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Fukuyama, Katusi

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Almost sure invariance principles for lacunary trigonometric series

Katusi FUKUYAMA

Department of mathematics, Kobe University, Rokko Kobe 657-8501, Japan
 E-mail: fukuyama@math.kobe-u.ac.jp

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Abstract. We prove almost sure invariance principles for lacunary trigonometric series. The result includes the series presented by P. Erdős. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Principe d'invariance pour séries trigonométriques lacunaires

Résumé. Nous montrons un principe d'invariance presque sûr pour des séries trigonométriques lacunaires. Le résultat inclut les séries présentées par P. Erdős. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

De multiples études ont été faites sur les théorèmes limites pour sommes $S_k(\omega) = \sum_{j=1}^k \sqrt{2} \cos 2\pi n_j \omega$ sur l'espace probabilisé de Lebesgue $([0, 1), \mathcal{B}[0, 1), d\omega)$, où la suite $\{n_k\}$ des nombres entiers satisfait

$$n_{k+1}/n_k \geq 1 + c/k^\alpha. \quad (1)$$

Erdős [2] montra le théorème limite central au cas de $\alpha < 1/2$, et conjectura que le théorème fut valable pour la suite $n_k = \lfloor \exp(k^\beta) \rfloor$, où $\beta > 0$ et $[x]$ désignant la part intégrale du nombre réel x . Parce que la suite satisfait (1) où $\alpha = 1 - \beta$, le cas $\beta > 1/2$ est un exemple du résultat par Erdős. Maintenant, on introduit quelques notations. $\#A$ désigne le nombre cardinal de l'ensemble A . On note

$$\Xi(G) = \{(m_1, m_2, m_3, m_4) \in (G \cup -G)^4 \mid m_1 + m_2 + m_3 + m_4 = 0, |m_i| \neq |m_j| (i \neq j)\}$$

pour $G \subset \mathbb{N}$. Murai [7] a montré le théorème limite central en supposant $\#\Xi(\{n_1, \dots, n_k\}) = o(k^2)$ et une condition plus faible que (1). Pour $n_k = \lfloor \exp(k^\beta) \rfloor$, où $\beta > 4/9$, Murai [7] a montré

$$\#\Xi(\{n_1, \dots, n_k\}) = o(k^{2-\delta}) \quad (2)$$

pour certain $\delta > 0$, et a résolu la conjecture partiellement.

Berkes [1] a obtenu un principe d'invariance presque sûr supposant (1) et $\#\Xi(\{n_\ell, \dots, n_k\}) = o(k^\delta(k - \ell)^2)$, où $0 < \delta < 1 - \alpha$. Mais cette condition est trop forte pour vérifier au cas de la suite $n_k = \lfloor \exp(k^\beta) \rfloor$.

Note présentée par Jean-Pierre KAHANE

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Le tableau ci-dessous présente les résultats connus pour la suite $n_k = \lfloor \exp(k^\beta) \rfloor$.

	TLC	LLI	PIPS
$\beta \in (1, \infty)$	Kac [5] (1939)	Erdős-Gál [3] (1955)	Philipp-Stout [8] (1975)
$\beta = 1$	Salem-Zygmund [9] (1947)	Erdős-Gál [3] (1955)	Philipp-Stout [8] (1975)
$\beta \in (1/2, 1)$	Erdős [2] (1962)	Takahashi [10] (1975)	Berkes [1] (1978)
$\beta \in (4/9, 1/2]$	Murai [7] (1982)	inconnu*	inconnu*
$\beta \in (0, 4/9]$	inconnu	inconnu	inconnu

L'objet de cette Note est de démontrer le théorème suivant.

