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AN ASYMPTOTIC PROPERTY OF GAP SERIES III

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1. Introduction

Let f be an $\mathbf{R}$-valued function on $\mathbf{R}$ satisfying

$$f(x+1) = f(x), \quad \int_0^1 f(x)\,dx = 0, \quad \|f\|^2 = \int_0^1 |f(x)|^2\,dx < \infty,$$

with Fourier series $f(x) \sim \sum_{\nu=1}^{\infty}(a_\nu \cos 2\pi\nu x + b_\nu \sin 2\pi\nu x)$. Denote $\|f\|_A = \sum_{\nu=1}^{\infty}\sqrt{a_\nu^2 + b_\nu^2}$. We investigate the distribution of values of

$$\Xi_f(x) = \varlimsup_{N\to\infty} \frac{1}{\sqrt{N \log\log N}} \sum_{k=1}^{N} f(n_k x)$$

where $\{n_k\}$ is a strictly increasing sequence of positive integers. For a brief survey on studies on Ξ_f, we refer the reader to our previous paper [1].

Takahashi [4] gave the concrete bound $\Xi_f \le \|f\|_A$ a.e. assuming $f \in \operatorname{Lip}\alpha$ $(\alpha > 1/2)$ and Takahashi's gap condition:

$$n_{k+1}/n_k \ge 1 + ck^{-\beta} \qquad (c > 0,\ \beta < 1/2).$$

In [1] we proved the following.

Theorem A.

(1) Takahashi's gap condition and $\|f\|_A < \infty$ imply $\Xi_f \le \|f\|_A$ a.e.

(2) Let us say that f has parallel Fourier coefficients if there exist $a, b \in \mathbf{R}$ with $a^2 + b^2 = 1$ such that $a_\nu = a\sqrt{a_\nu^2 + b_\nu^2}$ and $b_\nu = b\sqrt{a_\nu^2 + b_\nu^2}$ for all $\nu \in \mathbf{N}$. If $\|f\|_A < \infty$ and the Fourier coefficients of f are parallel, then for every $\varepsilon > 0$ there exist $\rho > 0$ and $\{n_k\}$ such that $n_{k+1}/n_k > 1 + \rho$ and $\Xi_f \ge \|f\|_A - \varepsilon$ a.e.

In this paper we consider the problem if there exists a sequence $\{n_k\}$ for which $\Xi_f = \|f\|_A$ holds, or in other words, if the bound $\|f\|_A$ can be achieved as the law of the iterated logarithm. Our main theorem below gives the affirmative answer to this question.

Theorem. *For any sequence $\{\rho_k\}$ of positive numbers converging to 0, there exists a sequence $\{n_k\}$ such that $n_{k+1}/n_k \ge 1 + \rho_k$ and $\Xi_f = \|f\|_A$ a.e. for arbitrary function f with parallel Fourier coefficients and $\|f\|_A < \infty$.*

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2. Construction of the sequence

We prove our theorem by assuming that $\{\rho_k\}$ monotonously decreases to 0 and satisfies $1 \geq \rho_k \geq k^{-1/3}$. Once we have proved under these assumptions, we can derive the general case in the following way.

Put $\rho'_k = \max\{k^{-1/3}, \min\{1, \sup_{i \geq k} \rho_i\}\}$. Then $\{\rho'_k\}$ monotonously decreases to 0 and satisfies $1 \geq \rho'_k \geq k^{-1/3}$. By $\rho_k \to 0$, there exists a K such that $\rho_k \leq 1$ for all $k \geq K$, and hence $\rho'_k \geq \rho_k$ for $k \geq K$. Firstly construct a sequence $\{n'_k\}$ satisfying $n'_{k+1}/n'_k \geq 1 + \rho'_k$ and $\Xi_f = \|f\|_A$ a.e. Secondly, take $n_1, \ldots, n_K$ with $n_{k+1}/n_k \geq 1 + \rho_k$ $(k = 1, \ldots, K-1)$, take $M \in \mathbf{N}$ large enough to satisfy $Mn'_{K+1}/n_K \geq 1 + \rho_K$, and put $n_k = Mn'_k$ for $k \geq K+1$. For $k \geq K+1$, we have $n_{k+1}/n_k = n'_{k+1}/n'_k \geq 1 + \rho'_k \geq 1 + \rho_k$ and hence the gap condition is satisfied for all k. By change of variable, we see that $\Xi_f = \|f\|_A$ holds for $\{Mn'_k\}$, and hence for $\{n_k\}$ because Ξ_f does not depend on finitely many terms. Therefore we verified the general case.

