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A metric discrepancy result for the sequence of powers of minus two

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Abstract

The law of the iterated logarithm for discrepancies of $\{(-2)^k t\}_k$ is proved. This result completes the concrete determination of the law of the iterated logarithm for discrepancies of the geometric progression with integer ratio, and reveals the fact that 2 is the only positive integer $\theta > 1$ such that fractional parts of $\{(-\theta)^k t\}_k$ converges to uniform distribution faster than those of $\{\theta^k t\}_k$ a.e. t.

Keywords: discrepancy, lacunary sequence, law of the iterated logarithm
2000 MSC: 11K38, 42A55, 60F15

1. Introduction

Kronecker [23] proved that the sequence of the fractional part of kt ($k = 1, 2, \dots$) is dense in the unit interval if and only if t is irrational, and it was more than twenty years later that Bohl [7], Sierpiński [29] and Weyl [32] proved independently that the sequence is uniformly distributed modulo one in the following sense: A sequence $\{x_k\}$ of real numbers is said to be uniformly distributed modulo one if ${}^{\#}\{k \le N \mid \langle x_k \rangle \in [\,a, b)\}/N \to b - a$ for all $[\,a, b) \subset [\,0, 1)$, where $\langle x \rangle$ denotes the fractional part $x - [\,x\,]$ of real number x. These results initiated the theory of uniform distribution.

We use the following discrepancies $D_N\{x_k\}$ and $D_N^*\{x_k\}$ to measure the speed of convergence (See [10]):

$$D_N\{x_k\} = \sup_{0 \le a < b < 1} \left| \frac{1}{N} {}^{\#}\{k \le N \mid \langle x_k \rangle \in [\,a, b)\} - (b - a) \right|,$$

$$D_N^*\{x_k\} = \sup_{0 \le a < 1} \left| \frac{1}{N} {}^{\#}\{k \le N \mid \langle x_k \rangle \in [\,0, a)\} - a \right|.$$

Weyl proved $D_N^*\{n_k t\} \to 0$ a.e. t under very mild condition $n_{k+1} - n_k > C > 0$ for all large k, and showed that the method of measure theory is effective in the research of the uniform distribution theory.

Various studies were done in this direction. For arithmetic progressions $\{kt\}$ and increasing functions g, Khintchine [21] proved that

$$ND_N^*\{kt\} = O\left((\log N) g(\log \log N)\right) \quad \text{a.e. } t$$

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holds if and only if the function g satisfies $\sum 1/g(n) < \infty$. When $\sum 1/g(n) < \infty$ is satisfied, we can easily derive a stronger result

$$ND_N^*\{kt\} = o\left((\log N)g(\log\log N)\right) \quad \text{a.e. } t,$$

and see that critical speed cannot be determined in almost everywhere sense. The critical speed was determined by Kesten [20] in the sense of convergence in measure:

$$\lim_{N\to\infty} \mathrm{Leb}\left\{t \in [0,1) \,\middle|\, \left|\frac{ND_N^*\{kt\}}{\log N \log\log N} - \frac{2}{\pi^2}\right| > \varepsilon\right\} = 0, \qquad (\varepsilon > 0).$$

In probability theory, the following beautiful result was proved by Chung [8] and Smirnov [30] independently, viz. the law of the iterated logarithm

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{U_k\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{U_k\}}{\sqrt{2N\log\log N}} = \frac{1}{2} \quad \text{a.s.}$$

where $\{U_k\}$ is the sequence of independent and uniformly distributed random variables.

After a number of studies on the behaviour of $D_N\{n_k t\}$ for increasing $\{n_k\}$, Erdős [11] conjectured $ND_N\{n_k t\} = O((N\log\log N)^{1/2})$ a.e. assuming the Hadamard gap condition $n_{k+1}/n_k \ge q > 1$. Since the law of the iterated logarithm

$$\varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \sum_{k=1}^{N} \cos 2\pi n_k t = \frac{1}{\sqrt{2}} \quad \text{a.e. } t$$

was proved under the Hadamard gap condition by Erdős-Gál [12], it was natural to expect the analogue of the Chung-Smirnov result above.

By using Takahashi's method [31], Philipp [26] solved the conjecture by showing the bounded law of the iterated logarithm

$$\frac{1}{4\sqrt{2}} \le \varlimsup_{N\to\infty} \frac{ND_N^*\{n_k t\}}{\sqrt{2N\log\log N}} \le \varlimsup_{N\to\infty} \frac{ND_N\{n_k t\}}{\sqrt{2N\log\log N}} \le \frac{1}{\sqrt{2}}\left(166 + \frac{664}{q^{1/2}-1}\right) \quad \text{a.e. } t.$$

For a proof using martingales and another approach, see Philipp [25, 27]. Dhompongsa [9] assumed the very strong gap condition

$$\log(n_{k+1}/n_k)/\log\log k \to \infty \quad (k \to \infty)$$

and derived the Chung-Smirnov type result

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{n_k t\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{n_k t\}}{\sqrt{2N\log\log N}} = \frac{1}{2} \quad \text{a.e. } t.$$

The condition was relaxed later [1, 14] to $n_{k+1}/n_k \to \infty$.

On the other hand, Berkes-Philipp [5] proved that for any $\varepsilon_k \to 0$ there exists $\{n_k\}$ with $n_{k+1}/n_k \ge 1 + \varepsilon_k$ and

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{n_k t\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{n_k t\}}{\sqrt{2N\log\log N}} = \infty \quad \text{a.e. } t.$$

But there still exist sequences obeying the bounded law of the iterated logarithm which do not satisfy the Hadamard gap condition. Indeed, Philipp [28] proved that the multiplicative semigroups generated by finitely many coprime integers (Hardy-Littlewood-Pólya sequences) are such examples. For permutational law of the iterated logarithm for Hardy-Littlewood-Pólya sequences, we refer the reader to Aistleitner-Berkes-Tichy [2]. See also [6, 19]. For other permutational law of the iterated logarithm results, see Aistleitner-Berkes-Tichy [3].

The studies for geometric progressions $\{\theta^k t\}$ were not in the same detail as those for arithmetic progressions. But recently, many concrete values of the limsups were evaluated [13, 15, 17]. For any $\theta \notin [-1, 1]$, there exist real numbers Σ^*_θ and Σ_θ such that

$$\varlimsup_{N\to\infty} \frac{ND^*_N\{\theta^k t\}}{\sqrt{2N\log\log N}} = \Sigma^*_\theta \quad \text{and} \quad \varlimsup_{N\to\infty} \frac{ND_N\{\theta_k t\}}{\sqrt{2N\log\log N}} = \Sigma_\theta \qquad \text{a.e. } t.$$

When $\theta \notin [-1, 1]$ is not a root of a rational number, we have $\Sigma^*_\theta = \Sigma_\theta = \frac{1}{2}$, while we have $\Sigma^*_\theta = \Sigma_\theta > \frac{1}{2}$ when $\theta > 1$ is a root of a rational number.

We give the concrete expressions in case when θ is an integer. The constant Σ_θ is equal to $\frac{1}{2}\sqrt{\frac{\theta+1}{\theta-1}}$ when $\theta > 1$ is an odd integer, is equal to $\frac{1}{2}\sqrt{\frac{(\theta+1)\theta(\theta-2)}{(\theta-1)^3}}$ when $\theta \geq 4$ is an even integer, and is equal to $\frac{\sqrt{42}}{9}$ when $\theta = 2$. When $\theta \leq -3$ is an integer, then $\Sigma_\theta = \Sigma_{-\theta}$. When $\theta < -1$ is an odd integer then $\Sigma^*_\theta = \frac{1}{2}\sqrt{\frac{|\theta|(|\theta|^3+2\theta^2-|\theta|+2)}{(|\theta|-1)(|\theta|+1)^3}}$, while $\Sigma^*_\theta = \frac{1}{2}$ when $\theta < -1$ is even. Among these, the concrete value of Σ_{-2} is missing.

We deal with the case $\theta = -2$ in this paper. In [17], we already proved the law of the iterated logarithm as follows. When r is even,

$$\varlimsup_{N\to\infty} \frac{ND^*_N\{(-\sqrt[r]{2}\,)^k t\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{(-\sqrt[r]{2}\,)^k t\}}{\sqrt{2N\log\log N}} = \Sigma_2 = \frac{\sqrt{42}}{9} \quad \text{a.e. } t,$$

while, when r is odd

$$\varlimsup_{N\to\infty} \frac{ND^*_N\{(-\sqrt[r]{2}\,)^k t\}}{\sqrt{2N\log\log N}} = \frac{1}{2} < \varlimsup_{N\to\infty} \frac{ND_N\{(-\sqrt[r]{2}\,)^k t\}}{\sqrt{2N\log\log N}} = \Sigma_{-2} \quad \text{a.e. } t. \tag{1.1}$$

We shall compute Σ_{-2}.

Theorem 1. *For odd r, we have*

$$\varlimsup_{N\to\infty} \frac{ND_N\{(-\sqrt[r]{2}\,)^k t\}}{\sqrt{2N\log\log N}} = \frac{\sqrt{910}}{49} \quad \text{a.e. } t. \tag{1.2}$$

We can conclude that 2 is the only positive integer θ such that $\Sigma_{-\theta} \neq \Sigma_\theta$. Our evaluation $\Sigma_{-2}/\Sigma_2 = 0.85...$ proves that the distribution of the fractional parts of $\{(-2)^k t\}$ tends to the uniform distribution about 15 % faster than that of $\{2^k t\}$ for a.e. t.

Before closing the introduction, we call attention to our recent result [4] proving the exact law of the iterated logarithm for the discrepancies of sequences $\{n_k t\}$ satisfying the Hadamard gap condition $n_{k+1}/n_k \geq q > 1$ and a very mild Diophantine condition. The constants appearing there are bounded from above by $\frac{1}{2}\sqrt{\frac{q+1}{q-1}}$, which is identical with Σ_q in case q is an odd integer. Without any Diophantine condition, we [18] were able to derive the slightly loose upperbound $\frac{1}{2}\sqrt{1+\frac{4}{\sqrt{3}(q-1)}}$. For other relating results, see [16].

