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# The central limit theorem for $\sum f(\theta^n x) g(\theta^{n^2} x)$

Dedicated to Professor Norio Kôno on his sixtieth birthday

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*Abstract.* In this note, it is proved that the distribution of values of $N^{-1/2} \sum_{n=1}^{N} f_1(\theta^{p_1(n)} x) \dots f_K(\theta^{p_K(n)} x)$ converges to normal distribution. Here $p_k(n)$ are polynomials.

## 1. *Introduction*

The study of this paper has been motivated by the polynomial ergodic theorem, which states

$$\frac{1}{N} \sum_{n=1}^{N} T^{p_1(n)} f_1 \, T^{p_2(n)} f_2 \dots T^{p_k(n)} f_k \xrightarrow{L^2} \prod_{i=1}^{k} \int f_i \, d\mu,$$

where $T$ is weakly mixing transformation, $p_k$'s polynomials and $f_k \in L^\infty$. In case $p_k$ are linear, the mean convergence was proved by Furstenberg, Katznelson and Ornstein [**9**], and for general polynomials by Bergelson [**2**].

Considering the transform $\theta \mapsto \theta x$ on unit interval $[0,1]$, the result of Furstenberg and Weiss [**8**] gives

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\theta^k x) g(\theta^{k^2} x) \to \int_0^1 f(x)\, dx \int_0^1 g(x)\, dx \quad \text{a.e. } x, \quad \theta = 2, 3 \dots$$

for bounded functions $f$ and $g$ with period 1. By comparing this with the following results on Riesz-Raikov sums, we can expect that the following central limit theorem holds:

$$\left| \left\{ x \in [0,1] \;\middle|\; \frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(\theta^k x) g(\theta^{k^2} x) \in [a,b] \right\} \right| \to \frac{1}{\sqrt{2\pi v}} \int_a^b e^{-x^2/2v}\, dx,$$

for all $a < b$, when $\int_0^1 f(x)\, dx = \int_0^1 g(x)\, dx = 0$. ($|\cdot|$ denotes the Lebesgue measure.)

We here give a brief survey on the probabilistic studies on Riesz-Raikov sums to explain the context above.

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Let $f$ be a real-valued locally integrable function on $\mathbf{R}$ with period 1. Raikov [**15**] proved that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(\theta^k x)=\int_0^1 f(x)\,dx,\quad \text{a.e. } x,\quad (\theta=2,3,\dots),$$

and Riesz [**17**] pointed out that this result is an example of the ergodic theorem.

Having investiged more precise limiting behaviour of Riesz-Raikov sums, Kac [**12**] proved the following central limit theorem: If $f$ is Hölder continuous and $\int_0^1 f(x)\,dx=0$, then

$$\left|\left\{x\in[\,0,1\,]\;\middle|\;\frac{1}{\sqrt{n}}\sum_{k=1}^{n} f(\theta^k x)\in[\,a,b\,]\right\}\right|\to\frac{1}{\sqrt{2\pi v}}\int_a^b e^{-x^2/2v}\,dx, \tag{1.1}$$

for all $a<b$ and $\theta=2,3,\dots$, where $v=\int_0^1 f^2(x)\,dx+2\sum_{k=1}^{\infty}\int_0^1 f(x)f(\theta^k x)\,dx$. Ibragimov [**11**] extended the above results to locally square integrable function $f$ with the $L^2$-Dini condition below:

$$\int_0^1\frac{\omega_2(y)}{y}\,dy<\infty\quad\text{where}\quad \omega_2(\delta)=\sup_{|h|\le\delta}\left(\int_0^1|f(x+h)-f(x)|^2\,dx\right)^{1/2}.$$

Although Takahashi [**20**] gave some results under the relative measure, we must wait until Berkes [**3**] and [**4**] to have the central limit theorem (1.1) for non integral $\theta>1$. He proved (1.1) for all $\theta>1$ as a corollary of his general result, while the limiting variance $v$ was not explicitely determined. Kaufman [**13**] proved $v=\int_0^1 f^2(x)\,dx$ for almost all $\theta>1$ with respect to Lebesgue measure, and also for all algebraic $\theta>1$ with

$$\theta^r\notin\mathbf{Q}\quad\text{for all } r\in\mathbf{N}. \tag{1.2}$$

Petit [**14**] treated the case when the algebraic number $\theta>1$ does not satisfy (1.2). When $r$ is the minimum positive integer with $\theta^r\in\mathbf{Q}$, while $\theta^r=\mu/\nu$ is an irreducible fraction of integers, he proved $v=\int_0^1 f^2(x)\,dx+2\sum_{s=1}^{\infty}\int_0^1 f(\mu^s x)f(\nu^s x)\,dx$. Fukuyama [**7**] proved that $v=\int_0^1 f^2(x)\,dx$ for all $\theta>1$ with (1.2), and the limiting variance was completely determined.

In the above studies, Hölder's condition, Dini's condition, or some variation of these are assumed.

We are now in a position to state our theorem. Note that $\theta>1$ is not necessarily an integer.

Theorem 1. *Let us assume that polynomials $p_k$ $(1\le k\le K)$ satisfy $p_k(\infty)=\infty$ and $(p_k-p_{k'})(\infty)=\infty$ $(k>k')$, and functions $f_1,\dots,f_K$ on $\mathbf{R}$ with period 1 satisfy*

$$\int_0^1 f_k(x)\,dx=0\quad\textit{and}\quad\int_0^1|f_k(x)|^{2K-2}\,dx<\infty\quad(k=1,\dots,K), \tag{1.3}$$

$$\int_0^1|f_k(x)-s_{f_k,n}(x)|^2\,dx=o(1/\log n)\quad(n\to\infty,\ k=1,\dots,K), \tag{1.4}$$

*and*

$$\sum_{n=0}^{\infty}\Big(\int_0^1 |s_{f_K,2^{n+1}}(x) - s_{f_K,2^n}(x)|^2\,dx\Big)^{1/2} < \infty \tag{1.5}$$

*where* $s_{f,n}$ *denotes the* $n$*-th subsum of Fourier series of* $f$*. Let* $P$ *be a probability measure on* $\mathbf{R}$ *which is absolutely continuous with respect to the Lebesgue measure. Then we have the central limit theorem*

$$P\Big\{x \in \mathbf{R} \;\Big|\; \frac{1}{\sqrt{N}}\sum_{n=1}^{N}\prod_{k=1}^{K} f_k(\theta^{p_k(n)}x) \in [\,a, b\,]\Big\} \to \frac{1}{\sqrt{2\pi v}}\int_a^b e^{-x^2/2v}\,dx, \tag{1.6}$$

*as* $N \to \infty$ *for all* $a < b$*,* $K \geq 2$ *and* $\theta > 1$*. The limiting variance* $v$ *in* (1.6) *is determined as follows:*

(1) *If* $\max_k \deg p_k \geq 2$*, then*

$$v = \prod_{k=1}^{K}\int_0^1 f_k^2(x)\,dx. \tag{1.7}$$

(2) *When all* $p_k$ *are linear, i.e.,* $p_k(x) = a_k x + b_k$*, and if the condition*

$$\theta^{a_k n} \notin \mathbf{Q} \quad \text{for all} \quad n \in \mathbf{N} \tag{1.8}$$

*is satisfied for at lease one of* $k = 1, \ldots, K$*, then* $v$ *is given by* (1.7)*.*

(3) *In case* (1.8) *is not true for all* $k = 1, \ldots, K$*, let us take the smallest* $n \geq 1$ *satisfying* $\theta^{a_k n} \in \mathbf{Q}$ *for all* $k$*, and write* $\theta^{a_k n} = q_k/r_k$ *by using* $q_k$*,* $r_k \in \mathbf{N}$*. Then*

$$v = \prod_{k=1}^{K}\int_0^1 f_k^2(x)\,dx + 2\sum_{n=1}^{\infty}\prod_{k=1}^{K}\int_0^1 f_k(q_k^n x) f_k(r_k^n x)\,dx. \tag{1.9}$$

Conditions (1.4) and (1.5) follows from the next condition

$$\sum_{n=1}^{\infty}\Big(\int_0^1 |f_k(x) - s_{f_k,2^n}(x)|^2\,dx\Big)^{1/2} < \infty \tag{1.10}$$

which is equivalent to $L^2$-Dini condition. The equivalence is proved by (3.3) of pp. 241 of Zygmund [**21**] and by (2.6) of pp. 160 of Bari [**1**].

## 2. *Preliminaries*

Set $\|f\|_\infty = \operatorname{ess\,sup}_{-\infty<x<\infty} |f(x)|$ and $\|f\|_p = (\int_0^1 |f(x)|^p\,dx)^{1/p}$. Let $[\,x\,]$ denote the integer part of $x$. In this note we abuse notation and denote $s_{f,[\,a\,]}$ simply by $s_{f,a}$ when $a$ is not an integer.

To prove the central limit theorem, we use the next lemma, whose idea is essentially due to P. Hartman [**10**].