Let $p_0 = 2$, $p_1 = 3$, $p_2 = 5$, ... be the sequence consisting of whole prime numbers. Put

$$P(l, m, n) = \{p_0^{i_0} p_1^{i_1} \ldots p_m^{i_m} \mid (l-1)^2 \leq i_0 < l^2, 0 \leq i_1, \ldots, i_m < n\}$$

$$\Pi(\lambda, \Lambda, j) = \bigcup_{l=\lambda}^{\Lambda} P(l, j, 2j) \quad \text{and} \quad \gamma(j) = \inf\Big\{\frac{n}{m} \;\Big|\; n > m, n, m \in \Pi(0, \infty, j)\Big\}.$$

We see that $2 = \gamma(0) \geq \gamma(1) \geq \gamma(2) \geq \cdots$.

Proposition. *There exists a non-decreasing divergent sequence $\{\widetilde{J}(l)\}$ of non-negative integers such that the arrangement $\{\nu_k\}$ in increasing order of $\bigcup_{l=1}^{\infty} P(l, \widetilde{J}(l), 2\widetilde{J}(l))$ satisfies $\nu_{k+1}/\nu_k \geq 1 + \rho_k$.*

Proof: Firstly, put $l_0 = 0$ and take k_1 such that $\gamma(1) \geq 1 + \rho_{k_1}$. Let us arrange $\Pi(l_0, \infty, 0)$ in increasing order and denote the first k_1 numbers by ν_1, ..., ν_{k_1}. Let us take l_1 large enough such that $\nu_1, \ldots, \nu_{k_1} \in \Pi(l_0, l_1 - 1, 0)$. If $n \in \Pi(l_0, l_1 - 1, 0)$ and $m \in \Pi(l_1, \infty, 1) \setminus \Pi(l_0, \infty, 0)$, then $n < m$ holds. Thus $\nu_1, \ldots, \nu_{k_1}$ are the least k_1 numbers in $\Pi(l_0, l_1 - 1, 0) \cup \Pi(l_1, \infty, 1)$.

Secondly, take k_2 as $\gamma(2) \geq 1 + \rho_{k_2}$ and arrange $\Pi(l_0, l_1 - 1, 0) \cup \Pi(l_1, \infty, 1)$ in increasing order. The first k_1 numbers are $\nu_1, \ldots, \nu_{k_1}$. Denote the next $k_2 - k_1$ numbers by $\nu_{k_1+1}, \ldots, \nu_{k_2}$. Let us take l_2 large enough such that $\nu_1, \ldots, \nu_{k_2} \in \Pi(l_0, l_1 - 1, 0) \cup \Pi(l_1, l_2 - 1, 1)$. Then $\nu_1, \ldots, \nu_{k_2}$ are the least k_2 numbers in $\Pi(l_0, l_1 - 1, 0) \cup \Pi(l_1, l_2 - 1, 1) \cup \Pi(l_2, \infty, 2)$.

In this way, we take $l_0 < l_1 < l_2 < \cdots$ and $k_1 < k_2 < \cdots$ such that the arrangement $\{\nu_k\}$ in increasing order of $\bigcup_{i=0}^{\infty} \Pi(l_i, l_{i+1} - 1, i)$ satisfies $\nu_k \in \Pi(0, \infty, i)$ for $k \leq k_{i+1}$.

Let us verify the gap condition on this sequence. For any k, let us take i such that $k_i \leq k < k_{i+1}$. Then we have $\nu_k, \nu_{k+1} \in \Pi(0, \infty, i)$ and hence $\nu_{k+1}/\nu_k \geq \gamma(i) \geq 1 + \rho_{k_i} \geq 1 + \rho_k$.