2. Preliminaries

First we recall the expression

$$\Sigma_{-2} = \left(\sup_{0\le x<y<1} v(x,y)\right)^{1/2}, \tag{2.1}$$

which is proved in [17]. Here $v(x,y)$ is a continuous function on $\mathbf{R}^2$ defined as the following absolutely and uniformly convergent series.

$$\widetilde{V}(\langle x\rangle,\langle y\rangle,\langle x\rangle,\langle y\rangle) + 2\sum_{k=1}^{\infty}\left(\frac{\widetilde{V}(\langle -2^{2k-1}y\rangle,\langle -2^{2k-1}x\rangle,\langle x\rangle,\langle y\rangle)}{2^{2k-1}} + \frac{\widetilde{V}(\langle 2^{2k}x\rangle,\langle 2^{2k}y\rangle,\langle x\rangle,\langle y\rangle)}{2^{2k}}\right),$$

where the function $\widetilde{V}$ is defined as

$$V(\xi,x) = \xi\wedge x - \xi x \tag{2.2}$$

$$\widetilde{V}(\xi,\eta,x,y) = V(\xi,x) + V(\eta,y) - V(\eta,x) - V(\xi,y) \tag{2.3}$$

for x, y, ξ, $\eta\in[0,1)$. Here we write $a\vee b = \max\{a,b\}$ and $a\wedge b = \min\{a,b\}$. Clearly we have $V(\xi,x) = V(x,\xi)$ and

$$\widetilde{V}(\xi,\eta,x,y) = \widetilde{V}(x,y,\xi,\eta) = -\widetilde{V}(\xi,\eta,y,x) = -\widetilde{V}(\eta,\xi,x,y) = \widetilde{V}(\eta,\xi,y,x). \tag{2.4}$$

Therefore we can express $v(x,y)$ simply by

$$v(x,y) = -\widetilde{V}(\langle x\rangle,\langle y\rangle,\langle x\rangle,\langle y\rangle) + 2\sum_{k=0}^{\infty}\frac{\widetilde{V}(\langle(-2)^k x\rangle,\langle(-2)^k y\rangle,\langle x\rangle,\langle y\rangle)}{(-2)^k}. \tag{2.5}$$

We prove the following formulas for ξ, η, x, $y\in\mathbf{R}$ and $c\in\mathbf{R}$:

$$\widetilde{V}(\langle\xi+c\rangle,\langle\eta+c\rangle,\langle x+c\rangle,\langle y+c\rangle) = \widetilde{V}(\langle\xi\rangle,\langle\eta\rangle,\langle x\rangle,\langle y\rangle), \tag{2.6}$$

$$\widetilde{V}(\langle-\eta\rangle,\langle-\xi\rangle,\langle-y\rangle,\langle-x\rangle) = \widetilde{V}(\langle\xi\rangle,\langle\eta\rangle,\langle x\rangle,\langle y\rangle), \tag{2.7}$$

$$\widetilde{V}(\langle\xi\rangle,\langle\eta\rangle,\langle x\rangle,\langle y\rangle) = V(\langle\eta-x\rangle,\langle y-x\rangle) - V(\langle\xi-x\rangle,\langle y-x\rangle), \tag{2.8}$$

$$\widetilde{V}(\langle x\rangle,\langle y\rangle,\langle x\rangle,\langle y\rangle) = -\langle y-x\rangle^2 + \langle y-x\rangle, \tag{2.9}$$

$$V(\xi,x) = V(x,\xi) = \xi(1-x) - (\xi-x)^+, \tag{2.10}$$

where a^+ denotes $a\vee 0$. First, for x, y, $t\in\mathbf{R}$ satisfying $0\le y-x<1$, we put $\mathbf{I}_{x,y}(t) = \sum_{n\in\mathbf{Z}}\mathbf{1}_{[x,y)}(t+n)$ and $\widetilde{\mathbf{I}}_{x,y}(t) = \mathbf{I}_{x,y}(t) - (y-x)$, where $\mathbf{1}_{[x,y)}$ is the indicator function of $[x,y)$. If $0\le x<y<1$, we see $\mathbf{I}_{x,y}(t) = \mathbf{1}_{[x,y)}(\langle t\rangle)$. We have

$$\widetilde{\mathbf{I}}_{x,y}(t) = \widetilde{\mathbf{I}}_{0,\langle y\rangle}(t) - \widetilde{\mathbf{I}}_{0,\langle x\rangle}(t) \qquad (y-x\in[0,1)). \tag{2.11}$$

Actually, it is almost trivial when $\langle x\rangle\le\langle y\rangle$, and otherwise it is verified by $\mathbf{I}_{x,y} = 1-\mathbf{I}_{\langle y\rangle,\langle x\rangle}$ and $y-x = \langle y\rangle - \langle x\rangle + 1$. We can show

$$\int_0^1\widetilde{\mathbf{I}}_{\xi,\eta}(t)\widetilde{\mathbf{I}}_{x,y}(t)\,dt = \widetilde{V}(\langle\xi\rangle,\langle\eta\rangle,\langle x\rangle,\langle y\rangle) \quad (\eta-\xi, y-x\in[0,1)). \tag{2.12}$$

Indeed, $\int_0^1 \mathbf{I}_{0,\langle\xi\rangle}(t)\mathbf{I}_{0,\langle x\rangle}(t)\,dt = \langle\xi\rangle \wedge \langle x\rangle$ implies $\int_0^1 \widetilde{\mathbf{I}}_{0,\langle\xi\rangle}(t)\widetilde{\mathbf{I}}_{0,\langle x\rangle}(t)\,dt = V(\langle\xi\rangle, \langle x\rangle)$, and hence (2.11) yields (2.12).

We have $\int_0^1 \widetilde{\mathbf{I}}_{\xi,\eta}(t-c)\widetilde{\mathbf{I}}_{x,y}(t-c)\,dt = \int_0^1 \widetilde{\mathbf{I}}_{\xi,\eta}(t)\widetilde{\mathbf{I}}_{x,y}(t)\,dt$ since the integrand has period 1. By noting $\widetilde{\mathbf{I}}_{\xi,\eta}(t-c) = \widetilde{\mathbf{I}}_{\xi+c,\eta+c}(t)$ and $\widetilde{\mathbf{I}}_{x,y}(t-c) = \widetilde{\mathbf{I}}_{x+c,y+c}(t)$, we can verify (2.6) assuming $\eta-\xi$, $y-x \in [\,0,1)$.

For general $\xi, \eta, x, y \in \mathbf{R}$, we can take $\widehat{\xi}, \widehat{\eta}, \widehat{x}$, and $\widehat{y} \in \mathbf{R}$ such that $0 \le \widehat{\eta}-\widehat{\xi} < 1$, $0 \le \widehat{y}-\widehat{x} < 1$, and $\widehat{\xi}-\xi, \widehat{\eta}-\eta, \widehat{x}-x, \widehat{y}-y \in \mathbf{Z}$. By $\widetilde{V}(\langle\xi+c\rangle, \langle\eta+c\rangle, \langle x+c\rangle, \langle y+c\rangle) = \widetilde{V}(\langle\widehat{\xi}+c\rangle, \langle\widehat{\eta}+c\rangle, \langle\widehat{x}+c\rangle, \langle\widehat{y}+c\rangle)$ and $\widetilde{V}(\langle\xi\rangle, \langle\eta\rangle, \langle x\rangle, \langle y\rangle) = \widetilde{V}(\langle\widehat{\xi}\rangle, \langle\widehat{\eta}\rangle, \langle\widehat{x}\rangle, \langle\widehat{y}\rangle)$ we see that (2.6) holds for any $\xi, \eta, x, y \in \mathbf{R}$.

Since we have $\widetilde{\mathbf{I}}_{\xi,\eta}(t) = \widetilde{\mathbf{I}}_{-\eta,-\xi}(-t)$ and $\widetilde{\mathbf{I}}_{x,y}(t) = \widetilde{\mathbf{I}}_{-y,-x}(-t)$ for almost every t, we have $\int_0^1 \widetilde{\mathbf{I}}_{\xi,\eta}(t)\widetilde{\mathbf{I}}_{x,y}(t)\,dt = \int_0^1 \widetilde{\mathbf{I}}_{-\eta,-\xi}(-t)\widetilde{\mathbf{I}}_{-y,-x}(-t)\,dt$. By changing the variable t by $-t$, and by noting (2.12) we have (2.7) for $\eta-\xi$, $y-x \in [\,0,1)$. We can verify (2.7) for general $\xi, \eta, x, y \in \mathbf{R}$ in the same way as above. By applying (2.6), we have $\widetilde{V}(\langle\xi\rangle, \langle\eta\rangle, \langle x\rangle, \langle y\rangle) = \widetilde{V}(\langle\xi-x\rangle, \langle\eta-x\rangle, 0, \langle y-x\rangle)$ and (2.8). Clearly, (2.8) implies (2.9). The convenient expression (2.10) is proved by $\xi - x - (\xi - x)^+ = (\xi - x) \wedge 0 = \xi \wedge x - x$.