THÉORÈME. – Soient $\{n_k\}$ une suite de nombres entiers positifs satisfaisant (1) pour un $\alpha \in [0, 1)$ et $c > 0$, et (2) pour un $\delta > 0$. Alors, on peut élargir l'espace probabilisé et construire une suite de variables aléatoires indépendantes standard normales $\{Z_k\}$, telles que

$$S_k = Z_1 + \cdots + Z_k + o(k^{\eta/2}) \text{ a.s. } (k \rightarrow \infty)$$

pour un $\eta < 1$.

En vertu d'un résultat de Murai [7], on peut vérifier un principe d'invariance presque sûr pour $n_k = \lfloor \exp(k^\beta) \rfloor$, où $\beta > 4/9$. Par conséquent, les cas avec * dans le tableau ci-dessus sont résolus.

1. Introduction

In the theory of the lacunary trigonometric series, there are many studies on limit theorems for $S_k(\omega) = \sum_{j=1}^k \sqrt{2} \cos 2\pi n_j \omega$ on Lebesgue probability space $([0, 1), \mathcal{B}[0, 1), d\omega)$, where the sequence $\{n_k\}$ of integers satisfies

$$n_{k+1}/n_k \geq 1 + c/k^\alpha. \quad (1)$$

Erdős [2] proved the central limit theorem for $\alpha < 1/2$ and conjectured that for $n_k = \lfloor \exp(k^\beta) \rfloor$ with $\beta > 0$ the theorem holds. Here $[x]$ denotes the integer part of real number x . Since this sequence satisfies (1) with $\alpha = 1 - \beta$, the case $\beta > 1/2$ is an example of Erdős's result. To state further results, we need the following notation. Denote $\#A$ the cardinal number of set A . Set

$$\Xi(G) = \{(m_1, m_2, m_3, m_4) \in (G \cup -G)^4 \mid m_1 + m_2 + m_3 + m_4 = 0, \|m_i\| \neq \|m_j\| (i \neq j)\}$$

for $G \subset \mathbb{N}$. Murai [7] proved the central limit theorem assuming $\#\Xi(\{n_1, \dots, n_k\}) = o(k^2)$ and a weaker condition as compared to (1). For $n_k = \lfloor \exp(k^\beta) \rfloor$ with $\beta > 4/9$, Murai [7] proved

$$\#\Xi(\{n_1, \dots, n_k\}) = o(k^{2-\delta}) \quad (2)$$

for some $\delta > 0$, and partially solved the conjecture.

Berkes [1] obtained the almost sure invariance principle assuming (1) and $\#\Xi(\{n_\ell, \dots, n_k\}) = o(k^\delta(k - \ell)^2)$ for some $0 < \delta < 1 - \alpha$. Unfortunately, this condition is too strong and hard to be verified for $n_k = \lfloor \exp(k^\beta) \rfloor$ with $\beta \leq 1/2$.

The following table shows the known limit theorems for $n_k = \lfloor \exp(k^\beta) \rfloor$.

Almost sure invariance principles for lacunary trigonometric series

	CLT	LIL	ASIP
$\beta \in (1, \infty)$	Kac [5] (1939)	Erdős–Gál [3] (1955)	Philipp–Stout [8] (1975)
$\beta = 1$	Salem–Zygmund [9] (1947)	Erdős–Gál [3] (1955)	Philipp–Stout [8] (1975)
$\beta \in (1/2, 1)$	Erdős [2] (1962)	Takahashi [10] (1975)	Berkes [1] (1978)
$\beta \in (4/9, 1/2]$	Murai [7] (1982)	unknown*	unknown*
$\beta \in (0, 4/9]$	unknown	unknown	unknown

Our result is as follows.

THEOREM. – *Let $\{n_k\}$ be a sequence of positive integers satisfying (1) for some $\alpha \in [0, 1)$ and $c > 0$, and (2) for some $\delta > 0$. Then we can enlarge the probability space and construct a standard normal i.i.d. $\{Z_k\}$ such that*

$$S_k = Z_1 + \cdots + Z_k + o(k^{\eta/2}) \text{ a.s. } (k \rightarrow \infty)$$

for some $\eta < 1$.