For $l_i \leq l < l_{i+1}$, put $\widetilde{J}(l) = i$. Then $\bigcup_{i=0}^{\infty} \Pi(l_i, l_{i+1}-1, i) = \bigcup_{i=0}^{\infty} \Pi(l_i, l_{i+1}-1, i)$ and hence $\{\nu_k\}$ is a desired sequence. ■

Let us take a sequence $\{J(l)\}$ of non-negative integers such that

$$J(l) \leq J(l+1), \quad J(l) \leq \frac{1}{2}\log_2 l, \quad J(l) \leq \widetilde{J}(l), \quad J(l)^{J(l)} \leq l^{1/2}, \quad J(l) \to \infty,$$

where we interpret 0^0 as 1. Let us put $D(L) = \sum_{l=1}^{L}(2l-1)J(l)^{J(l)}$. Thanks to $LJ(L)^{J(L)} \leq L^{3/2}$ and $D(L) \geq J(1)^{J(1)}L^2$, we have $(2L-1)J(L)^{J(L)} = o(D(L))$ and $D(L-1) \sim D(L)$.

Now we are in a position to construct our sequence $\{n_k\}$ which satisfies $\Xi_f = \|f\|_A$. Let $\{n_k\}$ be an arrangement in increasing order of $\bigcup_{l=1}^{\infty} P(l, J(l), J(l))$. Since it is a subsequence of $\{\nu_k\}$, it satisfies $n_{k+1}/n_k \geq 1+\rho_k$.

Let us take a positive integer I arbitrarily, and let $\{m_k\}$ be an arrangement in increasing order of the set theoretical union $\{n_k\}\cup\{2n_k\}\cup\ldots\cup\{In_k\}$. Given K, let $\widehat{K}$ denote the integer k such that $n_K = m_k$.

Lemma.

(0) $\{m_k\}$ *satisfies* $m_{k+1}/m_k \geq 1 + ck^{-1/3}$ *for some* $c > 0$.

(1) ${}^{\#}\{k \mid n_K \leq m_k < n_{K+1}\} = O(K^{1/3})$.

(2) $L_K = {}^{\#}\{k \leq K \mid In_k \geq n_K\} = O(K^{1/3})$.

(3) $K \leq \widehat{K} \leq IK$

Proof: (0) We prove that $\{m_k\}$ is a subsequence of $\{\nu_k\}$ except for finitely many terms, which implies $m_{k+1}/m_k \geq 1+\rho_{k+c} \geq 1+c'k^{-1/3}$ for some $c \in \mathbf{N}$ and $c' > 0$. Since $J(l)$ is non-decreasing, by definition, we see that $2\{n_k\} \subset \{n_k\}$. Thus it is sufficient to prove $on_k \in \{\nu_k\}$ for all odd numbers o less than I, except for finitely many k. If l is large enough, then o is a factor of $p_1^{J(l)} \ldots p_{J(l)}^{J(l)}$ and hence $oP(l, J(l), J(l)) \subset P(l, J(l), 2J(l)) \subset P(l, \widetilde{J}(l), 2\widetilde{J}(l))$. Thus on_k belongs to $\{\nu_k\}$ except for finitely many k.

(1) Let $i = 1, \ldots, I$, and let k' and k'' be the minimum and maximum of k for which $n_K \leq in_k < n_{K+1}$ holds. Thanks to $n_{K+1}/n_K \leq 2$ and $k'' \leq K$, we have

$$2 \geq \frac{n_{K+1}}{n_K} \geq \frac{in_{k''}}{in_{k'}} \geq \prod_{k=k'}^{k''-1}(1+ck^{-1/3}) \geq (k''-k')cK^{-1/3}.$$

(2) Denote k' the minimum of $k \leq K$ satisfying $In_k \geq n_K$. Then we have

$$I \geq \frac{n_K}{n_{k'}} \geq \prod_{k=k'}^{K-1}(1+ck^{-1/3}) \geq (K-k')cK^{-1/3}.$$

(3) Since $n_1, \ldots, n_K \le n_K$, there exist at least K many m_k such that $m_k \le n_K$. Thus $K \le \widehat{K}$. If $1 \le i \le I$ and $in_j \le n_K$, then $j \le K$. Thus there exist at most IK many k such that $m_k \le n_K$, and thereby $\widehat{K} \le IK$.

