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Fukuyama, Katusi

(Citation)

Ergodic Theory and Dynamical Systems, 20(5):1335-1353

(Issue Date)

2000-10

(Resource Type)

journal article

(Version)

Accepted Manuscript

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(URL)

<https://hdl.handle.net/20.500.14094/90003840>



The central limit theorem for $\sum f(\theta^n x)g(\theta^{n^2} x)$

Dedicated to Professor Norio Kôno on his sixtieth birthday

KATUSI FUKUYAMA†

Department of Mathematics, Kobe University, Rokko Kobe 657-8501 Japan

(Received 14 July 1998, and revised ??? Apr 1999)

Abstract. In this note, it is proved that the distribution of values of $N^{-1/2} \sum_{n=1}^N f_1(\theta^{p_1(n)} x) \dots f_K(\theta^{p_K(n)} x)$ converges to normal distribution. Here $p_k(n)$ are polynomials.

1. Introduction

The study of this paper has been motivated by the polynomial ergodic theorem, which states

$$\frac{1}{N} \sum_{n=1}^N T^{p_1(n)} f_1 T^{p_2(n)} f_2 \dots T^{p_k(n)} f_k \xrightarrow{L^2} \prod_{i=1}^k \int f_i d\mu,$$

where T is weakly mixing transformation, p_k 's polynomials and $f_k \in L^\infty$. In case p_k are linear, the mean convergence was proved by Furstenberg, Katznelson and Ornstein [9], and for general polynomials by Bergelson [2].

Considering the transform $\theta \mapsto \theta x$ on unit interval $[0, 1]$, the result of Furstenberg and Weiss [8] gives

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(\theta^k x) g(\theta^{k^2} x) \rightarrow \int_0^1 f(x) dx \int_0^1 g(x) dx \quad \text{a.e. } x, \quad \theta = 2, 3, \dots$$

for bounded functions f and g with period 1. By comparing this with the following results on Riesz-Raikov sums, we can expect that the following central limit theorem holds:

$$\left| \left\{ x \in [0, 1] \mid \frac{1}{\sqrt{n}} \sum_{k=1}^n f(\theta^k x) g(\theta^{k^2} x) \in [a, b] \right\} \right| \rightarrow \frac{1}{\sqrt{2\pi v}} \int_a^b e^{-x^2/2v} dx,$$

for all $a < b$, when $\int_0^1 f(x) dx = \int_0^1 g(x) dx = 0$. ($|\cdot|$ denotes the Lebesgue measure.)

We here give a brief survey on the probabilistic studies on Riesz-Raikov sums to explain the context above.

† This research was partially supported by Grant-in-Aid for Scientific Research (No. 09640268), Ministry of Education, Science and Culture.

Let f be a real-valued locally integrable function on \mathbf{R} with period 1. Raikov [15] proved that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(\theta^k x) = \int_0^1 f(x) dx, \quad \text{a.e. } x, \quad (\theta = 2, 3, \dots),$$

and Riesz [17] pointed out that this result is an example of the ergodic theorem.

Having investigated more precise limiting behaviour of Riesz-Raikov sums, Kac [12] proved the following central limit theorem: If f is Hölder continuous and $\int_0^1 f(x) dx = 0$, then

$$\left| \left\{ x \in [0, 1] \mid \frac{1}{\sqrt{n}} \sum_{k=1}^n f(\theta^k x) \in [a, b] \right\} \right| \rightarrow \frac{1}{\sqrt{2\pi v}} \int_a^b e^{-x^2/2v} dx, \quad (1.1)$$

for all $a < b$ and $\theta = 2, 3, \dots$, where $v = \int_0^1 f^2(x) dx + 2 \sum_{k=1}^{\infty} \int_0^1 f(x) f(\theta^k x) dx$. Ibragimov [11] extended the above results to locally square integrable function f with the L^2 -Dini condition below:

$$\int_0^1 \frac{\omega_2(y)}{y} dy < \infty \quad \text{where} \quad \omega_2(\delta) = \sup_{|h| \leq \delta} \left(\int_0^1 |f(x+h) - f(x)|^2 dx \right)^{1/2}.$$

Although Takahashi [20] gave some results under the relative measure, we must wait until Berkes [3] and [4] to have the central limit theorem (1.1) for non integral $\theta > 1$. He proved (1.1) for all $\theta > 1$ as a corollary of his general result, while the limiting variance v was not explicitly determined. Kaufman [13] proved $v = \int_0^1 f^2(x) dx$ for almost all $\theta > 1$ with respect to Lebesgue measure, and also for all algebraic $\theta > 1$ with

$$\theta^r \notin \mathbf{Q} \quad \text{for all } r \in \mathbf{N}. \quad (1.2)$$

Petit [14] treated the case when the algebraic number $\theta > 1$ does not satisfy (1.2). When r is the minimum positive integer with $\theta^r \in \mathbf{Q}$, while $\theta^r = \mu/\nu$ is an irreducible fraction of integers, he proved $v = \int_0^1 f^2(x) dx + 2 \sum_{s=1}^{\infty} \int_0^1 f(\mu^s x) f(\nu^s x) dx$. Fukuyama [7] proved that $v = \int_0^1 f^2(x) dx$ for all $\theta > 1$ with (1.2), and the limiting variance was completely determined.

In the above studies, Hölder's condition, Dini's condition, or some variation of these are assumed.

We are now in a position to state our theorem. Note that $\theta > 1$ is not necessarily an integer.

Theorem 1. *Let us assume that polynomials p_k ($1 \leq k \leq K$) satisfy $p_k(\infty) = \infty$ and $(p_k - p_{k'}) (\infty) = \infty$ ($k > k'$), and functions f_1, \dots, f_K on \mathbf{R} with period 1 satisfy*

$$\int_0^1 f_k(x) dx = 0 \quad \text{and} \quad \int_0^1 |f_k(x)|^{2K-2} dx < \infty \quad (k = 1, \dots, K), \quad (1.3)$$

$$\int_0^1 |f_k(x) - s_{f_k, n}(x)|^2 dx = o(1/\log n) \quad (n \rightarrow \infty, k = 1, \dots, K), \quad (1.4)$$

and

$$\sum_{n=0}^{\infty} \left(\int_0^1 |s_{f_K, 2^{n+1}}(x) - s_{f_K, 2^n}(x)|^2 dx \right)^{1/2} < \infty \quad (1.5)$$

where $s_{f,n}$ denotes the n -th subsum of Fourier series of f . Let P be a probability measure on \mathbf{R} which is absolutely continuous with respect to the Lebesgue measure. Then we have the central limit theorem

$$P \left\{ x \in \mathbf{R} \mid \frac{1}{\sqrt{N}} \sum_{n=1}^N \prod_{k=1}^K f_k(\theta^{p_k(n)} x) \in [a, b] \right\} \rightarrow \frac{1}{\sqrt{2\pi v}} \int_a^b e^{-x^2/2v} dx, \quad (1.6)$$

as $N \rightarrow \infty$ for all $a < b$, $K \geq 2$ and $\theta > 1$. The limiting variance v in (1.6) is determined as follows:

(1) If $\max_k \deg p_k \geq 2$, then

$$v = \prod_{k=1}^K \int_0^1 f_k^2(x) dx. \quad (1.7)$$

(2) When all p_k are linear, i.e., $p_k(x) = a_k x + b_k$, and if the condition

$$\theta^{a_k n} \notin \mathbf{Q} \quad \text{for all } n \in \mathbf{N} \quad (1.8)$$

is satisfied for at least one of $k = 1, \dots, K$, then v is given by (1.7).

(3) In case (1.8) is not true for all $k = 1, \dots, K$, let us take the smallest $n \geq 1$ satisfying $\theta^{a_k n} \in \mathbf{Q}$ for all k , and write $\theta^{a_k n} = q_k/r_k$ by using $q_k, r_k \in \mathbf{N}$. Then

$$v = \prod_{k=1}^K \int_0^1 f_k^2(x) dx + 2 \sum_{n=1}^{\infty} \prod_{k=1}^K \int_0^1 f_k(q_k^n x) f_k(r_k^n x) dx. \quad (1.9)$$

Conditions (1.4) and (1.5) follows from the next condition

$$\sum_{n=1}^{\infty} \left(\int_0^1 |f_k(x) - s_{f_k, 2^n}(x)|^2 dx \right)^{1/2} < \infty \quad (1.10)$$

which is equivalent to L^2 -Dini condition. The equivalence is proved by (3.3) of pp. 241 of Zygmund [21] and by (2.6) of pp. 160 of Bari [1].

2. Preliminaries

Set $\|f\|_{\infty} = \text{ess sup}_{-\infty < x < \infty} |f(x)|$ and $\|f\|_p = (\int_0^1 |f(x)|^p dx)^{1/p}$. Let $[x]$ denote the integer part of x . In this note we abuse notation and denote $s_{f,[a]}$ simply by $s_{f,a}$ when a is not an integer.

To prove the central limit theorem, we use the next lemma, whose idea is essentially due to P. Hartman [10].

