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(Citation)

Acta Mathematica Hungarica, 129(4):303-313

(Issue Date)

2010-12

(Resource Type)

journal article

(Version)

Accepted Manuscript

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(URL)

<https://hdl.handle.net/20.500.14094/90003838>



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# Pure gaussian limit distributions of trigonometric series with bounded gaps

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Received: date / Accepted: date

**Abstract** This paper mainly concerned with the limit distribution of  $(\cos 2\pi n_1 x + \cdots + \cos 2\pi n_N x)/\sqrt{N}$  on unit interval when increasing sequence  $\{n_k\}$  has bounded gaps, i.e.,  $1 \leq n_{k+1} - n_k = O(1)$ . By Bobkov-Götze, it was proved that the limiting variance must be less than  $1/2$  in this case. It was proved in the previous paper that the centered gaussian distribution with variance  $1/4$  together with mixtures of gaussian distributions belonging to a huge class can be limit distributions. In this paper it is proved that any gaussian distribution with variance less than  $1/2$  can be a limit distribution.

## 1 Introduction

In this note we prove the existence of an increasing sequence  $\{n_k\}$  of integers with bounded gaps  $n_{k+1} - n_k = O(1)$  obeying the central limit theorem

$$\frac{1}{\sqrt{N}} \sum_{k=1}^N \cos 2\pi n_k x \xrightarrow{\mathcal{D}} \mathfrak{N}_{0,\sigma^2} \text{ on } (E, \mathcal{B}(E), dx/|E|) \text{ for all } E \text{ with } |E| > 0, \quad (1)$$

for  $\sigma^2 < 1/2$ . Here,  $\xrightarrow{\mathcal{D}}$  denotes the convergence in distribution, and  $\mathfrak{N}_{m,\sigma^2}$  denotes the gaussian distribution with mean  $m$  and variance  $\sigma^2$ .

The central limit theorem with  $\sigma^2 = 1/2$  for lacunary sequence satisfying  $n_{k+1}/n_k \rightarrow \infty$  was first proved by Kac [8] in the special case  $E = [0, 1]$  and was extended to the Hadamard's gap case  $n_{k+1}/n_k \geq q > 1$  and general  $E$  by Salem-Zygmund [12].

The gap condition was weakened to  $n_{k+1}/n_k > 1 + c_k/\sqrt{k}$  with  $c_k \rightarrow \infty$  by Erdős [5] for  $E = [0, 1]$ , and by Takahashi [15] for general  $E$ . This gap condition is best possible in the following sense: There exists a sequence  $\{n_k\}$  with  $n_{k+1}/n_k > 1 + c/\sqrt{k}$

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The author was supported in part by KAKENHI 19204008

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for some  $c > 0$  which fails to obey the central limit theorem. Such examples were constructed by Erdős [5] without any proof and by Takahashi [16] with a proof.

As to the existence of lacunary series with small gaps obeying the central limit theorem, Berkes [1] proved the following: For all  $A(k) \uparrow \infty$ , there exists a sequence  $\{n_k\}$  satisfying  $n_{k+1} - n_k = O(A(k))$  and obeying the central limit theorem on  $E = [0, 1]$ .

It was an open problem if there exists a sequence  $\{n_k\}$  with bounded gaps  $n_{k+1} - n_k = O(1)$  and obeying the central limit theorem. As to sequences with bounded gaps, Bobkov-Götze [4] proved that the limiting variance must be less than  $1/2$  and it is impossible to have the central limit theorem with limiting variance  $1/2$ . Instead of having pure gaussian limit, they constructed a sequence whose limit distribution is a mixture of gaussian distribution.

As to this problem, our previous paper [6] proved that there exists a sequence of bounded gaps obeying the central limit theorem with  $\sigma^2 = 1/4$ . It was also proved that the class of limit distributions of bounded gaps sequences contains huge class of mixtures of gaussian distributions. But there still remains the problem asking if any centered pure gaussian distribution with variance less than  $1/2$  can be a limit distribution of the central limit theorem for bounded gaps sequence. Now we are in a position to state the main result.