Let us put

$$h_\lambda(x) = \Big(\frac{\sin \lambda x}{\lambda x}\Big)^2 \quad \text{and} \quad h(x) = \frac{h_{1/2}(x) + h_{1/2\sqrt{2}}(x)}{1+\sqrt{2}}.$$

It is easily verified that $h$ is a positive integrable analytic function on $\mathbf{R}$ which satisfies the following conditions (Cf. L. Breiman [**5**] pp.218):

$$\widehat{h}(u) = 0 \quad (|u| > 1), \quad \text{and} \quad |\widehat{h}(u)| \le 1 \quad (u \in \mathbf{R}).$$

For $u \in \mathbf{R}$, let us define a measure $\mu_u$ on $\mathbf{R}$ by $\mu_u(dx) = e^{\sqrt{-1}\,ux} h(x)\,dx$. Clearly $\mu_0$ is a probability measure. We frequently use the inequality

$$\left| \int_{-\infty}^{\infty} f(x)\, \mu_u(dx) \right| \le \int_{-\infty}^{\infty} |f(x)|\, \mu_0(dx),$$

and the relation

$$\int_{-\infty}^{\infty} e^{\sqrt{-1}\,\lambda x}\, \mu_u(dx) = 0 \quad \text{if} \quad |\lambda| \ge U = |u| + 1.$$

Lemma 1. *Suppose that a sequence* $\{X_N\}$ *of real functions on* $\mathbf{R}$ *satisfies*

$$\int_{-\infty}^{\infty} e^{\sqrt{-1}\,tX_N(x)}\, \mu_u(dx) \to e^{-t^2 v/2} \widehat{h}(u) \qquad (t \in \mathbf{R},\ N \to \infty), \tag{2.1}$$

*for all* $u \in \mathbf{R}$, *then the central limit theorem*

$$P\{\, x \in \mathbf{R} \mid X_N(x) \in [\,a, b\,]\,\} \to \frac{1}{\sqrt{2\pi v}} \int_a^b e^{-x^2/2v}\, dx \qquad (a < b,\ N \to \infty),$$

*holds where* $P$ *is any probability measure on* $\mathbf{R}$ *which is absolutely continuous with respect to the Lebesgue measure.*

*Proof.* Let us first suppose that $P(dx) = g(x)\,dx$ and $g \in C^\infty$ has compact support. Since $g/h$ is rapidly decreasing, both $g/h$ and $(g/h)\widehat{\ }$ are integrable. Therefore we can apply the inversion formula for $g/h$, Fubini's theorem and Lebesgue's convergence theorem in turn to prove the following convergence which is equivalent to (2.1):

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{\sqrt{-1}\,tX_N(x)}\, P(dx) &= \int_{-\infty}^{\infty} e^{\sqrt{-1}\,tX_N(x)} \frac{g(x)}{h(x)} h(x)\, dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} (g/h)\widehat{\ }(\gamma)\, d\gamma \int_{-\infty}^{\infty} e^{\sqrt{-1}\,tX_N(x)} e^{-\sqrt{-1}\,x\gamma} h(x)\, dx \\
&\to \frac{e^{-t^2 v/2}}{2\pi} \int_{-\infty}^{\infty} (g/h)\widehat{\ }(\gamma) \widehat{h}(-\gamma)\, d\gamma \\
&= e^{-t^2 v/2} ((g/h) * h(-\,\cdot\,))(0) = e^{-t^2 v/2}.
\end{aligned}$$

For the last equality, we have used integrability and continuity of $(g/h) * h(-\,\cdot\,)$ and its Fourier transform.

Next, we treat the general case. For all $\varepsilon > 0$, we can take $g_\varepsilon \in C^\infty$ with compact support such that

$$g_\varepsilon(x) \ge 0, \quad \int_{-\infty}^{\infty} g_\varepsilon(x)\, dx = 1, \quad \text{and} \quad \int_{-\infty}^{\infty} |g(x) - g_\varepsilon(x)|\, dx \le \varepsilon.$$

By using $g_\varepsilon$, we have

$$\begin{aligned}\Big|\int_{-\infty}^{\infty} & e^{\sqrt{-1}\,tX_N(x)} g(x)\,dx - e^{-t^2 v/2}\Big| \\ &\le \Big|\int_{-\infty}^{\infty} e^{\sqrt{-1}\,tX_N(x)} g(x)\,dx - \int_{-\infty}^{\infty} e^{\sqrt{-1}\,tX_N(x)} g_\varepsilon(x)\,dx\Big| \\ &\quad + \Big|\int_{-\infty}^{\infty} e^{\sqrt{-1}\,tX_N(x)} g_\varepsilon(x)\,dx - e^{-t^2 v/2}\Big|.\end{aligned}$$

The second term tends to zero as $N \to \infty$, and the first term is dominated by $\int |g - g_\varepsilon| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have the conclusion. □

The following Lemma is a variation of the result by P. Révész [**16**] and its argument is essentially due to Salem-Zygmund [**18**].

Lemma 2. *Let $u \in \mathbf{R}$ and $\{\xi_{m,N}\}_{1\le m\le M_N,\, N\ge 1}$ be an array of functions on $\mathbf{R}$. If*

$$B_N = \sup_{1\le m\le M_N} \|\xi_{m,N}\|_\infty \to 0 \quad (N\to\infty), \tag{2.2}$$

$$\int_{-\infty}^{\infty} \xi_{m_1,N}\ldots\xi_{m_r,N}\,d\mu_u = 0 \quad (N\in\mathbf{N},\ r\in\mathbf{N},\ m_0 \le m_1 < \cdots < m_r), \tag{2.3}$$

$$V_N = \sum_{m=m_0}^{M_N} \xi_{m,N}^2 \longrightarrow v \quad \textit{in measure } \mu_0 \quad (N\to\infty), \tag{2.4}$$

*and*

$$B_0 = \sup_{N\ge 1} \|V_N\|_\infty < \infty, \tag{2.5}$$

*are satisfied for some $m_0$, then* (2.1) *holds for $X_N = \sum_{m=1}^{M_N} \xi_{m,N}$.*

*Proof.* If we put $Y_N = \sum_{m=m_0}^{N_M} \xi_{m,N}$, by $|X_N - Y_N| \le m_0 B_N \to 0$, we have

$$\Big|\int_{-\infty}^{\infty} e^{\sqrt{-1}\,tX_N}\,d\mu_u - \int_{-\infty}^{\infty} e^{\sqrt{-1}\,tY_N}\,d\mu_u\Big| \le \int_{-\infty}^{\infty} |e^{\sqrt{-1}\,t(X_N - Y_N)} - 1|\,d\mu_0 \to 0.$$

Thus it is sufficient for us to prove (2.1) for $Y_N$. Let us recall the following asymptotic formula. (Cf. Salem-Zygmund [**18**])

$$e^{\sqrt{-1}\,x} = (1+\sqrt{-1}\,x)e^{-x^2/2 + R(x)} \quad \text{where} \quad |R(x)| \le |x|^3,$$

By applying this, we have

$$e^{\sqrt{-1}\,tY_N} = T_N \exp(-t^2 V_N/2 + R_N)$$

where

$$T_N = \prod_{m=m_0}^{M_N} (1+\sqrt{-1}\,t\xi_{m,N}) \quad \text{and} \quad R_N = \sum_{m=m_0}^{M_N} R(t\xi_{m,N}).$$

$R_N$ and $T_N$ have the following bounds:

$$R_N \le \sum_{m=m_0}^{M_N} |t\xi_{m,N}|^3 \le |t|^3 B_N V_N \le |t|^3 B_N B_0 \to 0 \quad \text{and} \quad |T_N| \le e^{t^2 V_N/2}.$$

By (2.3) we have $\int_{-\infty}^{\infty} T_N\, \mu_u(dx) = \widehat{h}(u)$ and hence we get

$$\left|\int_{-\infty}^{\infty} e^{\sqrt{-1}\,tY_N}\, d\mu_u - \widehat{h}(u)e^{-t^2v/2}\right|$$
$$= \left|\int_{-\infty}^{\infty} T_N\Big(\exp(-t^2V_N/2 + R_N) - \exp(-t^2v/2)\Big)\, d\mu_u\right|$$
$$\le \int_{-\infty}^{\infty} |\exp(R_N) - 1|\, d\mu_0 + \int_{-\infty}^{\infty} |\exp(t^2(V_N - v)/2) - 1|\, d\mu_0 \to 0,$$

since the integrand is bounded and tends to 0 in measure $\mu_0$. □

3. *The case* $\max\deg f_k \ge 2$

In this section we assume $\max\deg f_k \ge 2$ or $\deg f_K \ge 2$. If we denote $\beta_n^{(k)} = \theta^{p_k(n)}$, it is clear that there exists $q > 1$ and $N_0$ such that

$$\beta_{n+1}^{(k)}/\beta_n^{(k)} > q \quad (n \ge N_0,\ k = 1, \dots, K), \tag{3.1}$$
$$\beta_{n+1}^{(K)}/\beta_n^{(K)} \to \infty \quad (n \to \infty), \tag{3.2}$$
$$\beta_n^{(k+1)}/\beta_n^{(k)} \to \infty \quad (n \to \infty,\ k = 1, \dots, K-1). \tag{3.3}$$

Assuming only these three conditions, we can prove the central limit theorem for trigonometric polynomials.

