



# A metric discrepancy result for the Hardy-Littlewood-Pólya sequences

Fukuyama, Katusi  
Nakata, Keisuke

---

**(Citation)**

Monatshefte für Mathematik, 160(1):41-49

**(Issue Date)**

2010-04

**(Resource Type)**

journal article

**(Version)**

Accepted Manuscript

**(Rights)**

©Springer Verlag 2008. The final publication is available at Springer via <http://dx.doi.org/10.1007/s00605-008-0051-5>

**(URL)**

<https://hdl.handle.net/20.500.14094/90003844>



# A METRIC DISCREPANCY RESULT FOR THE HARDY-LITTLEWOOD-PÓLYA SEQUENCES

KATUSI FUKUYAMA (KOBE) AND KEISUKE NAKATA (KOBE)

## ABSTRACT

The exact law of the iterated logarithm for discrepancies of the Hardy-Littlewood-Pólya sequences is proved. The exact constant in the law of the iterated logarithm is explicitly computed in the case when the Hardy-Littlewood-Pólya sequence consists of odd numbers.

## 1. INTRODUCTION

In the theory of the uniform distribution, the following discrepancies  $D_N$  and  $D_N^*$  of sequence  $\{x_k\}$  of real numbers are frequently used:

$$D_N\{x_k\} = \sup_{0 \leq a' < a < 1} \left| \frac{1}{N} \sum_{k=1}^N f_{a',a}(x_k) \right|; \quad D_N^*\{x_k\} = \sup_{0 \leq a < 1} \left| \frac{1}{N} \sum_{k=1}^N f_{0,a}(x_k) \right|,$$

where  $f_{a',a}(x) = \mathbf{1}_{[a',a)}(\langle x \rangle) - (a - a')$ ,  $\mathbf{1}_{[a',a)}$  denotes the indicator function of  $[a', a)$  and  $\langle x \rangle$  denotes the fractional part  $x - [x]$  of the real number  $x$ .

Assuming that the sequence  $\{n_k\}$  grows exponentially, i.e.,  $n_{k+1}/n_k > q > 1$ , Philipp [9, 10] proved the following bounded law of the iterated logarithm and solved the Erdős-Gál conjecture:

$$\frac{1}{4\sqrt{2}} \leq \overline{\lim}_{N \rightarrow \infty} \frac{ND_N\{n_k x\}}{\sqrt{2N \log \log N}} \leq C < \infty, \quad \text{a.e.} \quad (1)$$

This result has been extended to the case of sub-exponential growth. The first case was the Hardy-Littlewood-Pólya sequences, which are defined as follows: Let us take a finite sequence  $q_1, \dots, q_\tau$  of coprime positive integers and let  $\{n_k\}$  be an arrangement in increasing order of

$$\{q_1^{i_1} \dots q_\tau^{i_\tau} \mid i_1, \dots, i_\tau \in \mathbf{N}_0\}$$

---

Keywords: discrepancy, Hardy-Littlewood-Pólya sequences, law of the iterated logarithm.

Subject class: Primary 11K38, 42A55, 60F15.

The first author was supported in part by the Grant-in-Aid for Scientific Research (B) 17340029 from JSPS.

where  $\mathbf{N}_0 = \{0, 1, 2, \dots\}$ . Although these sequences do not grow exponentially, Philipp [11] proved (1) and solved Baker's conjecture.

Recently, Berkes-Philipp-Tichy [1] proved (1) assuming three conditions (B), (C), and (G), which include the Hardy-Littlewood-Pólya sequences. We state these conditions in the next section.

Now let us focus on a case where a sharper result can be proved. For a binary sequence, Philipp [8] proved the law of the iterated logarithm:

$$\overline{\lim}_{N \rightarrow \infty} \frac{ND_N\{2^k x\}}{\sqrt{2N \log \log N}} = \sigma, \quad \text{a.e.}$$

This was the first result proving the exact law of the iterated logarithm for the discrepancy of the sequence of type  $\{n_k x\}$ , but unfortunately the value of  $\sigma$  was not calculated explicitly.

