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A LAW OF THE ITERATED LOGARITHM FOR DISCREPANCIES: NON-CONSTANT LIMSUP

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Abstract

We prove the existence of a sequence $\{n_k x\}$ whose discrepancies obey a bounded law of the iterated logarithm with a non-constant limsup.

1. INTRODUCTION

Let us define the discrepancies D_N and D_N^* of a sequence $\{x_k\}$ of real numbers by

$$D_N\{x_k\} = \sup_{0 \le a' < a < 1} \left| \frac{1}{N} \sum_{k=1}^N \widetilde{\mathbf{1}}_{a',a}(\langle x_k \rangle) \right|; \ D_N^*\{x_k\} = \sup_{0 \le a < 1} \left| \frac{1}{N} \sum_{k=1}^N \widetilde{\mathbf{1}}_{0,a}(\langle x_k \rangle) \right|.$$

where $\widetilde{\mathbf{1}}_{a',a}(x) = \mathbf{1}_{[a',a)}(x) - (a-a')$, $\mathbf{1}_{[a',a)}$ denotes the indicator function of [a', a) and $\langle x \rangle$ denotes the fractional part x - [x] of x.

For $\{n_k\}$ with exponential growth $n_{k+1}/n_k > q > 1$, Philipp [13, 14] proved the following bounded law of the iterated logarithm

$$\frac{1}{4\sqrt{2}} < \lim_{N \to \infty} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} \le C < \infty, \quad \text{a.e.}$$
(1)

The result was extended to the case of sub-exponential growth by Berkes, Philipp, and Tichy [15, 3] assuming some extra conditions. The limsup in (1) is explicitly calculated only in a few cases: the case when $n_{k+1}/n_k \to \infty$ [6, 4, 8]; the case when $n_k = \theta^k$ for $\theta > 1$ [12, 7]; the case when $\{n_k\}$ is a Hardy-Littlewood-Pólya sequence [9]. As is pointed out by Aistleitner and Berkes [1, 2], it is not known if the limsup in (1) is a constant almost everywhere.

In this paper we show the existence of a sequence $\{n_k\}$ of linear growth which obeys a bounded law of the iterated logarithm

$$0 < C_1 \le \lim_{N \to \infty} \frac{N D_N \{n_k x\}}{\sqrt{2N \log \log N}} \le C_2 < \infty, \quad \text{a.e.},$$
(2)

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and the limsup in (2) is not a constant a.e.

For a given strictly increasing sequence $\{n_k\}$ of positive integers, denote by $\{n_k^\circ\}$ the arrangement in increasing order of $\mathbf{N} \setminus \{n_k\}$.

Theorem 1. There exists a strictly increasing sequence $\{n_k\}$ of positive integers satisfying $n_{k+1} - n_k \leq 5$ and $n_{k+1}^{\circ} - n_k^{\circ} \leq 5$ such that

$$\frac{\overline{\lim}}{N\to\infty} \frac{ND_N\{n_kx\}}{\sqrt{2N\log\log N}} = \overline{\lim}_{N\to\infty} \frac{ND_N^*\{n_kx\}}{\sqrt{2N\log\log N}} = \sigma(x) \quad a.e. \quad x,$$

$$\frac{\overline{\lim}}{N\to\infty} \frac{ND_N\{n_k^\circ x\}}{\sqrt{2N\log\log N}} = \overline{\lim}_{N\to\infty} \frac{ND_N^*\{n_k^\circ x\}}{\sqrt{2N\log\log N}} = \sigma(x) \quad a.e. \quad x,$$

where

$$\sigma^{2}(x) = (4x\mathbf{1}_{[0,1/4)}(x) + \mathbf{1}_{[1/4,3/4)}(x) + 4(1-x)\mathbf{1}_{[3/4,1)}(x))/9 + 1/24.$$
(3)

We use a method of random series which originated with Salem-Zygmund [16] and Bobkov-Götze [5].