**Theorem 1** *For any  $0 < \sigma^2 < 1/2$ , there exists a sequence  $\{n_k\}$  of positive integers with  $1 \leq n_{k+1} - n_k \leq G(\sigma^2)$  obeying the central limit theorem (1) with limiting variance  $\sigma^2$ , where*

$$G(\sigma^2) = \begin{cases} 1 & \text{if } \sigma^2 = 0, \\ 2 & \text{if } 0 < \sigma^2 \leq 3/2 - \sqrt{2} = 0.085 \dots, \\ 3 & \text{if } 3/2 - \sqrt{2} < \sigma^2 \leq 9/2 - \sqrt{19} = 0.141 \dots, \\ 4 & \text{if } 9/2 - \sqrt{19} < \sigma^2 \leq 7/2 - \sqrt{11} = 0.183 \dots, \\ 5 & \text{if } 7/2 - \sqrt{11} < \sigma^2 \leq 11/2 - 2\sqrt{7} = 0.208 \dots, \end{cases}$$

and  $G(\sigma^2) = 2 \lceil (-4\sigma^4 + 12\sigma^2 + 3)/4(1 - 2\sigma^2) \rceil$  if  $11/2 - 2\sqrt{7} < \sigma^2 < 1/2$ .

The proof is by randomization which originated with Salem-Zygmund [13], and our model of randomization is given by a modification of those introduced by Berkes [2] and by Bobkov-Götze [4].

## 2 The central limit theorem

Let  $d \geq 2$  and  $\{X_n\}$  be a sequence of random variables defined as below: Let

$$\{(X_{\nu(d^2+2)+\mu d+1}, \dots, X_{\nu(d^2+2)+(\mu+1)d}), X_{\nu(d^2+2)+d^2+1}, X_{(\nu+1)(d^2+2)}\}_{\substack{\mu=0,1,\dots,d-1; \\ \nu=0,1,\dots}}$$

be an independent system whose distribution is defined by

$$\begin{aligned} P((X_{\nu(d^2+2)+\mu d+1}, \dots, X_{\nu(d^2+2)+(\mu+1)d}) &= (\delta_{1,j}, \dots, \delta_{d,j})) = 1/d, \quad (j = 1, \dots, d), \\ P((X_{\nu(d^2+2)+d(d-1)+1}, \dots, X_{\nu(d^2+2)+d^2}) &= (0, \dots, 0)) = (d-1)/d, \\ P((X_{\nu(d^2+2)+d(d-1)+1}, \dots, X_{\nu(d^2+2)+d^2}) &= (1, \dots, 1)) = 1/d, \\ P(X_{\nu(d^2+2)+d^2+1} &= 1) = 1, \\ P(X_{(\nu+1)(d^2+2)} &= 1) = 1 - P(X_{(\nu+1)(d^2+2)} = 0) = p, \end{aligned}$$

for  $\mu = 0, \dots, d-2$ , and  $\nu = 0, 1, \dots$ . We introduce notation. Denote  $c(x) = \cos 2\pi x$ ,  $S_L^{L'}(x) = \sum_{k=L+1}^{L'} X_k c(kx)$ ,  $\xi_{\nu,\mu} = S_{\nu(d^2+2)+\mu d}^{\nu(d^2+2)+(\mu+1)d}$ ,  $\eta_\nu = S_{\nu(d^2+2)+(d-1)d}^{\nu(d^2+2)+d^2}$ ,  $\zeta_\nu = S_{\nu(d^2+2)+d^2}^{(\nu+1)(d^2+2)}$  and  $\Xi_\nu = S_{\nu(d^2+2)}^{(\nu+1)(d^2+2)}$ . We have  $\Xi_\nu = \sum_{\mu=0}^{d-2} \xi_{\nu,\mu} + \eta_\nu + \zeta_\nu$ . Clearly the system  $\{\xi_{\nu,\mu}, \eta_\nu, \zeta_\nu\}_{\mu=0,1,\dots,d-2, \nu=0,1,\dots}$  is independent, and the sequence  $\{\Xi_\nu\}_{\nu=0,1,\dots}$  is also independent. Denote  $\tilde{\Xi}_\nu = \Xi_\nu - E\Xi_\nu$ ,  $\tilde{\xi}_{\nu,\mu} = \xi_{\nu,\mu} - E\xi_{\nu,\mu}$ ,  $\tilde{\eta}_\nu = \eta_\nu - E\eta_\nu$ ,  $\tilde{\zeta}_\nu = \zeta_\nu - E\zeta_\nu$ ,  $\tilde{S}_L^{L'} = S_L^{L'} - ES_L^{L'}$ ,  $T_N = \sum_{\nu=0}^{N-1} \tilde{\Xi}_\nu(x)/\sqrt{N} = \tilde{S}_0^{N(d^2+2)}/\sqrt{N}$ , and  $\sigma_{d,p}^2 = (d-1+p(1-p))/2$ .