By noting $\langle(-2)^k(x+\frac{1}{3})\rangle = \langle(-2)^k x + \frac{1}{3}\rangle$, $\langle(-2)^k(y+\frac{1}{3})\rangle = \langle(-2)^k y + \frac{1}{3}\rangle$, and by applying (2.6), we have $\widetilde{V}(\langle(-2)^k(x+\frac{1}{3})\rangle, \langle(-2)^k(y+\frac{1}{3})\rangle, \langle x+\frac{1}{3}\rangle, \langle y+\frac{1}{3}\rangle) = \widetilde{V}(\langle(-2)^k x\rangle, \langle(-2)^k y\rangle, \langle x\rangle, \langle y\rangle)$. It proves

$$v(x+\tfrac{1}{3}, y+\tfrac{1}{3}) = v(x,y). \tag{2.13}$$

By $\langle x+1\rangle = \langle x\rangle$, $\langle(-2)^k(x+1)\rangle = \langle(-2)^k x\rangle$, etc., and by (2.4), (2.7), we have

$$v(x+1,y) = v(x,y+1) = v(1-x,1-y) = v(1-y,1-x) = v(y,x) = v(x,y). \tag{2.14}$$

Put

$$\Delta = \{(x,y) \mid y \ge x,\ x+y \le 1,\ 2x+y \ge 1\} \text{ and } \Delta^{\#} = \{(x,y) \mid y \ge x,\ x+2y \le 2,\ 2x+y \ge 1\}.$$

Clearly, $\Delta \subset \Delta^{\#} \subset [\,0,1\,]^2$. We here prove

$$\sup_{0\le x\le y<1} v(x,y) = \sup_{0\le x<y<1} v(x,y) = \sup_{(x,y)\in\Delta^{\#}} v(x,y) = \sup_{(x,y)\in\Delta} v(x,y). \tag{2.15}$$

Because of $v(x,x) = 0$ for all $0 \le x < 1$, the first equality is trivial. By (2.14) we see $\sup_{0\le x\le y<1} v(x,y) = \sup_{x,y\in\mathbf{R}^2} v(x,y)$. Set $\vec{d} = (\frac{2}{3}, -\frac{1}{3})$, $\vec{e} = (-\frac{1}{3}, \frac{2}{3})$, and $\Delta^{\&} = \{(s+1)\vec{d} + (t+1)\vec{e} \mid 0 \le s,t \le 1\}$. Any $\vec{x} \in \mathbf{R}^2$ can be written as $\vec{x} = \delta\vec{d} + \epsilon\vec{e} = ([\,\delta\,]-[\,\epsilon\,])(2\vec{d}+\vec{e}) + (2[\,\epsilon\,]-[\,\delta\,]-1)(\vec{d}+\vec{e}) + (\langle\delta\rangle+1)\vec{d} + (\langle\epsilon\rangle+1)\vec{e}$ by using real δ and ϵ. Hence for any $\vec{x} \in \mathbf{R}$, there exist $n, m \in \mathbf{Z}$ and $\vec{y} \in \Delta^{\&}$ such that $\vec{x} = n(1,0) + m(\frac{1}{3}, \frac{1}{3}) + \vec{y}$. By (2.14), we see $\sup_{(x,y)\in\mathbf{R}^2} v(x,y) = \sup_{(x,y)\in\Delta^{\&}} v(x,y)$. Note that $\Delta = \{(x,y) \in \Delta^{\&} \mid y \ge x,\ x+y \le 1\}$ and $\Delta^{\#} = \{(x,y) \in \Delta^{\&} \mid y \ge x\}$. By (2.14), we see that $\sup_{(x,y)\in\Delta^{\&}} v(x,y) = \sup_{(x,y)\in\Delta^{\#}} v(x,y) = \sup_{(x,y)\in\Delta} v(x,y)$, and hence we completed the proof of (2.15).

Note that we have $x \le \frac{1}{2}$, $y \ge \frac{1}{3}$, $1 \le 2y + x \le 2$, $1 \le 2x + y \le 2$ for $(x,y) \in \Delta$, and $x \le \frac{2}{3}$,

$y \geq \frac{1}{3}$, $1 \leq 2y + x \leq 2$, $1 \leq 2x + y \leq 2$ for $(x, y) \in \Delta^{\#}$. (See Fig. 1.) Put

$$\begin{aligned} F(\xi, \eta, x, y) &= \widetilde{V}(\langle\xi\rangle, \langle\eta\rangle, \langle x\rangle, \langle y\rangle) \\ &\quad - \frac{1}{2}\widetilde{V}(\langle -2\xi\rangle, \langle -2\eta\rangle, \langle x\rangle, \langle y\rangle) + \frac{1}{4}\widetilde{V}(\langle 4\xi\rangle, \langle 4\eta\rangle, \langle x\rangle, \langle y\rangle), \end{aligned} \tag{2.16}$$

$$\Psi(z) = \begin{cases} (-12z^2 + 7z)/4 & \text{if } z \leq 1/3, \\ (-12z^2 + 10z - 1)/4 & \text{if } 1/3 < z \leq 1/2, \\ (-12z^2 + 14z - 3)/4 & \text{if } 1/2 < z \leq 2/3, \\ (-12z^2 + 17z - 5)/4 & \text{if } 2/3 \leq z, \end{cases} \tag{2.17}$$

$$\Phi(x, y) = \Psi(|\langle y\rangle - \langle x\rangle|). \tag{2.18}$$

By (2.4), we have

$$F(\xi, \eta, x, y) = -F(\eta, \xi, x, y) = -F(\xi, \eta, y, x). \tag{2.19}$$

It is clear that

$$F(\xi + i, \eta + j, x + k, y + l) = F(\xi, \eta, x, y) \quad \text{and} \quad \Phi(x + k, y + l) = \Phi(x, y) \tag{2.20}$$

hold for any integers i, j, k, and l. We can easily have invariance relations:

$$F(\xi + \tfrac{1}{3}, \eta + \tfrac{1}{3}, x + \tfrac{1}{3}, y + \tfrac{1}{3}) = F(1 - \eta, 1 - \xi, 1 - y, 1 - x) = F(\eta, \xi, y, x) = F(\xi, \eta, x, y). \tag{2.21}$$

We can also easily verify $\Psi(1 - z) = \Psi(z)$ and $|\langle 1 - x\rangle - \langle 1 - y\rangle| = |\langle y\rangle - \langle x\rangle|$, and see that $|\langle y + \frac{1}{3}\rangle - \langle x + \frac{1}{3}\rangle|$ equals to either $|\langle y\rangle - \langle x\rangle|$ or $1 - |\langle y\rangle - \langle x\rangle|$. By combining these we have the invariance relations for Φ:

$$\Phi(1 - y, 1 - x) = \Phi(y, x) = \Phi(x + \tfrac{1}{3}, y + \tfrac{1}{3}) = \Phi(x, y). \tag{2.22}$$

By $3x = -2(y - x) + 2y + x = -(y - x) + y + 2x$, we have

$$-(y - x) + 1 \leq 3x \leq -2(y - x) + 2 \quad ((x, y) \in \Delta^{\#}), \tag{2.23}$$

$$(1 - 3x)^+ \leq y - x, \quad (3y - 2)^+ \leq y - x \quad ((x, y) \in \Delta^{\#}). \tag{2.24}$$

Actually, (2.24) is clear from (2.23) by noting $1 - 3x \leq y - x$ and $3y - 2 = 3(y - x) + 3x - 2 \leq y - x$.

For $z \in [\,0, 1\,]$, we can verify

$$\begin{aligned} \Psi(z) &= \frac{-12z^2 + 7z}{4} \vee \frac{-12z^2 + 10z - 1}{4} \vee \frac{-12z^2 + 14z - 3}{4} \vee \frac{-12z^2 + 17z - 5}{4}, \\ \Psi(z) &\geq \frac{-6z^2 + 5z}{4} \vee \frac{-6z^2 + 7z - 1}{4} \geq -z^2 + z \geq \frac{-3z^2 + 3z}{4}, \\ \Psi(z) &\geq \frac{-12z^2 + 12z - 2}{4}. \end{aligned} \tag{2.25}$$

The next inequality is easily verified.

$$-z^2 \leq -\frac{10}{7}z + \frac{25}{49}. \tag{2.26}$$

We here state one of the key inequalities: For any $x, y, \xi, \eta \in \mathbf{R}$, it holds that

$$F(\xi, \eta, x, y) \leq \Phi(x, y). \tag{2.27}$$

Preparation for the proof of key inequality (2.27). It is enough to prove (2.27) for every ξ, $\eta \in \mathbf{R}^2$ and for every $(x, y) \in \Delta$ in view of the invariance relations (2.20), (2.21) and (2.22). For $(x, y) \in \Delta$, we have

$$\begin{aligned}
\frac{\partial}{\partial \xi}\widetilde{V}(\langle\xi\rangle, \langle\eta\rangle, \langle x\rangle, \langle y\rangle) &= \frac{\partial}{\partial \xi}V(\langle\xi\rangle, \langle x\rangle) - \frac{\partial}{\partial \xi}V(\langle\xi\rangle, \langle y\rangle) = \mathbf{1}_{[0,x)}(\langle\xi\rangle) - x - \mathbf{1}_{[0,y)}(\langle\xi\rangle) + y \\
&= -\mathbf{1}_{[x,y)}(\langle\xi\rangle) + y - x, \\
\frac{\partial}{\partial \xi}\frac{-1}{2}\widetilde{V}(\langle -2\xi\rangle, \langle -2\eta\rangle, \langle x\rangle, \langle y\rangle) &= -\mathbf{1}_{[x,y)}(\langle -2\xi\rangle) + y - x, \\
\frac{\partial}{\partial \xi}\frac{1}{4}\widetilde{V}(\langle 4\xi\rangle, \langle 4\eta\rangle, \langle x\rangle, \langle y\rangle) &= -\mathbf{1}_{[x,y)}(\langle 4\xi\rangle) + y - x,
\end{aligned}$$

and thereby we can verify $\frac{\partial F}{\partial \xi}(\xi, \eta, x, y) = 3(y-x) - \mathbf{1}_{[x,y)}(\langle\xi\rangle) - \mathbf{1}_{[x,y)}(\langle -2\xi\rangle) - \mathbf{1}_{[x,y)}(\langle 4\xi\rangle)$. Hence $\frac{\partial F}{\partial \xi}$ decreases in ξ from positive to negative only if $\langle\xi\rangle = x$, $\langle -2\xi\rangle = y$, or $\langle 4\xi\rangle = x$. The last conditions hold if and only if $\langle\xi\rangle = \xi_i$ ($i = 1, \dots, 7$) where

$$\xi_1 = x,\ \xi_2 = \frac{1-y}{2},\ \xi_3 = \frac{2-y}{2},\ \ \xi_4 = \frac{x}{4},\ \xi_5 = \frac{1+x}{4},\ \xi_6 = \frac{2+x}{4},\ \xi_7 = \frac{3+x}{4}.$$

And $\frac{\partial F}{\partial \eta}$ decreases in η from positive to negative only if $\langle\eta\rangle = \eta_i$ ($i = 1, \dots, 7$) where

$$\eta_1 = y,\ \eta_2 = \frac{1-x}{2},\ \eta_3 = \frac{2-x}{2},\ \eta_4 = \frac{y}{4},\ \eta_5 = \frac{1+y}{4},\ \eta_6 = \frac{2+y}{4},\ \eta_7 = \frac{3+y}{4}.$$

For given $(x, y) \in \Delta$, $F(\xi_i, \eta_j, x, y)$ (i, $j = 1, \dots, 7$) are the candidates of maximum of $F(\xi, \eta, x, y)$ in (ξ, η). Hence it is enough to prove

$$F(\xi_i, \eta_j, x, y) \le \Phi(x, y) \quad ((x, y) \in \Delta^\#)$$

for i, $j = 1, \dots, 7$. (Although it is enough to prove in Δ, we prove in $\Delta^\#$ to make the proof short.) We call it the inequality of type ij. We denote $F(\xi_i, \eta_j, x, y)$ simply by F_{ij}.