The first result proving the law of the iterated logarithm and determining the constant was given by Dhompongsa [3]. His result implies that

$$\overline{\lim}_{N \rightarrow \infty} \frac{ND_N\{n_k x\}}{\sqrt{2N \log \log N}} = \overline{\lim}_{N \rightarrow \infty} \frac{ND_N^*\{n_k x\}}{\sqrt{2N \log \log N}} = \frac{1}{2}, \quad \text{a.e.}$$

under the rather strong gap condition

$$\log(n_{k+1}/n_k)/\log \log k \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

Recently ([5]) the same conclusion was proved for all sequences  $\{n_k\}$  of positive real numbers satisfying

$$n_{k+1}/n_k \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

Besides these super-exponential cases, the exact law of the iterated logarithm

$$\overline{\lim}_{N \rightarrow \infty} \frac{ND_N\{\theta^k x\}}{\sqrt{2N \log \log N}} = \overline{\lim}_{N \rightarrow \infty} \frac{ND_N^*\{\theta^k x\}}{\sqrt{2N \log \log N}} = \sigma_\theta, \quad \text{a.e.} \quad (\theta > 1)$$

was proved in [4]. The value of  $\sigma_\theta$  depends on  $\theta$  in a delicate manner. When  $\theta^r \notin \mathbf{Q}$  for all  $r \in \mathbf{N}$ , we have  $\sigma_\theta = 1/2$ . When  $\theta = \sqrt[r]{p/q}$  with  $p, q \in \mathbf{N}$ ,  $r = \min\{k \in \mathbf{N} \mid \theta^k \in \mathbf{Q}\}$ , and  $\gcd(p, q) = 1$ , we have the estimate  $1/2 \leq \sigma_\theta \leq \sqrt{(pq+1)/(pq-1)}/2$ , and

$$\sigma_\theta = \begin{cases} \sqrt{(pq+1)/(pq-1)}/2 & \text{if } p \text{ and } q \text{ are odd,} \\ \sqrt{(p+1)p(p-2)/(p-1)^3}/2 & \text{if } p = 4, 6, 8, \dots \text{ and } q = 1, \\ \sqrt{42}/9 & \text{if } p = 2 \text{ and } q = 1, \\ \sqrt{22}/9 & \text{if } p = 5 \text{ and } q = 2. \end{cases}$$

Explicit values of  $\sigma_\theta$  for other  $\theta$  are unknown.

Since the central limit theorem and the law of the iterated logarithm of  $\sum f(n_k x)$  were proved in [2, 6, 7] for the Hardy-Littlewood-Pólya sequences that grow sub-exponentially, it is very natural to ask whether we have the exact law of the iterated logarithm for discrepancies. The answer is given in this theorem:

**Theorem 1.** *The discrepancies of a Hardy-Littlewood-Pólya sequence  $\{n_k\} = \{q_1^{i_1} \dots q_\tau^{i_\tau} \mid i_1, \dots, i_\tau \in \mathbf{N}_0\}$  obey the law of the iterated logarithm:*

$$\overline{\lim}_{N \rightarrow \infty} \frac{ND_N\{n_k x\}}{\sqrt{2N \log \log N}} = \overline{\lim}_{N \rightarrow \infty} \frac{ND_N^*\{n_k x\}}{\sqrt{2N \log \log N}} = \sup_{0 \leq a < 1} \sigma_{q_1, \dots, q_\tau; 0, a}, \quad a.e.,$$

where

$$\sigma_{q_1, \dots, q_\tau; 0, a} = \left( \sum_{i, j: \gcd(n_i, n_j)=1} \frac{\langle n_i a \rangle \wedge \langle n_j a \rangle - \langle n_i a \rangle \langle n_j a \rangle}{n_i n_j} \right)^{1/2}. \quad (2)$$

If  $q_1, \dots, q_\tau$  are odd numbers, then

$$\sup_{0 \leq a < 1} \sigma_{q_1, \dots, q_\tau; 0, a} = \frac{1}{2} \left( \prod_{i=1}^{\tau} \frac{q_i + 1}{q_i - 1} \right)^{1/2}.$$

If  $\tau \geq 2$  and  $q_1$  is even, we have the estimate

$$\frac{1}{2} \left( \prod_{i=2}^{\tau} \frac{q_i + 1}{q_i - 1} \right)^{1/2} \leq \sup_{0 \leq a < 1} \sigma_{q_1, \dots, q_\tau; 0, a} \leq \frac{1}{2} \left( \prod_{i=1}^{\tau} \frac{q_i + 1}{q_i - 1} \right)^{1/2}.$$

## 2. BERKES-PHILIPP-TICHY THEORY

We apply the following results by Berkes-Philipp-Tichy. Following [1], we will say that a sequence  $\{n_k\}$  satisfies

**Condition (B)**, if there exist constants  $0 < \alpha < 1/2$  and  $C > 0$  such that for each positive integer  $b$  and for each  $R \geq 1$  the number of solutions  $(h, \nu)$  of the Diophantine equation  $hn_\nu = b$  with  $h \in \mathbf{N}$ ,  $1 \leq h \leq R$ , does not exceed  $CR^\alpha$ .