Let us put

$$h_{\lambda}(x) = \left(\frac{\sin \lambda x}{\lambda x} \right)^2 \quad \text{and} \quad h(x) = \frac{h_{1/2}(x) + h_{1/2\sqrt{2}}(x)}{1 + \sqrt{2}}.$$

It is easily verified that h is a positive integrable analytic function on \mathbf{R} which satisfies the following conditions (Cf. L. Breiman [5] pp.218):

$$\widehat{h}(u) = 0 \quad (|u| > 1), \quad \text{and} \quad |\widehat{h}(u)| \leq 1 \quad (u \in \mathbf{R}).$$

For $u \in \mathbf{R}$, let us define a measure μ_u on \mathbf{R} by $\mu_u(dx) = e^{\sqrt{-1}ux} h(x) dx$. Clearly μ_0 is a probability measure. We frequently use the inequality

$$\left| \int_{-\infty}^{\infty} f(x) \mu_u(dx) \right| \leq \int_{-\infty}^{\infty} |f(x)| \mu_0(dx),$$

and the relation

$$\int_{-\infty}^{\infty} e^{\sqrt{-1}\lambda x} \mu_u(dx) = 0 \quad \text{if} \quad |\lambda| \geq U = |u| + 1.$$

Lemma 1. *Suppose that a sequence $\{X_N\}$ of real functions on \mathbf{R} satisfies*

$$\int_{-\infty}^{\infty} e^{\sqrt{-1}tX_N(x)} \mu_u(dx) \rightarrow e^{-t^2v/2} \widehat{h}(u) \quad (t \in \mathbf{R}, N \rightarrow \infty), \quad (2.1)$$

for all $u \in \mathbf{R}$, then the central limit theorem

$$P\{x \in \mathbf{R} \mid X_N(x) \in [a, b]\} \rightarrow \frac{1}{\sqrt{2\pi v}} \int_a^b e^{-x^2/2v} dx \quad (a < b, N \rightarrow \infty),$$

holds where P is any probability measure on \mathbf{R} which is absolutely continuous with respect to the Lebesgue measure.

Proof. Let us first suppose that $P(dx) = g(x) dx$ and $g \in C^\infty$ has compact support. Since g/h is rapidly decreasing, both g/h and $(g/h)^\wedge$ are integrable. Therefore we can apply the inversion formula for g/h , Fubini's theorem and Lebesgue's convergence theorem in turn to prove the following convergence which is equivalent to (2.1):

$$\begin{aligned} \int_{-\infty}^{\infty} e^{\sqrt{-1}tX_N(x)} P(dx) &= \int_{-\infty}^{\infty} e^{\sqrt{-1}tX_N(x)} \frac{g(x)}{h(x)} h(x) dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (g/h)^\wedge(\gamma) d\gamma \int_{-\infty}^{\infty} e^{\sqrt{-1}tX_N(x)} e^{-\sqrt{-1}x\gamma} h(x) dx \\ &\rightarrow \frac{e^{-t^2v/2}}{2\pi} \int_{-\infty}^{\infty} (g/h)^\wedge(\gamma) \widehat{h}(-\gamma) d\gamma \\ &= e^{-t^2v/2} ((g/h) * h(-\cdot))(0) = e^{-t^2v/2}. \end{aligned}$$

For the last equality, we have used integrability and continuity of $(g/h) * h(-\cdot)$ and its Fourier transform.

Next, we treat the general case. For all $\varepsilon > 0$, we can take $g_\varepsilon \in C^\infty$ with compact support such that

$$g_\varepsilon(x) \geq 0, \quad \int_{-\infty}^{\infty} g_\varepsilon(x) dx = 1, \quad \text{and} \quad \int_{-\infty}^{\infty} |g(x) - g_\varepsilon(x)| dx \leq \varepsilon.$$

By using g_ε , we have

$$\begin{aligned} & \left| \int_{-\infty}^{\infty} e^{\sqrt{-1}tX_N(x)} g(x) dx - e^{-t^2v/2} \right| \\ & \leq \left| \int_{-\infty}^{\infty} e^{\sqrt{-1}tX_N(x)} g(x) dx - \int_{-\infty}^{\infty} e^{\sqrt{-1}tX_N(x)} g_\varepsilon(x) dx \right| \\ & \quad + \left| \int_{-\infty}^{\infty} e^{\sqrt{-1}tX_N(x)} g_\varepsilon(x) dx - e^{-t^2v/2} \right|. \end{aligned}$$

The second term tends to zero as $N \rightarrow \infty$, and the first term is dominated by $\int |g - g_\varepsilon| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have the conclusion. \square

The following Lemma is a variation of the result by P. Révész [16] and its argument is essentially due to Salem-Zygmund [18].

Lemma 2. *Let $u \in \mathbf{R}$ and $\{\xi_{m,N}\}_{1 \leq m \leq M_N, N \geq 1}$ be an array of functions on \mathbf{R} . If*

$$B_N = \sup_{1 \leq m \leq M_N} \|\xi_{m,N}\|_\infty \rightarrow 0 \quad (N \rightarrow \infty), \quad (2.2)$$

$$\int_{-\infty}^{\infty} \xi_{m_1,N} \cdots \xi_{m_r,N} d\mu_u = 0 \quad (N \in \mathbf{N}, r \in \mathbf{N}, m_0 \leq m_1 < \cdots < m_r) \quad (2.3)$$

$$V_N = \sum_{m=m_0}^{M_N} \xi_{m,N}^2 \longrightarrow v \quad \text{in measure } \mu_0 \quad (N \rightarrow \infty), \quad (2.4)$$

and

$$B_0 = \sup_{N \geq 1} \|V_N\|_\infty < \infty, \quad (2.5)$$

are satisfied for some m_0 , then (2.1) holds for $X_N = \sum_{m=1}^{M_N} \xi_{m,N}$.

Proof. If we put $Y_N = \sum_{m=m_0}^{M_N} \xi_{m,N}$, by $|X_N - Y_N| \leq m_0 B_N \rightarrow 0$, we have

$$\left| \int_{-\infty}^{\infty} e^{\sqrt{-1}tX_N} d\mu_u - \int_{-\infty}^{\infty} e^{\sqrt{-1}tY_N} d\mu_u \right| \leq \int_{-\infty}^{\infty} |e^{\sqrt{-1}t(X_N - Y_N)} - 1| d\mu_0 \rightarrow 0.$$

Thus it is sufficient for us to prove (2.1) for Y_N . Let us recall the following asymptotic formula. (Cf. Salem-Zygmund [18])

$$e^{\sqrt{-1}x} = (1 + \sqrt{-1}x)e^{-x^2/2 + R(x)} \quad \text{where} \quad |R(x)| \leq |x|^3,$$

By applying this, we have

$$e^{\sqrt{-1}tY_N} = T_N \exp(-t^2V_N/2 + R_N)$$

where

$$T_N = \prod_{m=m_0}^{M_N} (1 + \sqrt{-1}t\xi_{m,N}) \quad \text{and} \quad R_N = \sum_{m=m_0}^{M_N} R(t\xi_{m,N}).$$

R_N and T_N have the following bounds:

$$R_N \leq \sum_{m=m_0}^{M_N} |t\xi_{m,N}|^3 \leq |t|^3 B_N V_N \leq |t|^3 B_N B_0 \rightarrow 0 \quad \text{and} \quad |T_N| \leq e^{t^2V_N/2}.$$

By (2.3) we have $\int_{-\infty}^{\infty} T_N \mu_u(dx) = \widehat{h}(u)$ and hence we get

$$\begin{aligned} & \left| \int_{-\infty}^{\infty} e^{\sqrt{-1}tY_N} d\mu_u - \widehat{h}(u)e^{-t^2v/2} \right| \\ &= \left| \int_{-\infty}^{\infty} T_N \left(\exp(-t^2V_N/2 + R_N) - \exp(-t^2v/2) \right) d\mu_u \right| \\ &\leq \int_{-\infty}^{\infty} |\exp(R_N) - 1| d\mu_0 + \int_{-\infty}^{\infty} |\exp(t^2(V_N - v)/2) - 1| d\mu_0 \rightarrow 0, \end{aligned}$$

since the integrand is bounded and tends to 0 in measure μ_0 . \square

3. The case $\max \deg f_k \geq 2$

In this section we assume $\max \deg f_k \geq 2$ or $\deg f_K \geq 2$. If we denote $\beta_n^{(k)} = \theta^{p_k(n)}$, it is clear that there exists $q > 1$ and N_0 such that

$$\beta_{n+1}^{(k)} / \beta_n^{(k)} > q \quad (n \geq N_0, k = 1, \dots, K), \quad (3.1)$$

$$\beta_{n+1}^{(K)} / \beta_n^{(K)} \rightarrow \infty \quad (n \rightarrow \infty), \quad (3.2)$$

$$\beta_n^{(k+1)} / \beta_n^{(k)} \rightarrow \infty \quad (n \rightarrow \infty, k = 1, \dots, K-1). \quad (3.3)$$

Assuming only these three conditions, we can prove the central limit theorem for trigonometric polynomials.

Proposition 1. *Let f_1, \dots, f_K be trigonometric polynomials without constant term. If sequences of positive numbers $\{\beta_n^{(1)}\}, \dots, \{\beta_n^{(K)}\}$ satisfy (3.1), (3.2) and (3.3), then for all $u \in \mathbf{R}$, (2.1) holds for*

$$X_N(x) = \frac{1}{\sqrt{N}} \sum_{n=1}^N \prod_{k=1}^K f_k(\beta_n^{(k)} x) \quad \text{and} \quad v = \prod_{k=1}^K \int_0^1 f_k^2(x) dx,$$

that is the central limit theorem holds.