3. Limiting variance

In this section we assume that f has parallel Fourier coefficients and prove $\int_0^1 (\sum_{k=1}^K f(n_k x)) \sim K\|f\|_A^2/2$. We prepare notation. Denote $\mathbf{Z}^{*n} = \mathbf{Z}^n \times \{0\} \times \{0\} \times \cdots$, $\mathbf{Z}^{*\infty} = \mathbf{Z}^{*1} \cup \mathbf{Z}^{*2} \cup \ldots$, and $\phi(j) = \max\{j, 0\}$. For $\mathbf{j} = (j_0, j_1, \ldots) \in \mathbf{Z}^{*\infty}$, put $\mathbf{p}^{\mathbf{j}} = p_0^{j_0} p_1^{j_1} \ldots$, $\phi(\mathbf{j}) = (\phi(j_0), \phi(j_1), \ldots)$, and

$$\Phi(l, \mathbf{j}) = \frac{\phi(2l - |j_0|)}{2l} \prod_{i=1}^{\infty} \frac{\phi(J(l) - |j_i|)}{J(l)}.$$

For $\mathbf{j} \in \mathbf{Z}^{*\infty}$ we have $\Phi(l, \mathbf{j}) \uparrow 1$ $(l \to \infty)$. In case $\gcd(n, n') = 1$, since f has parallel Fourier coefficients, we have

$$\int_0^1 f(nx) f(n'x)\, dx = \frac{1}{2} \sum_{i, i' \in \mathbf{N}:\ in = i'n'} (a_i a_{i'} + b_i b_{i'})$$
$$= \frac{1}{2} \sum_{m \in \mathbf{N}} (a_{nm} a_{n'm} + b_{nm} b_{n'm}) = \frac{1}{2} \sum_{m \in \mathbf{N}} \sqrt{(a_{nm}^2 + b_{nm}^2)(a_{n'm}^2 + b_{n'm}^2)} \ge 0.$$

Since the last summation has non-negative terms, we can change the order of summation when we sum up this integrals for n and n'. Therefore

$$\sum_{\mathbf{j} \in \mathbf{Z}^{*\infty}} \int_0^1 f(\mathbf{p}^{\phi(\mathbf{j})} x) f(\mathbf{p}^{\phi(-\mathbf{j})} x)\, dx = \sum_{n, n' \in \mathbf{N}:\ \gcd(n, n') = 1} \int_0^1 f(nx) f(n'x)\, dx.$$
$$= \frac{1}{2} \sum_{n, n' \in \mathbf{N}} \sqrt{(a_n^2 + b_n^2)(a_{n'}^2 + b_{n'}^2)} = \frac{\|f\|_A^2}{2}.$$

Let us put

$$\xi_l(x) = \sum_{i_0 = (l-1)^2}^{l^2 - 1} \sum_{i_1, \ldots, i_{J(l)} = 0}^{J(l) - 1} f(p_0^{i_0} \ldots p_{J(l)}^{i_{J(l)}} x) \qquad \text{and}$$

$$V(L) = \int_0^1 \left(\sum_{l=1}^L \xi_l(x) \right)^2 dx = \sum_{l=1}^L \int_0^1 \xi_l^2(x)\, dx + 2 \sum_{1 \le l' < l \le L} \int_0^1 \xi_{l'}(x) \xi_l(x)\, dx.$$

By changing variable by $p^{\min\{i, i'\}} x = y$ in each integrals, we have

$$\sum_{i, i' = 0}^{I-1} \int_0^1 f(qp^i x) f(q'p^{i'} x)\, dx = \sum_{j \in \mathbf{Z}} \phi(I - |j|) \int_0^1 f(qp^{\phi(j)} x) f(q'p^{\phi(-j)} x)\, dx.$$