**Condition (C)**, if there exist constants  $0 < \beta < 1/2$  and  $C_0 > 0$  such that for each  $N \geq 1$  and for fixed integers  $h_i$  with  $0 < |h_i| \leq N^3$ ,  $i = 1, 2, 3, 4$ , the number of solutions  $(\nu_1, \nu_2, \nu_3, \nu_4)$  of the Diophantine equation  $h_1 n_{\nu_1} + h_2 n_{\nu_2} + h_3 n_{\nu_3} + h_4 n_{\nu_4} = 0$  subject to  $1 \leq \nu_i \leq N$ ,  $i = 1, 2, 3, 4$ , does not exceed  $C_0 N^{1+\beta}$ , provided that no proper subsums of  $h_1 n_{\nu_1} + h_2 n_{\nu_2} + h_3 n_{\nu_3} + h_4 n_{\nu_4}$  vanish.

**Condition (G)**, if there exists a constant  $0 < \eta < 1$  such that for all  $k \geq k_0(\eta)$  we have  $n_{[k+k^{1-\eta}]} / n_k \geq k$ .

Now we are in a position to state their results. Denote

$$\tilde{F}_N(a) = \tilde{F}_N(a; x) = \frac{1}{N} \sum_{k=1}^N f_{0,a}(n_k x).$$

**Theorem A** (Berkes-Philipp-Tichy [1]). *Let  $\{n_k\}$  be an increasing sequence of positive integers satisfying conditions (B), (C), and (G). Then there exist constants  $\delta > 0$  and  $D$ , depending only on  $\alpha, \beta, \eta, C$ , and  $C_0$ , with the following property: For almost all  $x \in [0, 1)$ , there is an  $N_0 = N_0(x, \alpha, \beta, \eta, C, C_0)$  such that for all  $N \geq N_0$  and all  $a'$  and  $a$  with  $0 \leq a' < a \leq 1$ ,*

$$\max_{n \leq N} n |\tilde{F}_N(a; x) - \tilde{F}_N(a'; x)| \leq D(a - a')^\delta \sqrt{2N \log \log N} + \sqrt{N}.$$

**Theorem B** (Berkes-Philipp-Tichy [2]). *Let  $\{n_k\}$  be an increasing sequence of positive integers satisfying conditions (C) and (G). Assume that  $f(x+1) = f(x)$ ,  $\int_0^1 f(x) dx = 0$ ,  $f \in \mathbf{BV}(0, 1)$  and*

$$\int_0^1 \left( \sum_{k=1}^N f(n_k x) \right)^2 dx \sim \sigma^2 N \quad (3)$$

for some  $\sigma > 0$ . Then

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{\sqrt{2N \log \log N}} \sum_{k=1}^N f(n_k x) = \sigma, \quad a.e.$$

Since the Hardy-Littlewood-Pólya sequence satisfies the above conditions ([1, 2]), we can apply these theorems.

### 3. LIL FOR DISCREPANCIES

We now prove the law of the iterated logarithm in our theorem. In the proof, we omit the phrase 'a.e.'. We use the following result.

**Lemma 2** (Lemma 1 of [4]). *For coprime positive integers  $\mu$  and  $\nu$ ,*

$$r^{a', a} = \int_0^1 f_{a', a}(\mu t) f_{a', a}(\nu t) dt = \tilde{V}(\langle \mu a' \rangle, \langle \mu a \rangle, \langle \nu a' \rangle, \langle \nu a \rangle) / \mu \nu, \quad (4)$$

where  $\tilde{V}(x, y, \xi, \eta) = V(x, \xi) + V(y, \eta) - V(x, \eta) - V(y, \xi)$  and  $V(x, \xi) = x \wedge \xi - x\xi$ .