We prove

$$P\left(T_N \xrightarrow{\mathcal{D}} \mathfrak{N}_{0,\sigma_{d,p}^2} \text{ on } (E, \mathcal{B}(E), dx/|E|) \text{ for all } E \text{ with } |E| > 0.\right) = 1. \quad (2)$$

First, we prove bounded convergence for the law of large numbers for  $\{\tilde{\Xi}_\nu^2\}$ :

$$\frac{1}{N} \sum_{\nu=0}^{N-1} \tilde{\Xi}_\nu^2(x) \leq 4(d^2+2)^2 \quad \text{and} \quad P\left(\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\nu=0}^{N-1} \tilde{\Xi}_\nu^2(x) = \sigma_{d,p}^2, \text{ a.e. } x\right) = 1. \quad (3)$$

Because of  $|\Xi_\nu| \leq d^2+2$ , we have  $|\tilde{\Xi}_\nu| \leq 2(d^2+2)$ , and hence the first inequality is clear. By noting  $E\tilde{\xi}_{\nu,\mu}^2 = E\xi_{\nu,\mu}^2 - (E\xi_{\nu,\mu})^2$ , we have

$$E\tilde{\xi}_{\nu,\mu}^2 = \frac{1}{d} \sum_{k=1}^d c^2((\nu(d^2+2) + \mu d + k)x) - \left(\frac{1}{d} \sum_{k=1}^d c((\nu(d^2+2) + \mu d + k)x)\right)^2.$$

Since  $\{\nu(d^2+2)x\}_{\nu=0,1,\dots}$  is uniformly distributed mod 1 when  $x \notin \mathbf{Q}$ , we have

$$\begin{aligned} \frac{1}{N} \sum_{\nu=0}^{N-1} E\tilde{\xi}_{\nu,\mu}^2 &\rightarrow \frac{1}{d} \sum_{k=1}^d \int_0^1 c^2(y + (\mu d + k)x) dy - \frac{1}{d^2} \int_0^1 \left(\sum_{k=1}^d c(y + (\mu d + k)x)\right)^2 dy \\ &= \frac{1}{2} - \frac{1}{d^2} \int_0^1 \left(\sum_{k=1}^d c(y + kx)\right)^2 dy. \end{aligned}$$

And similarly, by  $E\tilde{\eta}_\nu^2 = ((d-1)/d^2) \left(\sum_{k=1}^d c((\nu(d^2+2) + (d-1)d + k)x)\right)^2$  and  $E\tilde{\zeta}_\nu^2 = p(1-p)c^2((\nu(d^2+2) + d^2+2)x)$ , we have

$$\begin{aligned} \frac{1}{N} \sum_{\nu=0}^{N-1} E\tilde{\eta}_\nu^2 &\rightarrow \frac{d-1}{d^2} \int_0^1 \left(\sum_{k=1}^d c(y + ((d-1)d + k)x)\right)^2 dy \\ &= \frac{d-1}{d^2} \int_0^1 \left(\sum_{k=1}^d c(y + kx)\right)^2 dy. \\ \frac{1}{N} \sum_{\nu=0}^{N-1} E\tilde{\zeta}_\nu^2 &\rightarrow p(1-p) \int_0^1 c^2(y + (d^2+2)x) dy = \frac{p(1-p)}{2}. \end{aligned}$$