Repeated application of the above equality yields

$$\frac{\int_0^1 \xi_l^2(x)\,dx}{(2l-1)J(l)^{J(l)}} = \frac{1}{(2l-1)J(l)^{J(l)}} \int_0^1 \Big(\sum_{i_0=0}^{2l-2} \sum_{i_1,\dots,i_{J(l)}=0}^{J(l)-1} f(p_0^{i_0} \dots p_{J(l)}^{i_{J(l)}} x) \Big)^2 dx$$

$$= \sum_{j_0,\dots,j_{J(l)} \in \mathbf{Z}} \frac{\phi(2l-1-|j_0|)\phi(J(l)-|j_1|)\dots\phi(J(l)-|j_{J(l)}|)}{(2l-1)J(l)^{J(l)}}$$

$$\times \int_0^1 f(p_0^{\phi(j_0)} \dots p_{J(l)}^{\phi(j_{J(l)})} x) f(p_0^{\phi(-j_0)} \dots p_{J(l)}^{\phi(-j_{J(l)})} x)\,dx$$

$$= \sum_{\mathbf{j} \in \mathbf{Z}^{*(J(l)+1)}} \Phi(l, \mathbf{j}) \int_0^1 f(\mathbf{p}^{\phi(\mathbf{j})} x) f(\mathbf{p}^{\phi(-\mathbf{j})} x)\,dx$$

$$\uparrow \sum_{\mathbf{j} \in \mathbf{Z}^{*\infty}} \int_0^1 f(\mathbf{p}^{\phi(\mathbf{j})} x) f(\mathbf{p}^{\phi(-\mathbf{j})} x)\,dx = \frac{\|f\|_A^2}{2} \quad (l \to \infty),$$

where the last limiting procedure is by monotone convergence theorem. Hence

$$\frac{1}{D(L)} \sum_{l=1}^{L} \int_0^1 \xi_l^2(x)\,dx = \frac{1}{D(L)} \sum_{l=1}^{L} \frac{\int_0^1 \xi_l^2(x)\,dx}{(2l-1)J(l)^{J(l)}} (2l-1)J(l)^{J(l)} \to \frac{\|f\|_A^2}{2}.$$

By the non-negativity of the integrals, $\sum_{1 \le l' < l \le L} \int_0^1 \xi_{l'}(x)\xi_l(x)\,dx$ is less than

$$\sum_{1 \le l' < l \le L} \sum_{i_0=(l-1)^2}^{l^2-1} \sum_{i_0=(l'-1)^2}^{l'^2-1} \sum_{i_1, i'_1, \dots, i_{J(L)}, i'_{J(L)}=0}^{J(L)-1}$$

$$\int_0^1 f(p_0^{i_0} \dots p_{J(l)}^{i_{J(L)}} x) f(p_0^{i'_0} \dots p_{J(l')}^{i'_{J(L)}} x)\,dx$$

$$= \sum_{1 \le l' < l \le L} \sum_{i_0=(l-1)^2}^{l^2-1} \sum_{i'_0=(l'-1)^2}^{l'^2-1} \sum_{j_1,\dots,j_{J(L)} \in \mathbf{Z}} \phi(J(L)-|j_1|) \cdots \phi(J(L)-|j_{J(L)}|)$$

$$\times \int_0^1 f(p_0^{i_0-i'_0} p_1^{\phi(j_1)} \dots p_{J(L)}^{\phi(j_{J(L)})} x) f(p_1^{\phi(-j_1)} \dots p_{J(L)}^{\phi(-j_{J(L)})} x)\,dx$$

Let us fix j_0 and estimate the number of (i_0, i'_0) such that $i_0 - i'_0 = j_0$, $(l-1)^2 \le i_0 < l^2$, $(l'-1)^2 \le i_0 < l'^2$, and $1 \le l' < l \le L$. For l and l' with $l' \le l-2$, we have $j_0 = i_0 - i'_0 \ge 2l-2$, which is impossible for large l. Thus such (i_0, i'_0) is at most finite. For l and l' with $l' = l-1$, the solutions (i_0, i'_0) are included in $\{((l-1)^2, (l-1)^2 - j_0), ((l-1)^2+1, (l-1)^2-j_0+1), \dots, ((l-1)^2+j_0-1, (l-1)^2)\}$, and hence at most j_0 many. Thus the total number of (i_0, i_0') is $j_0 L + C = O(L)$. Thus we can write