We first verify that (3) holds for  $f = f_{a',a}$ . Although it is already proved in [6], we present here a simple proof for convenience. Denote  $H = \{n_k\}$  and  $P = \{(n_\nu, n_\mu) \in H^2 \mid \gcd(n_\nu, n_\mu) = 1\}$ . For  $j, k \in \mathbf{N}$ , put  $m_{j,k}^0 = \gcd(n_j, n_k)$ ,  $m_{j,k}^1 = n_j/m_{j,k}^0$ , and  $m_{j,k}^2 = n_k/m_{j,k}^0$ . We have  $\int_0^1 f(n_j x) f(n_k x) dx = \int_0^1 f(m_{j,k}^1 x) f(m_{j,k}^2 x) dx$ . For  $\nu, \mu \leq K$  with  $\gcd(n_\nu, n_\mu) = 1$ , set

$$A_{K,\nu,\mu} =^\# \{(j, k) \in \mathbf{N}^2 \mid j \leq K, k \leq K, (m_{j,k}^1, m_{j,k}^2) = (n_\nu, n_\mu)\}.$$

Since  $j \leq K$  is equivalent to  $n_j \leq n_K$  or  $m_{j,k}^0 m_{j,k}^1 \leq n_K$ , and  $k \leq K$  is equivalent to  $m_{j,k}^0 m_{j,k}^2 \leq n_K$ , we see that  $A_{K,\nu,\mu}$  equals to

$$^\# \{(n_j, n_k) \in H^2 \mid m_{j,k}^0 m_{j,k}^1 \leq n_K, m_{j,k}^0 m_{j,k}^2 \leq n_K, (m_{j,k}^1, m_{j,k}^2) = (n_\nu, n_\mu)\}.$$

Since the mapping  $H^2 \ni (n_j, n_k) \mapsto (m_{j,k}^0, m_{j,k}^1, m_{j,k}^2) \in H \times P$  is bijective, we have

$$\begin{aligned} A_{K,\nu,\mu} &=^\# \{(a, b, c) \in H \times P \mid ab \leq n_K, ac \leq n_K, (b, c) = (n_\nu, n_\mu)\} \\ &=^\# \{a \in H \mid a \leq n_K/n_\nu, n_K/n_\mu\}. \end{aligned}$$

Elementary calculus shows that  $^\# \{(i_1, \dots, i_\tau) \in \mathbf{N}_0^\tau \mid i_1 \log q_1 + \dots + i_\tau \log q_\tau \leq \log L\} \sim C_*(\log L)^\tau$  where  $C_* = \tau! \log q_1 \dots \log q_\tau$ , and hence  $^\# \{k \mid n_k \leq L\} \sim C_*(\log L)^\tau$ . Hence we have

$$1 \geq \frac{A_{K,\nu,\mu}}{K} = \frac{^\# \{k \mid n_k \leq n_K/(n_\mu \vee n_\nu)\}}{^\# \{k \mid n_k \leq n_K\}} \sim \frac{(\log(n_K/(n_\mu \vee n_\nu)))^\tau}{(\log n_K)^\tau} \sim 1,$$

as  $K \rightarrow \infty$ . By this we have

$$\begin{aligned} \frac{1}{K} \int_0^1 \left( \sum_{k=1}^K f_{a',a}(n_k x) \right)^2 dx &= \frac{1}{K} \sum_{j,k: n_j, n_k \leq n_K} \int_0^1 f_{a',a}(n_j x) f_{a',a}(n_k x) dx \\ &= \sum_{\nu, \mu \leq K: \gcd(n_\nu, n_\mu)=1} \frac{A_{K,\nu,\mu}}{K} \int_0^1 f_{a',a}(n_\nu x) f_{a',a}(n_\mu x) dx \\ &\rightarrow \sum_{\nu, \mu: \gcd(n_\nu, n_\mu)=1} \int_0^1 f_{a',a}(n_\nu x) f_{a',a}(n_\mu x) dx, \quad (K \rightarrow \infty). \\ &= \sum_{\nu, \mu: \gcd(n_\nu, n_\mu)=1} \frac{\tilde{V}(\langle n_\nu a' \rangle, \langle n_\nu a \rangle, \langle n_\mu a' \rangle, \langle n_\mu a \rangle)}{n_\nu n_\mu} =: \sigma_{q_1, \dots, q_\tau; a', a}^2 \end{aligned} \tag{5}$$

In the last limiting procedure, we use the fact that the last series is absolutely summable, which follows from (4). As is proved in [6], the value  $\sigma_{q_1, \dots, q_\tau; a', a}^2$  of the last summation is positive if  $a'$  or  $a$  is irrational.

