(i^{-3}),\quad \|Y_i^2 - T_i^2\|_\infty = O(i^{-2}),\quad \|Y_i^4 - T_i^4\|_\infty = O(1),\quad EY_i^4 = O(i^2), \tag{11}$$

where the implied constants are depending only on h.

Let $k\in\Delta_i$ and $z\in J = J_{i,j,j'}$. We have $|h(\theta^k z) - E(h(\theta^k\,\cdot)\mid\mathcal{F}_i)| = \big|\frac{1}{\mu(J)}\int_J(h(\theta^k z)-h(\theta^k\zeta))\,\mu(d\zeta)\big| \le \max_{\zeta\in J}|h(\theta^k z)-h(\theta^k\zeta)| = O(|\theta^k|2^{-\rho(i)}) = O(|\theta^k|/|\theta^{i^+}|i^4) = O(i^{-4})$. It implies $T_i - E(T_i\mid\mathcal{F}_i) = O(i^{-3})$. On $J = J_{i-1,j,j'}$ we have $E(h(\theta^k\,\cdot\,)\mid\mathcal{F}_{i-1}) = \frac{1}{\mu(J)}\int_J h(\theta^k z)\,\mu(dz)$. Changing variable by $2^{\rho(i-1)}z=\zeta$ and by applying Lemma 1.1, we have $E(h(\theta^k\,\cdot)\mid\mathcal{F}_{i-1}) = \int_{D+w}h(\theta^k 2^{-\rho(i-1)}\zeta)\,\mu(d\zeta) = O(2^{\rho(i-1)}/|\theta^k|) = O(2(i-1)^4|\theta^{(i-1)^+}/\theta^{i^-}|) = O(i^{-5})$. Hence $E(T_i\mid\mathcal{F}_{i-1}) = O(i^{-4})$. By combining these we have the first estimate of (11). By $\|T_i\|_\infty = O(i)$, we have $\|Y_i\|_\infty = O(i)$. By $\|T_i^2 - Y_i^2\|_\infty \le \|T_i - Y_i\|_\infty\|T_i + Y_i\|_\infty$, we have the second. The third is proved similarily. The third and Lemma 1.2 implies the fourth.

Put $v_i = \int_{\mathbf{C}}T_i^2(z)\,\mu_R(dz)$, $l_M = \sum_{i=1}^{M}{^\#\Delta_i}$, $\beta_M = \sum_{i=1}^{M}v_i$, and $V_M = \sum_{i=1}^{M}E(Y_i^2\mid\mathcal{F}_{i-1})$. We show

$$\beta_M = \sigma^2(\theta,h)l_M + O(l_M^{1/2}),\qquad \beta_M\sim\sigma^2(\theta,h)l_M,\qquad \|V_M-\beta_M\|_\infty = O(1). \tag{12}$$

The first and the second is clear from (10), and we show the third. Express $T_i^2 - v_i$ as a linear combination of $\exp(2\pi\sqrt{-1}\,\Re((\overline{n}\theta^k + \overline{m}\theta^l)z))$ where $|n|$, $|m| \le d$ and k, $l \in \Delta_i$. Since $\overline{n}\theta^k + \overline{m}\theta^l \ne 0$ in this expression, we see by Lemma 1.3 that $|\overline{n}\theta^k + \overline{m}\theta^l| \ge C_{\theta,d}|\theta^{i^-}|$. Changing variable by $2^{\rho(i-1)}z = \zeta$, and by applying Lemma 1.1, we have $E(T_i^2 - v_i \mid \mathcal{F}_{i-1}) = \int_{D+w}(T_i^2(2^{-\rho(i-1)}\zeta) - v_i)\mu(d\zeta) = O(i^2 2^{\rho(i-1)}/|\theta^{i^-}|) = O(1/i^3)$ and $\| \sum_{i=1}^M E(T_i^2 \mid \mathcal{F}_{i-1}) - \beta_M\|_\infty = O(1)$. By (11), we have $\| \sum_{i=1}^M E(T_i^2 \mid \mathcal{F}_{i-1}) - V_M\|_\infty = O(1)$.

We use the following version of Strassen's Theorem.

Theorem 4 (Monrad-Philipp [12]) *Suppose that a square integrable martingale difference sequence $\{Y_i, \mathcal{F}_i\}$ satisfies $V_M = \sum_{i=1}^M E(Y_i^2 \mid \mathcal{F}_{i-1}) \to \infty$ a.s. and $\sum_{i=1}^\infty E\big(Y_i^2 \mathbf{1}_{\{Y_i^2 \ge \psi(V_i)\}}/\psi(V_i)\big) < \infty$ for some non-decreasing function ψ with $\psi(x) \to \infty$ $(x \to \infty)$ such that $\psi(x)(\log x)^\alpha/x$ is non-increasing for some $\alpha > 50$. If there exists a uniformly distributed random variable U which is independent of $\{Y_n\}$, there exists a standard normal i.i.d. $\{Z_i\}$ such that $\sum_{i\ge1} Y_i \mathbf{1}_{\{V_i \le t\}} = \sum_{i\le t} Z_i + o\big(t^{1/2}(\psi(t)/t)^{1/50}\big)$, a.s.*