Proposition 1. *Let $f_1, \dots, f_K$ be trigonometric polynomials without constant term. If sequences of positive numbers $\{\beta_n^{(1)}\}, \dots, \{\beta_n^{(K)}\}$ satisfy* (3.1)*,* (3.2) *and* (3.3)*, then for all $u \in \mathbf{R}$,* (2.1) *holds for*

$$X_N(x) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \prod_{k=1}^{K} f_k(\beta_n^{(k)} x) \quad \text{and} \quad v = \prod_{k=1}^{K} \int_0^1 f_k^2(x)\, dx,$$

*that is the central limit theorem holds.*

*Proof.* Let $I \in \mathbf{N}$ and assume that degrees of $f_1, \dots, f_K$ are less than $I$. Let us take $u$ arbitrary and put $U = |u| + 1$. Let us set $M_N = N$ and $\xi_{n,N}(x) = f_1(\beta_n^{(1)}x)\dots f_K(\beta_n^{(K)}x)/\sqrt{N}$ and prove that $\{\xi_{n,N}\}$ satisfy the conditions of Lemma 2. If we put $B = \max_k \|f_k\|_\infty$, we have $\|\xi_{n,N}\|_\infty \le B^K/\sqrt{N}$ and $\|V_N\|_\infty \le B^K$, and hence (2.2) and (2.5) are clear. $\xi_{n,N}$ can be expanded to

$$\xi_{n,N}(x) = \frac{1}{\sqrt{N}} \sum_{\substack{|i_k| \le I,\\ k=1,\dots,K}} \widehat{f}_1(i_1)\dots\widehat{f}_K(i_K) \exp\Big(2\pi\sqrt{-1}\,(i_1\beta_n^{(1)} + \dots + i_K\beta_n^{(K)})x\Big).$$

By (3.2) and (3.3), there exists $N_0$ such that, for any $n \ge N_0$ and $k = 1, \dots, K$,

$$\beta_{n+1}^{(K)}/\beta_n^{(K)} \ge 10I, \qquad \beta_n^{(k)} \ge 6U \tag{3.4}$$

and

$$|i_1\beta_n^{(1)} + \dots + i_k\beta_n^{(k)}| \in (\beta_n^{(k)}/2, 3I\beta_n^{(k)}) \quad \text{if} \quad |i_1|, \dots, |i_k| \le 2I,\ i_k \ne 0. \tag{3.5}$$

Let us now assume $N_0 \le n_1 < \cdots < n_r$ and verify multiple orthogonality (2.3). If we expand $\xi_{n_1,N} \ldots \xi_{n_r,N}$ into trigonometric polynomial, frequencies can be written as $\lambda_{n_r} + \cdots + \lambda_{r_1}$ where $\lambda_{n_i}$ is a frequency of $\xi_{n_i,N}$. Thanks to (3.4) and (3.5) we have

$$\begin{aligned}|\lambda_{n_r} + \cdots + \lambda_{r_1}| &\ge |\lambda_{n_r}| - |\lambda_{n_{r-1}}| - \cdots - |\lambda_{r_1}| \\ &\ge \beta^{(K)}_{n_r}/2 - 3I(\beta^{(K)}_{n_{r-1}} + \cdots + \beta^{(K)}_{n_1}) \\ &\ge \beta^{(K)}_{n_r}/2 - 3I\beta^{(K)}_{n_{r-1}}(1 + 1/10I + 1/(10I)^2 + \cdots) \\ &\ge \beta^{(K)}_{n_r}/2 - (3I/10I)\beta^{(K)}_{n_r}\{1/(1-1/10)\} \ge \beta^{(K)}_{n_r}/6 \ge U.\end{aligned}$$

By this estimate, we see $\int_{-\infty}^{\infty} \exp(2\pi\sqrt{-1}\,(\lambda_{n_r}+\cdots+\lambda_{r_1})x)\,\mu_u(dx) = 0$, and hence we have (2.3).

Lastly, let us verify (2.4). Let us take an $r$ satisfying $q^r \ge 12I$. To prove (2.4), it is sufficient to prove

$$\sum_{n=N_0}^{N} (\xi^2_{nr+j,N} - v/N) \to 0 \quad \text{in measure } \mu_0,$$

for each $j = 0, \ldots, r-1$. Let us put $\sigma_k^2 = \int_0^1 f_k^2(x)\,dx$. Since $v = \sigma_1^2 \ldots \sigma_K^2$, we have

$$\xi^2_{nr+j,N}(x) - \frac{v}{N} = \frac{1}{N}\sum_{\kappa=1}^{K}\prod_{k=1}^{\kappa-1} f_k^2(\beta^{(k)}_{nr+j}x)(f_\kappa^2(\beta^{(\kappa)}_{nr+j}x) - \sigma_\kappa^2)\prod_{k=\kappa+1}^{K}\sigma_k^2.$$

Thus it is sufficient to prove the following convergence in measure $\mu_0$:

$$\frac{1}{N}\sum_{n=N_0}^{N}\prod_{k=1}^{\kappa-1} f_k^2(\beta^{(k)}_{nr+j}x)(f_\kappa^2(\beta^{(\kappa)}_{nr+j}x) - \sigma_\kappa^2) \to 0.$$

Let $\zeta_n$ denote the summand. Note that the trigonometric polynomial expansion of $f_k^2(\beta^{(\kappa)}_{nr+j}x) - \sigma_\kappa^2$ has no constant term. Thus frequencies of the trigonometric expansion of $\zeta_n$ can be written as the right hand side of (3.5), and hence belong to $(\beta^{(\kappa)}_{nr+j}/2, 3I\beta^{(\kappa)}_{nr+j})$. Thanks to $\beta^{(\kappa)}_{nr+j}/2 \ge 6U/2 > 1$, we have

$$\int_{-\infty}^{\infty} \zeta_n\,d\mu_0 = 0. \tag{3.6}$$

By $(\beta^{(\kappa)}_{(n+1)r+j}/2)/(3I\beta^{(\kappa)}_{nr+j}) \ge q^r/6I \ge 2$, we have

$$\beta^{(\kappa)}_{(n+1)r+j}/2 - 3I\beta^{(\kappa)}_{nr+j} \ge (\beta^{(\kappa)}_{(n+1)r+j}/2)(1-1/2) \ge 6U/4 > 1$$

and hence frequencies of $\zeta_n$ and $\zeta_{n'}$ differ by at least 1 if $n \ne n'$. Thereby we conclude that $\int_{-\infty}^{\infty}\zeta_n(x)\zeta_{n'}(x)\,\mu_0(dx) = 0$, and we have

$$\int_{-\infty}^{\infty}\Big(\frac{1}{N}\sum_{n=N_0}^{N}\zeta_n\Big)^2 d\mu_0 = \frac{1}{N^2}\sum_{n=N_0}^{N}\int_{-\infty}^{\infty}\zeta_n^2\,d\mu_0 \le \frac{\max_n \|\zeta_n\|_\infty^2}{N} \to 0,$$

which implies the convergence in measure. □

Since $p_{k+1} - p_k$ is a polynomial diverging to infinity, there exists $a_0 > 0$ and $N_0 \in \mathbf{N}$ such that $p_{k+1}(n) - p_k(n) \geq 2a_0 n$ for all $n \geq N_0$. Since $p_K$ is not linear, there also exist $a_0 > 0$ and $N_0 \in \mathbf{N}$ such that $p_K(n+1) - p_K(n) \geq 2a_0 n$ for all $n \geq N_0$. Thus we have

$$\beta_n^{(k+1)}/\beta_n^{(k)} \geq \theta^{2a_0 n} \quad (n \geq N_0,\ k = 1, \ldots, K-1), \tag{3.7}$$

$$\beta_{n+1}^{(K)}/\beta_n^{(K)} \geq \theta^{2a_0 n} \quad (n \geq N_0). \tag{3.8}$$

Proposition 2. *Let functions $f_1, \ldots, f_K$ with period* 1 *satisfy* (1.3) *and* (1.4)*, and sequences of positive numbers $\{\beta_n^{(1)}\}, \ldots, \{\beta_n^{(K)}\}$ satisfy* (3.1), (3.7) *and* (3.8)*. Then the conclusion of Proposition 1 holds.*

Remark 1. *In this proposition, the condition* (1.5) *is not assumed. Thus the part (1) of Theorem 1 holds without* (1.5).

*Proof.* Because of $\beta_n^{(k)}\theta^{a_0 n} = o(\beta_n^{(k+1)})$, it can be proved in the same way that there exists $N_0$ such that, for all $n \geq N_0$,

$$|i_1\beta_n^{(1)} + \cdots + i_k\beta_n^{(k)}| \in (\beta_n^{(k)}/2, 3\theta^{a_0 n}\beta_n^{(k)}) \text{ if } |i_1|, \ldots, |i_k| \leq 2\theta^{a_0 n},\ i_k \neq 0. \tag{3.9}$$

Firstly, we have the following estimate:

$$\begin{aligned}
E_1^2 &= \left(\int_{-\infty}^{\infty}\left|\frac{1}{\sqrt{N}}\sum_{n=N_0}^{N}\prod_{k=1}^{K} s_{f_k,\theta^{a_0 n}}(\beta_n^{(k)}x) - \frac{1}{\sqrt{N}}\sum_{n=N_0}^{N}\prod_{k=1}^{K} s_{f_k,I}(\beta_n^{(k)}x)\right|\mu_0(dx)\right)^2 \\
&\leq \int_{-\infty}^{\infty}\left\{\frac{1}{\sqrt{N}}\sum_{n=N_0}^{N}\left(\prod_{k=1}^{K} s_{f_k,\theta^{a_0 n}}(\beta_n^{(k)}x) - \prod_{k=1}^{K} s_{f_k,I}(\beta_n^{(k)}x)\right)\right\}^2\mu_0(dx) \\
&\leq \frac{\sqrt{K}}{N}\sum_{\kappa=1}^{K}\int_{-\infty}^{\infty}\left(\begin{array}{c}\displaystyle\sum_{n=N_0}^{N}\prod_{k=1}^{\kappa-1} s_{f_k,\theta^{a_0 n}}(\beta_n^{(k)}x)\prod_{k=\kappa+1}^{K} s_{f_k,I}(\beta_n^{(k)}x) \\ \times\,(s_{f_\kappa,\theta^{a_0 n}} - s_{f_\kappa,I})(\beta_n^{(\kappa)}x)\end{array}\right)^2\mu_0(dx)
\end{aligned}$$