Let us prove that  $\sigma_{q_1, \dots, q_\tau; a', a}$  is continuous in  $a'$  and  $a$ . Since  $a \mapsto \langle \mu a \rangle$  is a continuous mapping from  $\mathbf{R}$  to the 1-dimensional torus  $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ , and since  $V(x, y)$  is a continuous function of  $\mathbf{T} \times \mathbf{T}$ , we see that  $r^{a', a}$  is continuous in  $(a', a)$ . By  $|\tilde{V}(x, y, \xi, \eta)| \leq 1$ , we see that (5) is uniformly convergent and hence  $\sigma_{q_1, \dots, q_\tau; a', a}$  is uniformly continuous in  $(a', a)$ .

Let us take a positive integer  $L$  arbitrarily and take a division  $0 = t_0 < t_1 < \dots < t_L < t_{L+1} = 1$  with  $t_{i+1} - t_i < 1/L$  ( $i = 0, 1, \dots, L$ ) and  $t_i \notin \mathbf{Q}$  ( $i = 1, \dots, L$ ). If  $t_i$  or  $t_j$  is irrational, we have  $\sigma_{q_1, \dots, q_\tau; t_i, t_j} > 0$ , and hence by Theorem B,

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{\sqrt{2N \log \log N}} \left| \sum_{k=1}^N f_{t_i, t_j}(n_k x) \right| = \sigma_{q_1, \dots, q_\tau; t_i, t_j}.$$

Therefore, we have

$$\overline{\lim}_{N \rightarrow \infty} \max_{0 \leq i < j \leq L} \frac{1}{\sqrt{2N \log \log N}} \left| \sum_{k=1}^N f_{t_i, t_j}(n_k x) \right| = \max_{0 \leq i < j \leq L} \sigma_{q_1, \dots, q_\tau; t_i, t_j}, \quad (6)$$

$$\overline{\lim}_{N \rightarrow \infty} \max_{i=1}^L \frac{1}{\sqrt{2N \log \log N}} \left| \sum_{k=1}^N f_{0, t_i}(n_k x) \right| = \max_{i=1}^L \sigma_{q_1, \dots, q_\tau; 0, t_i}. \quad (7)$$

On the other hand, by applying Theorem A, we have

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{\sqrt{2N \log \log N}} \sup_{|a-a'| < \varepsilon} N |\tilde{F}_N(a) - \tilde{F}_N(a')| \leq D\varepsilon^\delta. \quad (8)$$

For given  $0 \leq a' < a < 1$  let us take integers  $0 \leq i \leq j \leq L$  satisfying  $a' \in [t_i, t_{i+1})$  and  $a \in [t_j, t_{j+1})$ . Then we have

$$\tilde{F}_N(a) - \tilde{F}_N(a') = \tilde{F}_N(t_j) - \tilde{F}_N(t_i) + \tilde{F}_N(a) - \tilde{F}_N(t_j) + \tilde{F}_N(t_i) - \tilde{F}_N(a')$$

and thereby

$$\begin{aligned} \max_{0 \leq i < j \leq L} |\tilde{F}_N(t_j) - \tilde{F}_N(t_i)| &\leq \sup_{0 \leq a' < a < 1} |\tilde{F}_N(a) - \tilde{F}_N(a')| \\ &\leq \max_{0 \leq i < j \leq L} |\tilde{F}_N(t_j) - \tilde{F}_N(t_i)| + 2 \sup_{|a-a'| < 1/L} |\tilde{F}_N(a) - \tilde{F}_N(a')| \end{aligned} \quad (9)$$

By (6), (8) and (9), we have

$$\max_{0 \leq i < j \leq L} \sigma_{q_1, \dots, q_\tau; t_i, t_j} \leq \overline{\lim}_{N \rightarrow \infty} \frac{ND_N\{n_k x\}}{\sqrt{2N \log \log N}} \leq \max_{0 \leq i < j \leq L} \sigma_{q_1, \dots, q_\tau; t_i, t_j} + \frac{2D}{L^\delta}.$$

Letting  $L \rightarrow \infty$ , by continuity of  $\sigma_{q_1, \dots, q_\tau; a', a}$ , we have

$$\overline{\lim}_{N \rightarrow \infty} \frac{ND_N\{n_k x\}}{\sqrt{2N \log \log N}} = \sup_{0 \leq a' < a < 1} \sigma_{q_1, \dots, q_\tau; a', a}.$$