Proof. Let $I \in \mathbf{N}$ and assume that degrees of f_1, \dots, f_K are less than I . Let us take u arbitrary and put $U = |u| + 1$. Let us set $M_N = N$ and $\xi_{n,N}(x) = f_1(\beta_n^{(1)}x) \dots f_K(\beta_n^{(K)}x) / \sqrt{N}$ and prove that $\{\xi_{n,N}\}$ satisfy the conditions of Lemma 2. If we put $B = \max_k \|f_k\|_{\infty}$, we have $\|\xi_{n,N}\|_{\infty} \leq B^K / \sqrt{N}$ and $\|V_N\|_{\infty} \leq B^K$, and hence (2.2) and (2.5) are clear. $\xi_{n,N}$ can be expanded to

$$\xi_{n,N}(x) = \frac{1}{\sqrt{N}} \sum_{\substack{|i_k| \leq I, \\ k=1, \dots, K}} \widehat{f}_1(i_1) \dots \widehat{f}_K(i_K) \exp\left(2\pi\sqrt{-1}(i_1\beta_n^{(1)} + \dots + i_K\beta_n^{(K)})x\right).$$

By (3.2) and (3.3), there exists N_0 such that, for any $n \geq N_0$ and $k = 1, \dots, K$,

$$\beta_{n+1}^{(K)} / \beta_n^{(K)} \geq 10I, \quad \beta_n^{(k)} \geq 6U \quad (3.4)$$

and

$$|i_1\beta_n^{(1)} + \dots + i_k\beta_n^{(k)}| \in (\beta_n^{(k)}/2, 3I\beta_n^{(k)}) \quad \text{if} \quad |i_1|, \dots, |i_k| \leq 2I, \quad i_k \neq 0. \quad (3.5)$$

Let us now assume $N_0 \leq n_1 < \dots < n_r$ and verify multiple orthogonality (2.3). If we expand $\xi_{n_1, N} \dots \xi_{n_r, N}$ into trigonometric polynomial, frequencies can be written as $\lambda_{n_r} + \dots + \lambda_{n_1}$ where λ_{n_i} is a frequency of $\xi_{n_i, N}$. Thanks to (3.4) and (3.5) we have

$$\begin{aligned} |\lambda_{n_r} + \dots + \lambda_{n_1}| &\geq |\lambda_{n_r}| - |\lambda_{n_{r-1}}| - \dots - |\lambda_{n_1}| \\ &\geq \beta_{n_r}^{(K)}/2 - 3I(\beta_{n_{r-1}}^{(K)} + \dots + \beta_{n_1}^{(K)}) \\ &\geq \beta_{n_r}^{(K)}/2 - 3I\beta_{n_{r-1}}^{(K)}(1 + 1/10I + 1/(10I)^2 + \dots) \\ &\geq \beta_{n_r}^{(K)}/2 - (3I/10I)\beta_{n_r}^{(K)}\{1/(1 - 1/10)\} \geq \beta_{n_r}^{(K)}/6 \geq U. \end{aligned}$$

By this estimate, we see $\int_{-\infty}^{\infty} \exp(2\pi\sqrt{-1}(\lambda_{n_r} + \dots + \lambda_{n_1})x) \mu_u(dx) = 0$, and hence we have (2.3).

Lastly, let us verify (2.4). Let us take an r satisfying $q^r \geq 12I$. To prove (2.4), it is sufficient to prove

$$\sum_{n=N_0}^N (\xi_{nr+j, N}^2 - v/N) \rightarrow 0 \quad \text{in measure } \mu_0,$$

for each $j = 0, \dots, r-1$. Let us put $\sigma_k^2 = \int_0^1 f_k^2(x) dx$. Since $v = \sigma_1^2 \dots \sigma_K^2$, we have

$$\xi_{nr+j, N}^2(x) - \frac{v}{N} = \frac{1}{N} \sum_{\kappa=1}^K \prod_{k=1}^{\kappa-1} f_k^2(\beta_{nr+j}^{(k)} x) (f_{\kappa}^2(\beta_{nr+j}^{(\kappa)} x) - \sigma_{\kappa}^2) \prod_{k=\kappa+1}^K \sigma_k^2.$$

Thus it is sufficient to prove the following convergence in measure μ_0 :

$$\frac{1}{N} \sum_{n=N_0}^N \prod_{k=1}^{\kappa-1} f_k^2(\beta_{nr+j}^{(k)} x) (f_{\kappa}^2(\beta_{nr+j}^{(\kappa)} x) - \sigma_{\kappa}^2) \rightarrow 0.$$

Let ζ_n denote the summand. Note that the trigonometric polynomial expansion of $f_{\kappa}^2(\beta_{nr+j}^{(\kappa)} x) - \sigma_{\kappa}^2$ has no constant term. Thus frequencies of the trigonometric expansion of ζ_n can be written as the right hand side of (3.5), and hence belong to $(\beta_{nr+j}^{(\kappa)}/2, 3I\beta_{nr+j}^{(\kappa)})$. Thanks to $\beta_{nr+j}^{(\kappa)}/2 \geq 6U/2 > 1$, we have

$$\int_{-\infty}^{\infty} \zeta_n d\mu_0 = 0. \quad (3.6)$$

By $(\beta_{(n+1)r+j}^{(\kappa)}/2)/(3I\beta_{nr+j}^{(\kappa)}) \geq q^r/6I \geq 2$, we have

$$\beta_{(n+1)r+j}^{(\kappa)}/2 - 3I\beta_{nr+j}^{(\kappa)} \geq (\beta_{(n+1)r+j}^{(\kappa)}/2)(1 - 1/2) \geq 6U/4 > 1$$

and hence frequencies of ζ_n and $\zeta_{n'}$ differ by at least 1 if $n \neq n'$. Thereby we conclude that $\int_{-\infty}^{\infty} \zeta_n(x) \zeta_{n'}(x) \mu_0(dx) = 0$, and we have

$$\int_{-\infty}^{\infty} \left(\frac{1}{N} \sum_{n=N_0}^N \zeta_n \right)^2 d\mu_0 = \frac{1}{N^2} \sum_{n=N_0}^N \int_{-\infty}^{\infty} \zeta_n^2 d\mu_0 \leq \frac{\max_n \|\zeta_n\|_{\infty}^2}{N} \rightarrow 0,$$

which implies the convergence in measure. \square

Since $p_{k+1} - p_k$ is a polynomial diverging to infinity, there exists $a_0 > 0$ and $N_0 \in \mathbf{N}$ such that $p_{k+1}(n) - p_k(n) \geq 2a_0n$ for all $n \geq N_0$. Since p_K is not linear, there also exist $a_0 > 0$ and $N_0 \in \mathbf{N}$ such that $p_K(n+1) - p_K(n) \geq 2a_0n$ for all $n \geq N_0$. Thus we have

$$\beta_n^{(k+1)} / \beta_n^{(k)} \geq \theta^{2a_0n} \quad (n \geq N_0, k = 1, \dots, K-1), \quad (3.7)$$

$$\beta_{n+1}^{(K)} / \beta_n^{(K)} \geq \theta^{2a_0n} \quad (n \geq N_0). \quad (3.8)$$

Proposition 2. *Let functions f_1, \dots, f_K with period 1 satisfy (1.3) and (1.4), and sequences of positive numbers $\{\beta_n^{(1)}\}, \dots, \{\beta_n^{(K)}\}$ satisfy (3.1), (3.7) and (3.8). Then the conclusion of Proposition 1 holds.*

Remark 1. *In this proposition, the condition (1.5) is not assumed. Thus the part (1) of Theorem 1 holds without (1.5).*

Proof. Because of $\beta_n^{(k)} \theta^{a_0n} = o(\beta_n^{(k+1)})$, it can be proved in the same way that there exists N_0 such that, for all $n \geq N_0$,

$$|i_1 \beta_n^{(1)} + \dots + i_k \beta_n^{(k)}| \in (\beta_n^{(k)} / 2, 3\theta^{a_0n} \beta_n^{(k)}) \text{ if } |i_1|, \dots, |i_k| \leq 2\theta^{a_0n}, i_k \neq 0. \quad (3.9)$$

Firstly, we have the following estimate:

$$\begin{aligned} E_1^2 &= \left(\int_{-\infty}^{\infty} \left| \frac{1}{\sqrt{N}} \sum_{n=N_0}^N \prod_{k=1}^K s_{f_k, \theta^{a_0n}}(\beta_n^{(k)} x) - \frac{1}{\sqrt{N}} \sum_{n=N_0}^N \prod_{k=1}^K s_{f_k, I}(\beta_n^{(k)} x) \right| \mu_0(dx) \right)^2 \\ &\leq \int_{-\infty}^{\infty} \left\{ \frac{1}{\sqrt{N}} \sum_{n=N_0}^N \left(\prod_{k=1}^K s_{f_k, \theta^{a_0n}}(\beta_n^{(k)} x) - \prod_{k=1}^K s_{f_k, I}(\beta_n^{(k)} x) \right) \right\}^2 \mu_0(dx) \\ &\leq \frac{\sqrt{K}}{N} \sum_{\kappa=1}^K \int_{-\infty}^{\infty} \left(\sum_{n=N_0}^N \prod_{k=1}^{\kappa-1} s_{f_k, \theta^{a_0n}}(\beta_n^{(k)} x) \prod_{k=\kappa}^K s_{f_k, I}(\beta_n^{(k)} x) \right. \\ &\quad \left. \times (s_{f_\kappa, \theta^{a_0n}} - s_{f_\kappa, I})(\beta_n^{(\kappa)} x) \right)^2 \mu_0(dx) \end{aligned}$$