$$\sum_{1 \le l' < l \le L} \int_0^1 \xi_{l'}(x)\xi_l(x)\, dx \le \sum_{\mathbf{j} \in \mathbf{Z}^{*J(L)}} \Psi(\mathbf{j}, L) \int_0^1 f(\mathbf{p}^{\phi(\mathbf{j})} x) f(\mathbf{p}^{\phi(-\mathbf{j})} x)\, dx,$$

where $\Psi(\mathbf{j}, L) = O(LJ(L)^{J(L)}) = o(D(L))$. By the dominated convergence theorem, we have

$$\frac{1}{D(L)} \sum_{1 \le l' < l \le L} \int_0^1 \xi_{l'}(x)\xi_l(x)\, dx \to 0 \quad \text{and hence} \quad \frac{V(L)}{D(L)} \to \frac{\|f\|_A^2}{2}.$$

Put $\Pi(l) = P(1, J(1), J(1)) \cup \ldots \cup P(l, J(l), J(l))$. Note that ${}^{\#}\Pi(L) = D(L)$ and $V(L) = \int_0^1 (\sum_{n_k \in \Pi(L)} f(n_k x))^2\, dx$. If we divide $\min P(l+3, J(l+3), J(l+3))$ by $\max P(l, J(l), J(l))$, thanks to $p_n < 2^{2^{n+1}}$ and $J(l) \le \frac{1}{2} \log_2 l/2$, we have

$$\frac{p_0^{(l+2)^2}}{p_0^{l^2-1} p_1^{J(l)-1} \cdots p_{J(l)}^{J(l)-1}} \ge 2^{4l+5-(2^1+\cdots+2^{J(l)+1})(J(l)-1)} \ge 2^{4l+5-2^{J(l)+2}J(l)} > 1.$$

For given K, let us take l_K such that $n_K \in P(l_K, J(l_K), J(l_K))$. Then

$$\begin{aligned} &\max P(l_K - 1, J(l_K) - 1, J(l_K) - 1) < n_K \le \max P(l_K, J(l_K), J(l_K)) \\ &< \min P(l_K + 3, J(l_K + 3), J(l_K + 3)) \end{aligned}$$

and hence

$$\Pi(l_K - 1) \subset \{n_1, \ldots, n_K\} \subset \Pi(l_K + 2)$$

holds. Clearly we have $D(l_K - 1) \le K \le D(l_K + 2)$ and $D(l_K - 1) \sim D(l_K + 2)$, and thereby, $D(l_K - 1) \sim K \sim D(l_K + 2)$. By the positivity of the integrals $\int_0^1 f(nx)f(mx)\, dx$, we have $V(l_K - 1) \le \int_0^1 (\sum_{k=1}^K f(n_k x))^2\, dx \le V(l_K + 2)$. Dividing this inequality by K and letting $K \to \infty$, we have

$$\frac{1}{K} \int_0^1 \left(\sum_{k=1}^K f(n_k x) \right)^2 dx \to \frac{\|f\|_A^2}{2}.$$

4. LIL: trigonometric polynomial case

In this section we prove $\Xi_{f_I} = \|f_I\|_A$ for trigonometric polynomial $f_I(x) = \sum_{\nu=1}^{I}(a_\nu \cos 2\pi\nu x + b_\nu \sin 2\pi\nu x)$ with parallel Fourier coefficients. If $\|f_I\|_A = 0$, then $f_I = 0$, and hence $\Xi_{f_I} = 0$, which completes our proof. Therefore we assume $\|f_I\|_A > 0$.

Let us consider the formal Fourier expansion

$$\sum_{k=1}^{\infty} f_I(n_k x) = \sum_{\nu=1}^{\infty}(c_k \cos 2\pi m_k x + d_k \sin 2\pi m_k x)$$

and denote its N-th subsum by S_N. Put $v_n^2 = \int_0^1 S_N^2$. Note that this $\{m_k\}$ is a sequence given in section 2.

Let M be the maximum of modulus of all finite sums of the Fourier coefficients of f_I. It is clear that $|c_k|$, $|d_k| \le M$.