In the same way, we have

$$\max_{0 \leq j \leq L} |\tilde{F}_N(t_j)| \leq \sup_{0 \leq a < 1} |\tilde{F}_N(a)| \leq \max_{0 \leq j \leq L} |\tilde{F}_N(t_j)| + \sup_{|a-a'| < 1/L} |\tilde{F}_N(a) - \tilde{F}_N(a')|,$$

and hence

$$\overline{\lim}_{N \rightarrow \infty} \frac{ND_N^*\{n_k x\}}{\sqrt{2N \log \log N}} = \sup_{0 \leq a < 1} \sigma_{q_1, \dots, q_\tau; 0, a}.$$

Note that the proof of Lemma 2 of [4] proves the following:

**Lemma 3.** *We have  $r^{a', a} \leq r^{0, a-a'}$  for  $0 \leq a' < a < 1$ .*

By this, we have  $\sigma_{q_1, \dots, q_\tau; a', a} \leq \sigma_{q_1, \dots, q_\tau; 0, a-a'}$  and thereby we have  $\sup_{0 \leq a' < a < 1} \sigma_{q_1, \dots, q_\tau; a', a} \leq \sup_{0 \leq a < 1} \sigma_{q_1, \dots, q_\tau; 0, a}$ . Since the inequality of opposite direction is clear, we see

$$\sup_{0 \leq a' < a < 1} \sigma_{q_1, \dots, q_\tau; a', a} = \sup_{0 \leq a < 1} \sigma_{q_1, \dots, q_\tau; 0, a}.$$

This proves the first assertion of our Theorem.

#### 4. EVALUATION OF CONSTANTS

We evaluate

$$\Sigma_{q_1, \dots, q_\tau}^2 = \sup_{0 \leq a < 1} \sigma_{q_1, \dots, q_\tau; 0, a}^2 = \sup_{0 \leq a < 1} \sum_{k, l: (n_k, n_l)=1} \frac{V(\langle n_k a \rangle, \langle n_l a \rangle)}{n_k n_l}.$$

Because of  $0 \leq V(x, \xi) \leq V(1/2, 1/2) = 1/4$ , we have

$$\Sigma_{q_1, \dots, q_\tau}^2 \leq \frac{1}{4} \sum_{k, l: (n_k, n_l)=1} \frac{1}{n_k n_l}.$$

If  $q_1, \dots, q_\tau$  are odd, we have  $\langle n_k/2 \rangle = 1/2$  for all  $k$ , and hence the right hand side of the above inequality equals to  $\sigma_{q_1, \dots, q_\tau; 0, 1/2}^2$ . In this case we have

$$\Sigma_{q_1, \dots, q_\tau}^2 = \sigma_{q_1, \dots, q_\tau; 0, 1/2}^2 = \frac{1}{4} \sum_{k, l: (n_k, n_l)=1} \frac{1}{n_k n_l}.$$

By the series expression (2) of  $\sigma_{q_1, \dots, q_\tau; 0, a}$ , we easily have  $\sigma_{q_1, \dots, q_\tau; 0, a} \geq \sigma_{q_2, \dots, q_\tau; 0, a}$ , and hence  $\Sigma_{q_1, \dots, q_\tau} \geq \Sigma_{q_2, \dots, q_\tau}$ .

Put  $\mathcal{E} = \{(n_k, n_l) \mid \gcd(n_k, n_l) = 1\}$  and  $\varphi(i) = i \vee 0$ . For  $\mathbf{k} = (k_1, \dots, k_\tau) \in \mathbf{N}_0^\tau$ , denote  $\mathbf{q}^{\mathbf{k}} = q_1^{k_1} \dots q_\tau^{k_\tau}$ ,  $\varphi(\mathbf{k}) = (\varphi(k_1), \dots, \varphi(k_\tau))$ , and  $|\mathbf{k}| = (|k_1|, \dots, |k_\tau|)$ . We have  $\varphi(\mathbf{k}) + \varphi(-\mathbf{k}) = |\mathbf{k}|$  and hence  $\mathbf{q}^{\varphi(\mathbf{k})} \mathbf{q}^{\varphi(-\mathbf{k})} = \mathbf{q}^{\varphi(\mathbf{k}) + \varphi(-\mathbf{k})} = \mathbf{q}^{|\mathbf{k}|}$ .