Hierarchy of the system of inequalities. Denote

$$\psi(t) = V(\langle t - x\rangle, \langle y - x\rangle) \quad \text{and} \quad \phi(t) = 4\psi(t) - 2\psi(-2t) + \psi(4t). \tag{2.28}$$

By (2.8), we have

$$4F(\xi, \eta, x, y) = -\phi(\xi) + \phi(\eta). \tag{2.29}$$

Put $\xi_0 = -2y$, $\xi_* = 4x$, $\eta_0 = -2x$, and $\eta_* = 4y$. For simplicity, we denote $y - x$ by z. We have

$$\begin{aligned}
&\psi(4\xi_4) = \psi(4\xi_5) = \psi(4\xi_6) = \psi(4\xi_7) = \psi(-2\eta_2) = \psi(-2\eta_3) = \psi(\xi_1), \\
&\psi(4\eta_4) = \psi(4\eta_5) = \psi(4\eta_6) = \psi(4\eta_7) = \psi(-2\xi_2) = \psi(-2\xi_3) = \psi(\eta_1), \\
&\psi(-2\xi_4) = \psi(-2\xi_6) = \psi(\eta_3), \qquad \psi(-2\eta_4) = \psi(-2\eta_6) = \psi(\xi_3), \\
&\psi(-2\xi_5) = \psi(-2\xi_7) = \psi(\eta_2), \qquad \psi(-2\eta_5) = \psi(-2\eta_7) = \psi(\xi_2), \\
&\psi(4\xi_2) = \psi(4\xi_3) = \psi(-2\eta_1) = \psi(\xi_0), \qquad \psi(4\eta_2) = \psi(4\eta_3) = \psi(-2\xi_1) = \psi(\eta_0), \\
&\psi(4\xi_1) = \psi(\xi_*), \qquad \psi(4\eta_1) = \psi(\eta_*).
\end{aligned} \tag{2.30}$$

We give concrete expressions of $\psi(\xi_i)$ and $\psi(\eta_i)$. By $1 \le 2y + x \le 2$, we have $\langle -2y - x\rangle = 2 - 2y - x$, and by (2.10) we have $\psi(\xi_0) = V(2 - 2y - x, z) = z(2y + x - 1) - (3y - 2)^+$.

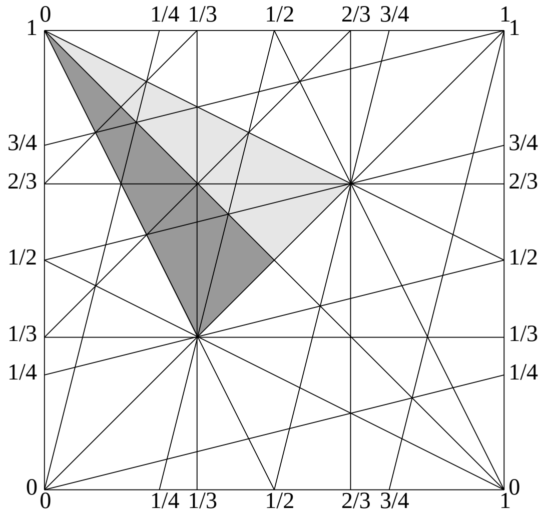

Figure 1: The unit square is divided into several pieces by lines $y-x=0$, $y-x=\frac{1}{3}$, $y-x=\frac{2}{3}$, $y+x=1$, $x=\frac{1}{3}$, $x=\frac{2}{3}$, $y=\frac{1}{3}$, $y=\frac{2}{3}$, $x+2y=1$, $x+2y=2$, $2x+y=1$, $2x+y=2$, $x-4y=0$, $x-4y=-1$, $x-4y=-2$, $x-4y=-3$, $4x-y=0$, $4x-y=1$, $4x-y=2$, and $4x-y=3$. The set Δ is painted in dark gray, and $\Delta^{\#}\setminus\Delta$ is painted in light gray.

Remark 1. When $2y+x=1$, we actually have $\langle -2y-x\rangle=0$, but the above evaluation gives $\langle -2y-x\rangle=2-2y-x=1$. But this error does not matter while calculating $V(\langle -2y-x\rangle, y-x)$ since $V(0,t)=V(1,t)=0$. In this paper we frequently abuse 1 instead of 0 in this way to avoid non-essential complexity.

By $x\le\frac{2}{3}$ and $1\le y+2x\le 2$, we have for $(x,y)\in\Delta^{\#}$

$$\psi(\eta_0)=\left\{\begin{array}{ll} V(2-3x,z)=z-(2-3x)z=-(1-3x)z & \text{if } x>\frac{1}{3} \\ V(1-3x,z)=(1-3x)-(1-3x)z & \text{if } x\le\frac{1}{3}\end{array}\right\}=(3x-1)z+(1-3x)^{+}.$$

Next, we consider $\psi(\eta_*)$. In $\Delta^{\#}$, we have $1\le 4y-x<4$. Actually, $4y-x=3y+(y-x)\ge 1$ by $y\ge\frac{1}{3}$, and $4y-x<4$ is trivial. If $3\le 4y-x<4$, by $y\le 1$, we have $\psi(\eta_*)=V(4y-x-3,z)=(4y-x-3)-(4y-x-3)z$. Note that we have $y\ge\frac{2}{3}$ in this case. (See Fig. 1). If $2\le 4y-x<3$, (2.10) yields $\psi(\eta_*)=V(4y-x-2,z)=(3+x-4y)z-(2-3y)^{+}$. If $1\le 4y-x<2$, by $\frac{1}{3}\le y$, we have $\psi(\eta_*)=V(4y-x-1,y-x)=z-(4y-x-1)z=(3+x-4y)z-(2-3y)+(x-4y+2)$. Note that we have $y\le\frac{2}{3}$ in this case.

We evaluate $\psi(\xi_*)$ as below:

$$\psi(\xi_*)=\begin{cases} V(3x,z)=z(1-3x)-(y-4x)^{+} & \text{if } x<\frac{1}{3}, \\ V(3x-1,z)=(3x-1)(1-z)-(4x-y-1)^{+} & \text{if } x\ge\frac{1}{3}.\end{cases}$$

These give the the following expressions in $\Delta^{\#}$:

$$\begin{aligned}\psi(\xi_*)&=(1-3x)z+(3x-1)^{+}-(4x-y-1)^{+}-(y-4x)^{+},\\ \psi(\eta_*)&=(3+x-4y)z-(2-3y)^{+}+(x-4y+2)^{+}+(4y-x-3)^{+}.\end{aligned}$$

Similarly, we have

$$\psi(\xi_0) = (2y + x - 1)z - (3y - 2)^+, \qquad \psi(\xi_1) = 0,$$
$$\psi(\xi_2) = \frac{1}{2}(2x + y - 1)z, \qquad \psi(\xi_3) = \frac{1}{2}(2x + y)z - \frac{1}{2}(3y - 2)^+,$$
$$\psi(\xi_4) = \frac{3}{4}xz, \qquad \psi(\xi_5) = \frac{3x - 1}{4}z + \frac{1}{4}(1 - 3x)^+,$$
$$\psi(\xi_6) = \frac{3x + 2}{4}z - \frac{1}{4}(4y - x - 2)^+, \qquad \psi(\xi_7) = \frac{3x + 1}{4}z - \frac{1}{4}(4y - x - 3)^+,$$

$$\psi(\eta_0) = (3x - 1)z + (1 - 3x)^+, \qquad \psi(\eta_1) = -z^2 + z,$$
$$\psi(\eta_2) = \frac{1}{2}(3x - 1)z + \frac{1}{2}(1 - 3x)^+, \qquad \psi(\eta_3) = \frac{3}{2}xz,$$
$$\psi(\eta_4) = \frac{1}{4}(4x - y)z + \frac{1}{4}(y - 4x)^+, \qquad \psi(\eta_5) = \frac{1}{4}(4x - y - 1)z + \frac{1}{4}(y - 4x + 1)^+,$$
$$\psi(\eta_6) = \frac{1}{4}(4x - y + 2)z - \frac{1}{4}(3y - 2)^+, \qquad \psi(\eta_7) = \frac{1}{4}(4x - y + 1)z.$$

We have the inequalities

$$\psi(\eta_2) \le \psi(\eta_3), \qquad \psi(\eta_4), \psi(\eta_5) \le \psi(\eta_7) \le \psi(\eta_6),$$
$$\psi(\xi_2) \le \psi(\xi_3), \qquad \psi(\xi_5) \le \psi(\xi_4) \le \psi(\xi_6), \psi(\xi_7).$$

Actually, (2.24) implies $\psi(\eta_2) \le \psi(\eta_3)$, $\psi(\xi_5) \le \psi(\xi_4)$, $\psi(\xi_2) \le \psi(\xi_3)$, and $\psi(\eta_7) \le \psi(\eta_6)$. By $y - 4x \le 4y - 4x$, we have $\frac{1}{4}(y - 4x)^+ \le z$ and $\psi(\eta_4) \le \psi(\eta_7)$. By (2.23), we have $y - 4x + 1 = z - 3x + 1 \le 2z$ and $(y - 4x + 1)^+ \le 2z$, and hence we have $\psi(\eta_5) \le \psi(\eta_7)$. By $4y - x - 2 = 2z + x + 2y - 2 \le 2z$, we have $(4y - x - 2)^+ \le 2z$ and $\psi(\xi_4) \le \psi(\xi_6)$. By $4y - x - 3 = z + 3(y - 1) \le z$, we have $(4y - x - 3)^+ \le z$ and $\psi(\xi_4) \le \psi(\xi_7)$.