By (3.9), moduli of frequencies of summand in the above integrand belong to $(\beta_n^{(K)}/2, 3\theta^{a_0 n}\beta_n^{(K)})$. Because $(\beta_n^{(K)}/2)/(3\theta^{a_0 n}\beta_n^{(K)}) \geq \theta^{a_0 n}/6 \to \infty$, the distance between these intervals is greater than 1, and therefore these summands are orthogonal. Thus

$$E_1^2 \leq \frac{\sqrt{K}}{N}\sum_{\kappa=1}^{K}\sum_{n=N_0}^{N}\int_{-\infty}^{\infty}\left(\begin{array}{c}\displaystyle\prod_{k=1}^{\kappa-1} s_{f_k,\theta^{a_0 n}}^2(\beta_n^{(k)}x)\prod_{k=\kappa+1}^{K} s_{f_k,I}^2(\beta_n^{(k)}x) \\ \times\,(s_{f_\kappa,\theta^{a_0 n}} - s_{f_\kappa,I})^2(\beta_n^{(\kappa)}x)\end{array}\right)\mu_0(dx). \tag{3.10}$$

If functions $g_k$ ($k = 1, \ldots, K$) are trigonometric polynomials whose degrees are less than $\theta^{a_0 n}$, then by applying a similar argument as the derivation of (3.6) to the decomposition

$$\begin{aligned}\int_{-\infty}^{\infty} \prod_{k=1}^{K} g_k^2(\beta_n^{(k)}x)\,\mu_0(dx) - \prod_{k=1}^{K} \|g_k\|_2^2 \\ = \sum_{\kappa=1}^{K} \int_{-\infty}^{\infty} \prod_{k=1}^{\kappa-1} g_k^2(\beta_n^{(k)}x)(g_\kappa^2(\beta_n^{(\kappa)}x) - \|g_\kappa\|_2^2)\,\mu_0(dx) \prod_{k=\kappa+1}^{K} \|g_k\|_2^2,\end{aligned}$$

we have $\int_{-\infty}^{\infty} \prod_{k=1}^{K} g_k^2(\beta_n^{(k)}x)\,\mu_0(dx) = \prod_{k=1}^{K} \|g_k\|_2^2$ for large $n$. By applying this to (3.10), we have

$$\begin{aligned}E_1^2 &\le \sqrt{K} \sum_{\kappa=1}^{K} \prod_{k=1}^{\kappa-1} \|s^2_{f_k,\theta^{a_0 n}}\|_2^2 \prod_{k=\kappa+1}^{K} \|s^2_{f_k,I}\|_2^2 \|s_{f_\kappa,\theta^{a_0 n}} - s_{f_\kappa,I}\|_2^2 \\ &\le \sqrt{K} \sum_{\kappa=1}^{K} \|f_\kappa - s_{f_\kappa,I}\|_2^2 \prod_{k\ne\kappa} \|f_k\|_2^2.\end{aligned}$$

Therefore, for any $\varepsilon > 0$, there exists $I_0$ such that $I \ge I_0$ implies $E_1 < \varepsilon/2$.

Before proceeding further, let us prepare some inequalities. For a function $F$ with period 1 and $\|F\|_{2p} < \infty$, $\|s_n\|_{2p} \le C\|F\|$ and $\|F - s_n\|_{2p} \le C\|F\|$ holds. (Cf. A. Zygmund [**21**].) We here prove $\int_{-\infty}^{\infty} |F(\theta x)|^{2p}\,\mu_0(dx) = \|F\|_{2p}^{2p}$ for $\theta > 1$ and $p \in \mathbf{N}$. For trigonometric polynomial, it is proved by the direct calculation as below:

$$\begin{aligned}\int_{-\infty}^{\infty} |F(\theta x)|^{2p}\,\mu_0(dx) &= \sum_{l_1,\dots,l_{2p}} \widehat{F}(l_1)\dots\widehat{F}(l_{2p})\widehat{h}(2\pi(l_1+\dots+l_p)\theta) \\ &= \sum_{l_1,\dots,l_{2p}} \widehat{F}(l_1)\dots\widehat{F}(l_{2p})\delta_{l_1+\dots+l_p,0} \\ &= \|F(x)\|_{2p}^{2p},\end{aligned}$$

because $\widehat{h}(2\pi l\theta) \ne 0$ and $l \in \mathbf{Z}$ is equivalent to $l = 0$. Since $s_n(\theta x)$ converges to $F(\theta x)$ in measure $dx$, hence in measure $\mu_0$. By Fatou lemma,

$$\int_{-\infty}^{\infty} |F(\theta x)|^{2p}\,\mu_0(dx) \le \liminf_{n\to\infty} \int_{-\infty}^{\infty} |s_n(\theta x)|^{2p}\,\mu_0(dx) = \lim_{n\to\infty} \int_0^1 |s_n(x)|^{2p}\,dx = \|F\|_{2p}^{2p}.$$

Hence we have that $s_n(\theta x)$ converges to $F(\theta x)$ in $L^{2p}(\mathbf{R}, \mu_0)$-sense. Thereby we can conclude that

$$\int_{-\infty}^{\infty} |F(\theta x)|^{2p}\,\mu_0(dx) = \lim_{n\to\infty} \int_{-\infty}^{\infty} |s_n(\theta x)|^{2p}\,\mu_0(dx) = \lim_{n\to\infty} \|s_n\|_{2p}^{2p} = \|F\|_{2p}^{2p}.$$

By using these, we have

$$E_2 = \int_{-\infty}^{\infty} \left| \frac{1}{\sqrt{N}} \sum_{n=N_0}^{N} \prod_{k=1}^{K} f_k(\beta_n^{(k)}x) - \frac{1}{\sqrt{N}} \sum_{n=N_0}^{N} \prod_{k=1}^{K} s_{\theta^{a_0 n},f_k}(\beta_n^{(k)}x) \right| \mu_0(dx)$$

$$\le \frac{1}{\sqrt{N}} \sum_{n=N_0}^{N} \sum_{\kappa=1}^{K} \int_{-\infty}^{\infty} \left| \begin{array}{c} \prod_{k=1}^{\kappa-1} f_k(\beta_n^{(k)} x) \prod_{k=\kappa+1}^{K} s_{\theta^{a_0 n}, f_k}(\beta_n^{(k)} x) \\ \times (f_\kappa - s_{\theta_0^n, f_\kappa})(\beta_n^{(\kappa)} x) \end{array} \right| \mu_0(dx)$$

$$\le \frac{1}{\sqrt{N}} \sum_{n=N_0}^{N} \sum_{\kappa=1}^{K} \prod_{k=1}^{\kappa-1} \|f_k\|_{2(K-1)} \|f_\kappa - s_{\theta^{a_0 n}, f_\kappa}\|_2 \prod_{k=\kappa+1}^{K} \|s_{\theta^{a_0 n}, f_k}\|_{2(K-1)}$$

$$\le \sum_{\kappa=1}^{K} \prod_{k \ne \kappa} \|f_k\|_{2(K-1)} \frac{1}{\sqrt{N}} \sum_{n=N_0}^{N} \|f_\kappa - s_{\theta^{a_0 n}, f_\kappa}\|_2 \to 0$$

because of the fact that $\frac{1}{2(K-1)}(K-1) + \frac{1}{2} = 1$, Hölder's inequality and (1.4).

Thanks to the estimate of $E_1$ and $E_2$ above, we have

$$\limsup_{N\to\infty} \int_{-\infty}^{\infty} |X_N(x) - X_N^{(I)}(x)| \, \mu_0(dx) \le \varepsilon$$

for large $I$, where $X_N^{(I)}(x) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \prod_{k=1}^{K} s_{I, f_k}(\beta_n^{(k)} x)$. By putting $v^{(I)} = \prod_{k=1}^{K} \|s_{I, f_k}\|_2^2$, we have

$$|v - v^{(I)}| \le \varepsilon$$

for large $I$. Noting Proposition 1, the fact that $|e^x - 1| \le |x|$ $(x \le 0)$ and that $|e^{\sqrt{-1}\,x} - 1| \le |x|$ $(x \in \mathbf{R})$, if we take $I$ large enough, we have

$$\begin{aligned}
&\limsup_{N\to\infty} \left| \int_{-\infty}^{\infty} e^{\sqrt{-1}\, t X_N} \, d\mu_u - e^{-t^2 v/2} \widehat{h}(u) \right| \\
&\quad \le \limsup_{N\to\infty} \left| \int_{-\infty}^{\infty} e^{\sqrt{-1}\, t X_N} \, \mu_u(dx) - \int_{-\infty}^{\infty} e^{\sqrt{-1}\, t X_N^{(I)}} \, d\mu_u \right| \\
&\qquad + \limsup_{N\to\infty} \left| \int_{-\infty}^{\infty} e^{\sqrt{-1}\, t X_N^{(I)}} \, d\mu_u - e^{-t^2 v^{(I)}/2} \widehat{h}(u) \right| + \limsup_{N\to\infty} |e^{-t^2 v^{(I)}/2} - e^{-t^2 v/2}| \\
&\quad \le |t| \int_{-\infty}^{\infty} |X_N - X_N^{(I)}| \, d\mu_0 + t^2 |v^{(I)} - v|/2 \le (|t| + t^2/2)\varepsilon,
\end{aligned}$$

which implies the conclusion. □

## 4. *The case when $p_k$ are linear*

Let $\{\mu_l^{(k)}\}_{l\in\mathbf{N}}$ be an arrangement of the set $\bigcup_{i_k=1}^{I} \{i_k \theta^{p_k(n)} \mid n \in \mathbf{N}\}$ in increasing order, and $\{\lambda_j\}_{j\in\mathbf{Z}}$ be an arrangement of the set

$$\{\, i_1 \theta^{p_1(n)} + \cdots + i_K \theta_K^{p_K(n)} \mid n \in \mathbf{N}, 1 \le |i_k| \le I \ (k = 1, \ldots, K) \,\} \cup \{0\}$$

in the order $\cdots < \lambda_{-2} < \lambda_{-1} < \lambda_0 = 0 < \lambda_1 < \lambda_2 < \cdots$. It is easily known that $-\lambda_j = \lambda_{-j}$ holds, and that $\{\mu_l^{(k)}\}$ has Hadamard gaps; i.e., there exists $q > 1$ such that

$$\mu_{l+1}^{(k)} / \mu_l^{(k)} > q > 1 \qquad (l \in \mathbf{N},\ k = 1, \ldots, K.) \tag{4.1}$$

Because $a_1 < a_2 < \cdots < a_K$, we can take $a_0 > 0$ satisfying $a_0 < (a_{k+1} - a_k)/2$ $(k = 1, \ldots, K-1)$. Clearly, $p_{k+1}(n) - p_k(n) > 2a_0 n$ for large $n$.