Hence by  $E\tilde{\Xi}_\nu^2 = \sum_{\mu=0}^{d-2} E\tilde{\xi}_{\nu,\mu}^2 + E\tilde{\eta}_\nu^2 + E\tilde{\zeta}_\nu^2$ , we have  $\sum_{\nu=0}^{N-1} E\tilde{\Xi}_\nu^2(x)/N \rightarrow \sigma_{d,p}^2$  for all  $x \notin \mathbf{Q}$ , and hence for a.e.  $x$ .

Because  $\{\tilde{\Xi}_\nu^2\}$  is a sequence of uniformly bounded independent random variables, by the law of large numbers, we have  $\sum_{\nu=0}^{N-1} (\tilde{\Xi}_\nu^2(x) - E\tilde{\Xi}_\nu^2(x))/N \rightarrow 0$  a.s., for all  $x$ . By Fubini's theorem, we see that it holds for a.e.  $x$ , a.s., that is (3).

Secondly, we put  $\Pi_N(x) = \prod_{\nu=0}^{N-1} (1 + i\lambda \tilde{\Xi}_\nu(x)/\sqrt{N})$  and prove

$$P(\Pi_{N^{3/2}} \rightarrow 1, L^2([0, 1], dx)\text{-weakly, as } N \rightarrow \infty.) = 1. \quad (4)$$

By  $|1 + it|^2 = 1 + t^2 \leq e^{t^2}$ , we have

$$|\Pi_N|^2 = \prod_{\nu=0}^{N-1} \left(1 + \frac{\lambda^2 \tilde{\Xi}_\nu^2(x)}{N}\right) \leq \exp\left(\sum_{\nu=0}^{N-1} \frac{\lambda^2 \tilde{\Xi}_\nu^2(x)}{N}\right) \leq e^{4\lambda^2(d^2+1)^2}. \quad (5)$$

Hence we have  $\sup_N \int_0^1 |\Pi_N(x)|^2 dx < \infty$  and thereby it is sufficient to prove that

$$P\left(\lim_{N \rightarrow \infty} \int_0^1 e^{inx} \Pi_{N^{3/2}}(x) dx = \int_0^1 e^{inx} dx = \delta_{n,0} \text{ for all } n \in \mathbf{Z}.\right) = 1 \quad (6)$$

Put  $J_N = \int_0^1 e^{inx} \Pi_N dx$ . Clearly we have  $EJ_N = \int_0^1 e^{inx} dx = \delta_{n,0}$  and

$$\begin{aligned} E|J_N - \delta_{n,0}|^2 &= E \int_0^1 \int_0^1 e^{in(x-y)} \prod_{\nu=0}^{N-1} \left(1 + \frac{i\lambda}{\sqrt{N}} \tilde{\Xi}_\nu(x)\right) \left(1 - \frac{i\lambda}{\sqrt{N}} \tilde{\Xi}_\nu(y)\right) dx dy - \delta_{n,0} \\ &= \int_0^1 \int_0^1 e^{in(x-y)} \prod_{\nu=0}^{N-1} \left(1 + \frac{\lambda^2}{N} E \tilde{\Xi}_\nu(x) \tilde{\Xi}_\nu(y)\right) dx dy - \delta_{n,0}. \end{aligned}$$