By applying the inequalities above and relations (2.30), we can prove

$$\phi(\alpha_2) \le \phi(\alpha_3), \quad \phi(\alpha_5) \le \phi(\alpha_7), \quad \phi(\alpha_4) \le \phi(\alpha_6), \quad (\alpha = \xi, \eta).$$

Actually, by $\phi(\xi_2) = 4\psi(\xi_2) - 2\psi(\eta_1) + \psi(\xi_0)$ and $\phi(\xi_3) = 4\psi(\xi_3) - 2\psi(\eta_1) + \psi(\xi_0)$, we have $\phi(\xi_2) \le \phi(\xi_3)$. By $\phi(\xi_5) = 4\psi(\xi_5) - 2\psi(\eta_2) + \psi(\xi_1)$ and $\phi(\xi_7) = 4\psi(\xi_7) - 2\psi(\eta_2) + \psi(\xi_1)$, we have $\phi(\xi_5) \le \phi(\xi_7)$, and other inequalities can be proved in the same way. Moreover, $4\psi(\eta_6) - 2\psi(\xi_3) = 2(1 - z)z = 4\psi(\eta_7) - 2\psi(\xi_2)$ implies $\phi(\eta_6) = \phi(\eta_7)$, and $4\psi(\xi_4) - 2\psi(\eta_3) = 0 = 4\psi(\xi_5) - 2\psi(\eta_2)$ implies $\phi(\xi_4) = \phi(\xi_5)$. Hence we have

$$\phi(\xi_4) \le \phi(\xi_5), \phi(\xi_6), \phi(\xi_7), \quad \text{and} \quad \phi(\eta_4), \phi(\eta_5), \phi(\eta_6) \le \phi(\eta_7).$$

By noting (2.29) and the inequalities above, we have

$$F(\xi_i, \eta_j, x, y) \le F(\xi_k, \eta_l, x, y) \quad (i \in \Xi_k, j \in H_l, k \in \{1, 2, 4\}, l \in \{1, 3, 7\}),$$

where $\Xi_1 = H_1 = \{1\}$, $\Xi_2 = H_3 = \{2, 3\}$, and $\Xi_4 = H_7 = \{4, 5, 6, 7\}$. It means that it is sufficient to prove the inequalities of types 11, 13, 17, 21, 23, 27, 41, 43, and 47.

Recall that $\Delta^\#$ is invariant under the transformation $(x, y) \mapsto (1 - y, 1 - x)$. By noting (2.21) and denoting $(X, Y) = (1 - y, 1 - x)$, we have $F(x, \frac{2-x}{2}, x, y) = F(1 - \frac{2-x}{2}, 1 - x, 1 - y, 1 - x) =$

$F(\frac{1-Y}{2}, Y, X, Y)$. Thus, if we prove the inequality of type 13, it can be transformed to the inequality of type 21. In the same way, by $F(x, \frac{3+y}{4}, x, y) = F(1 - \frac{3+y}{4}, 1-x, 1-y, 1-x) = F(\frac{X}{4}, Y, X, Y)$ and $F(\frac{1-y}{2}, \frac{3+y}{4}, x, y) = F(1 - \frac{3+y}{4}, 1 - \frac{1-y}{2}, 1-y, 1-x) = F(\frac{X}{4}, \frac{2-X}{2}, X, Y)$, the inequalities of types 17 and 27 are transformed to the inequalities of types 41 and 43.

Hence it is sufficient to prove the inequalities of types 11, 13, 17, 23, 27, and 47.

The inequality of type 11. First, we note that it is sufficient to prove $F(x, y, x, y) \leq \Phi(x, y)$ for $(x, y) \in \Delta$. Actually, if we prove this, by (2.21) and (2.22), we have $F(1-y, 1-x, 1-y, 1-x) \leq \Phi(1-y, 1-x)$ for $(x, y) \in \Delta$. Since $\{(1-y, 1-x) \mid (x, y) \in \Delta\}$ covers $\Delta^{\#} \setminus \Delta$, we have $F(x, y, x, y) \leq \Phi(x, y)$ for $(x, y) \in \Delta^{\#}$.

By noting (2.8) and calculating $\psi(\eta_0) - \psi(\xi_0)$ on Δ, we can verify

$$-\widetilde{V}(\langle -2x\rangle, \langle -2y\rangle, \langle x\rangle, \langle y\rangle) = \psi(\eta_0) - \psi(\xi_0) = -2z^2 + (1-3x)^+ + (3y-2)^+. \tag{2.31}$$

Δ is divided into eight pieces $A_1, \ldots, A_8$ by lines $x = \frac{1}{3}$, $y = \frac{2}{3}$, $x - 4y = -2$, $x - 4y = -3$, $4x - y = 0$, and $4x - y = 1$ (Cf. Figure 1). By calculating $\psi(\eta_*) - \psi(\xi_*)$, we have

$$\begin{aligned}
&\widetilde{V}(\langle 4x\rangle, \langle 4y\rangle, x, y) = \psi(\eta_*) - \psi(\xi_*) \qquad\qquad (2.32)\\
&= \begin{cases}
-4z^2 + 7z - 3 & \text{on } A_1 = \{(x, y) \mid x < \frac{1}{3}, 3 \leq 4y - x, 0 \leq y - 4x, \frac{2}{3} \leq y\},\\
-4z^2 - 6x + 3y & \text{on } A_2 = \{(x, y) \mid x < \frac{1}{3}, 2 \leq 4y - x < 3, 0 \leq y - 4x, \frac{2}{3} \leq y\},\\
-4z^2 + 2z & \text{on } A_3 = \{(x, y) \mid x < \frac{1}{3}, 2 \leq 4y - x < 3, -1 \leq y - 4x \leq 0, \frac{2}{3} \leq y\},\\
-4z^2 - 2x + 5y - 2 & \text{on } A_4 = \{(x, y) \mid x < \frac{1}{3}, 2 \leq 4y - x < 3, -1 \leq y - 4x \leq 0, y \leq \frac{2}{3}\},\\
-4z^2 + z & \text{on } A_5 = \{(x, y) \mid x < \frac{1}{3}, 1 \leq 4y - x < 2, -1 \leq y - 4x \leq 0, y \leq \frac{2}{3}\},\\
-4z^2 + 5z - 1 & \text{on } A_6 = \{(x, y) \mid \frac{1}{3} \leq x, 2 \leq 4y - x < 3, -1 \leq y - 4x \leq 0, y \leq \frac{2}{3}\},\\
-4z^2 - 4x + y + 1 & \text{on } A_7 = \{(x, y) \mid \frac{1}{3} \leq x, 1 \leq 4y - x < 2, -1 \leq y - 4x \leq 0, y \leq \frac{2}{3}\},\\
-4z^2 & \text{on } A_8 = \{(x, y) \mid \frac{1}{3} \leq x, 1 \leq 4y - x < 2, y - 4x \leq -1, y \leq \frac{2}{3}\}.
\end{cases}
\end{aligned}$$

Hence by (2.9), (2.31), (2.32), and (2.25), we can verify the inequality of type 11 as below.

On A_1, $4F_{11} = -12z^2 + 17z - 5 \leq 4\Psi(z) = 4\Phi(x, y)$. On A_2, $4F_{11} = -12z^2 + 14z - 3 + (1 - 2x - y) \leq -12z^2 + 14z - 3 \leq 4\Psi(z)$. On A_3, $4F_{11} = -12z^2 + 12z - 2 \leq 4\Psi(z)$. On A_4, $4F_{11} = -12z^2 + 10z - 1 + (1 - 2x - y) \leq -12z^2 + 10z - 1 \leq 4\Psi(z)$. On A_5, $4F_{11} = -12z^2 + 7z + 2(1 - 2x - y) \leq -12z^2 + 7z \leq 4\Psi(z)$. On A_6, by $z \leq \frac{1}{3}$, we have $4F_{11} = -12z^2 + 7z + 2z - 1 < -12z^2 + 7z \leq 4\Psi(z)$. On A_7, $4F_{11} = -12z^2 + 7z + (1 - x - 2y) \leq -12z^2 + 7z \leq 4\Psi(z)$. On A_8, $4F_{11} = -12z^2 + 7z - 3z \leq -12z^2 + 7z \leq 4\Psi(z)$.

The inequality of type 13. By $4F_{13} = 4\psi(\eta_3) - 6\psi(\xi_1) + 3\psi(\eta_0) - \psi(\xi_*)$, we have

$$4F_{13} = (18x - 4)z + 1 - 3x + 2(1 - 3x)^+ + (-y + 4x - 1)^+ + (y - 4x)^+,$$

where we used $-(1 - 3x)^+ + (3x - 1)^+ = 3x - 1$. We divide $\Delta^{\#}$ into a few pieces.

When $y - 4x \geq 0$, we have $x \leq \frac{1}{3}$, and hence $4F_{13} = 3x(6z - 4) - 3z + 3$. If $z \geq \frac{2}{3}$, by (2.23) we have $4F_{13} \leq (-2z + 2)(6z - 4) - 3z + 3 = -12z^2 + 17z - 5 \leq 4\Psi(z)$, and otherwise we have $4F_{13} \leq (-z + 1)(6z - 4) - 3z + 3 = -6z^2 + 7z - 1 \leq 4\Psi(z)$.

If $y - 4x \leq 0$ and $x \leq \frac{1}{3}$, we have $4F_{13} = 9x(2z - 1) - 4z + 3$. In case $z \geq \frac{1}{2}$, we see $4F_{13} \leq 3(-2z + 2)(2z - 1) - 4z + 3 = -12z^2 + 14z - 3 \leq 4\Psi(z)$, and otherwise, we see $4F_{13} \leq 3(-z + 1)(2z - 1) - 4z + 3 = -6z^2 + 5z \leq 4\Psi(z)$.

If $x \geq \frac{1}{3}$ and $y - 4x \geq -1$, we have $4F_{13} = 3x(6z - 1) - 4z + 1$. In case $z \geq \frac{1}{6}$, we see $4F_{13} \leq (-2z - 2)(6z - 1) - 4z + 1 = -12z^2 + 10z - 1 \leq 4\Psi(z)$, and otherwise we see $4F_{13} \leq (-z + 1)(6z - 1) - 4z + 1 = -6z^2 + 3z \leq -6z^2 + 5z \leq 4\Psi(z)$.

If $y - 4x \leq -1$, we have $4F_{13} = (18x - 5)z \leq (6(-2z + 2) - 5)z = -12z^2 + 7z \leq 4\Psi(z)$.

