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# AN ASYMPTOTIC PROPERTY OF GAP SERIES III

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## 1. Introduction

Let  $f$  be an  $\mathbf{R}$ -valued function on  $\mathbf{R}$  satisfying

$$f(x+1) = f(x), \quad \int_0^1 f(x) dx = 0, \quad \|f\|^2 = \int_0^1 |f(x)|^2 dx < \infty,$$

with Fourier series  $f(x) \sim \sum_{\nu=1}^{\infty} (a_{\nu} \cos 2\pi\nu x + b_{\nu} \sin 2\pi\nu x)$ . Denote  $\|f\|_A = \sum_{\nu=1}^{\infty} \sqrt{a_{\nu}^2 + b_{\nu}^2}$ . We investigate the distribution of values of

$$\Xi_f(x) = \overline{\lim}_{N \rightarrow \infty} \frac{1}{\sqrt{N \log \log N}} \sum_{k=1}^N f(n_k x)$$

where  $\{n_k\}$  is a strictly increasing sequence of positive integers. For a brief survey on studies on  $\Xi_f$ , we refer the reader to our previous paper [1].

Takahashi [4] gave the concrete bound  $\Xi_f \leq \|f\|_A$  a.e. assuming  $f \in \text{Lip } \alpha$  ( $\alpha > 1/2$ ) and Takahashi's gap condition:

$$n_{k+1}/n_k \geq 1 + ck^{-\beta} \quad (c > 0, \beta < 1/2).$$

In [1] we proved the following.

### Theorem A.

- (1) *Takahashi's gap condition and  $\|f\|_A < \infty$  imply  $\Xi_f \leq \|f\|_A$  a.e.*
- (2) *Let us say that  $f$  has parallel Fourier coefficients if there exist  $a, b \in \mathbf{R}$  with  $a^2 + b^2 = 1$  such that  $a_{\nu} = a\sqrt{a_{\nu}^2 + b_{\nu}^2}$  and  $b_{\nu} = b\sqrt{a_{\nu}^2 + b_{\nu}^2}$  for all  $\nu \in \mathbf{N}$ . If  $\|f\|_A < \infty$  and the Fourier coefficients of  $f$  are parallel, then for every  $\varepsilon > 0$  there exist  $\rho > 0$  and  $\{n_k\}$  such that  $n_{k+1}/n_k > 1 + \rho$  and  $\Xi_f \geq \|f\|_A - \varepsilon$  a.e.*

In this paper we consider the problem if there exists a sequence  $\{n_k\}$  for which  $\Xi_f = \|f\|_A$  holds, or in other words, if the bound  $\|f\|_A$  can be achieved as the law of the iterated logarithm. Our main theorem below gives the affirmative answer to this question.

**Theorem.** *For any sequence  $\{\rho_k\}$  of positive numbers converging to 0, there exists a sequence  $\{n_k\}$  such that  $n_{k+1}/n_k \geq 1 + \rho_k$  and  $\Xi_f = \|f\|_A$  a.e. for arbitrary function  $f$  with parallel Fourier coefficients and  $\|f\|_A < \infty$ .*

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## 2. Construction of the sequence

We prove our theorem by assuming that  $\{\rho_k\}$  monotonously decreases to 0 and satisfies  $1 \geq \rho_k \geq k^{-1/3}$ . Once we have proved under these assumptions, we can derive the general case in the following way.

Put  $\rho'_k = \max\{k^{-1/3}, \min\{1, \sup_{i \geq k} \rho_i\}\}$ . Then  $\{\rho'_k\}$  monotonously decreases to 0 and satisfies  $1 \geq \rho'_k \geq k^{-1/3}$ . By  $\rho_k \rightarrow 0$ , there exists a  $K$  such that  $\rho_k \leq 1$  for all  $k \geq K$ , and hence  $\rho'_k \geq \rho_k$  for  $k \geq K$ . Firstly construct a sequence  $\{n'_k\}$  satisfying  $n'_{k+1}/n'_k \geq 1 + \rho'_k$  and  $\Xi_f = \|f\|_A$  a.e. Secondly, take  $n_1, \dots, n_K$  with  $n_{k+1}/n_k \geq 1 + \rho_k$  ( $k = 1, \dots, K-1$ ), take  $M \in \mathbf{N}$  large enough to satisfy  $Mn'_{K+1}/n_K \geq 1 + \rho_K$ , and put  $n_k = Mn'_k$  for  $k \geq K+1$ . For  $k \geq K+1$ , we have  $n_{k+1}/n_k = n'_{k+1}/n'_k \geq 1 + \rho'_k \geq 1 + \rho_k$  and hence the gap condition is satisfied for all  $k$ . By change of variable, we see that  $\Xi_f = \|f\|_A$  holds for  $\{Mn'_k\}$ , and hence for  $\{n_k\}$  because  $\Xi_f$  does not depend on finitely many terms. Therefore we verified the general case.