Let  $\mathcal{D} = \{(\mathbf{k}, \mathbf{l}) \in \mathbf{N}_0^\tau \times \mathbf{N}_0^\tau \mid \mathbf{k} \cdot \mathbf{l} = 0\}$ . Then the mappings

$$\Lambda : \mathcal{E} \ni (\mathbf{q}^{\mathbf{k}}, \mathbf{q}^{\mathbf{l}}) \mapsto (\mathbf{k}, \mathbf{l}) \in \mathcal{D} \quad \text{and} \quad \Psi : \mathcal{D} \ni (\mathbf{k}, \mathbf{l}) \mapsto \mathbf{k} - \mathbf{l} \in \mathbf{Z}^\tau$$

are bijective, and hence we have the change of variable formula

$$\sum_{(n_k, n_l) \in \mathcal{E}} g(n_k, n_l) = \sum_{\mathbf{m} \in \mathbf{Z}^\tau} g((\Psi \circ \Lambda)^{-1}(\mathbf{m}))$$

for any non-negative function  $g$ . We have  $\Psi^{-1}(\mathbf{m}) = (\varphi(\mathbf{m}), \varphi(-\mathbf{m}))$  and hence  $(\Psi \circ \Lambda)^{-1}(\mathbf{m}) = (\mathbf{q}^{\varphi(\mathbf{m})}, \mathbf{q}^{\varphi(-\mathbf{m})})$ . By these, we have

$$\sum_{(n_k, n_l) \in \mathcal{E}} \frac{1}{n_k n_l} = \sum_{\mathbf{m} \in \mathbf{Z}^\tau} \frac{1}{\mathbf{q}^{\varphi(\mathbf{m})} \mathbf{q}^{\varphi(-\mathbf{m})}} = \sum_{\mathbf{m} \in \mathbf{Z}^\tau} \frac{1}{\mathbf{q}^{|\mathbf{m}|}} = \prod_{i=1}^{\tau} \frac{q_i + 1}{q_i - 1}.$$

#### ACKNOWLEDGEMENT

The authors thank the referee for his valuable comments.

#### REFERENCES

- [1] Berkes I, Philipp W, Tichy R (2006) Empirical processes in probabilistic number theory: The LIL for the discrepancy of  $(n_k \omega) \bmod 1$ . *Illinois Jour Math* 50: 107-145
- [2] Berkes I, Philipp W, Tichy R (2008) Metric discrepancy results for sequence  $\{n_k x\}$  and diophantine equations, *Diophantine Approximations*. In Tichy R, Schlickewei H, Schmidt K (eds) *Festschrift for Wolfgang Schmidt*, *Development in Mathematics* 17, Springer.
- [3] Dhompongsa S (1987) Almost sure invariance principles for the empirical process of lacunary sequences. *Acta Math Hungar* 49: 83-102
- [4] Fukuyama K (2008) The law of the iterated logarithm for discrepancies of  $\{\theta^n x\}$ . *Acta Math Hungar* 118: 155-170.
- [5] Fukuyama K (to appear) The law of the iterated logarithm for the discrepancies of a permutation of  $\{n_k x\}$ . *Acta Math Hungar*
- [6] Fukuyama K, Petit B (2001) Le théorème limite central pour les suites de R.C. Baker. *Ergod Th Dyn Sys* 21: 479-492
- [7] Fukuyama K, Petit B (2003) An asymptotic property of gap series II. *Acta Math Hungar* 98: 245-258
- [8] Philipp W (1971) Mixing sequences of random variables and probabilistic number theory. *Memoirs Amer Math Soc* 111
- [9] Philipp W (1975) Limit theorems for lacunary series and uniform distribution mod 1. *Acta Arith* 26: 241-251
- [10] Philipp W (1977) A functional law of the iterated logarithm for empirical distribution functions of weakly dependent random variables. *Ann Probab* 5: 319-350
- [11] Philipp W (1994) Empirical distribution functions and strong approximation theorems for dependent random variables. A problem of Baker in probabilistic number theory. *Trans Amer Math Soc* 345: 705-727

DEPARTMENT OF MATHEMATICS, KOBE UNIVERSITY, ROKKO, KOBE, 657-8501, JAPAN, FUKUYAMA@MATH.KOBE-U.AC.JP