2. Proof

Let $J = \{-1, 1\}^3 \setminus \{(1, 1, 1), (-1, -1, -1)\}$ and $\{\xi_n\}_{n=1}^\infty$ be a *J*-valued i.i.d. such that ξ_n is uniformly distributed over *J*. We define the sequence $\{X_n\}$ of random variables by $(X_{3n-2}, X_{3n-1}, X_{3n}) = \xi_n$ $(n \in$ **N**). Clearly $\{X_{3n-2}\}_{n=1}^\infty$, $\{X_{3n-1}\}_{n=1}^\infty$, and $\{X_{3n}\}_{n=1}^\infty$ are fair $\{-1, 1\}$ valued i.i.d.

Denote $\mathbf{1}_{[a',a)}(\langle kx \rangle)$ and $\widetilde{\mathbf{1}}_{[a',a)}(\langle kx \rangle)$ simply by $\mathbf{1}_{[a',a)}\langle kx \rangle$ and $\widetilde{\mathbf{1}}_{[a',a)}\langle kx \rangle$. Put $i_k = \widetilde{\mathbf{1}}_{[a',a)}\langle kx \rangle$, $Y_k = i_{3k-2}X_{3k-2} + i_{3k-1}X_{3k-1} + i_{3k}X_{3k}$, $r(x,a) = \int_0^1 \widetilde{\mathbf{1}}_{[0,a)}\langle y \rangle \widetilde{\mathbf{1}}_{[0,a)}\langle y - x \rangle \, dy$, and $\varphi(x) = \sqrt{2x \log \log x}$.

Lemma 2. For $x \notin \mathbf{Q}$, we have

$$\frac{1}{3N} \sum_{k=1}^{N} EY_k^2 \to \sigma^2(x, a - a'),$$
(4)

where $\sigma^2(x, a) = r(0, a) - 4r(x, a)/9 - 2r(2x, a)/9$. The function $\sigma^2(x, a)$ is continuous in (x, a) and satisfies

$$\sigma^2(x,a) = \sigma^2(1-x,a). \tag{5}$$

Proof: By $EX_n = 0$, $EX_n^2 = 1$, and $EX_{3n-2}X_{3n-1} = EX_{3n-2}X_{3n} = EX_{3n-1}X_{3n} = -1/3$, we have

$$EY_k^2 = i_{3k-2}^2 + i_{3k-1}^2 + i_{3k}^2 - \frac{2}{3}i_{3k-2}i_{3k-1} - \frac{2}{3}i_{3k-1}i_{3k} - \frac{2}{3}i_{3k-2}i_{3k}.$$

Since the sequence $\{3kx\}$ is uniformly distributed mod 1, we have

$$\frac{1}{N}\sum_{k=1}^{N}i_{3k-2}i_{3k} = \frac{1}{N}\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[a',a)}\langle(3k-2)x\rangle\widetilde{\mathbf{1}}_{[a',a)}\langle3kx\rangle$$
$$\rightarrow \int_{0}^{1}\widetilde{\mathbf{1}}_{[a',a)}\langle y-2x\rangle\widetilde{\mathbf{1}}_{[a',a)}\langle y\rangle\,dy = \int_{0}^{1}\widetilde{\mathbf{1}}_{[0,a-a')}\langle y-2x\rangle\widetilde{\mathbf{1}}_{[0,a-a')}\langle y\rangle\,dy.$$

In the same way, we have

$$\frac{1}{N} \sum_{k=1}^{N} i_{3k-j} i_{3k-j-1} \to r(x, a - a') \quad \text{and} \quad \frac{1}{N} \sum_{k=1}^{N} i_{3k-j}^2 \to r(0, a - a')$$

for j = 0, 1, 2. Therefore we have (4).