Let  $p_0 = 2, p_1 = 3, p_2 = 5, \dots$  be the sequence consisting of whole prime numbers. Put

$$P(l, m, n) = \{p_0^{i_0} p_1^{i_1} \dots p_m^{i_m} \mid (l-1)^2 \leq i_0 < l^2, 0 \leq i_1, \dots, i_m < n\}$$

$$\Pi(\lambda, \Lambda, j) = \bigcup_{l=\lambda}^{\Lambda} P(l, j, 2j) \quad \text{and} \quad \gamma(j) = \inf \left\{ \frac{n}{m} \mid n > m, n, m \in \Pi(0, \infty, j) \right\}.$$

We see that  $2 = \gamma(0) \geq \gamma(1) \geq \gamma(2) \geq \dots$ .

**Proposition.** *There exists a non-decreasing divergent sequence  $\{\tilde{J}(l)\}$  of non-negative integers such that the arrangement  $\{\nu_k\}$  in increasing order of  $\bigcup_{l=1}^{\infty} P(l, \tilde{J}(l), 2\tilde{J}(l))$  satisfies  $\nu_{k+1}/\nu_k \geq 1 + \rho_k$ .*

*Proof:* Firstly, put  $l_0 = 0$  and take  $k_1$  such that  $\gamma(1) \geq 1 + \rho_{k_1}$ . Let us arrange  $\Pi(l_0, \infty, 0)$  in increasing order and denote the first  $k_1$  numbers by  $\nu_1, \dots, \nu_{k_1}$ . Let us take  $l_1$  large enough such that  $\nu_1, \dots, \nu_{k_1} \in \Pi(l_0, l_1 - 1, 0)$ . If  $n \in \Pi(l_0, l_1 - 1, 0)$  and  $m \in \Pi(l_1, \infty, 1) \setminus \Pi(l_0, \infty, 0)$ , then  $n < m$  holds. Thus  $\nu_1, \dots, \nu_{k_1}$  are the least  $k_1$  numbers in  $\Pi(l_0, l_1 - 1, 0) \cup \Pi(l_1, \infty, 1)$ .

Secondly, take  $k_2$  as  $\gamma(2) \geq 1 + \rho_{k_2}$  and arrange  $\Pi(l_0, l_1 - 1, 0) \cup \Pi(l_1, \infty, 1)$  in increasing order. The first  $k_1$  numbers are  $\nu_1, \dots, \nu_{k_1}$ . Denote the next  $k_2 - k_1$  numbers by  $\nu_{k_1+1}, \dots, \nu_{k_2}$ . Let us take  $l_2$  large enough such that  $\nu_1, \dots, \nu_{k_2} \in \Pi(l_0, l_1 - 1, 0) \cup \Pi(l_1, l_2 - 1, 1)$ . Then  $\nu_1, \dots, \nu_{k_2}$  are the least  $k_2$  numbers in  $\Pi(l_0, l_1 - 1, 0) \cup \Pi(l_1, l_2 - 1, 1) \cup \Pi(l_2, \infty, 2)$ .

In this way, we take  $l_0 < l_1 < l_2 < \dots$  and  $k_1 < k_2 < \dots$  such that the arrangement  $\{\nu_k\}$  in increasing order of  $\bigcup_{i=0}^{\infty} \Pi(l_i, l_{i+1} - 1, i)$  satisfies  $\nu_k \in \Pi(0, \infty, i)$  for  $k \leq k_{i+1}$ .

Let us verify the gap condition on this sequence. For any  $k$ , let us take  $i$  such that  $k_i \leq k < k_{i+1}$ . Then we have  $\nu_k, \nu_{k+1} \in \Pi(0, \infty, i)$  and hence  $\nu_{k+1}/\nu_k \geq \gamma(i) \geq 1 + \rho_{k_i} \geq 1 + \rho_k$ .