By (3.9), moduli of frequencies of summand in the above integrand belong to $(\beta_n^{(K)} / 2, 3\theta^{a_0n} \beta_n^{(K)})$. Because $(\beta_n^{(K)} / 2) / (3\theta^{a_0n} \beta_n^{(K)}) \geq \theta^{a_0n} / 6 \rightarrow \infty$, the distance between these intervals is greater than 1, and therefore these summands are orthogonal. Thus

$$E_1^2 \leq \frac{\sqrt{K}}{N} \sum_{\kappa=1}^K \sum_{n=N_0}^N \int_{-\infty}^{\infty} \left(\prod_{k=1}^{\kappa-1} s_{f_k, \theta^{a_0n}}^2(\beta_n^{(k)} x) \prod_{k=\kappa}^K s_{f_k, I}^2(\beta_n^{(k)} x) \right. \\ \left. \times (s_{f_\kappa, \theta^{a_0n}} - s_{f_\kappa, I})^2(\beta_n^{(\kappa)} x) \right) \mu_0(dx). \quad (3.10)$$

If functions g_k ($k = 1, \dots, K$) are trigonometric polynomials whose degrees are

less than $\theta^{a_0 n}$, then by applying a similar argument as the derivation of (3.6) to the decomposition

$$\begin{aligned} & \int_{-\infty}^{\infty} \prod_{k=1}^K g_k^2(\beta_n^{(k)} x) \mu_0(dx) - \prod_{k=1}^K \|g_k\|_2^2 \\ &= \sum_{\kappa=1}^K \int_{-\infty}^{\infty} \prod_{k=1}^{\kappa-1} g_k^2(\beta_n^{(k)} x) (g_{\kappa}^2(\beta_n^{(\kappa)} x) - \|g_{\kappa}\|_2^2) \mu_0(dx) \prod_{k=\kappa+1}^K \|g_k\|_2^2, \end{aligned}$$

we have $\int_{-\infty}^{\infty} \prod_{k=1}^K g_k^2(\beta_n^{(k)} x) \mu_0(dx) = \prod_{k=1}^K \|g_k\|_2^2$ for large n . By applying this to (3.10), we have

$$\begin{aligned} E_1^2 &\leq \sqrt{K} \sum_{\kappa=1}^K \prod_{k=1}^{\kappa-1} \|s_{f_k, \theta^{a_0 n}}^2\|_2^2 \prod_{k=\kappa+1}^K \|s_{f_k, I}^2\|_2^2 \|s_{f_{\kappa}, \theta^{a_0 n}} - s_{f_{\kappa}, I}\|_2^2 \\ &\leq \sqrt{K} \sum_{\kappa=1}^K \|f_{\kappa} - s_{f_{\kappa}, I}\|_2^2 \prod_{k \neq \kappa} \|f_k\|_2^2. \end{aligned}$$

Therefore, for any $\varepsilon > 0$, there exists I_0 such that $I \geq I_0$ implies $E_1 < \varepsilon/2$.

Before proceeding further, let us prepare some inequalities. For a function F with period 1 and $\|F\|_{2p} < \infty$, $\|s_n\|_{2p} \leq C\|F\|$ and $\|F - s_n\|_{2p} \leq C\|F\|$ holds. (Cf. A. Zygmund [21].) We here prove $\int_{-\infty}^{\infty} |F(\theta x)|^{2p} \mu_0(dx) = \|F\|_{2p}^{2p}$ for $\theta > 1$ and $p \in \mathbf{N}$. For trigonometric polynomial, it is proved by the direct calculation as below:

$$\begin{aligned} \int_{-\infty}^{\infty} |F(\theta x)|^{2p} \mu_0(dx) &= \sum_{l_1, \dots, l_{2p}} \widehat{F}(l_1) \dots \widehat{F}(l_{2p}) \widehat{h}(2\pi(l_1 + \dots + l_p)\theta) \\ &= \sum_{l_1, \dots, l_{2p}} \widehat{F}(l_1) \dots \widehat{F}(l_{2p}) \delta_{l_1 + \dots + l_p, 0} \\ &= \|F(x)\|_{2p}^{2p}, \end{aligned}$$

because $\widehat{h}(2\pi l\theta) \neq 0$ and $l \in \mathbf{Z}$ is equivalent to $l = 0$. Since $s_n(\theta x)$ converges to $F(\theta x)$ in measure dx , hence in measure μ_0 . By Fatou lemma,

$$\int_{-\infty}^{\infty} |F(\theta x)|^{2p} \mu_0(dx) \leq \liminf_{n \rightarrow \infty} \int_{-\infty}^{\infty} |s_n(\theta x)|^{2p} \mu_0(dx) = \lim_{n \rightarrow \infty} \int_0^1 |s_n(x)|^{2p} dx = \|F\|_{2p}^{2p}.$$

Hence we have that $s_n(\theta x)$ converges to $F(\theta x)$ in $L^{2p}(\mathbf{R}, \mu_0)$ -sense. Thereby we can conclude that

$$\int_{-\infty}^{\infty} |F(\theta x)|^{2p} \mu_0(dx) = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |s_n(\theta x)|^{2p} \mu_0(dx) = \lim_{n \rightarrow \infty} \|s_n\|_{2p}^{2p} = \|F\|_{2p}^{2p}.$$

By using these, we have

$$E_2 = \int_{-\infty}^{\infty} \left| \frac{1}{\sqrt{N}} \sum_{n=N_0}^N \prod_{k=1}^K f_k(\beta_n^{(k)} x) - \frac{1}{\sqrt{N}} \sum_{n=N_0}^N \prod_{k=1}^K s_{\theta^{a_0 n}, f_k}(\beta_n^{(k)} x) \right| \mu_0(dx)$$

$$\begin{aligned}
&\leq \frac{1}{\sqrt{N}} \sum_{n=N_0}^N \sum_{\kappa=1}^K \int_{-\infty}^{\infty} \left| \prod_{k=1}^{\kappa-1} f_k(\beta_n^{(k)} x) \prod_{k=\kappa+1}^K s_{\theta^{a_0 n}, f_k}(\beta_n^{(k)} x) \right. \\
&\quad \left. \times (f_{\kappa} - s_{\theta^{a_0 n}, f_{\kappa}})(\beta_n^{(\kappa)} x) \right| \mu_0(dx) \\
&\leq \frac{1}{\sqrt{N}} \sum_{n=N_0}^N \sum_{\kappa=1}^K \prod_{k=1}^{\kappa-1} \|f_k\|_{2(K-1)} \|f_{\kappa} - s_{\theta^{a_0 n}, f_{\kappa}}\|_2 \prod_{k=\kappa+1}^K \|s_{\theta^{a_0 n}, f_k}\|_{2(K-1)} \\
&\leq \sum_{\kappa=1}^K \prod_{k \neq \kappa} \|f_k\|_{2(K-1)} \frac{1}{\sqrt{N}} \sum_{n=N_0}^N \|f_{\kappa} - s_{\theta^{a_0 n}, f_{\kappa}}\|_2 \rightarrow 0
\end{aligned}$$

because of the fact that $\frac{1}{2(K-1)}(K-1) + \frac{1}{2} = 1$, Hölder's inequality and (1.4).

Thanks to the estimate of E_1 and E_2 above, we have

$$\limsup_{N \rightarrow \infty} \int_{-\infty}^{\infty} |X_N(x) - X_N^{(I)}(x)| \mu_0(dx) \leq \varepsilon$$

for large I , where $X_N^{(I)}(x) = \frac{1}{\sqrt{N}} \sum_{n=1}^N \prod_{k=1}^K s_{I, f_k}(\beta_n^{(k)} x)$. By putting $v^{(I)} = \prod_{k=1}^K \|s_{I, f_k}\|_2^2$, we have

$$|v - v^{(I)}| \leq \varepsilon$$

for large I . Noting Proposition 1, the fact that $|e^x - 1| \leq |x|$ ($x \leq 0$) and that $|e^{\sqrt{-1}x} - 1| \leq |x|$ ($x \in \mathbf{R}$), if we take I large enough, we have

$$\begin{aligned}
&\limsup_{N \rightarrow \infty} \left| \int_{-\infty}^{\infty} e^{\sqrt{-1}tX_N} d\mu_u - e^{-t^2v/2} \widehat{h}(u) \right| \\
&\leq \limsup_{N \rightarrow \infty} \left| \int_{-\infty}^{\infty} e^{\sqrt{-1}tX_N} \mu_u(dx) - \int_{-\infty}^{\infty} e^{\sqrt{-1}tX_N^{(I)}} d\mu_u \right| \\
&\quad + \limsup_{N \rightarrow \infty} \left| \int_{-\infty}^{\infty} e^{\sqrt{-1}tX_N^{(I)}} d\mu_u - e^{-t^2v^{(I)}/2} \widehat{h}(u) \right| + \limsup_{N \rightarrow \infty} |e^{-t^2v^{(I)}/2} - e^{-t^2v/2}| \\
&\leq |t| \int_{-\infty}^{\infty} |X_N - X_N^{(I)}| d\mu_0 + t^2|v^{(I)} - v|/2 \leq (|t| + t^2/2)\varepsilon,
\end{aligned}$$

which implies the conclusion. \square

4. The case when p_k are linear

Let $\{\mu_l^{(k)}\}_{l \in \mathbf{N}}$ be an arrangement of the set $\bigcup_{i_k=1}^I \{i_k \theta^{p_k(n)} \mid n \in \mathbf{N}\}$ in increasing order, and $\{\lambda_j\}_{j \in \mathbf{Z}}$ be an arrangement of the set

$$\{i_1 \theta^{p_1(n)} + \cdots + i_K \theta_K^{p_K(n)} \mid n \in \mathbf{N}, 1 \leq |i_k| \leq I \ (k = 1, \dots, K)\} \cup \{0\}$$

in the order $\cdots < \lambda_{-2} < \lambda_{-1} < \lambda_0 = 0 < \lambda_1 < \lambda_2 < \cdots$. It is easily known that $-\lambda_j = \lambda_{-j}$ holds, and that $\{\mu_l^{(k)}\}$ has Hadamard gaps; i.e., there exists $q > 1$ such that

$$\mu_{l+1}^{(k)} / \mu_l^{(k)} > q > 1 \quad (l \in \mathbf{N}, k = 1, \dots, K.) \quad (4.1)$$

Because $a_1 < a_2 < \cdots < a_K$, we can take $a_0 > 0$ satisfying $a_0 < (a_{k+1} - a_k)/2$ ($k = 1, \dots, K-1$). Clearly, $p_{k+1}(n) - p_k(n) > 2a_0 n$ for large n .