By noting  $|\tilde{\Xi}_\nu(x) \tilde{\Xi}_\nu(y)| \leq (2(d^2 + 1))^2$  and  $1 + x = \exp(x + O(x^2))$  for  $|x| \leq 1$ , we have

$$\begin{aligned} \prod_{\nu=0}^{N-1} \left(1 + \frac{\lambda^2}{N} E \tilde{\Xi}_\nu(x) \tilde{\Xi}_\nu(y)\right) &= \exp(G_N(x, y) + NO(1/N^2)) \\ &= 1 + G_N(x, y) + O(G_N^2(x, y)) + O(1/N), \end{aligned}$$

where  $G_N(x, y) = (\lambda^2/N) \sum_{\nu=0}^{N-1} E \tilde{\Xi}_\nu(x) \tilde{\Xi}_\nu(y)$  and the orders denoted by  $O$  are uniform in  $x$ . Note that  $|G_N(x, y)| \leq (2(d^2 + 2))^2 \lambda^2$ . If  $\nu \neq \nu'$ , then  $\int_0^1 \tilde{\Xi}_\nu(x) \tilde{\Xi}_{\nu'}(x) dx = 0$ . Hence  $\{E \tilde{\Xi}_\nu(x) \tilde{\Xi}_\nu(y)\}$  is orthogonal on  $[0, 1]^2$  and we have

$$\int_0^1 \int_0^1 G_N^2(x, y) dx dy = \frac{\lambda^4}{N^2} \sum_{\nu=0}^{N-1} \int_0^1 \int_0^1 (E \tilde{\Xi}_\nu(x) \tilde{\Xi}_\nu(y))^2 dx dy \leq \frac{(2(d^2 + 2))^4 \lambda^4}{N}.$$

Hence we have verified

$$\left| \int_0^1 \int_0^1 e^{in(x-y)} G_N^L(x, y) dx dy \right| \leq \frac{(2(d^2 + 1))^{2L} \lambda^{2L}}{N}$$

for  $L = 2$ . Since  $\int_0^1 e^{inx} \tilde{\Xi}_\nu(x) dx = 0$  except for at most one  $\nu$ , we can verify the above inequality for  $L = 1$ . Hence we have

$$\int_0^1 \int_0^1 e^{in(x-y)} \prod_{\nu=0}^{N-1} \left(1 + \frac{\lambda^2}{N} E \tilde{\Xi}_\nu(x) \tilde{\Xi}_\nu(y)\right) dx dy = \delta_{n,0} + O(1/N)$$

or  $E|J_N - \delta_{n,0}|^2 = O(1/N)$  for all  $n \in \mathbf{Z}$ . Since the right hand side is summable in  $N$  if we replace  $N$  by  $N^{3/2}$ , by applying Beppo Levi's theorem we have (6) and hence (4).

Now we prove (2). We first prove (2) holds for subsequence  $\{T_{N^{3/2}}\}$ . By applying  $e^{it} = (1 + it) \exp(-t^2/2 + r(t))$  where  $r(t) = O(|t|^3)$ , we have

$$\exp(i\lambda T_N(x)) = \Pi_N(x) \exp\left(-\frac{\lambda^2}{2N} \sum_{\nu=0}^{N-1} \tilde{\Xi}_\nu^2(x) + \sum_{\nu=0}^{N-1} r(i\lambda \tilde{\Xi}_\nu(x)/\sqrt{N})\right).$$

Here we have the estimate  $R_N(x) = \sum_{\nu=0}^{N-1} r(i\lambda \tilde{\Xi}_\nu(x)/\sqrt{N}) = O(N^{-1/2})$  uniformly in  $x$ . By (5) and (3), we have

$$\begin{aligned} & \left| \frac{1}{|E|} \int_E \exp(i\lambda T_N(x)) dx - \frac{e^{-\sigma_{d,p}^2 \lambda^2 / 2}}{|E|} \int_E \Pi_N(x) dx \right| \\ & \leq \frac{1}{|E|} \int_E |\Pi_N(x)| \left| \exp\left(-\frac{\lambda^2}{2N} \sum_{\nu=0}^{N-1} \tilde{\Xi}_\nu^2(x) + R_N(x)\right) - e^{-\sigma_{d,p}^2 \lambda^2 / 2} \right| dx \\ & \leq \frac{1}{|E|} \int_0^1 \left| \exp(R_N(x)) - \exp\left(\frac{\lambda^2}{2N} \sum_{\nu=0}^{N-1} \tilde{\Xi}_\nu^2(x) - \frac{\sigma_{d,p}^2 \lambda^2}{2}\right) \right| dx \rightarrow 0. \end{aligned}$$