For  $l_i \leq l < l_{i+1}$ , put  $\tilde{J}(l) = i$ . Then  $\bigcup_{i=0}^{\infty} \Pi(l_i, l_{i+1} - 1, i) = \bigcup_{i=0}^{\infty} \Pi(l_i, l_{i+1} - 1, i)$  and hence  $\{\nu_k\}$  is a desired sequence. ■

Let us take a sequence  $\{J(l)\}$  of non-negative integers such that

$$J(l) \leq J(l+1), \quad J(l) \leq \frac{1}{2} \log_2 l, \quad J(l) \leq \tilde{J}(l), \quad J(l)^{J(l)} \leq l^{1/2}, \quad J(l) \rightarrow \infty,$$

where we interpret  $0^0$  as 1. Let us put  $D(L) = \sum_{l=1}^L (2l-1)J(l)^{J(l)}$ . Thanks to  $LJ(L)^{J(L)} \leq L^{3/2}$  and  $D(L) \geq J(1)^{J(1)}L^2$ , we have  $(2L-1)J(L)^{J(L)} = o(D(L))$  and  $D(L-1) \sim D(L)$ .

Now we are in a position to construct our sequence  $\{n_k\}$  which satisfies  $\Xi_f = \|f\|_A$ . Let  $\{n_k\}$  be an arrangement in increasing order of  $\bigcup_{l=1}^{\infty} P(l, J(l), J(l))$ . Since it is a subsequence of  $\{\nu_k\}$ , it satisfies  $n_{k+1}/n_k \geq 1 + \rho_k$ .

Let us take a positive integer  $I$  arbitrarily, and let  $\{m_k\}$  be an arrangement in increasing order of the set theoretical union  $\{n_k\} \cup \{2n_k\} \cup \dots \cup \{In_k\}$ . Given  $K$ , let  $\hat{K}$  denote the integer  $k$  such that  $n_K = m_k$ .

**Lemma.**

- (0)  $\{m_k\}$  satisfies  $m_{k+1}/m_k \geq 1 + ck^{-1/3}$  for some  $c > 0$ .
- (1)  $\#\{k \mid n_K \leq m_k < n_{K+1}\} = O(K^{1/3})$ .
- (2)  $L_K = \#\{k \leq K \mid In_k \geq n_K\} = O(K^{1/3})$ .
- (3)  $K \leq \hat{K} \leq IK$

*Proof:* (0) We prove that  $\{m_k\}$  is a subsequence of  $\{\nu_k\}$  except for finitely many terms, which implies  $m_{k+1}/m_k \geq 1 + \rho_{k+c} \geq 1 + c'k^{-1/3}$  for some  $c \in \mathbf{N}$  and  $c' > 0$ . Since  $J(l)$  is non-decreasing, by definition, we see that  $2\{n_k\} \subset \{n_k\}$ . Thus it is sufficient to prove  $on_k \in \{\nu_k\}$  for all odd numbers  $o$  less than  $I$ , except for finitely many  $k$ . If  $l$  is large enough, then  $o$  is a factor of  $p_1^{J(l)} \dots p_{J(l)}^{J(l)}$  and hence  $oP(l, J(l), J(l)) \subset P(l, J(l), 2J(l)) \subset P(l, \tilde{J}(l), 2\tilde{J}(l))$ . Thus  $on_k$  belongs to  $\{\nu_k\}$  except for finitely many  $k$ .

(1) Let  $i = 1, \dots, I$ , and let  $k'$  and  $k''$  be the minimum and maximum of  $k$  for which  $n_K \leq in_k < n_{K+1}$  holds. Thanks to  $n_{K+1}/n_K \leq 2$  and  $k'' \leq K$ , we have

$$2 \geq \frac{n_{K+1}}{n_K} \geq \frac{in_{k''}}{in_{k'}} \geq \prod_{k=k'}^{k''-1} (1 + ck^{-1/3}) \geq (k'' - k')cK^{-1/3}.$$

(2) Denote  $k'$  the minimum of  $k \leq K$  satisfying  $In_k \geq n_K$ . Then we have

$$I \geq \frac{n_K}{n_{k'}} \geq \prod_{k=k'}^{K-1} (1 + ck^{-1/3}) \geq (K - k')cK^{-1/3}.$$

(3) Since  $n_1, \dots, n_K \leq n_K$ , there exist at least  $K$  many  $m_k$  such that  $m_k \leq n_K$ . Thus  $K \leq \widehat{K}$ . If  $1 \leq i \leq I$  and  $in_j \leq n_K$ , then  $j \leq K$ . Thus there exist at most  $IK$  many  $k$  such that  $m_k \leq n_K$ , and thereby  $\widehat{K} \leq IK$ .