It is clear that we can enlarge the probability space again and embed i.i.d. into some Wiener process. Thus the ordinary statement of almost sure invariance principle ‘we can enlarge the probability space and construct a Wiener process such that $S_k = W(k) + o(k^{\eta/2})$ a.s.’ is derived from our result.

Thanks to the previous work by Murai [7], we see that the almost sure invariance principle holds for $n_k = [\exp(k^\beta)]$ with $\beta > 4/9$. Thus the fields having * in the above table are solved by this result.

Our proof almost follows Berkes’s line. Hunt inequality and Abel transform make it possible to show a new result.

2. Proof

Take $\gamma > 1/\delta$ and $0 < \rho < 1$ satisfying $\alpha < \rho - 1/\gamma$, and put

$$\begin{aligned} X_j(\omega) &= \sqrt{2} \cos 2\pi n_j \omega, \quad v(k) = [k^\gamma], \quad \tilde{v}(k) = [k^{\gamma\rho}], \\ J(k) &= (v(k-1) + \tilde{v}(k-1), v(k) + \tilde{v}(k-1)], \quad \tilde{J}(k) = (v(k) + \tilde{v}(k-1), v(k) + \tilde{v}(k)], \\ \Delta_k &= \sum_{j \in J(k)} X_j \quad \text{and} \quad \tilde{\Delta}_k = \sum_{j \in \tilde{J}(k)} X_j. \end{aligned}$$

Note that $\gamma > 1$. Denote by $\mathcal{F}(\ell)$ the sigma field on $[0, 1)$ generated by $\{[i2^{-\ell}, (i+1)2^{-\ell}) \mid i = 0, \dots, 2^\ell - 1\}$. Because $E(X_j \mid \mathcal{F}(\ell)) = 2^\ell \int_{i2^{-\ell}}^{(i+1)2^{-\ell}} X_j(\omega) d\omega$ on $[i2^{-\ell}, (i+1)2^{-\ell})$, we have

$$|E(X_j \mid \mathcal{F}(\ell)) - X_j| \leq C n_j / 2^\ell \quad \text{and} \quad |E(X_j \mid \mathcal{F}(\ell))| \leq C 2^\ell / n_j.$$

Here, and now on, C denotes deterministic constant which may change one by one. Put

$$\ell(k) = [\log(k^{2\gamma} n_{v(k)+\tilde{v}(k-1)})], \quad \tilde{\ell}(k) = [\log(k^{2\gamma} n_{v(k)+\tilde{v}(k)})], \quad \mathcal{F}_k = \mathcal{F}(\ell(k)) \quad \text{and} \quad \tilde{\mathcal{F}}_k = \mathcal{F}(\tilde{\ell}(k)).$$

Then, by $v(k) - v(k-1), \tilde{v}(k) - \tilde{v}(k-1) \leq 2\gamma k^{\gamma-1}$, we have

$$|E(\Delta_k \mid \mathcal{F}_k) - \Delta_k| \leq 2\gamma k^{\gamma-1} C n_{v(k)+\tilde{v}(k-1)} 2^{-\ell(k)} \leq C k^{-1-\gamma} \quad \text{and} \quad |E(\tilde{\Delta}_k \mid \tilde{\mathcal{F}}_k) - \tilde{\Delta}_k| \leq C k^{-1-\gamma}.$$

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Since we have $n_{k+1}/n_k \geq \exp(c/2k^\alpha)$ and $\gamma\rho - 1 - \gamma\alpha > 0$, we have

$$\frac{n_{v(k-1)+\tilde{v}(k-1)}}{n_{v(k-1)+\tilde{v}(k-2)}} \geq \exp\left(\frac{c(\tilde{v}(k-1) - \tilde{v}(k-2))}{2(v(k-1) + \tilde{v}(k-1))^\alpha}\right) \geq \exp(C k^{\gamma\rho-1-\gamma\alpha}) \geq C k^{5\gamma}$$

for some $C > 0$. Thus we have

$$|E(\Delta_k | \mathcal{F}_{k-1})| \leq 2\gamma k^{\gamma-1} C \frac{2^{\ell(k-1)}}{n_{v(k-1)+\tilde{v}(k-1)}} \leq C k^{-2\gamma-1} \quad \text{and} \quad |E(\tilde{\Delta}_k | \tilde{\mathcal{F}}_{k-1})| \leq C \gamma k^{-2\gamma-1}.$$