Lemma 3. *Let $I$ and $K$ be a positive integers.*
(1) *For any $\varepsilon > 0$, there exists $N_0$ such that for $k = 1, \ldots, K$, $n \geq N_0$, and $0 < |i_k| \leq \theta^{a_0 n}$ we have*

$$\frac{i_1\theta^{p_1(n)} + \cdots + i_k\theta^{p_k(n)}}{i_k\theta^{p_k(n)}} \in (1-\varepsilon, 1+\varepsilon). \tag{4.2}$$

(2) *There exist $C_1$, $C_2 > 0$ and $N_0$ such that for $n$, $n' \geq N_0$, $|i_j| \leq \theta^{a_0 n}$, $|i'_j| \leq \theta^{a_0 n'}$ $(j = 1, \ldots, k-1)$, $|i_k|$, $|i'_k| \leq I$, and $|i_k\theta^{p_k(n)}| > |i'_k\theta^{p_k(n')}|$ we have*

$$\left|\frac{(i_1\theta^{p_1(n)} + \cdots + i_k\theta^{p_k(n)}) \pm (i'_1\theta^{p_1(n')} + \cdots + i'_k\theta^{p_k(n')})}{i_k\theta^{p_k(n)}}\right| \in (C_1, C_2). \tag{4.3}$$

(3) *For any $U > 0$ and $\kappa = 0, 1, \ldots, K$ there exists $N_0$ such that if $n \geq n' \geq N_0$, $0 < |i_k| \leq \theta^{a_0 n}$, $0 < |i'_k| \leq \theta^{a_0 n'}$ $(1 \leq k \leq \kappa)$, $0 < |i_k|$, $|i'_k| \leq I$ $(\kappa < k \leq K)$ and*

$$|(i_1\theta^{p_1(n)} + \cdots + i_K\theta^{p_K(n)}) + (i'_1\theta^{p_1(n')} + \cdots + i'_K\theta^{p_K(n')})| \leq U \tag{4.4}$$

*we have*

$$i_k\theta^{p_k(n)} + i'_k\theta^{p_k(n')} = 0 \quad \textit{for} \quad k > \kappa, \quad \textit{and} \tag{4.5}$$

$$|(i_k\theta^{p_k(n)} + \cdots + i_\kappa\theta^{p_\kappa(n)}) + (i'_k\theta^{p_k(n')} + \cdots + i'_\kappa\theta^{p_\kappa(n')})| \leq \frac{\theta^{p_k(n')}}{3} \tag{4.6}$$

*for $k \leq \kappa$.*

*Proof.* (4.2) is clear from $\theta^{a_0 n}\theta^{p_k(n)} \leq \theta^{p_{k+1}(n)}/\theta^{2a_0 n} = o(\theta^{p_{k+1}(n)})$. Let us prove (4.3). Take $\varepsilon > 0$ small enough such that $(1-\varepsilon) - (1+\varepsilon)/q$ is positive. By using (4.2), we have

$$\begin{aligned}
&|(i_1\theta^{p_1(n)} + \cdots + i_k\theta^{p_k(n)}) \pm (i'_1\theta^{p_1(n')} + \cdots + i'_k\theta^{p_k(n')})| \\
&\geq \left|\frac{i_1\theta^{p_1(n)} + \cdots + i_k\theta^{p_k(n)}}{i_k\theta^{p_k(n)}} i_k\theta^{p_k(n)}\right| - \left|\frac{i'_1\theta^{p_1(n')} + \cdots + i'_k\theta^{p_k(n')}}{i'_k\theta^{p_k(n')}} i'_k\theta^{p_k(n')}\right| \\
&\geq (1-\varepsilon)|i_k|\theta^{p_k(n)} - (1+\varepsilon)|i'_k|\theta^{p_k(n')} \\
&\geq \{(1-\varepsilon) - (1+\varepsilon)/q\}|i_k|\theta^{p_k(n)}.
\end{aligned}$$

The upper estimate

$$|(i_1\theta^{p_1(n)} + \cdots + i_k\theta^{p_k(n)}) \pm (i'_1\theta^{p_1(n')} + \cdots + i'_k\theta^{p_k(n')})| \leq 2(1+\varepsilon)|i_k|\theta^{p_k(n)}$$

can be proved in a similar way.

Lastly, let us prove (4.5) and (4.6). Assume that (4.5) is not true and take the largest of $k$ such that $i_k\theta^{p_k(n)} + i'_k\theta^{p_k(n')} \neq 0$ for large $n$ and $n'$. Then, by using (4.3), we see that (4.4) cannot hold for large $n$ and $n'$. Thus we have (4.5). By this, we see that (4.4) is valid if we replace $K$ by $\kappa$. Noting this and (4.2) we have

$$\begin{aligned}
&|(i_k\theta^{p_k(n)} + \cdots + i_\kappa\theta^{p_\kappa(n)}) + (i'_k\theta^{p_k(n')} + \cdots + i'_\kappa\theta^{p_\kappa(n')})| \\
&\quad \leq |(i_1\theta^{p_1(n)} + \cdots + i_\kappa\theta^{p_\kappa(n)}) + (i'_1\theta^{p_1(n')} + \cdots + i'_\kappa\theta^{p_\kappa(n')})| \\
&\qquad + |(i_1\theta^{p_1(n)} + \cdots + i_{k-1}\theta^{p_{k-1}(n)}) + (i'_1\theta^{p_1(n')} + \cdots + i'_{k-1}\theta^{p_{k-1}(n')})| \qquad (4.7) \\
&\quad \leq U + 2\theta^{a_0 n + p_{k-1}(n)} \leq 3\theta^{a_0 n + p_{k-1}(n)},
\end{aligned}$$

for large $n$. By putting $k = \kappa$, we have $i_\kappa \theta^{p_\kappa(n)} + i'_\kappa \theta^{p_\kappa(n')} = o(\theta^{p_\kappa(n)})$ and hence $|i_\kappa|\theta^{p_\kappa(n)}/2 \le |i'_\kappa|\theta^{p_\kappa(n')}$ and thereby $\theta^{p_\kappa(n)}/2 \le \theta^{p_\kappa(n')+a_0 n'}$ or $n \le \frac{a_\kappa + a_0}{a_\kappa} n' + \alpha$. Thus we have $a_0 n + p_{k-1}(n) \le (a_0 + a_{k-1}) \frac{a_\kappa + a_0}{a_\kappa} n' + \alpha'$. Because

$$
\begin{aligned}
(a_0 + a_{k-1}) \frac{a_\kappa + a_0}{a_\kappa} < (a_k - a_0) \frac{a_\kappa + a_0}{a_\kappa} &= a_k \Big(1 - \frac{a_0}{a_k}\Big)\Big(1 + \frac{a_0}{a_\kappa}\Big) \\
&\le a_k \Big(1 - \frac{a_0}{a_\kappa}\Big)\Big(1 + \frac{a_0}{a_\kappa}\Big) < a_k,
\end{aligned}
$$

we have $\theta^{a_0 n + p_{k-1}(n)} = o(\theta^{a_k n' + \alpha'})$. Combining this with (4.7), we have the conclusion. □

Let us put $J(w) = [\,\theta^{a_K w}, \theta^{a_K (w+1)}) \cap \mathbf{N}$,

$$
\Delta_w(f) = \Big( \sum_{|j| \in J(w)} |\widehat{f}(j)|^2 \Big)^{1/2} \quad \text{and} \quad D(f) = \sum_{w=0}^{\infty} \Delta_w(f).
$$

It is easily seen that there exists a constant $C$ depending only on $\theta^{a_K}$ such that,

$$
D(f) \le C \sum_{n=0}^{\infty} \Big( \int_0^1 |s_{f,2^{n+1}}(x) - s_{f,2^n}(x)|^2 \, dx \Big)^{1/2}, \quad (f \in L^2).
$$

Lemma 4. *If $f_1, \ldots, f_K$ satisfy the condition of Theorem 1, then we have*

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \Big( \sum_{n=N_0}^{N} \prod_{k=1}^{\kappa} s_{f_k, \theta^{a_0 n}}(\theta^{a_k n} x) \prod_{k=\kappa+1}^{K} s_{f_k, I}(\theta^{a_k n} x) \Big)^2 \mu_0(dx) \\
&\qquad \le 2N \prod_{k=1}^{K-1} \|f_k\|_2^2 D(f_K) \qquad (\kappa = 0, 1, \ldots, K, \ N \ge N_0)
\end{aligned}
\tag{4.8}
$$

*where $N_0$ is given by (3) of Lemma 3 with $U = 1$. We also have*

$$
v^{(I)} = \prod_{k=1}^{K} \int_0^1 s_{f_k, I}^2(x)\, dx + 2 \sum_{n=1}^{\infty} \prod_{k=1}^{K} \int_0^1 s_{f_k, I}(q_k^n x) s_{f_k, I}(r_k^n x)\, dx \to v, \ (I \to \infty).
$$