Lemma 3. Let I and K be a positive integers.

(1) For any $\varepsilon > 0$, there exists N_0 such that for $k = 1, \dots, K$, $n \geq N_0$, and $0 < |i_k| \leq \theta^{a_0 n}$ we have

$$\frac{i_1 \theta^{p_1(n)} + \dots + i_k \theta^{p_k(n)}}{i_k \theta^{p_k(n)}} \in (1 - \varepsilon, 1 + \varepsilon). \quad (4.2)$$

(2) There exist $C_1, C_2 > 0$ and N_0 such that for $n, n' \geq N_0$, $|i_j| \leq \theta^{a_0 n}$, $|i'_j| \leq \theta^{a_0 n'}$ ($j = 1, \dots, k-1$), $|i_k|, |i'_k| \leq I$, and $|i_k \theta^{p_k(n)}| > |i'_k \theta^{p_k(n')}|$ we have

$$\left| \frac{(i_1 \theta^{p_1(n)} + \dots + i_k \theta^{p_k(n)}) \pm (i'_1 \theta^{p_1(n')} + \dots + i'_k \theta^{p_k(n')})}{i_k \theta^{p_k(n)}} \right| \in (C_1, C_2). \quad (4.3)$$

(3) For any $U > 0$ and $\kappa = 0, 1, \dots, K$ there exists N_0 such that if $n \geq n' \geq N_0$, $0 < |i_k| \leq \theta^{a_0 n}$, $0 < |i'_k| \leq \theta^{a_0 n'}$ ($1 \leq k \leq \kappa$), $0 < |i_k|, |i'_k| \leq I$ ($\kappa < k \leq K$) and

$$|(i_1 \theta^{p_1(n)} + \dots + i_K \theta^{p_K(n)}) + (i'_1 \theta^{p_1(n')} + \dots + i'_K \theta^{p_K(n')})| \leq U \quad (4.4)$$

we have

$$i_k \theta^{p_k(n)} + i'_k \theta^{p_k(n')} = 0 \quad \text{for } k > \kappa, \quad \text{and} \quad (4.5)$$

$$|(i_k \theta^{p_k(n)} + \dots + i_\kappa \theta^{p_\kappa(n)}) + (i'_k \theta^{p_k(n')} + \dots + i'_\kappa \theta^{p_\kappa(n')})| \leq \frac{\theta^{p_\kappa(n')}}{3} \quad (4.6)$$

for $k \leq \kappa$.

Proof. (4.2) is clear from $\theta^{a_0 n} \theta^{p_k(n)} \leq \theta^{p_{k+1}(n)} / \theta^{2a_0 n} = o(\theta^{p_{k+1}(n)})$. Let us prove (4.3). Take $\varepsilon > 0$ small enough such that $(1 - \varepsilon) - (1 + \varepsilon)/q$ is positive. By using (4.2), we have

$$\begin{aligned} & |(i_1 \theta^{p_1(n)} + \dots + i_k \theta^{p_k(n)}) \pm (i'_1 \theta^{p_1(n')} + \dots + i'_k \theta^{p_k(n')})| \\ & \geq \left| \frac{i_1 \theta^{p_1(n)} + \dots + i_k \theta^{p_k(n)}}{i_k \theta^{p_k(n)}} i_k \theta^{p_k(n)} \right| - \left| \frac{i'_1 \theta^{p_1(n')} + \dots + i'_k \theta^{p_k(n')}}{i'_k \theta^{p_k(n')}} i'_k \theta^{p_k(n')} \right| \\ & \geq (1 - \varepsilon) |i_k| \theta^{p_k(n)} - (1 + \varepsilon) |i'_k| \theta^{p_k(n')} \\ & \geq \{(1 - \varepsilon) - (1 + \varepsilon)/q\} |i_k| \theta^{p_k(n)}. \end{aligned}$$

The upper estimate

$$|(i_1 \theta^{p_1(n)} + \dots + i_k \theta^{p_k(n)}) \pm (i'_1 \theta^{p_1(n')} + \dots + i'_k \theta^{p_k(n')})| \leq 2(1 + \varepsilon) |i_k| \theta^{p_k(n)}$$

can be proved in a similar way.

Lastly, let us prove (4.5) and (4.6). Assume that (4.5) is not true and take the largest of k such that $i_k \theta^{p_k(n)} + i'_k \theta^{p_k(n')} \neq 0$ for large n and n' . Then, by using (4.3), we see that (4.4) cannot hold for large n and n' . Thus we have (4.5). By this, we see that (4.4) is valid if we replace K by κ . Noting this and (4.2) we have

$$\begin{aligned} & |(i_k \theta^{p_k(n)} + \dots + i_\kappa \theta^{p_\kappa(n)}) + (i'_k \theta^{p_k(n')} + \dots + i'_\kappa \theta^{p_\kappa(n')})| \\ & \leq |(i_1 \theta^{p_1(n)} + \dots + i_\kappa \theta^{p_\kappa(n)}) + (i'_1 \theta^{p_1(n')} + \dots + i'_\kappa \theta^{p_\kappa(n')})| \\ & \quad + |(i_1 \theta^{p_1(n)} + \dots + i_{k-1} \theta^{p_{k-1}(n)}) + (i'_1 \theta^{p_1(n')} + \dots + i'_{k-1} \theta^{p_{k-1}(n')})| \\ & \leq U + 2\theta^{a_0 n + p_{k-1}(n)} \leq 3\theta^{a_0 n + p_{k-1}(n)}, \end{aligned} \quad (4.7)$$

for large n . By putting $k = \kappa$, we have $i_\kappa \theta^{p_\kappa(n)} + i'_\kappa \theta^{p_\kappa(n')} = o(\theta^{p_\kappa(n)})$ and hence $|i_\kappa| \theta^{p_\kappa(n)}/2 \leq |i'_\kappa| \theta^{p_\kappa(n')}$ and thereby $\theta^{p_\kappa(n)}/2 \leq \theta^{p_\kappa(n') + a_0 n'}$ or $n \leq \frac{a_\kappa + a_0}{a_\kappa} n' + \alpha$. Thus we have $a_0 n + p_{k-1}(n) \leq (a_0 + a_{k-1}) \frac{a_\kappa + a_0}{a_\kappa} n' + \alpha'$. Because

$$\begin{aligned} (a_0 + a_{k-1}) \frac{a_\kappa + a_0}{a_\kappa} &< (a_k - a_0) \frac{a_\kappa + a_0}{a_\kappa} = a_k \left(1 - \frac{a_0}{a_k}\right) \left(1 + \frac{a_0}{a_\kappa}\right) \\ &\leq a_k \left(1 - \frac{a_0}{a_\kappa}\right) \left(1 + \frac{a_0}{a_\kappa}\right) < a_k, \end{aligned}$$

we have $\theta^{a_0 n + p_{k-1}(n)} = o(\theta^{a_k n' + \alpha'})$. Combining this with (4.7), we have the conclusion. \square

Let us put $J(w) = [\theta^{a_\kappa w}, \theta^{a_\kappa(w+1)}) \cap \mathbf{N}$,

$$\Delta_w(f) = \left(\sum_{|j| \in J(w)} |\hat{f}(j)|^2 \right)^{1/2} \quad \text{and} \quad D(f) = \sum_{w=0}^{\infty} \Delta_w(f).$$

It is easily seen that there exists a constant C depending only on θ^{a_κ} such that,

$$D(f) \leq C \sum_{n=0}^{\infty} \left(\int_0^1 |s_{f,2^{n+1}}(x) - s_{f,2^n}(x)|^2 dx \right)^{1/2}, \quad (f \in L^2).$$

Lemma 4. If f_1, \dots, f_K satisfy the condition of Theorem 1, then we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(\sum_{n=N_0}^N \prod_{k=1}^{\kappa} s_{f_k, \theta^{a_0 n}}(\theta^{a_\kappa n} x) \prod_{k=\kappa+1}^K s_{f_k, I}(\theta^{a_\kappa n} x) \right)^2 \mu_0(dx) \\ &\leq 2N \prod_{k=1}^{K-1} \|f_k\|_2^2 D(f_K) \quad (\kappa = 0, 1, \dots, K, N \geq N_0) \end{aligned} \quad (4.8)$$

where N_0 is given by (3) of Lemma 3 with $U = 1$. We also have

$$v^{(I)} = \prod_{k=1}^K \int_0^1 s_{f_k, I}^2(x) dx + 2 \sum_{n=1}^{\infty} \prod_{k=1}^K \int_0^1 s_{f_k, I}(q_k^n x) s_{f_k, I}(r_k^n x) dx \rightarrow v, \quad (I \rightarrow \infty).$$

Proof. Let us consider the case $N_0 \leq n' \leq n$ and (4.4) is valid. Then, for $k > \kappa$, we have $i'_k = \varphi_k(i_k) = [-i_k \theta^{a_\kappa(n-n')}]^*$ by (4.5), where $[x]^* = [x + 1/2]$. We also have $i'_\kappa = \varphi_\kappa(i_\kappa) = [-i_\kappa \theta^{a_\kappa(n-n')}]^*$ by (4.6) with $k = \kappa$. For $k < \kappa$, by (4.6), we have

$$\begin{aligned} i'_k &= \varphi_k(i_k, i_{k+1}, \dots, i_\kappa) \\ &= \left[-i_k \theta^{a_\kappa(n-n')} - \theta^{-p_k(n')} \left(i_{k+1} \theta^{p_{k+1}(n)} + \dots + i_\kappa \theta^{p_\kappa(n)} \right. \right. \\ &\quad \left. \left. + i'_{k+1} \theta^{p_{k+1}(n')} + \dots + i'_\kappa \theta^{p_\kappa(n)} \right) \right]^*. \end{aligned}$$

By $\theta^{a_\kappa(n-n')} \geq 1$, mappings $i_k \mapsto \varphi_k(i_k)$ ($k \geq \kappa$) and $i_k \mapsto \varphi_k(i_k, i_{k+1}, \dots, i_\kappa)$ ($k < \kappa$) are injective for any given $i_{k+1}, \dots, i_\kappa, n$ and n' .