### 3. Limiting variance

In this section we assume that  $f$  has parallel Fourier coefficients and prove  $\int_0^1 (\sum_{k=1}^K f(n_k x)) \sim K \|f\|_A^2 / 2$ . We prepare notation. Denote  $\mathbf{Z}^{*n} = \mathbf{Z}^n \times \{0\} \times \{0\} \times \dots$ ,  $\mathbf{Z}^{*\infty} = \mathbf{Z}^{*1} \cup \mathbf{Z}^{*2} \cup \dots$ , and  $\phi(j) = \max\{j, 0\}$ . For  $\mathbf{j} = (j_0, j_1, \dots) \in \mathbf{Z}^{*\infty}$ , put  $\mathbf{p}^{\mathbf{j}} = p_0^{j_0} p_1^{j_1} \dots$ ,  $\phi(\mathbf{j}) = (\phi(j_0), \phi(j_1), \dots)$ , and

$$\Phi(l, \mathbf{j}) = \frac{\phi(2l - |j_0|)}{2l} \prod_{i=1}^{\infty} \frac{\phi(J(l) - |j_i|)}{J(l)}.$$

For  $\mathbf{j} \in \mathbf{Z}^{*\infty}$  we have  $\Phi(l, \mathbf{j}) \uparrow 1$  ( $l \rightarrow \infty$ ). In case  $\gcd(n, n') = 1$ , since  $f$  has parallel Fourier coefficients, we have

$$\begin{aligned} \int_0^1 f(nx) f(n'x) dx &= \frac{1}{2} \sum_{i, i' \in \mathbf{N}: in=i'n'} (a_i a_{i'} + b_i b_{i'}) \\ &= \frac{1}{2} \sum_{m \in \mathbf{N}} (a_{nm} a_{n'm} + b_{nm} b_{n'm}) = \frac{1}{2} \sum_{m \in \mathbf{N}} \sqrt{(a_{nm}^2 + b_{nm}^2)(a_{n'm}^2 + b_{n'm}^2)} \geq 0. \end{aligned}$$

Since the last summation has non-negative terms, we can change the order of summation when we sum up this integrals for  $n$  and  $n'$ . Therefore

$$\begin{aligned} \sum_{\mathbf{j} \in \mathbf{Z}^{*\infty}} \int_0^1 f(\mathbf{p}^{\phi(\mathbf{j})} x) f(\mathbf{p}^{\phi(-\mathbf{j})} x) dx &= \sum_{n, n' \in \mathbf{N}: \gcd(n, n')=1} \int_0^1 f(nx) f(n'x) dx. \\ &= \frac{1}{2} \sum_{n, n' \in \mathbf{N}} \sqrt{(a_n^2 + b_n^2)(a_{n'}^2 + b_{n'}^2)} = \frac{\|f\|_A^2}{2}. \end{aligned}$$

Let us put

$$\begin{aligned} \xi_l(x) &= \sum_{i_0=(l-1)^2}^{l^2-1} \sum_{i_1, \dots, i_{J(l)}=0}^{J(l)-1} f(p_0^{i_0} \dots p_{J(l)}^{i_{J(l)}} x) \quad \text{and} \\ V(L) &= \int_0^1 \left( \sum_{l=1}^L \xi_l(x) \right)^2 dx = \sum_{l=1}^L \int_0^1 \xi_l^2(x) dx + 2 \sum_{1 \leq l' < l \leq L} \int_0^1 \xi_{l'}(x) \xi_l(x) dx. \end{aligned}$$

By changing variable by  $p^{\min\{i, i'\}} x = y$  in each integrals, we have

$$\sum_{i, i'=0}^{I-1} \int_0^1 f(q p^i x) f(q' p^{i'} x) dx = \sum_{j \in \mathbf{Z}} \phi(I - |j|) \int_0^1 f(q p^{\phi(j)} x) f(q' p^{\phi(-j)} x) dx.$$