Put $Y_k = E(\Delta_k | \mathcal{F}_k) - E(\Delta_k | \mathcal{F}_{k-1})$ and $\tilde{Y}_k = E(\tilde{\Delta}_k | \tilde{\mathcal{F}}_k) - E(\tilde{\Delta}_k | \tilde{\mathcal{F}}_{k-1})$. Then $\{Y_k\}$ and $\{\tilde{Y}_k\}$ are martingale difference sequences satisfying

$$|Y_k - \Delta_k|, |\tilde{Y}_k - \tilde{\Delta}_k| \leq C k^{-\gamma-1} \quad \text{and} \quad |Y_k^2 - \Delta_k^2|, |\tilde{Y}_k^2 - \tilde{\Delta}_k^2| \leq C k^{-2}. \quad (3)$$

First, two inequalities are trivial. The rest is derived from $|Y^2 - \Delta^2| \leq (2|\Delta| + |Y - \Delta|)|Y - \Delta|$ and $|\Delta_k| \leq 2\gamma k^{\gamma-1}$.

Now we prepare some lemmas.

LEMMA 1. – *There exists a positive integer L such that $|V_n - v(n)| < L$ and $|\tilde{V}_n - \tilde{v}(n)| < L$, where $V_n = \sum_{k=1}^n E(Y_k^2 | \mathcal{F}_{k-1})$ and $\tilde{V}_n = \sum_{k=1}^n E(\tilde{Y}_k^2 | \tilde{\mathcal{F}}_{k-1})$.*

Proof. – For a trigonometric polynomial S , denote by $\text{Spec}(S)$ the set of frequencies of S . We have

$$\min \text{Spec}(\Delta_k^2 - E\Delta_k^2) \geq \min_{j \in J(k)} (n_{j+1} - n_j) \geq c \min_{j \in J(k)} \frac{n_j}{j^\alpha} \geq c \frac{n_{v(k-1)+\tilde{v}(k-1)}}{(v(k) + \tilde{v}(k))^\alpha}.$$

As before we have

$$|E(\Delta_k^2 - E\Delta_k^2 | \mathcal{F}_{k-1})| \leq (2\gamma k^{\gamma-1})^2 C 2^{\ell(k-1)} \frac{(v(k) + \tilde{v}(k))^\alpha}{n_{v(k-1)+\tilde{v}(k-1)}} \leq C k^{-\gamma-2+\alpha\gamma} \leq C k^{-3}$$

and

$$|E(\tilde{\Delta}_{k+1}^2 - E\tilde{\Delta}_{k+1}^2 | \mathcal{F}_k)| \leq C k^{-3}.$$

From these we have the conclusion. □

LEMMA 2. – *If $a > 0$, $x_k \geq 0$ and $s_k = x_1 + \dots + x_k = O(k^a)$, then $\sum x_k/k^{a+\varepsilon} < \infty$ for $\varepsilon > 0$.*

Proof. – Denote $s_0 = 0$. By Abel transform, we have

$$\sum_{k=1}^K \frac{x_k}{k^{a+\varepsilon}} = \sum_{k=1}^K s_k \left(\frac{1}{k^{a+\varepsilon}} - \frac{1}{(k+1)^{a+\varepsilon}} \right) + \frac{s_K}{(K+1)^{a+\varepsilon}} = O\left(\sum_{k=1}^K \frac{k^a}{k^{a+1+\varepsilon}} + \frac{K^a}{K^{a+\varepsilon}} \right) = O(1).$$