By using (4.5) and (4.6), we have

$$\begin{aligned} & \int_{-\infty}^{\infty} \left(\sum_{n=N_0}^N \prod_{k=1}^{\kappa} s_{f_k, \theta^{a_0 n}}(\theta^{a_k n} x) \prod_{k=\kappa+1}^K s_{f_k, I}(\theta^{a_k n} x) \right)^2 \mu_0(dx) \\ & \leq \sum_{n', n=N_0}^N \sum_{*} |\widehat{f}_1(i_1) \widehat{f}_1(i'_1)| \dots |\widehat{f}_K(i_K) \widehat{f}_K(i'_K)| \\ & \leq 2 \sum_{n'=N_0}^N \sum_{n=n'}^N \prod_{k=\kappa}^K \sum_{i_k \neq 0} |\widehat{f}_k(i_k) \widehat{f}_k(\varphi_k(i_k))| \prod_{k=1}^{\kappa-1} \sum_{i_k \neq 0} |\widehat{f}_k(i_k) \widehat{f}_k(\varphi_k(i_k, \dots, i_{\kappa}))|, \end{aligned}$$

where \sum_{*} denotes summation for $(i_1, i'_1, \dots, i_K, i'_K)$ with $0 < |i_k| \leq \theta^{a_0 n}$, $0 < |i'_k| \leq \theta^{a_0 n'}$ ($1 \leq k \leq \kappa$), and $0 < |i_k|, |i'_k| \leq I$ ($\kappa < k \leq K$). If $k < \kappa$,

$$\begin{aligned} & \sum_{i_k \neq 0} |\widehat{f}_k(i_k) \widehat{f}_k(\varphi_k(i_k, \dots, i_{\kappa}))| \\ & \leq \left(\sum_{i_k \neq 0} |\widehat{f}_k(i_k)|^2 \right)^{1/2} \left(\sum_{i_k \neq 0} |\widehat{f}_k(\varphi_k(i_k, \dots, i_{\kappa}))|^2 \right)^{1/2} \leq \|f_k\|_2^2, \end{aligned}$$

since φ_k is injective. $\sum_{i_k \in \mathbf{Z}} |\widehat{f}_k(i_k) \widehat{f}_k(\varphi_k(i_k))| \leq \|f_k\|_2^2$ is also clear for $k = \kappa + 1, \dots, K - 1$. In case $k = K$, we have

$$\begin{aligned} & \sum_{i_K \neq 0} |\widehat{f}_K(i_K) \widehat{f}_K(\varphi_K(i_K))| = \sum_{w=0}^{\infty} \sum_{i_K \in J(w)} \left| \widehat{f}_K(i_K) \widehat{f}_K([-i_K \theta^{a_K(n-n')}]^*) \right| \\ & \leq \sum_{w=0}^{\infty} \left(\sum_{i_K \in J(w)} |\widehat{f}_K(i_K)|^2 \right)^{1/2} \left(\sum_{i_K \in J(w)} |\widehat{f}_K([-i_K \theta^{a_K(n-n')}]^*)|^2 \right)^{1/2} \\ & \leq \sum_{w=0}^{\infty} \Delta_w(f_K) \Delta_{w+n-n'}(f_K). \end{aligned}$$

Thus we have

$$\begin{aligned} & \int_{-\infty}^{\infty} \left(\sum_{n=N_0}^N \prod_{k=1}^{\kappa} s_{f_k, \theta^{a_0 n}}(\theta^{a_k n} x) \prod_{k=\kappa+1}^K s_{f_k, I}(\theta^{a_k n} x) \right)^2 \mu_0(dx) \\ & \leq 2 \prod_{k=1}^{K-1} \|f_k\|_2^2 \sum_{n'=N_0}^N \sum_{n=n'}^N \sum_{w=0}^{\infty} \Delta_w(f_K) \Delta_{w+n-n'}(f_K) \\ & \leq 2N \prod_{k=1}^{K-1} \|f_k\|_2^2 \sum_{m=1}^{\infty} \sum_{w=0}^{\infty} \Delta_w(f_K) \Delta_{w+m}(f_K) \\ & \leq 2N \prod_{k=1}^{K-1} \|f_k\|_2^2 \sum_{w=0}^{\infty} \Delta_w(f_K) \sum_{m=1}^{\infty} \Delta_{w+m}(f_K) \leq 2N \prod_{k=1}^{K-1} \|f_k\|_2^2 D(f_K)^2, \end{aligned}$$

which shows (4.8). Next let us verify the convergence of $v^{(I)}$. We may assume that q_K and r_K are relatively prime. By estimating in the same way as above,

we have $|\prod_{k=1}^K \int_0^1 s_{f_k, I}^2(x) dx| \leq \prod_{k=1}^K \|f_k\|_2^2$ and

$$\begin{aligned} \left| \prod_{k=1}^K \int_0^1 s_{f_k, I}(q_k^n x) s_{f_k, I}(r_k^n x) dx \right| &\leq \prod_{k=1}^{K-1} \|f_k\|_2^2 \sum_{i_K \neq 0} |\widehat{f}_K(i_K q_K^n) \widehat{f}_K(-i_K r_K^n)| \\ &\leq \prod_{k=1}^{K-1} \|f_k\|_2^2 \sum_{i_K \neq 0} |\widehat{f}_K(i_K) \widehat{f}_K([-i_K \theta^{a_K n}]^*)| \\ &\leq \prod_{k=1}^{K-1} \|f_k\|_2^2 \sum_{w=0}^{\infty} \Delta_w(f_K) \Delta_{w+n}(f_K). \end{aligned}$$

In the same way as above we have

$$\prod_{k=1}^K \|f_k\|_2^2 + 2 \sum_{n=1}^{\infty} \prod_{k=1}^{K-1} \|f_k\|_2^2 \sum_{w=0}^{\infty} \Delta_w(f_K) \Delta_{w+n}(f_K) < \infty.$$

Since each summand of $v^{(I)}$ converges to that of v , and is dominated as above by the summand of summable series, by Lebesgue's convergence theorem for series, we have the conclusion. \square

Let us here prove the central limit theorem in the case when f_1, \dots, f_K are trigonometric polynomials without constant term.

Let us define X_N and $c_{j,N}$ by

$$X_N(x) = \sum_{j=-\infty}^{\infty} c_{j,N} \exp(2\pi\sqrt{-1} \lambda_j x) = \frac{1}{\sqrt{N}} \sum_{n=1}^N \prod_{k=1}^K f_k(\theta^{p_k(n)} x).$$

Obviously $\{c_{j,N}\}_{-\infty < j < \infty, N \geq 1}$ satisfies $c_{-j,N} = \bar{c}_{j,N}$ and $c_{j,N} = 0$ ($|j| > J_N$) for some J_N .

Firstly, let us prove

$$\int_{-\infty}^{\infty} X_N(x)^2 \mu_R(dx) \rightarrow v, \quad (4.9)$$

where

$$\int_{-\infty}^{\infty} f(x) \mu_R(dx) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(x) dx.$$

The left hand side of above formula is a symbolic expression and does not mean an integral with respect to the measure μ_R . From now on, we frequently use the following relations:

$$\begin{aligned} \int_{-\infty}^{\infty} f(\theta x) \mu_R(dx) &= \int_{-\infty}^{\infty} f(x) \mu_R(dx) \quad \text{if } \theta \neq 0; \\ \int_{-\infty}^{\infty} f(x) \mu_R(dx) &= \int_0^{\alpha} f(x) dx \quad \text{if } f \text{ has period } \alpha; \text{ and } \int_{-\infty}^{\infty} e^{\sqrt{-1}tx} \mu_R(dx) = \delta_{t0}. \end{aligned}$$