*Proof.* Let us consider the case $N_0 \le n' \le n$ and (4.4) is valid. Then, for $k > \kappa$, we have $i'_k = \varphi_k(i_k) = [\,-i_k \theta^{a_k(n-n')}\,]^*$ by (4.5), where $[\,x\,]^* = [\,x + 1/2\,]$. We also have $i'_\kappa = \varphi_\kappa(i_\kappa) = [\,-i_\kappa \theta^{a_\kappa(n-n')}\,]^*$ by (4.6) with $k = \kappa$. For $k < \kappa$, by (4.6), we have

$$
\begin{aligned}
i'_k &= \varphi_k(i_k, i_{k+1}, \ldots, i_\kappa) \\
&= \left[ -i_k \theta^{a_k(n-n')} - \theta^{-p_k(n')} \begin{pmatrix} i_{k+1}\theta^{p_{k+1}(n)} + \cdots + i_\kappa \theta^{p_\kappa(n)} \\ + i'_{k+1}\theta^{p_{k+1}(n')} + \cdots + i'_\kappa \theta^{p_\kappa(n)} \end{pmatrix} \right]^*.
\end{aligned}
$$

By $\theta^{a_k(n-n')} \ge 1$, mappings $i_k \mapsto \varphi_k(i_k)$ $(k \ge \kappa)$ and $i_k \mapsto \varphi_k(i_k, i_{k+1}, \ldots, i_\kappa)$ $(k < \kappa)$ are injective for any given $i_{k+1}, \ldots, i_\kappa$, $n$ and $n'$.

By using (4.5) and (4.6), we have

$$\int_{-\infty}^{\infty}\Big(\sum_{n=N_0}^{N}\prod_{k=1}^{\kappa}s_{f_k,\theta^{a_0n}}(\theta^{a_kn}x)\prod_{k=\kappa+1}^{K}s_{f_k,I}(\theta^{a_kn}x)\Big)^2\mu_0(dx)$$
$$\le\sum_{n',n=N_0}^{N}\sum_{*}|\widehat{f}_1(i_1)\widehat{f}_1(i_1')|\dots|\widehat{f}_K(i_K)\widehat{f}_K(i_K')|$$
$$\le 2\sum_{n'=N_0}^{N}\sum_{n=n'}^{N}\prod_{k=\kappa}^{K}\sum_{i_k\ne 0}|\widehat{f}_k(i_k)\widehat{f}_k(\varphi_k(i_k))|\prod_{k=1}^{\kappa-1}\sum_{i_k\ne 0}|\widehat{f}_k(i_k)\widehat{f}_k(\varphi_k(i_k,\dots,i_\kappa))|,$$

where $\sum_*$ denotes summation for $(i_1,i_1',\dots,i_K,i_K')$ with $0<|i_k|\le\theta^{a_0n}$, $0<|i_k'|\le\theta^{a_0n'}$ $(1\le k\le\kappa)$, and $0<|i_k|$, $|i_k'|\le I$ $(\kappa<k\le K)$. If $k<\kappa$,

$$\sum_{i_k\ne 0}|\widehat{f}_k(i_k)\widehat{f}_k(\varphi_k(i_k,\dots,i_\kappa))|$$
$$\le\Big(\sum_{i_k\ne 0}|\widehat{f}_k(i_k)|^2\Big)^{1/2}\Big(\sum_{i_k\ne 0}\widehat{f}_k(\varphi_k(i_k,\dots,i_\kappa))|^2\Big)^{1/2}\le\|f_k\|_2^2,$$

since $\varphi_k$ is injective. $\sum_{i_k\in\mathbf{Z}}|\widehat{f}_k(i_k)\widehat{f}_k(\varphi_k(i_k))|\le\|f_k\|_2^2$ is also clear for $k=\kappa+1$, $\dots$, $K-1$. In case $k=K$, we have

$$\sum_{i_K\ne 0}|\widehat{f}_K(i_K)\widehat{f}_K(\varphi_K(i_K))|=\sum_{w=0}^{\infty}\sum_{i_K\in J(w)}\Big|\widehat{f}_K(i_K)\widehat{f}_K([-i_K\theta^{a_K(n-n')}]^*)\Big|$$
$$\le\sum_{w=0}^{\infty}\Big(\sum_{i_K\in J(w)}|\widehat{f}_K(i_K)|^2\Big)^{1/2}\Big(\sum_{i_K\in J(w)}\Big|\widehat{f}_K([-i_K\theta^{a_K(n-n')}]^*)\Big|^2\Big)^{1/2}$$
$$\le\sum_{w=0}^{\infty}\Delta_w(f_K)\Delta_{w+n-n'}(f_K).$$

Thus we have

$$\int_{-\infty}^{\infty}\Big(\sum_{n=N_0}^{N}\prod_{k=1}^{\kappa}s_{f_k,\theta^{a_0n}}(\theta^{a_kn}x)\prod_{k=\kappa+1}^{K}s_{f_k,I}(\theta^{a_kn}x)\Big)^2\mu_0(dx)$$
$$\le 2\prod_{k=1}^{K-1}\|f_k\|_2^2\sum_{n'=N_0}^{N}\sum_{n=n'}^{N}\sum_{w=0}^{\infty}\Delta_w(f_K)\Delta_{w+n-n'}(f_K)$$
$$\le 2N\prod_{k=1}^{K-1}\|f_k\|_2^2\sum_{m=1}^{\infty}\sum_{w=0}^{\infty}\Delta_w(f_K)\Delta_{w+m}(f_K)$$
$$\le 2N\prod_{k=1}^{K-1}\|f_k\|_2^2\sum_{w=0}^{\infty}\Delta_w(f_K)\sum_{m=1}^{\infty}\Delta_{w+m}(f_K)\le 2N\prod_{k=1}^{K-1}\|f_k\|_2^2D(f_K)^2,$$

which shows (4.8). Next let us verify the convergence of $v^{(I)}$. We may assume that $q_K$ and $r_K$ are relatively prime. By estimating in the same way as above, we have $|\prod_{k=1}^{K}\int_0^1 s_{f_k,I}^2(x)\,dx| \le \prod_{k=1}^{K}\|f_k\|_2^2$ and

$$\begin{aligned}\left|\prod_{k=1}^{K}\int_0^1 s_{f_k,I}(q_k^n x)s_{f_k,I}(r_k^n x)\,dx\right| &\le \prod_{k=1}^{K-1}\|f_k\|_2^2\sum_{i_K\ne 0}|\widehat{f}_K(i_K q_K^n)\widehat{f}_K(-i_K r_K^n)|\\ &\le \prod_{k=1}^{K-1}\|f_k\|_2^2\sum_{i_K\ne 0}|\widehat{f}_K(i_K)\widehat{f}_K([-i_K\theta^{a_K n}]^*)|\\ &\le \prod_{k=1}^{K-1}\|f_k\|_2^2\sum_{w=0}^{\infty}\Delta_w(f_K)\Delta_{w+n}(f_K).\end{aligned}$$

In the same way as above we have

$$\prod_{k=1}^{K}\|f_k\|_2^2 + 2\sum_{n=1}^{\infty}\prod_{k=1}^{K-1}\|f_k\|_2^2\sum_{w=0}^{\infty}\Delta_w(f_K)\Delta_{w+n}(f_K) < \infty.$$

Since each summand of $v^{(I)}$ converges to that of $v$, and is dominated as above by the summand of summable series, by Lebesgue's convergence theorem for series, we have the conclusion. □

Let us here prove the central limit theorem in the case when $f_1, \dots, f_K$ are trigonometric polynomials without constant term.

Let us define $X_N$ and $c_{j,N}$ by

$$X_N(x) = \sum_{j=-\infty}^{\infty} c_{j,N}\exp(2\pi\sqrt{-1}\,\lambda_j x) = \frac{1}{\sqrt{N}}\sum_{n=1}^{N}\prod_{k=1}^{K} f_k(\theta^{p_k(n)}x).$$

Obviously $\{c_{j,N}\}_{-\infty<j\infty,\,N\ge 1}$ satisfies $c_{-j,N} = \bar{c}_{j,N}$ and $c_{j,N} = 0$ ($|j| > J_N$) for some $J_N$.

Firstly, let us prove

$$\int_{-\infty}^{\infty} X_N(x)^2\,\mu_R(dx) \to v, \tag{4.9}$$

where

$$\int_{-\infty}^{\infty} f(x)\,\mu_R(dx) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)\,dx.$$

The left hand side of above formula is a symbolic expression and does not mean an integral with respect to the measure $\mu_R$. From now on, we frequently use the following relations:

$$\int_{-\infty}^{\infty} f(\theta x)\,\mu_R(dx) = \int_{-\infty}^{\infty} f(x)\,\mu_R(dx) \quad \text{if } \theta \ne 0;$$

$$\int_{-\infty}^{\infty} f(x)\,\mu_R(dx) = \int_0^{\alpha} f(x)\,dx \quad \text{if } f \text{ has period } \alpha;\text{ and} \quad \int_{-\infty}^{\infty} e^{\sqrt{-1}\,tx}\,\mu_R(dx) = \delta_{t0}.$$