□

LEMMA 3. – *For θ satisfying $(2\gamma - 1)/(2\gamma) < \theta < 1$, we have*

$$\sum_{k=1}^{\infty} \frac{E\Delta_k^4}{v(k)^{2\theta}} < \infty, \quad \sum_{k=1}^{\infty} \frac{E\tilde{\Delta}_k^4}{\tilde{v}(k)^{2\theta}} < \infty, \quad \sum_{k=1}^{\infty} \frac{EY_k^4}{v(k)^{2\theta}} < \infty, \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{E\tilde{Y}_k^4}{\tilde{v}(k)^{2\theta}} < \infty.$$

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Proof. – Let us first prove $ES^4 \leq 12(\#G)^2 + \#\Xi(G)$ for $S = \sum_{m \in G} \sqrt{2} \cos 2\pi m\omega$. By $ES^4 = \frac{1}{4} \# \{(m_1, m_2, m_3, m_4) \in (G \cup -G)^4 \mid m_1 + m_2 + m_3 + m_4 = 0\}$, we have

$$\begin{aligned} 0 &\leq ES^4 - \frac{1}{4} \#\Xi(G) \\ &\leq \frac{1}{4} \sum_{1 \leq i < j \leq 4} \# \left\{ (m_1, m_2, m_3, m_4) \in (G \cup -G)^4 \mid \begin{array}{l} m_1 + m_2 + m_3 + m_4 = 0 \\ |m_i| = |m_j| \end{array} \right\} \\ &\leq 3\#(G \cup -G)^2 = 12(\#G)^2 = 12(ES^2)^2. \end{aligned}$$

If G_1, \dots, G_n are mutually disjoint, we have $\#\Xi(G_1) + \dots + \#\Xi(G_n) \leq \#\Xi(G_1 \cup \dots \cup G_n)$. Thus

$$\begin{aligned} \sum_{k=1}^K E\Delta_k^4 &\leq 12 \sum_{k=1}^K (\#\text{Spec}(\Delta_k))^2 + \#\Xi\left(\text{Spec}\left(\sum_{k=1}^K \Delta_k\right)\right) \\ &\leq C \sum_{k=1}^K k^{2\gamma-2} + \left(\#\text{Spec}\left(\sum_{k=1}^K \Delta_k\right)\right)^{2-\delta} \\ &= O(K^{2\gamma-1} + K^{\gamma(2-\delta)}) = O(K^{2\gamma-1}). \end{aligned}$$

By Lemma 2 and $2\gamma - 1 < 2\theta\gamma$, we have the first inequality. $Y^4 \leq 8\Delta^4 + 8(Y - \Delta)^4$ and (3) yield the third. The rest is proved in the same way. \square

We use the embedding theorem below, which is a refinement of classical Strassen's work.

THEOREM A (Monrad–Philipp [6]). – *Let $\{Y_n, \mathcal{F}_n, n \geq 1\}$ be a real-valued square-integrable martingale difference sequence. Let f be a nondecreasing function with $f(x) \rightarrow \infty$ as $x \rightarrow \infty$ and such that $f(x)(\log x)^\alpha/x$ is nonincreasing for some $\alpha > 50$. Suppose that*

$$V_n = \sum_{j=1}^n E(Y_j^2 \mid \mathcal{F}_{j-1}) \rightarrow \infty \text{ a.s. and } \sum_{n=1}^{\infty} E\left(\frac{Y_n^2 \mathbf{1}_{\{Y_n^2 \geq f(V_n)\}}}{f(V_n)}\right) < \infty.$$

Suppose that there exists a random variable which is uniformly distributed over $[0, 1]$ and independent of the sequence $\{Y_n\}$. Then there exists a standard normal i.i.d. $\{Z_n\}$ such that with probability 1,

$$\sum_{k \geq 1} Y_k \mathbf{1}_{\{V_k \leq t\}} = \sum_{k \leq t} Z_k + o\left(t^{1/2} \left(\frac{f(t)}{t}\right)^{1/50}\right) \quad (t \rightarrow \infty).$$