Repeated application of the above equality yields

$$\begin{aligned}
\frac{\int_0^1 \xi_l^2(x) dx}{(2l-1)J(l)^{J(l)}} &= \frac{1}{(2l-1)J(l)^{J(l)}} \int_0^1 \left( \sum_{i_0=0}^{2l-2} \sum_{i_1, \dots, i_{J(l)}=0}^{J(l)-1} f(p_0^{i_0} \dots p_{J(l)}^{i_{J(l)}} x) \right)^2 dx \\
&= \sum_{j_0, \dots, j_{J(l)} \in \mathbf{Z}} \frac{\phi(2l-1-|j_0|) \phi(J(l)-|j_1|) \dots \phi(J(l)-|j_{J(l)}|)}{(2l-1)J(l)^{J(l)}} \\
&\quad \times \int_0^1 f(p_0^{\phi(j_0)} \dots p_{J(l)}^{\phi(j_{J(l)})} x) f(p_0^{\phi(-j_0)} \dots p_{J(l)}^{\phi(-j_{J(l)})} x) dx \\
&= \sum_{\mathbf{j} \in \mathbf{Z}^{*(J(l)+1)}} \Phi(l, \mathbf{j}) \int_0^1 f(\mathbf{p}^{\phi(\mathbf{j})} x) f(\mathbf{p}^{\phi(-\mathbf{j})} x) dx \\
&\uparrow \sum_{\mathbf{j} \in \mathbf{Z}^{*\infty}} \int_0^1 f(\mathbf{p}^{\phi(\mathbf{j})} x) f(\mathbf{p}^{\phi(-\mathbf{j})} x) dx = \frac{\|f\|_A^2}{2} \quad (l \rightarrow \infty),
\end{aligned}$$

where the last limiting procedure is by monotone convergence theorem. Hence

$$\frac{1}{D(L)} \sum_{l=1}^L \int_0^1 \xi_l^2(x) dx = \frac{1}{D(L)} \sum_{l=1}^L \frac{\int_0^1 \xi_l^2(x) dx}{(2l-1)J(l)^{J(l)}} (2l-1)J(l)^{J(l)} \rightarrow \frac{\|f\|_A^2}{2}.$$

By the non-negativity of the integrals,  $\sum_{1 \leq l' < l \leq L} \int_0^1 \xi_{l'}(x) \xi_l(x) dx$  is less than

$$\begin{aligned}
&\sum_{1 \leq l' < l \leq L} \sum_{i_0=(l-1)^2}^{l^2-1} \sum_{i_0=(l'-1)^2}^{l'^2-1} \sum_{i_1, i'_1, \dots, i_{J(L)}, i'_{J(L)}=0}^{J(L)-1} \\
&\quad \int_0^1 f(p_0^{i_0} \dots p_{J(l)}^{i_{J(l)}} x) f(p_0^{i'_0} \dots p_{J(l')}^{i'_{J(l')}} x) dx \\
&= \sum_{1 \leq l' < l \leq L} \sum_{i_0=(l-1)^2}^{l^2-1} \sum_{i'_0=(l'-1)^2}^{l'^2-1} \sum_{j_1, \dots, j_{J(L)} \in \mathbf{Z}} \phi(J(L)-|j_1|) \dots \phi(J(L)-|j_{J(L)}|) \\
&\quad \times \int_0^1 f(p_0^{i_0-i'_0} p_1^{\phi(j_1)} \dots p_{J(L)}^{\phi(j_{J(L)})} x) f(p_1^{\phi(-j_1)} \dots p_{J(L)}^{\phi(-j_{J(L)})} x) dx
\end{aligned}$$

Let us fix  $j_0$  and estimate the number of  $(i_0, i'_0)$  such that  $i_0 - i'_0 = j_0$ ,  $(l-1)^2 \leq i_0 < l^2$ ,  $(l'-1)^2 \leq i'_0 < l'^2$ , and  $1 \leq l' < l \leq L$ . For  $l$  and  $l'$  with  $l' \leq l-2$ , we have  $j_0 = i_0 - i'_0 \geq 2l-2$ , which is impossible for large  $l$ . Thus such  $(i_0, i'_0)$  is at most finite. For  $l$  and  $l'$  with  $l' = l-1$ , the solutions  $(i_0, i'_0)$  are included in  $\{((l-1)^2, (l-1)^2 - j_0), ((l-1)^2 + 1, (l-1)^2 - j_0 + 1), \dots, ((l-1)^2 + j_0 - 1, (l-1)^2)\}$ ,