Put $\psi(x) = x^{4/5}$. By $V_i \ge Ci^2$, we see $E\big(Y_i^2 \mathbf{1}_{\{Y_i^2 \ge \psi(V_i)\}}/\psi(V_i)\big) \le EY_i^4/\psi(Ci^2)^2 = O(i^{-6/5})$ is summable in i. Let us take a constant C_0 satisfying $\|V_M - \beta_M\|_\infty < C_0$. By $\beta_{M+1} - \beta_M = v_M \to \infty$, we see $V_M < \beta_M + C_0 < \beta_{M+1} - C_0 < V_{M+1}$. By putting $t = \beta_M + C_0$, Theorem 4 implies $\sum_{i=1}^M Y_i = \sum_{i\le\beta_M+C_0} Z_i + o(\beta_M^{124/250})$, a.s. By (11), the left hand side can be replaced by $\sum_{i=1}^M T_i$.

Put $\Delta''_M = \Delta'_1 \cup \cdots \cup \Delta'_M$. By Lemma 1.2, we have $E\big(l_M^{-2/5} \max_{l\in\Delta_M} \sum_{k\in\Delta_M:k\le l} h(\theta^k\,\cdot)\big)^4 = O(M^{-6/5})$ and $E\big(l_M^{-2/5} \max_{l\in\Delta''_M} \sum_{k\in\Delta''_M:k\le l} h(\theta^k\,\cdot\,)\big)^4 = O((\log M)^2 M^{-6/5})$. Since these are summable in M, by Beppo-Levi lemma, we have $\max_{l\in\Delta_M} \sum_{k\in\Delta_M:k\le l} h(\theta^k z) = o(l_M^{2/5})$ and $\max_{l\in\Delta''_M} \sum_{k\in\Delta''_M:k\le l} h(\theta^k z) = o(l_M^{2/5})$. Therefore we have $\sum_{k=1}^N h(\theta^k z) = \sum_{i\le\beta_M+C_0} Z_i + o(N^{124/250})$, a.s. for $N \in \Delta'_M \cup \Delta_M$. By $\beta_M + C_0 = \sigma^2(\theta,h)l_M + O(l_M^{1/2})$, ${}^\#(\Delta'_M \cup \Delta_M) \le 2M$, $M^+ = l_M + O(M\log M)$, we see $|\beta_M + C_0 - \sigma^2(\theta,h)N| \le KM\log M$ for some $K > 0$. Hence

$$P\Bigg(\max_{N\in\Delta'_M\cup\Delta_M}\Bigg|\sum_{i\le\beta_M+C_0} Z_i - \sum_{i\le\sigma^2(\theta,h)N} Z_i\Bigg| \ge \sqrt{4KM}\log M\Bigg) \le 2P(|N_{0,1}| \ge \sqrt{4\log M}) \le 4M^{-2}.$$

Since it is summable in M, we see $\max_{N\in\Delta'_M\cup\Delta_M}\big|\sum_{i\le\beta_M+C_0} Z_i - \sum_{i\le\sigma^2(\theta,h)N} Z_i\big| \le \sqrt{4KM}\log M = o(l_M^{2/5})$ for large M, a.s. By these, we have the conclusion. □

4. The central limit theorem and the metric discrepancy results

By Proposition 3, we see that the law of $\frac{1}{\sqrt{N}}\sum_{k=1}^N f_d(\theta^k z)$ converges weakly to $N(0,\sigma^2(\theta,f_d))$. By (9) and (8), we can take d such that $|\sigma^2(\theta,f_d) - \sigma^2(\theta,f)| < \varepsilon$ and $\int_D\big(\frac{1}{\sqrt{N}}\sum_{k=1}^N (f - f_d)(\theta^k z)\big)^2 \mu(dz) < \varepsilon$ for $N \ge 1$. By combining these, we can prove (3). By Proposition 3, we have

$$\varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \sum_{k=1}^N h(\theta^k z) = \sigma(\theta,h) \quad \mu\text{-a.e. } z. \tag{13}$$

for any trigonometric polynomial h satisfying (1). To prove (6), we use the following result. The one dimensional version is proved in [5,6,7]. The following version can be proved in the same way as [11].

Proposition 4.1 *Let $\{\lambda_k\}$ be a sequence of complex numbers satisfying the gap condition $|\lambda_{k+1}/\lambda_k| > Q > 1$. Then for any dense countable set $S \subset [0,1)$, we have*

$$\varlimsup_{N\to\infty} \frac{ND_N\{\lambda_k z\}}{\sqrt{2N\log\log N}} = \sup_{a,a',b,b'\in S:0\le a<a'\le 1,0\le b<b'\le 1} \lim_{d\to\infty} \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}}\Bigg|\sum_{k=1}^N \widetilde{\mathbf{1}}_{a,a',b,b',d}(\lambda_k z)\Bigg| \ a.e.\ z.$$

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