Continuity of $\sigma^2(x, a)$ is clear. By changing variable, we have

$$\int_0^1 \widetilde{\mathbf{1}}_{[0,a)} \langle y + x \rangle \widetilde{\mathbf{1}}_{[0,a)} \langle y \rangle \, dy = \int_0^1 \widetilde{\mathbf{1}}_{[0,a)} \langle y \rangle \widetilde{\mathbf{1}}_{[0,a)} \langle y - x \rangle \, dy,$$

or r(-x, a) = r(x, a). Hence $\sigma^2(x, a)$ is a even function of x with period one, and thereby we have (5).

Lemma 3. For $x \notin \mathbf{Q}$, we have

$$\sigma^{2}(x) = \max_{0 \le a < 1} \sigma^{2}(x, a) / 2 = \max_{0 \le a \le 1/2} \sigma^{2}(x, a) / 2.$$
(6)

 $\sigma^2(x,a) = 0$ implies a = 0.

Proof: Put $m(x) = x \land (1-x)$ and $M(x) = x \lor (1-x)$. $m(x) \le 1/2 \le M(x)$ is clear. For $a, x \in [0, 1)$, we have

$$r(x,a) = \begin{cases} -a^2 & 0 \le a \le m(x) \\ a - a^2 - m(x) & m(x) \le a \le M(x) \\ -(1-a)^2 & M(x) \le a < 1. \end{cases}$$
(7)

We prove this in the case $x \leq 1/2$ or m(x) = x. The other case can be proved by using (5). Note that $r(x, a) + a^2 = \int_0^1 \mathbf{1}_{[0,a)} \langle y - x \rangle \mathbf{1}_{[0,a)} \langle y \rangle dy$ equals to the measure of $A_x = [0, a) \cap ([0, a) + x \mod 1)$. We can easily see that $A_x = \emptyset$ if $0 \leq a \leq x$, $A_x = [x, a)$ if $x \leq a \leq 1 - x$, and $A_x = [0, a + x - 1) \cup [x, a)$ if $1 - x \leq a < 1$. (7) is verified by calculating the measure of A_x . Especially, we have $r(0, a) = a - a^2$.

By (5), it is sufficient to prove (6) for $x \in [0, 1/2]$. The second equality of (6) follows from r(x, a) = r(x, 1-a) which is clear from (7). By using (7), we calculate $9\sigma^2(x, a)$ for $a, x \in [0, 1/2]$.

If $0 \le x \le 1/4$, by noting $0 \le m(x) = x \le m(2x) = 2x \le 1/2$, we have $9\sigma^2(x,a) = 9a - 3a^2$ on [0,x), $9\sigma^2(x,a) = 5a - 3a^2 + 4x$ on [x, 2x), and $9\sigma^2(x,a) = 3a - 3a^2 + 8x$ on [2x, 1/2). If $1/4 \le x \le 1/3$, by noting $0 \le m(x) = x \le m(2x) = 1 - 2x \le 1/2$, we have $9\sigma^2(x, a) = 9a - 3a^2$ on [0, x), $9\sigma^2(x, a) = 5a - 3a^2 + 4x$ on [x, 1-2x), and $9\sigma^2(x, a) = 3a - 3a^2 + 2$ on [1 - 2x, 1/2).

If $1/3 \le x \le 1/2$, by noting $0 \le m(2x) = 1 - 2x \le m(x) = x \le 1/2$, we have $9\sigma^2(x, a) = 9a - 3a^2$ on [0, 1-2x), $9\sigma^2(x, a) = 7a - 3a^2 - 4x + 2$ on [1 - 2x, x), and $9\sigma^2(x, a) = 3a - 3a^2 + 2$ on [x, 1/2).

By differentiating by a, we have $(9a - 3a^2)' = 9 - 6a > 0$, $(5a - 3a^2 + 4x)' = 5 - 6a > 0$, $(7a - 3a^2 - 4x + 2)' = 7 - 6a > 0$, $(3a - 3a^2 + 4x)' = (3a - 3a^2 + 2)' = 3 - 6a > 0$ for $a \in [0, 1/2]$. Hence $9\sigma^2(x, a)$ strictly increases in $a \in [0, 1/2]$ and takes its maximum at a = 1/2. Hence (6) is proved. Clearly $\sigma^2(x, a) = 0$ only for a = 0.