Let us denote $F_{k,\nu}(x) = f_k(x) f_k(\theta^{a_k \nu} x)$ and $d_{k,\nu} = \int_{-\infty}^{\infty} F_{k,\nu}(x) \mu_R(dx)$. Let us prove that there exists N_0 and ν_0 such that

$$\int_{-\infty}^{\infty} \prod_{k=1}^K F_{k,\nu}(\theta^{p_k(n)} x) \mu_R(dx) = \prod_{k=1}^K d_{k,\nu} \quad (n \geq N_0) \quad (4.10)$$

$$= 0 \quad (\nu \geq \nu_0, n \geq N_0). \quad (4.11)$$

If $\theta^{a_K \nu} > I$, then the trigonometric polynomial expansion of $F_{K,\nu}$ has no constant term and thereby $d_{K,\nu} = 0$. Thus (4.11) follows from (4.10). Since the left hand side of (4.10) can be decomposed into

$$\int_{-\infty}^{\infty} \prod_{k=1}^K F_{k,\nu} d\mu_R = \sum_{\kappa=1}^K \int_{-\infty}^{\infty} \prod_{k=1}^{\kappa-1} F_{k,\nu} (F_{\kappa,\nu} - d_{\kappa,\nu}) d\mu_R \prod_{k=\kappa+1}^K d_{k,\nu} + \prod_{k=1}^K d_{k,\nu},$$

the proof of (4.10) reduced to the proofs of

$$\int_{-\infty}^{\infty} \prod_{k=1}^{\kappa-1} F_{k,\nu}(\theta^{p_k(n)} x) (F_{\kappa,\nu}(\theta^{p_\kappa(n)} x) - d_{\kappa,\nu}) \mu_R(dx) = 0, \quad (n \geq N_0). \quad (4.12)$$

Since $F_{k,\nu} - d_{k,\nu}$ have no constant term, we can take $0 < I_{k,\nu} < I'_{k,\nu}$ such that the frequencies of $F_{k,\nu} - d_{k,\nu}$ belong to $(I_{k,\nu}, I'_{k,\nu})$. Then frequencies of $F_{k,\nu}(\theta^{p_k(n)} x) - d_{k,\nu}$ belong to $(I_{k,\nu} \theta^{p_k(n)}, I'_{k,\nu} \theta^{p_k(n)})$. Since we have $I_{k+1,\nu} \theta^{p_{k+1}(n)} / I'_{k,\nu} \theta^{p_k(n)} \rightarrow \infty$ as $n \rightarrow \infty$, $I_{k+1,\nu} \theta^{p_{k+1}(n)} / I'_{k,\nu} \theta^{p_k(n)} > 3$ for large n . Thus, for large n , moduli of the frequencies of the integrand of (4.12) is greater than

$$\begin{aligned} I_{\kappa,\nu} \theta^{p_\kappa(n)} - I'_{\kappa-1,\nu} \theta^{p_{\kappa-1}(n)} - \dots - I'_{1,\nu} \theta^{p_1(n)} &\geq I_{\kappa,\nu} \theta^{p_\kappa(n)} (1 - 1/3 - 1/3^2 - \dots) \\ &\geq I_{\kappa,\nu} \theta^{p_\kappa(n)} / 2 > 0. \end{aligned}$$

Hence the integrand of (4.12) has no constant term and thereby (4.12) follows.

By noting (4.10) and (4.11), we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{N}} \sum_{n=N_0}^N \prod_{k=1}^K f_k(\theta^{p_k(n)} x) \right)^2 \mu_R(dx) \\ &= \sum_{\nu=0}^{N-N_0} (2 - \delta_{\nu 0}) \frac{1}{N} \sum_{n=N_0}^{N-\nu} \int_{-\infty}^{\infty} \prod_{k=1}^K f_k(\theta^{p_k(n)} x) f_k(\theta^{p_k(n+\nu)} x) \mu_R(dx). \\ &= \sum_{\nu=0}^{N-N_0} (2 - \delta_{\nu 0}) \frac{N - \nu - N_0 + 1}{N} \prod_{k=1}^K d_{k,\nu} \end{aligned}$$

Thanks to (4.11), the sum in ν is essentially up to ν_0 , and hence as $N \rightarrow \infty$

$$\rightarrow \sum_{\nu=0}^{\infty} (2 - \delta_{\nu 0}) \prod_{k=1}^K d_{k,\nu}.$$

Let us evaluate $d_{k,\nu}$. $d_{k,0} = \|f_k\|_2^2$ is clear. If $\nu \geq 1$ and (1.8) hold, frequency $i + i' \theta^{a_k \nu}$ of $F_{k,\nu}(x) = f_k(x) f_k(\theta^{a_k \nu} x)$ can not be zero and thereby $d_{k,\nu} = 0$. Thus we have (1.7) when (1.8) holds for some k . Suppose that (1.8) does not holds for all k . Note that the set $N(k) = \{n \mid \theta^{a_k n} \in \mathbf{Q}\}$ is $\{mn_k \mid m \in \mathbf{N}\}$ where $n_k = \min N(k)$. Thus $N(0) = N(1) \cap \dots \cap N(K)$ is $\{mn \mid m \in \mathbf{N}\}$ where $n = \gcd(n_1, \dots, n_K) = \min N(0)$. If ν is not a multiple of n , then $\nu \notin N(k)$ for some k , and thereby $\prod_k d_{k,\nu} = 0$. If $\nu = mn$, then we have $\theta^{a_k \nu} = (q_k/r_k)^m$, and

thereby

$$\begin{aligned} d_{k,\nu} &= \int_{-\infty}^{\infty} f_k(x) f_k((q_k/r_k)^m x) \mu_R(dx) = \int_{-\infty}^{\infty} f_k(r_k^m x) f_k(q_k^m x) \mu_R(dx) \\ &= \int_0^1 f_k(r_k^m x) f_k(q_k^m x) dx. \end{aligned}$$

Therefore (1.9) is proved, and consequently (4.9) is verified.

Let us here verify (2.1) by using Lemma 2. By applying (3) of Lemma 3 with $\kappa = 0$, we see that there exists J_0 such that $|\lambda_j - \lambda_{j'}| \leq U$ with $j, j' \geq J_0$ implies $j = j'$. We also see that if $j \geq J_0$, values of $i_1 \theta^{p_1(n)}, \dots, i_K \theta^{p_K(n)}$ satisfying $\lambda_j = i_1 \theta^{p_1(n)} + \dots + i_K \theta^{p_K(n)}$ are uniquely determined. Note that i_1, \dots, i_K , and n are not necessarily unique. Actually, $8 \times 2^0 = 4 \times 2^1 = 2 \times 2^2 = 1 \times 2^3$ gives an example for the case $\theta = 2$ and $K = 1$. But if we give the values i_1, \dots, i_K , then n is uniquely determined. Since the choice of values of i_1, \dots, i_K are at most $(2I)^K$ in number, there are at most $(2I)^K$ many (i_1, \dots, i_K, n) which satisfy $\lambda_j = i_1 \theta^{p_1(n)} + \dots + i_K \theta^{p_K(n)}$, if $j \geq J_0$.

Let H_l be the collection of j such that ‘leading term’ $i_K \theta^{p_K(n)}$ of $\lambda_j = i_1 \theta^{p_1(n)} + \dots + i_K \theta^{p_K(n)}$ equals to $\mu_l^{(K)}$, i.e.,

$$\begin{aligned} H_l &= \left\{ j \in \mathbf{Z} \mid \begin{array}{l} |j| \geq J_0, \lambda_j = i_1 \theta^{p_1(n)} + \dots + i_K \theta^{p_K(n)}, \\ n \in \mathbf{N}, |i_1|, \dots, |i_K| \leq I, i_K \theta^{p_K(n)} = \mu_l^{(K)} \end{array} \right\}, \quad (l \in \mathbf{N}), \\ H_0 &= \mathbf{Z} \setminus \bigcup_{l=1}^{\infty} H_l. \end{aligned}$$

As in the proof of Lemma 3, we take $\varepsilon > 0$ satisfying $(1 - \varepsilon) - (1 + \varepsilon)/q > 0$, and take $\Theta > 0$ such that $(1 - \varepsilon) - (1 + \varepsilon)(1/q + 1/(\Theta - 1)) > 0$. Let us put $\pi_0 = 0$, $\pi_m = \max\{l \geq \pi_{m-1} \mid \mu_l^{(K)} \leq \Theta^m\}$ ($m \geq 0$),

$$G_m = \bigcup_{l=\pi_{m-1}+1}^{\pi_m} H_l \quad \text{and} \quad \xi_{m,N} = \sum_{j \in G_m} c_{j,N} \exp(2\pi\sqrt{-1} \lambda_j x).$$

By definition and (4.2), if $j \in G_m$ then one can find l such that $|\lambda_j| \in ((1 - \varepsilon)\mu_l^{(K)}, (1 + \varepsilon)\mu_l^{(K)})$ and $\Theta^{m-1} < \mu_l^{(K)} \leq \Theta^m$. From now on, we verify the conditions (2.2), (2.3), (2.4) and (2.5).