The inequality of type 17. By $-(1 - 3x)^+ + (3x - 1)^+ = 3x - 1$, we have

$$4F_{17} = -3z^2 + 3z + (3x - 1)(3z - 1) + (1 - 3x)^+ + (y - 4x)^+ + (-y + 4x - 1)^+.$$

We divide $\Delta^\#$ into five parts.

If $y-4x \geq 0$, by (2.23), we have $4F_{17} = (1-3x)(3-3z)-3z^2+4z-1 \leq z(3-3z)-3z^2+4z-1 = -6z^2 + 7z - 1 \leq 4\Psi(z)$.

If $y - 4x \leq 0$ and $x \leq \frac{1}{3}$, we have $z \leq \frac{2}{3}$, and by applying (2.23), we have $4F_{17} = (1 - 3x)(2 - 3z) - 3z^2 + 3z \leq z(2 - 3z) - 3z^2 + 3z \leq -6z^2 + 5z \leq 4\Psi(z)$.

When $x \geq \frac{1}{3}$, $y - 4x \geq -1$, we have $4F_{17} = (3x - 1)(3z - 1) - 3z^2 + 3z$. If $z \geq \frac{1}{3}$, by applying (2.23), we have $(3x - 1)(3z - 1) \leq (-2z + 1)(3z - 1) \leq (-z + 1)(3z - 1) = -3z^2 + 4z - 1$ and $4F_{17} \leq -6z^2+7z-1 \leq 4\Psi(z)$. If $z \leq \frac{1}{3}$, we have $(3x-1)(3z-1) \leq 0$ and $4F_{17} \leq -3z^2+3z \leq 4\Psi(z)$.

If $y-4x \leq -1$, by applying (2.23), we have $4F_{17} = 3z(3x-1)-3z^2+2z \leq 3z(1-2z)-3z^2+2z = -9z^2 + 5z \leq -6z^2 + 5z \leq 4\Psi(z)$.

The inequality of type 23. We have $4F_{23} = -6z^2 + 4z + (3y - 2)^+ + (1 - 3x)^+$. If $(3y - 2)^+ > 0$ and $(1 - 3x)^+ > 0$, then $4F_{23} = -6z^2 + 7z - 1 \leq 4\Psi(z)$. Otherwise, by (2.24), we have $(3y - 2)^+ + (1 - 3x)^+ \leq z$ and $4F_{23} \leq -6z^2 + 5z \leq 4\Psi(z)$.

The inequality of type 27. We have $4F_{27} = -9z^2 + 8z - 9xz + (3y - 2)^+$. If $(3y - 2)^+ = 0$, we have $-9xz \leq -3(-z + 1)z$ and $4F_{27} \leq -6z^2 + 5z \leq 4\Psi(z)$. If $(3y - 2)^+ > 0$, then $4F_{27} = -9z^2 + 11z + 3x(1-3z)-2$. In case $z \leq \frac{1}{3}$, we have $4F_{27} \leq -9z^2+11z+(-2z+2)(1-3z)-2 = -3z^2+3z \leq 4\Psi(z)$. In case $z \geq \frac{1}{3}$, we have $4F_{27} \leq -9z^2 + 11z + (-z + 1)(1 - 3z) - 2 = -6z^2 + 7z - 1 \leq 4\Psi(z)$.

The inequality of type 47. We have $4F_{47} = -3z^2 + 3z \leq 4\Psi(z)$.

Thus (2.27) has been proved.

3. Proof of the Theorem

To prove (1.2), by (1.1), (2.1), and (2.15), it is enough to prove

$$\sup_{(x,y)\in\Delta} v(x, y) = \frac{130}{343}. \tag{3.1}$$

We prove that the above supremum is attained at

$$P_1 : \left(\frac{1}{7}, \frac{6}{7}\right).$$

The evaluation $v\left(\frac{1}{7}, \frac{6}{7}\right) = \frac{130}{343}$ can be found in [17]. Hence it suffices to prove

$$v(x, y) < v\left(\frac{1}{7}, \frac{6}{7}\right) = \frac{130}{343} \quad \text{for} \quad (x, y) \in \Delta \setminus \{P_1\}. \tag{3.2}$$

Write $\widetilde{v}_{-1}(x,y) = -\widetilde{V}(\langle x\rangle, \langle y\rangle, \langle x\rangle, \langle y\rangle)$ and

$$\widetilde{v}_L(x,y) = -\widetilde{V}(\langle x\rangle, \langle y\rangle, \langle x\rangle, \langle y\rangle) + 2\sum_{k=0}^{L} \frac{\widetilde{V}(\langle(-2)^k x\rangle, \langle(-2)^k y\rangle, \langle x\rangle, \langle y\rangle)}{(-2)^k} \quad (L = 0, 1, 2, \dots).$$

We shall prove

$$v(x,y) \le \widetilde{v}_L(x,y) + 2^{-L}\frac{8}{7}\Phi(x,y) =: v_L(x,y) \quad (L = -1, 0, 1, 2, \dots). \tag{3.3}$$

Note that $v(x,y) = \widetilde{v}_L(x,y) + 2\sum_{j=0}^{\infty}\sum_{i=1}^{3} \widetilde{V}(\langle(-2)^{L+3j+i}x\rangle, \langle(-2)^{L+3j+i}y\rangle, \langle x\rangle, \langle y\rangle)/(-2)^{L+3j+i}$. Put $\xi = (-2)^{L+3j+1}x$ and $\eta = (-2)^{L+3j+1}y$. If $L + 3j + 1$ is even, we have by (2.27),

$$\sum_{i=1}^{3} \frac{\widetilde{V}(\langle(-2)^{L+3j+i}x\rangle, \langle(-2)^{L+3j+i}y\rangle, \langle x\rangle, \langle y\rangle)}{(-2)^{L+3j+i}} = \frac{F(\xi,\eta,x,y)}{2^{L+3j+1}} \le \frac{\Phi(x,y)}{2^{L+3j+1}}.$$

If $L + 3j + 1$ is odd, by noting (2.19), we have

$$\sum_{i=1}^{3} \frac{\widetilde{V}(\langle(-2)^{L+3j+i}x\rangle, \langle(-2)^{L+3j+i}y\rangle, \langle x\rangle, \langle y\rangle)}{(-2)^{L+3j+i}} = \frac{F(\eta,\xi,x,y)}{2^{L+3j+1}} \le \frac{\Phi(x,y)}{2^{L+3j+1}}.$$

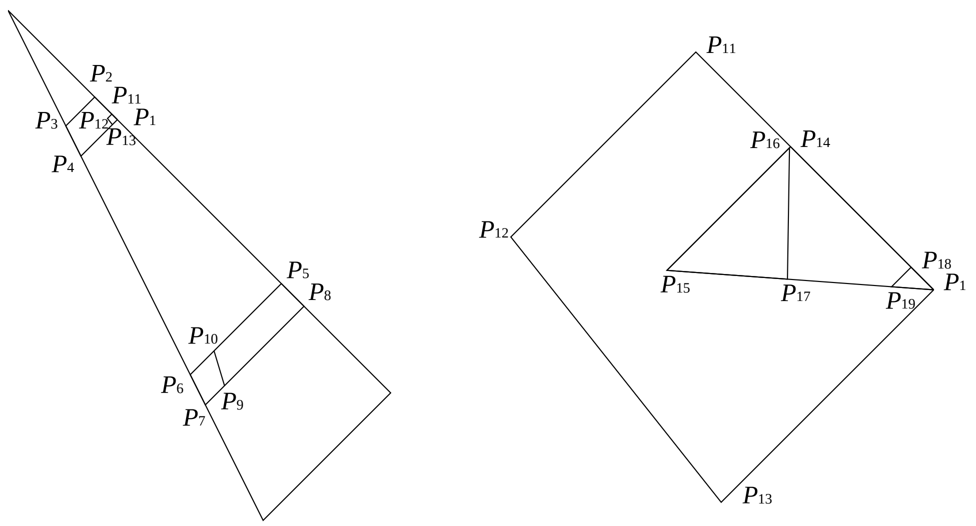

Figure 2: The left triangle is Δ. In Δ, we take the quadrangles Δ_{-1+}: $P_1P_2P_3P_4$, Δ_{-1-}: $P_5P_6P_7P_8$, Δ_1: $P_6P_7P_9P_{10}$, and Δ_3: $P_1P_{11}P_{12}P_{13}$. In Δ_3, we take the triangles Δ_5: $P_1P_{14}P_{15}$ and Δ_6: $P_1P_{18}P_{19}$.

In the sequel we split Δ in parts $\Delta \setminus \Delta_{-1}$, $\Delta_{-1-} \setminus \Delta_1$, Δ_1, $\Delta_{-1+} \setminus \Delta_3$, $\Delta_3 \setminus \Delta_5$, $\Delta_5 \setminus \Delta_6$, Δ_6 where $\Delta_{-1} = \Delta_{-1-} \cup \Delta_{-1+}$, and prove (3.2) for each part.

3.1. $\Delta \setminus \Delta_{-1}$ part

We put

$$\Delta_{-1-} = \left\{(x,y) \in \Delta \;\middle|\; \frac{65}{287} \le y - x \le \frac{2}{7}\right\}, \quad \Delta_{-1+} = \left\{(x,y) \in \Delta \;\middle|\; \frac{5}{7} \le y - x \le \frac{222}{287}\right\},$$

and $\Delta_{-1} = \Delta_{-1-} \cup \Delta_{-1+}$, and prove

$$v(x,y) \le v_{-1}(x,y) < \frac{130}{343} \quad \text{for} \quad (x,y) \in \Delta \setminus \Delta_{-1}. \tag{3.4}$$

By (2.17) and (3.3), we have

$$v_{-1}(x,y) = \widetilde{v}_{-1}(x,y) + \frac{16}{7}\Phi(x,y) = z^2 - z + \frac{4}{7}(-12z^2 + 17z - 5) = -\frac{41}{7}z^2 + \frac{61}{7}z - \frac{20}{7}$$

for $(x,y) \in \Delta$ with $\frac{2}{3} \le y - x < 1$. In this case the inequality $v_{-1}(x,y) \ge \frac{130}{343}$ is equivalent to $(x,y) \in \Delta_{-1+}$.