Lemma 4. For $l \in \mathbf{N}$, $i < 2^{l}$, and $x \notin \mathbf{Q}$, we have

$$\lim_{N \to \infty} \frac{1}{\varphi(N)} \sup_{a < 2^{-l}} \left| \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[2^{-l}i, 2^{-l}i+a)} \langle kx \rangle X_k \right| \le 6 \cdot 2^{-l/2} \quad a.s.$$

Proof: Take an integer N and $\lambda > 0$ arbitrarily and denote

$$\widetilde{S}_N = \max_{n \le N} \sup_{a < 2^{-l}} \sum_{k=1}^n \mathbf{1}_{[2^{-l}i, 2^{-l}i+a)} \langle 3kx \rangle X_{3k}, \quad b_N = \sum_{k=1}^N \mathbf{1}_{[2^{-l}i, 2^{-l}(i+1))} \langle 3kx \rangle.$$

Define k_1, \ldots, k_M by $\{3k_1, 3k_2, \ldots, 3k_M\} = \{3k \leq N \mid \langle 3kx \rangle \in [2^{-l}i, 2^{-l}(i+1))\}$ and $\langle 3k_1x \rangle < \langle 3k_2x \rangle < \cdots < \langle 3k_Mx \rangle$. By putting $J_{m,n} = \{j \leq m \mid 3k_j \leq n\}$ and $S_{m,n} = \sum_{j \in J_{m,n}} X_{3k_j}$, we have $\widetilde{S}_N = \max_{n \leq N} \max_{m \leq M} S_{m,n}$. Defining random variables n_0 and m_0 by $n_0 = \min\{n; \max_{m \leq M} S_{m,n} > \lambda\}$ and $m_0 = \min\{m; S_{m,n_0} > \lambda\}$, we have a disjoint decomposition $\{\widetilde{S}_N > \lambda\} = \bigcup_{n \leq N} \bigcup_{m \leq M} C_{n,m}$ where $C_{n,m} = \{n_0 = n, m_0 = m\}$. Since $C_{n,m}$ belongs to the sigma field generated by X_{3k_j} $(3k_j \leq n)$, it is independent of $S_{m,N} - S_{m,n}$ which is a function of X_{3k_j} $(3k_j > n)$. Hence by noting $P(S_{m,N} - S_{m,n} \geq 0) \geq 1/2$, we have

$$P(C_{n,m}) \leq 2P(C_{n,m} \cap \{S_{m,n} > \lambda\})P(S_{m,N} - S_{m,n} \geq 0)$$

$$\leq 2P(C_{n,m} \cap \{S_{m,N} > \lambda\}) \leq 2P(C_{n,m} \cap \{\max_{m' \leq M} S_{m',N} > \lambda\}).$$

By summing for $n \leq N$ and $m \leq M$, we have

$$P(\widetilde{S}_{N} > \lambda) \leq 2P(\widetilde{S}_{N} > \lambda, \max_{m' \leq M} S_{m',N} > \lambda) = 2P(\max_{m' \leq M} S_{m',N} > \lambda)$$
$$\leq 4P(S_{M,N} > \lambda) = 4P\left(\sum_{k=1}^{N} \mathbf{1}_{[2^{-l}i,2^{-l}(i+1))}\langle 3kx \rangle X_{3k} > \lambda\right),$$

where the last inequality is by reflection principle. For fair ± 1 valued i.i.d. $\{Z_n\}$, we have $E \exp(t(Z_1 + \cdots + Z_n)) \leq e^{nt^2/2}$ (Lemma 4.2.1 of [16]), which implies $P(|Z_1 + \cdots + Z_n| \geq \mu) \leq 2 \exp(-\mu^2/2n)$. By applying this we have $P(|\widetilde{S}_N| > \lambda) \leq 8 \exp(-\lambda^2/2b_N)$.