$S_{\widehat{K}}$ coincides with the subsum of the Fourier expansion of $\sum_{k=1}^{K} f_I(n_k x)$ up to frequency less than n_K. Hence $\sum_{k=1}^{K} f_I(n_k x) - S_{\widehat{K}}$ contains at most IL_K terms with coefficients less than M. Thus we have

$$\left\|\sum_{k=1}^{K} f_I(n_k x) - S_{\widehat{K}}\right\|_\infty = O(K^{1/3}), \qquad v_{\widehat{K}} \sim \sqrt{K/2}\,\|f_I\|_A \quad \text{and}$$

$$\varlimsup_{K\to\infty} \frac{S_{\widehat{K}}}{\sqrt{2v_{\widehat{K}}^2 \log\log v_{\widehat{K}}^2}} = \varlimsup_{K\to\infty} \frac{\sum_{k=1}^{K} f_I(n_k x)}{\sqrt{K\|f_I\|_A^2 \log\log K}}$$

If $\widehat{K} \le j < \widehat{K+1}$, then $v_{\widehat{K}} \le j < v_{\widehat{K+1}}$. By $v_{\widehat{K}} \sim \sqrt{K/2}\,\|f_I\|_A \sim v_{\widehat{K+1}}$, we have $v_j \sim \sqrt{K/2}\,\|f_I\|_A$ for $\widehat{K} \le j < \widehat{K+1}$. Thanks to (4) of Lemma, we have $v_j \sim \sqrt{K/2}\,\|f_I\|_A \asymp \widehat{K} \asymp j$, where $a_n \asymp b_n$ means $a_n = O(b_n)$ and $b_n = O(a_n)$ at once. Thus we have c_j, $d_j = O(1) = o(v_j/j^{1/3})$, and hence we can apply the Takahashi's law of the iterated logarithm below ([3]):

Theorem B. *Suppose that $\{m_k\}$ satisfies Takahashi's gap condition with $\beta < 1/2$, and define S_N and v_N as above. If c_j, $d_j = o(v_j/j^\beta)$, and $v_n^2 \to \infty$, then*

$$\varlimsup_{N\to\infty} \frac{S_j}{\sqrt{2v_j^2 \log\log v_j^2}} = 1, \qquad \text{a.e.}$$

By (1) of Lemma, we have $\max_{\widehat{K}\le j<\widehat{K+1}} \|S_j - S_{\widehat{K}}\|_\infty = O(K^{1/3}) = o(v_{\widehat{K}})$, and hence

$$\varlimsup_{j\to\infty} \frac{S_j}{\sqrt{2v_j^2 \log\log v_j^2}} = \varlimsup_{K\to\infty} \frac{S_{\widehat{K}}}{\sqrt{2v_{\widehat{K}}^2 \log\log v_{\widehat{K}}^2}}.$$

Combining these, we have $\Xi_{f_I} = \|f_I\|_A$.

5. LIL: functions with absolutely convergent Fourier series.

We proved $\Xi_{f_I} = \|f_I\|_A$ in the previous section. By (1) of Theorem A, we have $|\Xi_{f-f_I}| \le \|f - f_I\|_A$. By $\overline{\lim}\, a - \overline{\lim}(-b) \le \overline{\lim}(a+b) \le \overline{\lim}\, a + \overline{\lim}\, b$ and

$$\varlimsup_{N\to\infty} \frac{\sum_{k=1}^{N} f(n_k x)}{\sqrt{N \log\log N}} = \varlimsup_{N\to\infty} \left(\frac{\sum_{k=1}^{N} f_I(n_k x)}{\sqrt{N \log\log N}} + \frac{\sum_{k=1}^{N} (f - f_I)(n_k x)}{\sqrt{N \log\log N}} \right),$$

we have $\|f_I\|_A - \|f - f_I\|_A \le \Xi_f \le \|f_I\|_A + \|f - f_I\|_A$ a.e. By letting $I \to \infty$ we have $\|f - f_I\|_A \to 0$ and $\|f_I\|_A \to \|f\|_A$, and thereby $\Xi_f = \|f\|_A$ a.e.

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