Due to (4), we have

$$\frac{1}{|E|} \int_E \Pi_{N^{3/2}}(x) dx = \frac{1}{|E|} \int_0^1 \mathbf{1}_E \Pi_{N^{3/2}}(x) dx \rightarrow \frac{1}{|E|} \int_0^1 \mathbf{1}_E dx = 1, \quad \text{a.s.}$$

Hence we have

$$\frac{1}{|E|} \int_E \exp(i\lambda T_{N^{3/2}}(x)) dx \rightarrow e^{-\sigma_{d,p}^2 \lambda^2 / 2},$$

a.s., for all  $\lambda$ . By applying Fubini's Theorem again we see that it holds for a.e.  $\lambda$ , a.s. If a sequence of characteristic functions converges to a characteristic function a.e., then the corresponding sequence of probability measures converges weakly and characteristic function converges at every point. (This fact can be found as Corollary 2 of 203pp of [9]. It is also easily known by modifying the standard proof of Glivenko Theorem (e.g., Theorem 2.6.4 of [7] or Theorem 26.3 of [3].) Hence we have verified (2) for  $\{T_{N^{3/2}}\}$ .

Since  $N^{3/2}/(N-1)^{3/2} \rightarrow 1$ , to prove (2) for  $\{T_N\}$ , it is enough to prove the convergence of  $\{T'_M\}$  defined as below:

$$T'_M = \frac{1}{\sqrt{(N-1)^{3/2}}} \sum_{\nu=0}^{M-1} \tilde{\Xi}_\nu(x), \quad (N-1)^{3/2} < M \leq N^{3/2}.$$

Since  $|\tilde{\Xi}_\nu|$  is bounded by  $2(d^2 + 1)$ ,

$$|T'_M - T_{(N-1)^{3/2}}| \leq 2(d^2 + 1)(N^{3/2} - (N-1)^{3/2})/(N-1)^{3/4} = O(N^{-1/4}) = o(1),$$

which completes the proof of (2).

Since  $\tilde{\Xi}_\nu$  is a sum of  $d^2 + 2$  terms, by putting  $\sigma_\Delta^2 = \sigma_{d,p}^2/(d^2 + 2)$ , we have

$$P\left(\tilde{S}_0^K / \sqrt{K} \xrightarrow{\mathcal{D}} \mathfrak{N}_{0, \sigma_\Delta^2} \text{ on } (E, \mathcal{B}(E), dx/|E|) \text{ for all } E \text{ with } |E| > 0.\right) = 1 \quad (7)$$

By noting the property  $\sup_K |\sum_{k=1}^K c((Ak+B)x)| < \infty$  a.e.  $x$  of Dirichlet kernel, we have

$$\begin{aligned} S_0^K - \tilde{S}_0^K &= \frac{1}{d} \sum_{k=1}^K c(kx) + \left(1 - \frac{1}{d}\right) \sum_{\nu: \nu(d^2+2)+d^2+1 \leq K} c((\nu(d^2+2)+d^2+1)x) \\ &\quad + \left(p(1-p) - \frac{1}{d}\right) \sum_{\nu: \nu(d^2+2)+d^2+2 \leq K} c((\nu(d^2+2)+d^2+2)x) \\ &= O(1), \end{aligned}$$

and thereby (7) is valid if we replace  $\tilde{S}_0^K$  by  $S_0^K$ . Arrange the sequence  $\{k \mid X_k = 1\}$  in increasing order and denote it by  $\{n_k\}$ . If we put  $H_K = \#\{k \leq K \mid X_k = 1\} = \#\{k \in \mathbf{N} \mid n_k \leq K\}$ , by the law of large numbers, we have  $H_K \sim Ke_{d,p}$  where  $e_{d,p} = (d+1+p)/(d^2+2)$ . Hence we have

$$P\left(S_0^K / \sqrt{H_K} \xrightarrow{\mathcal{D}} \mathfrak{N}_{0, \sigma_*^2} \text{ on } (E, \mathcal{B}(E), dx/|E|) \text{ for all } E \text{ with } |E| > 0.\right) = 1 \quad (8)$$

where  $\sigma_*^2 = \sigma_{\Delta}^2 / e_{d,p} = (d-1+p(1-p))/2(d+1+p)$ .