Thanks to $b_N/N \to 2^{-l}$ we have $b_N < 2N2^{-l}$ for large N, and hence $P(|\tilde{S}_N| > \lambda) \leq 8 \exp(-\lambda^2/4N2^{-l})$. By putting $N = 2^j$ and $\lambda = \sqrt{5 \cdot 2^j 2^{-l} \log \log 2^j}$ we have

$$\sum_{j} P(|\widetilde{S}_{2^{j}}| > \sqrt{5 \cdot 2^{j} 2^{-l} \log \log 2^{j}}) \le \sum_{j} 8(j \log 2)^{-5/4} < \infty.$$

By Borel-Cantelli Lemma, we see $|\tilde{S}_{2^j}| \leq \sqrt{5 \cdot 2^j 2^{-l} \log \log 2^j}$ a.s. for large j, and hence

$$\lim_{N \to \infty} \frac{1}{\varphi(N)} \sup_{a < 2^{-l}} \left| \sum_{k=1}^{N} \mathbf{1}_{[2^{-l}i, 2^{-l}i+a)} \langle 3kx \rangle X_{3k} \right| \le \sqrt{10 \cdot 2^{-l}} \quad \text{a.s.}$$

It remains valid if we replace 3k by 3k - 1 or 3k - 2, and combining these, we have

$$\overline{\lim}_{N \to \infty} \frac{1}{\varphi(N)} \sup_{a < 2^{-l}} \left| \sum_{k=1}^{N} \mathbf{1}_{[2^{-l}i, 2^{-l}i+a)} \langle kx \rangle X_k \right| \le \sqrt{30 \cdot 2^{-l}} \quad \text{a.s}$$

By decomposing $\sum_{k=1}^{N} X_k$ into the sum of $\sum_{k \leq (N+2)/3} X_{3k-2}$, $\sum_{k \leq (N+1)/3} X_{3k-1}$, and $\sum_{k \leq N/3} X_{3k}$, and by applying the law of the iterated logarithm for each, we have

$$\overline{\lim}_{N \to \infty} \sup_{a < 2^{-l}} \frac{a}{\varphi(N)} \left| \sum_{k=1}^{N} X_k \right| \le \overline{\lim}_{N \to \infty} \frac{2^{-l}}{\varphi(N)} \left| \sum_{k=1}^{N} X_k \right| \le 2^{-l} \sqrt{3}.$$

By adding these we have the conclusion.

Lemma 5. For $x \notin \mathbf{Q}$ and $s = \pm 1$, we have

$$\overline{\lim}_{N \to \infty} \sup_{a < 1} \sum_{k=1}^{N} \frac{s \widetilde{\mathbf{1}}_{[0,a]} \langle kx \rangle X_k}{\varphi(N)} = \overline{\lim}_{N \to \infty} \sup_{a' < a < 1} \sum_{k=1}^{N} \frac{s \widetilde{\mathbf{1}}_{[a',a]} \langle kx \rangle X_k}{\varphi(N)} = \sqrt{2}\sigma(x) \quad a.s.$$

Proof: We prove for s = 1. The other case is proved in the same way. Denote $\sum_{a',a}^{N,x} = \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a',a]} \langle kx \rangle X_k / \varphi(N)$. Since $\{Y_k\}$ is an independent sequence satisfying $EY_k = 0$, $|Y_k| \leq 3$, $V_N = EY_1^2 + \cdots + EY_N^2 \sim 3N\sigma^2(x, a - a') \to \infty$ (a' < a), by the law of the iterated logarithm [11], we have $\overline{\lim_{N\to\infty} (Y_1 + \cdots + Y_N)} / \sqrt{2V_N \log \log V_N} = 1$ a.s. For a = a', we have $Y_k = 0$. Hence $\overline{\lim}_{N \to \infty} \Sigma_{a',a}^{N,x} = \sigma(x, a - a')$ a.s., and thereby

$$\overline{\lim}_{N \to \infty} \max_{j' < j < 2^l} \sum_{2^{-l} j', 2^{-l} j}^{N, x} = \overline{\lim}_{N \to \infty} \max_{j < 2^l} \sum_{0, 2^{-l} j}^{N, x} = \max_{j < 2^l} \sigma(x, 2^{-l} j) \quad \text{a.s.}$$