As we have verified, we have $\#H_l \leq (2I)^K$. Because

$$\Theta = \Theta^m / \Theta^{m-1} \geq \mu_{\pi_m}^{(K)} / \mu_{\pi_{m-1}}^{(K)} \geq q^{\pi_m - \pi_{m-1}},$$

we have $\pi_m - \pi_{m-1} \leq \log_q \Theta$ and thereby $\#G_m \leq (2I)^K \log_q \Theta$. Applying $|\widehat{f}_k(i)| \leq \|f_k\|_2$ to

$$c_{j,N} = \frac{1}{\sqrt{N}} \sum_{\substack{(i_1, \dots, i_K, n) \in [-I, I]^K \times (-\infty, \infty): \\ i_1 \theta^{p_1(n)} + \dots + i_K \theta^{p_K(n)} = \lambda_j}} \widehat{f}(i_1) \dots \widehat{f}(i_K),$$

we have $|c_{j,N}| \leq (2I)^K \|f_1\|_2 \dots \|f_K\|_2 / \sqrt{N}$. By this estimate, we have

$$\begin{aligned} \|\xi_{m,N}\|_\infty &\leq \left(\sum_{j \in G_m} |c_{j,N}|^2 \right)^{1/2} \left(\sum_{j \in G_m} 1^2 \right)^{1/2} \\ &\leq \#G_m (2I)^K \|f_1\|_2 \dots \|f_K\|_2 / \sqrt{N} \\ &\leq (2I)^{2K} \log_q \Theta \|f_1\|_2 \dots \|f_K\|_2 / \sqrt{N} = C / \sqrt{N}, \end{aligned} \quad (4.13)$$

which implies (2.2), i.e., $B_N \leq C / \sqrt{N} \rightarrow 0$. By (4.13) and (4.9), we have

$$\begin{aligned} \|V_N\|_\infty &\leq \sum_{m=1}^{\infty} \|\xi_{m,N}\|_\infty^2 \leq \sum_{m=1}^{\infty} \#G_m \sum_{j \in G_m} |c_{j,N}|^2 \\ &\leq (2I)^K \log_q \Theta \int_{-\infty}^{\infty} X_N^2(x) \mu_R(dx) \rightarrow (2I)^K (\log_q \Theta) v, \end{aligned}$$

which implies (2.5).

Next, let us verify (2.3). Let us take $r \in \mathbf{N}$ and $m_1 < \dots < m_r$. Let ϕ_j be a frequency of $\xi_{m_j,N}$. Since we have $|\phi_j| \in ((1-\varepsilon)\mu_{l_j}^{(K)}, (1+\varepsilon)\mu_{l_j}^{(K)})$ and $\Theta^{m-1} < \mu_{l_j}^{(K)} \leq \Theta^m$ for some l_j , we have the following estimate:

$$\begin{aligned} |\phi_r + \dots + \phi_1| &\geq |\phi_r| - |\phi_{r-1}| - \dots - |\phi_1| \\ &\geq (1-\varepsilon)\mu_{l_r}^{(K)} - (1+\varepsilon)(\mu_{l_{r-1}}^{(K)} + \dots + \mu_{l_1}^{(K)}) \\ &\geq (1-\varepsilon)\mu_{l_r}^{(K)} - (1+\varepsilon)\mu_{l_r}^{(K)}(1/q + 1/\Theta + 1/\Theta^2 + \dots) \\ &\geq \Theta^{m_r-1} \{(1-\varepsilon) - (1+\varepsilon)(1/q + 1/(\Theta-1))\} \rightarrow \infty. \end{aligned}$$

Thus there exists m_0 such that, if $m_r > m_0$, the last term is greater than U . This implies (2.3).

Lastly, we here verify (2.4). Let us denote by $B_{m,\kappa}$ the set of $(j, j') \in G_m^2$ such that $i_k \theta^{p_k(n)} + i'_k \theta^{p_k(n')} = 0$ for $\kappa < k \leq K$ and $i_k \theta^{p_k(n)} + i'_k \theta^{p_k(n')} \neq 0$, where $\lambda_j = i_1 \theta^{p_1(n)} + \dots + i_K \theta^{p_K(n)}$ and $\lambda_{j'} = i'_1 \theta^{p_1(n')} + \dots + i'_K \theta^{p_K(n')}$. Since we have $G_m = \bigcup_{\kappa=0}^K B_{m,\kappa}$ and $B_{m,k} \cap B_{m,k'} = \emptyset$ ($k \neq k'$), we have

$$\xi_{m,N}^2 = \zeta_{m,0,N} + \zeta_{m,1,N} + \dots + \zeta_{m,K,N},$$

where

$$\zeta_{m,\kappa,N} = \sum_{(j,j') \in B_{m,\kappa}} c_{j,N} c_{j',N} \exp(2\pi\sqrt{-1}(\lambda_j + \lambda_{j'})x).$$

Clearly we have

$$\zeta_{m,0,N} = \sum_{j \in G_m} |c_{j,N}|^2 \quad \text{and} \quad \sum_{m=1}^{\infty} \zeta_{m,0,N} = \sum_{j=-\infty}^{\infty} |c_{j,N}|^2 = \int_{-\infty}^{\infty} X_N^2(x) \mu_R(dx) = v_N.$$

By the way, $\Theta^{m-1} < i_K \theta^{p_K(n)} \Theta^m$ implies $\alpha m - \beta < n < \alpha m + \beta$ for some $\alpha, \beta > 0$. If $(j, j') \in B_{m,k}$, we have by (4.3) that

$$\begin{aligned} |\lambda_j + \lambda_{j'}| &\in ((1-\varepsilon)|i_k| \theta^{p_k(n)}, (1+\varepsilon)|i_k| \theta^{p_k(n)}) \\ &\subset ((1-\varepsilon)\theta^{p_k(\alpha m - \beta)}, (1+\varepsilon)I \theta^{p_k(\alpha m + \beta)}) = \Lambda_{k,m}. \end{aligned}$$

Since $(1 - \varepsilon)\theta^{p_k(\alpha(m+r)-\beta)}/(1 + \varepsilon)I\theta^{p_k(\alpha m + \beta)} \geq \{(1 - \varepsilon)/(1 + \varepsilon)I\}\theta^{a_k(\alpha r - 2\beta)} \geq 2$ for large r , the distance between $\Lambda_{k,m}$ and $\Lambda_{k,m'}$ is greater than 1 if $|m - m'| \geq r$. We also see that distance between arbitrary two of $\Lambda_{1,m}, \dots, \Lambda_{K,m}$ is also greater than 1. Thus we have

$$\begin{aligned} \int_{-\infty}^{\infty} (V_N - v_N)^2 d\mu_0 &= \int_{-\infty}^{\infty} \left(\sum_{m=1}^{\infty} \sum_{k=1}^K \zeta_{m,k,N} \right)^2 d\mu_0 \\ &\leq C \sum_{\substack{0 \leq j < r \\ 1 \leq k \leq K}} \int_{-\infty}^{\infty} \left(\sum_{m=1}^{\infty} \zeta_{mr+j,k,N} \right)^2 d\mu_0 = C \sum_{\substack{0 \leq j < r \\ 1 \leq k \leq K}} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \zeta_{mr+j,k,N}^2 d\mu_0 \\ &= C \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \xi_{m,N}^4 d\mu_0 \leq C \|\xi_{m,N}\|_{\infty}^2 \|V_N\|_{\infty} \sim C B_N v \rightarrow 0. \end{aligned}$$

Thus we have (2.4). Therefore we have the central limit theorem (2.1) if f_k are trigonometric polynomials without constant.

Let us define E_1 and E_2 as the previous section. By applying (4.8) for $f_1, \dots, f_{\kappa-1}, f_{\kappa} - s_{f_{\kappa}, I}, f_{\kappa+1}, \dots, f_K$, we have the following estimate of E_1^2 :

$$\begin{aligned} &\left(\int_{-\infty}^{\infty} \left| \frac{1}{\sqrt{N}} \sum_{n=N_0}^N \prod_{k=1}^K s_{\theta^{a_0 n}, f_k}(\theta^{p_k(n)} x) - \frac{1}{\sqrt{N}} \sum_{n=N_0}^N \prod_{k=1}^K s_{I, f_k}(\theta^{p_k(n)} x) \right| \mu_0(dx) \right)^2 \\ &\leq \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{N}} \sum_{n=N_0}^N \prod_{k=1}^K s_{\theta^{a_0 n}, f_k}(\theta^{p_k(n)} x) - \frac{1}{\sqrt{N}} \sum_{n=N_0}^N \prod_{k=1}^K s_{I, f_k}(\theta^{p_k(n)} x) \right)^2 \mu_0(dx) \\ &\leq \frac{\sqrt{K}}{N} \sum_{\kappa=1}^K \int_{-\infty}^{\infty} \left(\sum_{n=N_0}^N \prod_{k=1}^{\kappa-1} s_{\theta^{a_0 n}, f_k}(\theta^{p_k(n)} x) \prod_{k=\kappa+1}^K s_{I, f_k}(\theta^{p_k(n)} x) \right. \\ &\quad \left. \times (s_{\theta^{a_0 n}, f_{\kappa}} - s_{I, f_{\kappa}})(\theta^{p_{\kappa}(n)} x) \right)^2 \mu_0(dx) \\ &\leq 2\sqrt{K} \sum_{\kappa=1}^{K-1} \prod_{\substack{1 \leq k < K \\ k \neq \kappa}} \|f_k\|_2^2 \|f_{\kappa} - s_{I, f_{\kappa}}\|_2^2 D(f_K) + \prod_{k=1}^{K-1} \|f_k\|_2^2 D(f_K - s_{I, f_K}). \end{aligned}$$

Thus if we take I large enough, we have $E_1 < \varepsilon$. The estimate of E_2 can be done in the same way as section 3, we can conclude the proof also in the same way as before.

5. On implication of regularity conditions

If we assume (1.10), we have $\sum_{n=1}^N \|f_k - s_{f_k, \theta_0^n}\|_2 = o(\sqrt{N})$ ($n \rightarrow \infty, k = 1, \dots, K$). Because $\|f_k - s_{f_k, \theta_0^n}\|_2$ is decreasing in n , we have (1.4) as follows:

$$\|f_k - s_{f_k, \theta_0^n}\|_2 \leq \frac{1}{N} \sum_{n=1}^N \|f_k - s_{f_k, \theta_0^n}\|_2 = o(1/\sqrt{N}).$$

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