When $\frac{1}{2} \le y - x \le \frac{2}{3}$, we have $v_{-1}(x,y) = -\frac{41}{7}(z - \frac{49}{82})^2 + \frac{433}{1148} < \frac{130}{343}$.

Note that $v_{-1}(x,y) = -(-z^2 + z) + \frac{16}{7}\Psi(z)$ is symmetric around $z = \frac{1}{2}$. Hence for $(x,y) \in \Delta$ with $0 \le y - x \le \frac{1}{2}$, the inequality $v_{-1}(x,y) \ge \frac{130}{343}$ is equivalent to $(x,y) \in \Delta_{-1-}$. By combining these, we have proved (3.4).

Since Δ_{-1+} is determined by $\frac{5}{7} \le y - x \le \frac{222}{287}$, $y + x \le 1$, and $y + 2x \ge 1$, we see that it is the quadrangle with vertices P_1,

$$P_2 : \left(\frac{65}{574}, \frac{509}{574}\right), \qquad P_3 : \left(\frac{65}{861}, \frac{731}{861}\right), \quad \text{and} \quad P_4 : \left(\frac{2}{21}, \frac{17}{21}\right).$$

Since Δ_{-1-} is determined by $\frac{65}{287} \le y - x \le \frac{2}{7}$, $y + x \le 1$, and $y + 2x \ge 1$, we see that it is the quadrangle with vertices

$$P_5 : \left(\frac{5}{14}, \frac{9}{14}\right), \qquad P_6 : \left(\frac{5}{21}, \frac{11}{21}\right), \qquad P_7 : \left(\frac{74}{287}, \frac{139}{287}\right), \quad \text{and} \quad P_8 : \left(\frac{111}{287}, \frac{176}{287}\right).$$

3.2. $\Delta_{-1-} \setminus \Delta_1$ part

By noting (2.31), for $(x,y) \in \Delta_{-1}$, we have

$$\widetilde{v}_1(x,y) = -3z^2 + z + (1 - 3x)^+ + (3y - 2)^+. \tag{3.5}$$

We consider Δ_{-1-}. Since $y \le \frac{2}{3}$ holds on P_5, P_6, P_7, and P_8, it holds on Δ_{-1-}. Hence for $(x,y) \in \Delta_{-1-}$, we have $v_1(x,y) = -\frac{33}{7}z^2 + 2z + (1 - 3x)^+$. Put

$$\Delta_1 = \left\{(x,y) \in \Delta_{-1-} \;\middle|\; -\frac{34}{49}y - \frac{113}{49}x + \frac{345}{343} \ge 0\right\}.$$

On $(\Delta_{-1-} \setminus \Delta_1) \cap \{(x,y) \mid x \le \frac{1}{3}\}$. by $-z^2 \le -\frac{4}{7}z + \frac{4}{49}$, we have $v_1(x,y) = -\frac{33}{7}z^2 + 2z + (1 - 3x) \le (-\frac{34}{49}y - \frac{113}{49}x + \frac{345}{343}) + \frac{130}{343} < \frac{130}{343}$.

We see P_6, $P_7 \in \Delta_1$ and P_5, $P_8 \notin \Delta_1$. Hence Δ_1 is determined by $\frac{65}{287} \leq y - x \leq \frac{2}{7}$, $-\frac{34}{49}y - \frac{113}{49}x + \frac{345}{343} \geq 0$, and $y + 2x \geq 1$, which is the quadrangle with vertices P_6, P_7,

$$P_9 : \left(\frac{1705}{6027}, \frac{3070}{6027}\right), \quad \text{and} \quad P_{10} : \left(\frac{277}{1029}, \frac{571}{1029}\right).$$

We can verify $x < \frac{1}{3}$ on P_6, P_7, P_9, and P_{10}, and hence $x < \frac{1}{3}$ on Δ_1. Therefore we see $\Delta_{-1-} \cap \{(x, y) \mid x = \frac{1}{3}\}$ is outside of Δ_1 and $v_1(\frac{1}{3}, \frac{1}{3} + c) < \frac{130}{343}$ for $\frac{65}{287} \leq c \leq \frac{2}{7}$. If we consider on the line $y - x = c$, we see $v_1(x, y) = v_1(x, x + c) = v_1(\frac{1}{3}, \frac{1}{3} + c)$ for $x \geq \frac{1}{3}$, and consequently we have $v_1(x, y) < \frac{130}{343}$ for $(x, y) \in \Delta_{-1-} \cap \{(x, y) \mid x \geq \frac{1}{3}\}$. Hence we have proved

$$v(x, y) \leq v_1(x, y) < \frac{130}{343} \quad \text{for} \quad (x, y) \in \Delta_{-1-} \setminus \Delta_1. \tag{3.6}$$

Since we have $y \leq \frac{2}{3}$ in Δ_{-1-} and by $\Delta_1 \subset \Delta_{-1-}$, we have $y \leq \frac{2}{3}$ in Δ_1. We have also verified $x < \frac{1}{3}$ in Δ_1.

3.3. Δ_1 part

We consider Δ_1. We can verify P_6, P_7, P_9, and P_{10} satisfy $0 < 4x - y < 1$, $1 < 4y - x < 2$, $0 < 3 - 9x < 1$, $0 < 5 - 9y < 1$, $0 < 3 - 8x - y < 1$, and $0 < 5 - 8y - x < 1$. Hence we see that these inequalities hold in Δ_1. We have $\widetilde{v_1}(x, y) = -3z^2 + z + (1 - 3x)$ by (3.5) and $\widetilde{V}(\langle 4x\rangle, \langle 4y\rangle, \langle x\rangle, \langle y\rangle) = -4z^2 + z$ by (2.32). By (2.8) we have

$$\begin{aligned}
&-\widetilde{V}(\langle -8x\rangle, \langle -8y\rangle, \langle x\rangle, \langle y\rangle) = -V(5 - 8y - x, z) + V(3 - 9x, z) \\
&= -(z - (5 - 8y - x)z) + (z - (3 - 9x)z) = -8z^2 + 2z.
\end{aligned}$$

Hence by (2.23), $v_3(x, y) = -\frac{52}{7}z^2 + \frac{9}{4}z + (1 - 3x) \leq -\frac{52}{7}z^2 + \frac{13}{4}z = -\frac{52}{7}\left(z - \frac{7}{32}\right)^2 + \frac{91}{256} \leq \frac{91}{256} < \frac{130}{343}$. Therefore,

$$v(x, y) \leq v_3(x, y) < \frac{130}{343} \quad \text{for} \quad (x, y) \in \Delta_1. \tag{3.7}$$

3.4. $\Delta_{-1+} \setminus \Delta_3$ part

In Δ_{-1+}, we have $x \leq \frac{1}{3}$ and $y \geq \frac{2}{3}$. Hence by (3.5), we have $\widetilde{v_1}(x, y) = -3z^2 + 4z - 1$.

We can verify P_1, P_2, P_3, and P_4 satisfy $4x - y < 0$, $3 < 4y - x$, $0 \leq 8 - 9y < 1$, and $0 \leq 2 - 8x - y < 1$. Hence these inequalities holds on Δ_{-1+}. By (2.32), we have $\widetilde{V}(\langle 4x\rangle, \langle 4y\rangle, \langle x\rangle, \langle y\rangle) = -4z^2 + 7z - 3$. By (2.4), (2.8), and (2.10), we have

$$\begin{aligned}
&-\widetilde{V}(\langle -8x\rangle, \langle -8y\rangle, \langle x\rangle, \langle y\rangle) = \widetilde{V}(\langle -8x\rangle, \langle -8y\rangle, \langle y\rangle, \langle x\rangle) \\
&= -V(2 - 8x - y, 1 + x - y) + V(8 - 9y, 1 + x - y) \\
&= -((2 - 8x - y)z - (1 - 9x)^+) + ((8 - 9y)z - (7 - 8y - x)^+) \\
&= -8z^2 + 6z + (1 - 9x)^+ - (7 - 8y - x)^+.
\end{aligned}$$

Hence in Δ_{-1+}, we have

$$\widetilde{v_3}(x, y) = -7z^2 + 9z - \frac{5}{2} + \frac{1}{4}(1 - 9x)^+ - \frac{1}{4}(7 - 8y - x)^+, \tag{3.8}$$

and $v_3(x,y) = \frac{1}{28}(-208z^2 + 269z - 75 + 7(1-9x)^+ - 7(7-8y-x)^+)$.

Firstly, in $\{(x,y) \in \Delta_{-1+} \mid x \leq \frac{1}{9}\}$, by $-(7-8y-x)^+ \leq -(7-8y-x)$, we have $v_3(x,y) \leq \frac{1}{28}(-208z^2 + 325z - 117) =: v_3^*(z)$. Since $v_3^*(z)$ in increasing in $z \in (-\infty, \frac{325}{416})$, by $\frac{222}{287} < \frac{325}{416}$ and by noting $z \leq \frac{222}{287}$ in Δ_{-1+}, we have $v_3(x,y) \leq v_3^*(\frac{222}{287}) = \frac{818805}{2306332} < \frac{130}{343}$.