Let us denote $F_{k,\nu}(x) = f_k(x)f_k(\theta^{a_k\nu}x)$ and $d_{k,\nu} = \int_{-\infty}^{\infty} F_{k,\nu}(x)\,\mu_R(dx)$. Let us prove that there exists $N_0$ and $\nu_0$ such that

$$\int_{-\infty}^{\infty}\prod_{k=1}^{K} F_{k,\nu}(\theta^{p_k(n)}x)\,\mu_R(dx) = \prod_{k=1}^{K} d_{k,\nu} \qquad (n \ge N_0) \tag{4.10}$$

$$= 0 \qquad (\nu \geq \nu_0,\ n \geq N_0). \tag{4.11}$$

If $\theta^{a_K \nu} > I$, then the trigonometric polynomial expansion of $F_{K,\nu}$ has no constant term and thereby $d_{K,\nu} = 0$. Thus (4.11) follows from (4.10). Since the left hand side of (4.10) can be decomposed into

$$\int_{-\infty}^{\infty} \prod_{k=1}^{K} F_{k,\nu}\, d\mu_R = \sum_{\kappa=1}^{K} \int_{-\infty}^{\infty} \prod_{k=1}^{\kappa-1} F_{k,\nu}(F_{\kappa,\nu} - d_{\kappa,\nu})\, d\mu_R \prod_{k=\kappa+1}^{K} d_{k,\nu} + \prod_{k=1}^{K} d_{k,\nu},$$

the proof of (4.10) reduced to the proofs of

$$\int_{-\infty}^{\infty} \prod_{k=1}^{\kappa-1} F_{k,\nu}(\theta^{p_k(n)} x)(F_{\kappa,\nu}(\theta^{p_\kappa(n)} x) - d_{\kappa,\nu})\, \mu_R(dx) = 0, \quad (n \geq N_0). \tag{4.12}$$

Since $F_{k,\nu} - d_{k,\nu}$ have no constant term, we can take $0 < I_{k,\nu} < I'_{k,\nu}$ such that the frequencies of $F_{k,\nu} - d_{k,\nu}$ belong to $(I_{k,\nu}, I'_{k,\nu})$. Then frequencies of $F_{k,\nu}(\theta^{p_k(n)} x) - d_{k,\nu}$ belong to $(I_{k,\nu}\theta^{p_k(n)}, I'_{k,\nu}\theta^{p_k(n)})$. Since we have $I_{k+1,\nu}\theta^{p_{k+1}(n)}/I'_{k,\nu}\theta^{p_k(n)} \to \infty$ as $n \to \infty$, $I_{k+1,\nu}\theta^{p_{k+1}(n)}/I'_{k,\nu}\theta^{p_k(n)} > 3$ for large $n$. Thus, for large $n$, moduli of the frequencies of the integrand of (4.12) is greater than

$$\begin{aligned} I_{\kappa,\nu}\theta^{p_\kappa(n)} - I'_{\kappa-1,\nu}\theta^{p_{\kappa-1}(n)} - \cdots - I'_{1,\nu}\theta^{p_1(n)} &\geq I_{\kappa,\nu}\theta^{p_\kappa(n)}(1 - 1/3 - 1/3^2 - \cdots) \\ &\geq I_{\kappa,\nu}\theta^{p_\kappa(n)}/2 > 0. \end{aligned}$$

Hence the integrand of (4.12) has no constant term and thereby (4.12) follows.

By noting (4.10) and (4.11), we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \Big( \frac{1}{\sqrt{N}} \sum_{n=N_0}^{N} \prod_{k=1}^{K} f_k(\theta^{p_k(n)} x) \Big)^2 \mu_R(dx) \\ &= \sum_{\nu=0}^{N-N_0} (2 - \delta_{\nu 0}) \frac{1}{N} \sum_{n=N_0}^{N-\nu} \int_{-\infty}^{\infty} \prod_{k=1}^{K} f_k(\theta^{p_k(n)} x) f_k(\theta^{p_k(n+\nu)} x)\, \mu_R(dx). \\ &= \sum_{\nu=0}^{N-N_0} (2 - \delta_{\nu 0}) \frac{N - \nu - N_0 + 1}{N} \prod_{k=1}^{K} d_{k,\nu} \end{aligned}$$

Thanks to (4.11), the sum in $\nu$ is essentially up to $\nu_0$, and hence as $N \to \infty$

$$\to \sum_{\nu=0}^{\infty} (2 - \delta_{\nu 0}) \prod_{k=1}^{K} d_{k,\nu}.$$

Let us evaluate $d_{k,\nu}$. $d_{k,0} = \|f_k\|_2^2$ is clear. If $\nu \geq 1$ and (1.8) hold, frequency $i + i'\theta^{a_k \nu}$ of $F_{k,\nu}(x) = f_k(x) f_k(\theta^{a_k \nu} x)$ can not be zero and thereby $d_{k,\nu} = 0$. Thus we have (1.7) when (1.8) holds for some $k$. Suppose that (1.8) does not holds for all $k$. Note that the set $N(k) = \{\, n \mid \theta^{a_k n} \in \mathbf{Q} \,\}$ is $\{\, m n_k \mid m \in \mathbf{N} \,\}$ where $n_k = \min N(k)$. Thus $N(0) = N(1) \cap \ldots \cap N(K)$ is $\{\, mn \mid m \in \mathbf{N} \,\}$ where $n = \gcd(n_1, \ldots, n_K) = \min N(0)$. If $\nu$ is not a multiple of $n$, then $\nu \notin N(k)$ for some $k$, and thereby $\prod_k d_{k,\nu} = 0$. If $\nu = mn$, then we have $\theta^{a_k \nu} = (q_k/r_k)^m$, and thereby

$$d_{k,\nu} = \int_{-\infty}^{\infty} f_k(x) f_k((q_k/r_k)^m x)\, \mu_R(dx) = \int_{-\infty}^{\infty} f_k(r_k^m x) f_k(q_k^m x)\, \mu_R(dx)$$
$$= \int_0^1 f_k(r_k^m x) f_k(q_k^m x)\, dx.$$

Therefore (1.9) is proved, and consequently (4.9) is verified.

Let us here verify (2.1) by using Lemma 2. By applying (3) of Lemma 3 with $\kappa = 0$, we see that there exists $J_0$ such that $|\lambda_j - \lambda_{j'}| \le U$ with $j, j' \ge J_0$ implies $j = j'$. We also see that if $j \ge J_0$, values of $i_1\theta^{p_1(n)}, \dots, i_K\theta^{p_K(n)}$ satisfying $\lambda_j = i_1\theta^{p_1(n)} + \cdots + i_K\theta^{p_K(n)}$ are uniquely determined. Note that $i_1, \dots, i_K$, and $n$ are not necessarily unique. Actually, $8 \times 2^0 = 4 \times 2^1 = 2 \times 2^2 = 1 \times 2^3$ gives an example for the case $\theta = 2$ and $K = 1$. But if we give the values $i_1, \dots, i_K$, then $n$ is uniquely determined. Since the choice of values of $i_1, \dots, i_K$ are at most $(2I)^K$ in number, there are at most $(2I)^K$ many $(i_1, \dots, i_K, n)$ which satisfy $\lambda_j = i_1\theta^{p_1(n)} + \cdots + i_K\theta^{p_K(n)}$, if $j \ge J_0$.

Let $H_l$ be the collection of $j$ such that '*leading term*' $i_K\theta^{p_K(n)}$ of $\lambda_j = i_1\theta^{p_1(n)} + \cdots + i_K\theta^{p_K(n)}$ equals to $\mu_l^{(K)}$, i.e.,

$$H_l = \left\{ j \in \mathbf{Z} \;\middle|\; \begin{array}{l} |j| \ge J_0, \lambda_j = i_1\theta^{p_1(n)} + \cdots + i_K\theta^{p_K(n)}, \\ n \in \mathbf{N}, |i_1|, \dots, |i_K| \le I, i_K\theta^{p_K(n)} = \mu_l^{(K)} \end{array} \right\}, \quad (l \in \mathbf{N}),$$

$$H_0 = \mathbf{Z} \setminus \bigcup_{l=1}^{\infty} H_l.$$

As in the proof of Lemma 3, we take $\varepsilon > 0$ satisfying $(1-\varepsilon) - (1+\varepsilon)/q > 0$, and take $\Theta > 0$ such that $(1-\varepsilon) - (1+\varepsilon)(1/q + 1/(\Theta - 1)) > 0$. Let us put $\pi_0 = 0$, $\pi_m = \max\{\, l \ge \pi_{m-1} \mid \mu_l^{(K)} \le \Theta^m \,\}$ $(m \ge 0)$,

$$G_m = \bigcup_{l=\pi_{m-1}+1}^{\pi_m} H_l \quad \text{and} \quad \xi_{m,N} = \sum_{j \in G_m} c_{j,N} \exp(2\pi\sqrt{-1}\,\lambda_j x).$$

By definition and (4.2), if $j \in G_m$ then one can find $l$ such that $|\lambda_j| \in ((1-\varepsilon)\mu_l^{(K)}, (1+\varepsilon)\mu_l^{(K)})$ and $\Theta^{m-1} < \mu_l^{(K)} \le \Theta^m$. From now on, we verify the conditions (2.2), (2.3), (2.4) and (2.5).