and hence at most  $j_0$  many. Thus the total number of  $(i_0, i'_0)$  is  $j_0 L + C = O(L)$ . Thus we can write

$$\sum_{1 \leq l' < l \leq L} \int_0^1 \xi_{l'}(x) \xi_l(x) dx \leq \sum_{\mathbf{j} \in \mathbf{Z}^{*J(L)}} \Psi(\mathbf{j}, L) \int_0^1 f(\mathbf{p}^{\phi(\mathbf{j})} x) f(\mathbf{p}^{\phi(-\mathbf{j})} x) dx,$$

where  $\Psi(\mathbf{j}, L) = O(LJ(L)^{J(L)}) = o(D(L))$ . By the dominated convergence theorem, we have

$$\frac{1}{D(L)} \sum_{1 \leq l' < l \leq L} \int_0^1 \xi_{l'}(x) \xi_l(x) dx \rightarrow 0 \quad \text{and hence} \quad \frac{V(L)}{D(L)} \rightarrow \frac{\|f\|_A^2}{2}.$$

Put  $\Pi(l) = P(1, J(1), J(1)) \cup \dots \cup P(l, J(l), J(l))$ . Note that  $\#\Pi(L) = D(L)$  and  $V(L) = \int_0^1 (\sum_{n_k \in \Pi(L)} f(n_k x))^2 dx$ . If we divide  $\min P(l+3, J(l+3), J(l+3))$  by  $\max P(l, J(l), J(l))$ , thanks to  $p_n < 2^{2^{n+1}}$  and  $J(l) \leq \frac{1}{2} \log_2 l/2$ , we have

$$\frac{p_0^{(l+2)^2}}{p_0^{l^2-1} p_1^{J(l)-1} \dots p_{J(l)}^{J(l)-1}} \geq 2^{4l+5-(2^1+\dots+2^{J(l)+1})(J(l)-1)} \geq 2^{4l+5-2^{J(l)+2}J(l)} > 1.$$

For given  $K$ , let us take  $l_K$  such that  $n_K \in P(l_K, J(l_K), J(l_K))$ . Then

$$\begin{aligned} \max P(l_K - 1, J(l_K) - 1, J(l_K) - 1) &< n_K \leq \max P(l_K, J(l_K), J(l_K)) \\ &< \min P(l_K + 3, J(l_K + 3), J(l_K + 3)) \end{aligned}$$

and hence

$$\Pi(l_K - 1) \subset \{n_1, \dots, n_K\} \subset \Pi(l_K + 2)$$

holds. Clearly we have  $D(l_K - 1) \leq K \leq D(l_K + 2)$  and  $D(l_K - 1) \sim D(l_K + 2)$ , and thereby,  $D(l_K - 1) \sim K \sim D(l_K + 2)$ . By the positivity of the integrals  $\int_0^1 f(nx) f(mx) dx$ , we have  $V(l_K - 1) \leq \int_0^1 (\sum_{k=1}^K f(n_k x))^2 dx \leq V(l_K + 2)$ . Dividing this inequality by  $K$  and letting  $K \rightarrow \infty$ , we have

$$\frac{1}{K} \int_0^1 \left( \sum_{k=1}^K f(n_k x) \right)^2 dx \rightarrow \frac{\|f\|_A^2}{2}.$$

#### 4. LIL: trigonometric polynomial case

In this section we prove  $\Xi_{f_I} = \|f_I\|_A$  for trigonometric polynomial  $f_I(x) = \sum_{\nu=1}^I (a_\nu \cos 2\pi\nu x + b_\nu \sin 2\pi\nu x)$  with parallel Fourier coefficients. If  $\|f_I\|_A = 0$ , then  $f_I = 0$ , and hence  $\Xi_{f_I} = 0$ , which completes our proof. Therefore we assume  $\|f_I\|_A > 0$ .

Let us consider the formal Fourier expansion

$$\sum_{k=1}^{\infty} f_I(n_k x) = \sum_{\nu=1}^{\infty} (c_k \cos 2\pi m_k x + d_k \sin 2\pi m_k x)$$

and denote its  $N$ -th subsum by  $S_N$ . Put  $v_n^2 = \int_0^1 S_N^2$ . Note that this  $\{m_k\}$  is a sequence given in section 2.

Let  $M$  be the maximum of modulus of all finite sums of the Fourier coefficients of  $f_I$ . It is clear that  $|c_k|, |d_k| \leq M$ .