Firstly, let our probability space product with another Lebesgue probability space and regard all random variables on original space as variables on enlarged space. Take θ as in Lemma 3 and put $f(x) = x^\theta$. Choose $\eta < 1$ bigger than θ , ρ , $1 - (1 - \theta)/25$ and $1 - 1/(2\gamma)$. By Lemmas 1 and 3, we have

$$\sum E\left(\frac{Y_n^2 \mathbf{1}_{\{Y_n^2 \geq V_n^\theta\}}}{V_n^\theta}\right) \leq C \sum E\frac{(Y_n^4)}{v(n)^{2\theta}} < \infty$$

and hence we can apply Theorem A.

Since $\gamma > 1$, we have $v(n) - v(n-1) \rightarrow \infty$. Noting this together with Lemma 1 and putting $t = v(n) + L$, we have $Y_1 + \dots + Y_n = W(v(n) + L) + o(v(n)^{\eta/2})$, where $W(n) = Z_1 + \dots + Z_n$. Because of (3) and $|W(v(n) + L) - W(v(n))| = O(\log v(n))$, we have $\Delta_1 + \dots + \Delta_n = W(v(n)) + o(v(n)^{\eta/2})$.

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In the same way we have $\tilde{\Delta}_1 + \cdots + \tilde{\Delta}_n = \tilde{W}(\tilde{v}(n)) + o(\tilde{v}(n)^{\eta/2})$ by using other i.i.d. $\{\tilde{Z}_n\}$ and $\tilde{W}(n) = \tilde{Z}_1 + \cdots + \tilde{Z}_n$. Because of $\tilde{W}(\tilde{v}(n)) = O(\tilde{v}(n)^{1/2}(\log \tilde{v}(n))^{1/2}) = o(v(n)^{\eta/2})$ we have $\tilde{\Delta}_1 + \cdots + \tilde{\Delta}_n = o(v(n)^{\eta/2})$.

The equalities $|W(v(n)) - W(v(n) + \tilde{v}(n))| = O(\tilde{v}(n)^{1/2}(\log(v(n) + \tilde{v}(n)))^{1/2}) = o((v(n) + \tilde{v}(n))^{\eta/2})$ together with the above estimates yield $S_{u(n)} = W(u(n)) + o(u(n)^{\eta/2})$, where $u(k) = v(k) + \tilde{v}(k)$.

Lastly, we prove the following estimates, which conclude the proof. Put

$$A_k := \max_{n=u(k)}^{u(k+1)} |S_n - S_{u(k)}| = o(u(k)^{\eta/2})$$

and

$$B_k := \max_{n=u(k)}^{u(k+1)} |W(n) - W(u(k))| = o(u(k)^{\eta/2}).$$

By Hunt inequality (see [4]), we have $EA_k^4 \leq C E|S_{u(k+1)} - S_{u(k)}|^4$. This together with Lemma 3 yields

$$\sum E \frac{A_k^4}{u(k)^{2\eta}} \leq C \sum \frac{E\Delta_k^4 + E\tilde{\Delta}_k^4}{u(k)^{2\eta}} < \infty.$$

By Beppo–Levi theorem, we have the first estimate.

The second estimate follows from Levy's inequality: by $EB_k^4 \leq C E|W(u(k+1)) - W(u(k))|^4$, we have

$$\sum E \frac{B_k^4}{u(k)^{2\eta}} \leq C \sum \frac{(u(k+1) - u(k))^2}{u(k)^{2\eta}} \leq C \sum k^{2(\gamma-1-\eta\gamma)} < \infty$$

since $2(\gamma - 1 - \eta\gamma) < -1$. By Beppo–Levi theorem again, we have the second estimate.

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