As we have verified, we have ${}^{\#}H_l \le (2I)^K$. Because

$$\Theta = \Theta^m/\Theta^{m-1} \ge \mu_{\pi_m}^{(K)}/\mu_{\pi_{m-1}}^{(K)} \ge q^{\pi_m - \pi_{m-1}},$$

we have $\pi_m - \pi_{m-1} \le \log_q \Theta$ and thereby ${}^{\#}G_m \le (2I)^K \log_q \Theta$. Applying $|\widehat{f}_k(i)| \le \|f_k\|_2$ to

$$c_{j,N} = \frac{1}{\sqrt{N}} \sum_{\substack{(i_1,\dots,i_K,n) \in [-I,I]^K \times (-\infty,\infty): \\ i_1\theta^{p_1(n)} + \cdots + i_K\theta^{p_K(n)} = \lambda_j}} \widehat{f}(i_1) \dots \widehat{f}(i_K),$$

we have $|c_{j,N}| \le (2I)^K \|f_1\|_2 \dots \|f_K\|_2/\sqrt{N}$. By this estimate, we have

$$\begin{aligned} \|\xi_{m,N}\|_\infty &\le \Big(\sum_{j\in G_m} |c_{j,N}|^2\Big)^{1/2} \Big(\sum_{j\in G_m} 1^2\Big)^{1/2} \\ &\le {}^\# G_m (2I)^K \|f_1\|_2 \dots \|f_K\|_2/\sqrt{N} \\ &\le (2I)^{2K} \log_q \Theta \|f_1\|_2 \dots \|f_K\|_2/\sqrt{N} = C/\sqrt{N}, \end{aligned} \tag{4.13}$$

which implies (2.2), i.e., $B_N \le C/\sqrt{N} \to 0$. By (4.13) and (4.9), we have

$$\begin{aligned} \|V_N\|_\infty &\le \sum_{m=1}^\infty \|\xi_{m,N}\|_\infty^2 \le \sum_{m=1}^\infty {}^\# G_m \sum_{j\in G_m} |c_{j,N}|^2 \\ &\le (2I)^K \log_q \Theta \int_{-\infty}^\infty X_N^2(x)\, \mu_R(dx) \to (2I)^K (\log_q \Theta) v, \end{aligned}$$

which implies (2.5).

Next, let us verify (2.3). Let us take $r \in \mathbf{N}$ and $m_1 < \dots < m_r$. Let $\phi_j$ be a frequency of $\xi_{m_j,N}$. Since we have $|\phi_j| \in ((1-\varepsilon)\mu_{l_j}^{(K)}, (1+\varepsilon)\mu_{l_j}^{(K)})$ and $\Theta^{m-1} < \mu_{l_j}^{(K)} \le \Theta^m$ for some $l_j$, we have the following estimate:

$$\begin{aligned} |\phi_r + \dots + \phi_1| &\ge |\phi_r| - |\phi_{r-1}| - \dots - |\phi_1| \\ &\ge (1-\varepsilon)\mu_{l_r}^{(K)} - (1+\varepsilon)(\mu_{l_{r-1}}^{(K)} + \dots + \mu_{l_1}^{(K)}) \\ &\ge (1-\varepsilon)\mu_{l_r}^{(K)} - (1+\varepsilon)\mu_{l_r}^{(K)}(1/q + 1/\Theta + 1/\Theta^2 + \cdots) \\ &\ge \Theta^{m_r - 1}\{(1-\varepsilon) - (1+\varepsilon)(1/q + 1/(\Theta - 1))\} \to \infty. \end{aligned}$$

Thus there exists $m_0$ such that, if $m_r > m_0$, the last term is greater than $U$. This implies (2.3).

Lastly, we here verify (2.4). Let us denote by $B_{m,\kappa}$ the set of $(j,j') \in G_m^2$ such that $i_k\theta^{p_k(n)} + i'_k\theta^{p_k(n')} = 0$ for $\kappa < k \le K$ and $i_k\theta^{p_\kappa(n)} + i'_k\theta^{p_\kappa(n')} \ne 0$, where $\lambda_j = i_1\theta^{p_1(n)} + \dots + i_K\theta^{p_K(n)}$ and $\lambda_{j'} = i'_1\theta^{p_1(n')} + \dots + i'_K\theta^{p_K(n')}$. Since we have $G_m = \bigcup_{\kappa=0}^K B_{m,\kappa}$ and $B_{m,k} \cap B_{m,k'} = \emptyset$ $(k \ne k')$, we have

$$\xi_{m,N}^2 = \zeta_{m,0,N} + \zeta_{m,1,N} + \dots + \zeta_{m,K,N},$$

where

$$\zeta_{m,\kappa,N} = \sum_{(j,j')\in B_{m\kappa}} c_{j,N} c_{j',N} \exp(2\pi\sqrt{-1}\,(\lambda_j + \lambda_{j'})x).$$

Clearly we have

$$\zeta_{m,0,N} = \sum_{j\in G_m} |c_{j,N}|^2 \quad \text{and} \quad \sum_{m=1}^\infty \zeta_{m,0,N} = \sum_{j=-\infty}^\infty |c_{j,N}|^2 = \int_{-\infty}^\infty X_N^2(x)\, \mu_R(dx) = v_N.$$

By the way, $\Theta^{m-1} < i_K\theta^{p_K(n)}\Theta^m$ implies $\alpha m - \beta < n < \alpha m + \beta$ for some $\alpha$, $\beta > 0$. If $(j,j') \in B_{m,k}$, we have by (4.3) that

$$\begin{aligned} |\lambda_j + \lambda_{j'}| &\in ((1-\varepsilon)|i_k|\theta^{p_k(n)}, (1+\varepsilon)|i_k|\theta^{p_k(n)}) \\ &\subset ((1-\varepsilon)\theta^{p_k(\alpha m - \beta)}, (1+\varepsilon)I\theta^{p_k(\alpha m + \beta)}) = \Lambda_{k,m}. \end{aligned}$$

Since $(1-\varepsilon)\theta^{p_k(\alpha(m+r)-\beta)}/(1+\varepsilon)I\theta^{p_k(\alpha m+\beta)} \geq \{(1-\varepsilon)/(1+\varepsilon)I\}\theta^{a_k(\alpha r-2\beta)} \geq 2$ for large $r$, the distance between $\Lambda_{k,m}$ and $\Lambda_{k,m'}$ is greater than 1 if $|m-m'| \geq r$. We also see that distance between arbitrary two of $\Lambda_{1,m}, \dots, \Lambda_{K,m}$ is also greater than 1. Thus we have

$$\begin{aligned}\int_{-\infty}^{\infty}(V_N-v_N)^2\,d\mu_0 &= \int_{-\infty}^{\infty}\Big(\sum_{m=1}^{\infty}\sum_{k=1}^{K}\zeta_{m,k,N}\Big)^2\,d\mu_0\\ &\leq C\sum_{\substack{0\leq j<r\\1\leq k\leq K}}\int_{-\infty}^{\infty}\Big(\sum_{m=1}^{\infty}\zeta_{mr+j,k,N}\Big)^2\,d\mu_0 \quad = C\sum_{\substack{0\leq j<r\\1\leq k\leq K}}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}\zeta^2_{mr+j,k,N}\,d\mu_0\\ &= C\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}\xi^4_{m,N}\,d\mu_0 \leq C\|\xi_{m,N}\|^2_\infty\|V_N\|_\infty \sim CB_Nv \to 0.\end{aligned}$$

Thus we have (2.4). Therefore we have the central limit theorem (2.1) if $f_k$ are trigonometric polynomials without constant.

Let us define $E_1$ and $E_2$ as the previous section. By applying (4.8) for $f_1, \dots, f_{\kappa-1}, f_\kappa - s_{f_\kappa,I}, f_{\kappa+1}, \dots, f_K$, we have the following estimate of $E_1^2$:

$$\begin{aligned}&\Big(\int_{-\infty}^{\infty}\Big|\frac{1}{\sqrt{N}}\sum_{n=N_0}^{N}\prod_{k=1}^{K}s_{\theta^{a_0n},f_k}(\theta^{p_k(n)}x)-\frac{1}{\sqrt{N}}\sum_{n=N_0}^{N}\prod_{k=1}^{K}s_{I,f_k}(\theta^{p_k(n)}x)\Big|\,\mu_0(dx)\Big)^2\\ &\leq \int_{-\infty}^{\infty}\Big(\frac{1}{\sqrt{N}}\sum_{n=N_0}^{N}\prod_{k=1}^{K}s_{\theta^{a_0n},f_k}(\theta^{p_k(n)}x)-\frac{1}{\sqrt{N}}\sum_{n=N_0}^{N}\prod_{k=1}^{K}s_{I,f_k}(\theta^{p_k(n)}x)\Big)^2\,\mu_0(dx)\\ &\leq \frac{\sqrt{K}}{N}\sum_{\kappa=1}^{K}\int_{-\infty}^{\infty}\left(\begin{aligned}\sum_{n=N_0}^{N}\prod_{k=1}^{\kappa-1}s_{\theta^{a_0n},f_k}(\theta^{p_k(n)}x)&\prod_{k=\kappa+1}^{K}s_{I,f_k}(\theta^{p_k(n)}x)\\ \times\,(s_{\theta^{a_0n},f_\kappa}-s_{I,f_\kappa})&(\theta^{p_\kappa(n)}x)\end{aligned}\right)^2\mu_0(dx)\\ &\leq 2\sqrt{K}\sum_{\kappa=1}^{K-1}\prod_{\substack{1\leq k<K\\k\neq\kappa}}\|f_k\|_2^2\|f_\kappa-s_{I,f_\kappa}\|_2^2D(f_K)+\prod_{k=1}^{K-1}\|f_k\|_2^2D(f_K-s_{I,f_K}).\end{aligned}$$

Thus if we take $I$ large enough, we have $E_1 < \varepsilon$. The estimate of $E_2$ can be done in the same way as section 3, we can conclude the proof also in the same way as before.

## 5. *On implication of regularity conditions*

If we assume (1.10), we have $\sum_{n=1}^{N}\|f_k - s_{f_k,\theta_0^n}\|_2 = o(\sqrt{N})$ $(n\to\infty, k=1,\dots,K)$. Because $\|f_k - s_{f_k,\theta_0^n}\|_2$ is decreasing in $n$, we have (1.4) as follows:

$$\|f_k - s_{f_k,\theta_0^n}\|_2 \leq \frac{1}{N}\sum_{n=1}^{N}\|f_k - s_{f_k,\theta_0^n}\|_2 = o(1/\sqrt{N}).$$

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