By taking limsup in

 $\max_{j<2^{l}} \sum_{0,2^{-l}j}^{N,x} \leq \sup_{a<1} \sum_{0,a}^{N,x} \leq \sup_{a'<a<1} \sum_{a',a}^{N,x} \leq \max_{j'<j<2^{l}} \sum_{2^{-l}j',2^{-l}j}^{N,x} + 2\max_{i<2^{l}} \sup_{a<2^{-l}} \left| \sum_{2^{-l}i,2^{-l}i+a}^{N,x} \right|,$

we have

$$\max_{j<2^l} \sigma(x, 2^{-l}j) \leq \overline{\lim_{N \to \infty}} \sup_{a<1} \Sigma_{0,a}^{N,x} \leq \overline{\lim_{N \to \infty}} \sup_{a' < a < 1} \Sigma_{a',a}^{N,x} \leq \max_{j<2^l} \sigma(x, 2^{-l}j) + 12 \cdot 2^{-l/2} \quad \text{a.s.}$$

By letting $l \to \infty$, we have the conclusion. By applying the result $ND_N\{kx\} = O((\log N)(\log \log N)^{1+\varepsilon})$ a.e. x by Khintchine [10] together with Lemma 5 to

$$\begin{split} &\inf_{a<1}\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[0,a)}\langle kx\rangle\frac{1}{2} + \sup_{a<1}\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[0,a)}\langle kx\rangle\frac{sX_{k}}{2} \leq \sup_{a<1}\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[0,a)}\langle kx\rangle\frac{1+sX_{k}}{2} \\ &\leq \sup_{a<1}\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[0,a)}\langle kx\rangle\frac{1}{2} + \sup_{a<1}\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[0,a)}\langle kx\rangle\frac{sX_{k}}{2} \quad (s=\pm1), \end{split}$$

we have

$$\overline{\lim_{N \to \infty}} \frac{1}{\varphi(N)} \sup_{a < 1} \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[0,a)} \langle kx \rangle \frac{1 + sX_k(\omega)}{2} = \sigma(x)/\sqrt{2} \quad \text{a.e. } (x, \omega).$$

It remains valid if we replace $1 + sX_k(\omega)$ by $-1 - sX_k(\omega)$. Hence

$$\lim_{N \to \infty} \frac{1}{\varphi(N)} \sup_{a < 1} \left| \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[0,a)} \langle kx \rangle \frac{1 + sX_k(\omega)}{2} \right| = \sigma(x)/\sqrt{2} \quad \text{a.e. } (x, \omega).$$

The law of large numbers $B_N = \sum_{k=1}^N (1 + sX_k(\omega))/2 \sim N/2$ a.s. implies

$$\lim_{N \to \infty} \frac{1}{\varphi(B_N)} \sup_{a < 1} \left| \sum_{k=1}^N \widetilde{\mathbf{1}}_{[0,a)} \langle kx \rangle \frac{1 + sX_k(\omega)}{2} \right| = \sigma(x) \tag{8}$$

for almost every (x, ω) . By taking ω which satisfies the formula (8) for a.e. x, and by denoting $\{n_k\} = \{j \mid X_k(\omega) = 1\}$, (8) with s = 1 yields the law of the iterated logarithm for $D_N^*\{n_k\}$. The law for $D_N\{n_k\}$ can be proved in the same way. (8) with s = -1 yields the laws for $\{n_k^\circ\}$.

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