By definition, we see that  $n_{k+1} - n_k \leq 2d$ . As a continuous function of  $p \in [0, 1]$ , the limiting variance  $\sigma_*^2$  takes the maximum  $(2d+3)/2 - \sqrt{(d+1)^2+2}$  at  $p = -(d+1) + \sqrt{(d+1)^2+2}$  and the minimum  $(d-1)/2(d+2)$  at  $p = 1$ . Hence any  $\sigma^2 \in [(d-1)/2(d+2), (2d+3)/2 - \sqrt{(d+1)^2+2}] = I_d$  can be a limiting variance in (1) for a sequence with  $n_{k+1} - n_k \leq 2d$ .

By  $d/2(d+3) \leq (2d+3)/2 - \sqrt{(d+1)^2+2}$  we have  $I_d \cap I_{d+1} \neq \emptyset$  for  $d \geq 3$ . Therefore we see that  $\sigma^2 \in [1/5 = 0.2, 1/2) = I_3 \cup I_4 \cup \dots$  can be a limiting variance in (1) for a sequence with bounded gaps. From  $\sigma^2 \leq (2d+3)/2 - \sqrt{(d+1)^2+2}$ , we can deduce  $d \geq (-4\sigma^4 + 12\sigma^2 + 3)/4(1-2\sigma^2)$  and hence we can put  $G(\sigma^2) = 2[-4\sigma^4 + 12\sigma^2 + 3]/4(1-2\sigma^2)$  in this case.

The case  $d = 2$  proves the case  $\sigma^2 \in [1/8 = 0.125, 7/2 - \sqrt{11} = 0.183\dots]$ . We have  $n_{k+1} - n_k \leq 4$  and we can put  $G(\sigma^2) = 4$ .

Now we construct other model of randomization to have smaller limiting variances.

Firstly, consider an independent system  $\{(X_{12\nu+3\mu+1}, X_{12\nu+3\mu+2}, X_{12\nu+3\mu+3}), (X_{12\nu+8}, X_{12\nu+9}, X_{12\nu+10}), X_{12\nu+7}, X_{12\nu+11}, X_{12\nu+12}\}_{\nu=0,1,\dots; \mu=1,2}$  defined by

$$\begin{aligned} P((X_{12\nu+3\mu+1}, X_{12\nu+3\mu+2}, X_{12\nu+3\mu+3}) &= (\delta_{1,j}, \dots, \delta_{3,j})) = 1/3 \quad (j = 1, 2, 3), \\ P((X_{12\nu+8}, X_{12\nu+9}, X_{12\nu+10}) &= (0, 0, 0)) = 2/3, \\ P((X_{12\nu+8}, X_{12\nu+9}, X_{12\nu+10}) &= (1, 1, 1)) = 1/3, \\ P(X_{12\nu+7} = 1) &= P(X_{12\nu+11} = 1) = 1, \\ P(X_{12\nu+12} = 1) &= 1 - P(X_{12\nu+12} = 0) = p, \end{aligned}$$

for  $\mu = 0, 1$  and  $\nu = 0, 1, 2, \dots$ . In this case we have  $\sigma_*^2 = (2+p(1-p))/2(5+p)$  in (8) and  $n_{k+1} - n_k \leq 5$ . The variance  $\sigma_*^2$  takes its maximum  $11/2 - 2\sqrt{7}$  at  $p = -5 + 2\sqrt{7}$  and minimum  $1/6$  at  $p = 1$ . Hence for  $\sigma^2 \in [1/6 = 0.166\dots, 11/2 - 2\sqrt{7} = 0.208\dots]$  we can put  $G(\sigma^2) = 5$ .