$S_{\widehat{K}}$  coincides with the subsum of the Fourier expansion of  $\sum_{k=1}^K f_I(n_k x)$  up to frequency less than  $n_K$ . Hence  $\sum_{k=1}^K f_I(n_k x) - S_{\widehat{K}}$  contains at most  $IL_K$  terms with coefficients less than  $M$ . Thus we have

$$\left\| \sum_{k=1}^K f_I(n_k x) - S_{\widehat{K}} \right\|_{\infty} = O(K^{1/3}), \quad v_{\widehat{K}} \sim \sqrt{K/2} \|f_I\|_A \quad \text{and}$$

$$\overline{\lim}_{K \rightarrow \infty} \frac{S_{\widehat{K}}}{\sqrt{2v_{\widehat{K}}^2 \log \log v_{\widehat{K}}^2}} = \overline{\lim}_{K \rightarrow \infty} \frac{\sum_{k=1}^K f_I(n_k x)}{\sqrt{K \|f_I\|_A^2 \log \log K}}$$

If  $\widehat{K} \leq j < \widehat{K} + 1$ , then  $v_{\widehat{K}} \leq j < v_{\widehat{K}+1}$ . By  $v_{\widehat{K}} \sim \sqrt{K/2} \|f_I\|_A \sim v_{\widehat{K}+1}$ , we have  $v_j \sim \sqrt{K/2} \|f_I\|_A$  for  $\widehat{K} \leq j < \widehat{K} + 1$ . Thanks to (4) of Lemma, we have  $v_j \sim \sqrt{K/2} \|f_I\|_A \asymp \widehat{K} \asymp j$ , where  $a_n \asymp b_n$  means  $a_n = O(b_n)$  and  $b_n = O(a_n)$  at once. Thus we have  $c_j, d_j = O(1) = o(v_j/j^{1/3})$ , and hence we can apply the Takahashi's law of the iterated logarithm below ([3]):

**Theorem B.** Suppose that  $\{m_k\}$  satisfies Takahashi's gap condition with  $\beta < 1/2$ , and define  $S_N$  and  $v_N$  as above. If  $c_j, d_j = o(v_j/j^\beta)$ , and  $v_n^2 \rightarrow \infty$ , then

$$\overline{\lim}_{N \rightarrow \infty} \frac{S_j}{\sqrt{2v_j^2 \log \log v_j^2}} = 1, \quad \text{a.e.}$$

By (1) of Lemma, we have  $\max_{\widehat{K} \leq j < \widehat{K}+1} \|S_j - S_{\widehat{K}}\|_{\infty} = O(K^{1/3}) = o(v_{\widehat{K}})$ , and hence

$$\overline{\lim}_{j \rightarrow \infty} \frac{S_j}{\sqrt{2v_j^2 \log \log v_j^2}} = \overline{\lim}_{K \rightarrow \infty} \frac{S_{\widehat{K}}}{\sqrt{2v_{\widehat{K}}^2 \log \log v_{\widehat{K}}^2}}.$$

Combining these, we have  $\Xi_{f_I} = \|f_I\|_A$ .



## 5. LIL: functions with absolutely convergent Fourier series.

We proved  $\Xi_{f_I} = \|f_I\|_A$  in the previous section. By (1) of Theorem A, we have  $|\Xi_{f-f_I}| \leq \|f - f_I\|_A$ . By  $\overline{\lim} a - \overline{\lim}(-b) \leq \overline{\lim}(a+b) \leq \overline{\lim} a + \overline{\lim} b$  and

$$\overline{\lim}_{N \rightarrow \infty} \frac{\sum_{k=1}^N f(n_k x)}{\sqrt{N \log \log N}} = \overline{\lim}_{N \rightarrow \infty} \left( \frac{\sum_{k=1}^N f_I(n_k x)}{\sqrt{N \log \log N}} + \frac{\sum_{k=1}^N (f - f_I)(n_k x)}{\sqrt{N \log \log N}} \right),$$

we have  $\|f_I\|_A - \|f - f_I\|_A \leq \Xi_f \leq \|f_I\|_A + \|f - f_I\|_A$  a.e. By letting  $I \rightarrow \infty$  we have  $\|f - f_I\|_A \rightarrow 0$  and  $\|f_I\|_A \rightarrow \|f\|_A$ , and thereby  $\Xi_f = \|f\|_A$  a.e.

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