Secondly, on $\{(x,y) \in \Delta_{-1+} \mid x \geq \frac{1}{9}\}$, we have $v_3(x,y) = \frac{1}{28}(-208z^2+269z-75)-\frac{1}{4}(7-8y-x)^+$. On $\{(x,y) \in \Delta_{-1+} \mid x \geq \frac{1}{9}, y-x > \frac{1005}{1379}\}$, by (2.26) and $-(7-8y-x)^+ \leq 0$, we have $v_3(x,y) \leq -\frac{197}{196}z + \frac{1525}{1372} = -\frac{197}{196}(z - \frac{1005}{1379}) + \frac{130}{343} < \frac{130}{343}$. On $\{(x,y) \in \Delta_{-1+} \mid x \geq \frac{1}{9}, \frac{195}{196}y + \frac{123}{98}x - \frac{349}{343} < 0\}$, by (2.26) and $-(7-8y-x)^+ \leq -(7-8y-x)$, we have $v_3(x,y) \leq (\frac{195}{196}y + \frac{123}{98}x - \frac{349}{343}) + \frac{130}{343} < \frac{130}{343}$. Let

$$\Delta_3 := \left\{(x,y) \,\middle|\, y+x \leq 1, \quad \frac{5}{7} \leq y-x \leq \frac{1005}{1379}, \quad \frac{195}{196}y + \frac{123}{98}x - \frac{349}{343} \geq 0\right\}.$$

Since Δ_3 is the quadrangle having vertices P_1,

$$P_{11} : \left(\frac{187}{1379}, \frac{1192}{1379}\right), \qquad P_{12} : \left(\frac{1613}{12411}, \frac{10658}{12411}\right), \quad \text{and} \quad P_{13} : \left(\frac{421}{3087}, \frac{2626}{3087}\right),$$

we can verify $P_1, P_{11}, P_{12}, P_{13} \in \Delta_{-1+}$, and $\Delta_3 \subset \Delta_{-1+}$. Combining these, we have

$$v(x,y) \leq v_3(x,y) < \frac{130}{343} \quad \text{for} \quad (x,y) \in \Delta_{-1+} \setminus \Delta_3. \tag{3.9}$$

3.5. $\Delta_3 \setminus \Delta_5$ *part*

We consider Δ_3. By calculating values at P_1, P_{11}, P_{12}, and P_{13}, we have $1-9x < 0$, $0 < 15y-12 < 1$, $0 < 16x-y-1 < 1$, $0 < 16y-x-13 < 1$, $0 < -33x+5 < 1$, $0 < -33y+29 < 1$, $0 < -32x-y+6 < 1$, and $0 < -32y-x+28 < 1$ on Δ_3. By (3.8), we have $\widetilde{v_3}(x,y) = -7z^2 + 9z - \frac{5}{2} - \frac{1}{4}(7-8y-x)^+$. By (2.4) and (2.8), we have

$$\begin{aligned}
&\widetilde{V}(\langle 16x\rangle, \langle 16y\rangle, \langle x\rangle, \langle y\rangle) = -\widetilde{V}(\langle 16x\rangle, \langle 16y\rangle, \langle y\rangle, \langle x\rangle) \\
&= V(16x-y-1, 1+x-y) - V(15y-12, 1+x-y) \\
&= ((16x-y-1)z - (15x-2)^+) - ((1+x-y) - (15y-12)(1+x-y)) \\
&= -16z^2 + 12z + 15y - 13 - (15x-2)^+,
\end{aligned}$$

and

$$\begin{aligned}
&-\widetilde{V}(\langle -32x\rangle, \langle -32y\rangle, \langle x\rangle, \langle y\rangle) = -V(-32y-x+28, y-x) + V(-33x+5, y-x) \\
&= -((-32y-x+28) - (-32y-x+28)z) + ((-33x+5) - (-33x+5)z) = -32z^2 + 55z - 23.
\end{aligned}$$

Hence we have

$$\widetilde{v_5}(x,y) = -11z^2 + \frac{223}{16}z - \frac{89}{16} - \frac{1}{4}(7-8y-x)^+ + \frac{1}{8}\left(15y - (15x-2)^+\right),$$

and by $-(15x-2)^+ \leq -(15x-2)$ we have $v_5(x,y) \leq -\frac{311}{28}z^2 + \frac{447}{28}z - \frac{75}{14} - \frac{1}{4}(7-8y-x)^+$.

On $\{(x,y) \in \Delta_3 \mid y-x > \frac{1574}{2177}\}$, by $-\frac{1}{4}(7-8y-x)^+ \leq 0$, we have $v_5(x,y) \leq -\frac{311}{28}z^2 + \frac{447}{28}z - \frac{75}{14} = -\frac{311}{28}(z-\frac{5}{7})(z-\frac{1574}{2177}) + \frac{130}{343} < \frac{130}{343}$. Note that $\frac{1574}{2177} < \frac{1005}{1379}$.

On $\{(x,y) \in \Delta_3 \mid \frac{411}{196}y + \frac{15}{98}x - \frac{624}{343} < 0\}$, by (2.26) and $-(7-8y-x)^+ \le -(7-8y-x)$, we have $v_5(x,y) \le (\frac{411}{196}y + \frac{15}{98}x - \frac{624}{343}) + \frac{130}{343} < \frac{130}{343}$. Let

$$\Delta_5 := \left\{(x,y) \,\middle|\, y+x \le 1, \quad \frac{5}{7} \le y-x \le \frac{1574}{2177}, \quad \frac{411}{196}y + \frac{15}{98}x - \frac{624}{343} \ge 0\right\}.$$

It is the triangle with vertices P_1,

$$P_{14} : \left(\frac{603}{4354}, \frac{3751}{4354}\right), \quad \text{and} \quad P_{15} : \left(\frac{43114}{320019}, \frac{274492}{320019}\right).$$

P_{14} and P_{15} are located in Δ_3, and we see $\Delta_5 \subset \Delta_3$. We have proved

$$v(x,y) \le v_5(x,y) < \frac{130}{343} \quad \text{for} \quad (x,y) \in \Delta_3 \setminus \Delta_5. \tag{3.10}$$

3.6. $\Delta_5 \setminus \Delta_6$ part

We consider Δ_5. We can show that $0 < 15x-2$, $0 < 64y-x-54 < 1$, $0 < 63x-8 \le 1$, and $0 \le 63y-54 < 1$ hold at P_1, P_{14}, and P_{15}, and hence on Δ_5. We have $\widetilde{v}_5(x,y) = -11z^2 + \frac{253}{16}z - \frac{85}{16} - \frac{1}{4}(7-8y-x)^+$. By (2.8) we have

$$\begin{aligned}&\widetilde{V}(\langle 64x\rangle, \langle 64y\rangle, \langle x\rangle, \langle y\rangle) = V(64y-x-54, z) - V(z, 63x-8)\\&\quad = z - z(64y-x-54) - z(1-63x+8) + (y-64x+8)^+ = -64z^2 + 46z + (y-64x+8)^+.\end{aligned}$$

Therefore we have

$$\widetilde{v}_6(x,y) = -13z^2 + \frac{69}{4}z - \frac{85}{16} - \frac{1}{4}(7-8y-x)^+ + \frac{1}{32}(-64x+y+8)^+,$$

and by $-(7-8y-x)^+ \le 0$ we have $v_6(x,y) \le -\frac{731}{56}z^2 + \frac{3881}{224}z - \frac{1195}{224} + \frac{1}{32}(-64x+y+8)^+$. We show that $(x,y) \in \Delta_5$ and $y - x \ge \frac{10385}{14511}$ imply $v_6(x,y) \le \frac{130}{343}$. We divide Δ_5 into two parts.

In $\{(x,y) \in \Delta_5 \mid -64x+y+8 < 0\}$, by (2.26), we have $v_6(x,y) \le -\frac{2073}{1568}z + \frac{14545}{10976}$, and hence we have $v_6(x,y) < \frac{130}{343}$ if $y - x > \frac{10385}{14511}$.

In $\Delta_6^* := \{(x,y) \in \Delta_5 \mid -64x+y+8 \ge 0\}$, by (2.26), we have $v_6(x,y) \le -\frac{253}{196}y - \frac{1063}{1568}x + \frac{17289}{10976} =: v_6^*(x,y)$. By $P_1, P_{14} \notin \Delta_6^*$, and $P_{15} \in \Delta_6^*$, we find that Δ_6^* is the triangle with vertices P_{15},

$$P_{16} : \left(\frac{2110}{15239}, \frac{13128}{15239}\right), \quad \text{and} \quad P_{17} : \left(\frac{4252}{30723}, \frac{26344}{30723}\right).$$

We can verify that $v_6^*(x,y) < \frac{130}{343}$ holds on these points, and hence holds on Δ_6^*. Set

$$\Delta_6 := \left\{(x,y) \,\middle|\, y+x \le 1, \quad \frac{5}{7} \le y-x \le \frac{10385}{14511}, \quad \frac{411}{196}y + \frac{15}{98}x - \frac{624}{343} \ge 0\right\}.$$

By $\frac{10385}{14511} < \frac{1574}{2177}$, we see $\Delta_6 \subset \Delta_5$ and, we have seen that

$$v(x,y) \le v_6(x,y) < \frac{130}{343} \quad \text{for} \quad (x,y) \in \Delta_5 \setminus \Delta_6. \tag{3.11}$$

It is the triangle with vertices P_1,

$$P_{18} : \left(\frac{2063}{14511}, \frac{12448}{14511}\right), \quad \text{and} \quad P_{19} : \left(\frac{301991}{2133117}, \frac{1828586}{2133117}\right).$$

3.7. Δ_6 *part*

We consider Δ_6. We can show that $-64x+y+8<0$, $0<-129x+19<1$, $0<-128y-x+110<1$, $-129y+110<0$, $0<255x-36<1$, $0<256y-x-219<1$, $255y-219<0$, and $256x-y-36<0$ hold on P_1, P_{18}, and P_{19}, and hence hold on Δ_6. We have the estimate $\widetilde{v}_6(x,y)=-13z^2+\frac{69}{4}z-\frac{85}{16}-\frac{1}{4}(7-8y-x)^+\le -13z^2+\frac{69}{4}z-\frac{85}{16}$. By (2.8) we have

$$\begin{aligned}
&-\widetilde{V}(\langle -128x\rangle,\langle -128y\rangle,\langle x\rangle,\langle y\rangle)=-V(-128y-x+110,z)+V(-129x+19,z)\\
&=-(-128y-x+110)+(-128y-x+110)z+(-129x+19)(1-z)-(-128x-y+19)^+\\
&\le -128z^2+219z-91,\\
&\widetilde{V}(\langle 256x\rangle,\langle 256y\rangle,\langle x\rangle,\langle y\rangle)=V(256y-x-219,z)-V(255x-36,z)\\
&=(256y-x-219)-(256y-x-219)z-(255x-36)+(255x-36)z=-256z^2+439z-183.
\end{aligned}$$

We therefore have $\widetilde{v}_8(x,y)\le -17z^2+\frac{3085}{128}z-\frac{1045}{128}$, and by (2.26) we have

$$v_8(x,y)\le -\frac{3811}{224}z^2+\frac{5403}{224}z-\frac{915}{112}\le \frac{289}{1568}\left(\frac{5}{7}-z\right)+\frac{130}{343}<\frac{130}{343} \text{ for } (x,y)\in\Delta_6\setminus\{P_1\}. \quad (3.12)$$

By combining (3.4), (3.6), (3.7), (3.9), (3.10), (3.11), and (3.12), we have (3.2).

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