Secondly, consider an independent system  $\{(X_{7\nu+1}, X_{7\nu+2}), (X_{7\nu+4}, X_{7\nu+5}), X_{7\nu+3}, X_{7\nu+6}, X_{7\nu+7}\}_{\nu=0,1,\dots}$  defined by

$$\begin{aligned} P((X_{7\nu+1}, X_{7\nu+2}) &= (\delta_{1,j}, \delta_{2,j})) = 1/2 \quad (j = 1, 2), \\ P((X_{7\nu+4}, X_{7\nu+5}) &= (1, 1)) = P((X_{7\nu+4}, X_{7\nu+5}) = (0, 0)) = 1/2, \\ P(X_{7\nu+3} = 1) &= P(X_{7\nu+6} = 1) = 1, \quad P(X_{7\nu+7} = 1) = 1 - P(X_{7\nu+7} = 0) = p, \end{aligned}$$

for  $\nu = 0, 1, 2, \dots$ . In this case we have  $\sigma_*^2 = (1 + p(1 - p))/2(4 + p)$  in (8) and  $n_{k+1} - n_k \leq 3$ . The variance  $\sigma_*^2$  takes its maximum  $9/2 - \sqrt{19}$  at  $p = -4 + \sqrt{19}$  and minimum  $1/10$  at  $p = 1$ . Hence for  $\sigma^2 \in [1/10 = 0.1, 9/2 - \sqrt{19} = 0.141\dots]$ , we can put  $G(\sigma^2) = 3$ .

Thirdly, consider an independent system  $\{(X_{8\nu+1}, X_{8\nu+2}), (X_{8\nu+4}, X_{8\nu+5}), X_{8\nu+3}, X_{8\nu+6}, X_{8\nu+7}, X_{8\nu+8}\}_{\nu=0,1,\dots}$  defined by

$$\begin{aligned} P((X_{8\nu+1}, X_{8\nu+2}) = (\delta_{1,j}, \delta_{2,j})) &= 1/2 \quad (j = 1, 2), \\ P((X_{8\nu+4}, X_{8\nu+5}) = (1, 1)) &= P((X_{8\nu+4}, X_{8\nu+5}) = (0, 0)) = 1/2, \\ P(X_{8\nu+3} = 1) &= P(X_{8\nu+6} = 1) = P(X_{8\nu+8} = 1) = 1, \\ P(X_{8\nu+7} = 1) &= 1 - P(X_{8\nu+7} = 0) = p, \end{aligned}$$

for  $\nu = 0, 1, 2, \dots$ . In this case we have  $\sigma_*^2 = (1 + p(1 - p))/2(5 + p)$  in (8) and  $n_{k+1} - n_k \leq 3$ . The variance  $\sigma_*^2$  takes its maximum  $11/2 - \sqrt{29}$  at  $p = -6 + \sqrt{29}$  and minimum  $1/12$  at  $p = 1$ . Hence for  $\sigma^2 \in [1/12 = 0.083, 11/2 - \sqrt{29} = 0.114\dots]$  we can put  $G(\sigma^2) = 3$ .

Fourthly, consider an independent sequence  $\{X_n\}$  defined by

$$P(X_{2\nu+1} = 1) = 1, \quad P(X_{2\nu+2} = 1) = 1 - P(X_{2\nu+2} = 0) = p$$

for  $\nu = 0, 1, 2, \dots$ . In this case we have  $\sigma_*^2 = p(1 - p)/2(1 + p)$  in (8) and  $n_{k+1} - n_k \leq 2$ . The variance  $\sigma_*^2$  takes its maximum  $3/2 - \sqrt{2}$  at  $p = -1 + \sqrt{2}$  and minimum 0 at  $p = 1$ . Hence for  $\sigma^2 \in [0, 3/2 - \sqrt{2} = 0.085\dots]$ , we can put  $G(\sigma^2) = 2$ .

Finally, if we put  $n_k = k$ , then we have  $\sigma^2 = 0$  and hence  $G(0) = 1$ .

For  $\sigma^2$  which has multiple possibilities of values of  $G(\sigma^2)$ , by choosing the minimum among them, we have the conclusion.

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