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A LAW OF THE ITERATED LOGARITHM FOR DISCREPANCIES: NON-CONSTANT LIMSUP

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Abstract

We prove the existence of a sequence $\{n_k x\}$ whose discrepancies obey a bounded law of the iterated logarithm with a non-constant limsup.

1. Introduction

Let us define the discrepancies D_N and D_N^* of a sequence $\{x_k\}$ of real numbers by

$$D_N\{x_k\} = \sup_{0\le a'<a<1} \left| \frac{1}{N} \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{a',a}(\langle x_k\rangle) \right|; \; D_N^*\{x_k\} = \sup_{0\le a<1} \left| \frac{1}{N} \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{0,a}(\langle x_k\rangle) \right|,$$

where $\widetilde{\mathbf{1}}_{a',a}(x) = \mathbf{1}_{[a',a)}(x) - (a-a')$, $\mathbf{1}_{[a',a)}$ denotes the indicator function of $[\,a', a)$ and $\langle x\rangle$ denotes the fractional part $x - [\,x\,]$ of x.

For $\{n_k\}$ with exponential growth $n_{k+1}/n_k > q > 1$, Philipp [13, 14] proved the following bounded law of the iterated logarithm

$$\frac{1}{4\sqrt{2}} < \varlimsup_{N\to\infty} \frac{N D_N\{n_k x\}}{\sqrt{2N\log\log N}} \le C < \infty, \quad \text{a.e.} \tag{1}$$

The result was extended to the case of sub-exponential growth by Berkes, Philipp, and Tichy [15, 3] assuming some extra conditions. The limsup in (1) is explicitly calculated only in a few cases: the case when $n_{k+1}/n_k \to \infty$ [6, 4, 8]; the case when $n_k = \theta^k$ for $\theta > 1$ [12, 7]; the case when $\{n_k\}$ is a Hardy-Littlewood-Pólya sequence [9]. As is pointed out by Aistleitner and Berkes [1, 2], it is not known if the limsup in (1) is a constant almost everywhere.

In this paper we show the existence of a sequence $\{n_k\}$ of linear growth which obeys a bounded law of the iterated logarithm

$$0 < C_1 \le \varlimsup_{N\to\infty} \frac{N D_N\{n_k x\}}{\sqrt{2N\log\log N}} \le C_2 < \infty, \quad \text{a.e.}, \tag{2}$$

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and the limsup in (2) is not a constant a.e.

For a given strictly increasing sequence $\{n_k\}$ of positive integers, denote by $\{n_k^\circ\}$ the arrangement in increasing order of $\mathbf{N} \setminus \{n_k\}$.

Theorem 1. *There exists a strictly increasing sequence $\{n_k\}$ of positive integers satisfying $n_{k+1} - n_k \leq 5$ and $n_{k+1}^\circ - n_k^\circ \leq 5$ such that*

$$\varlimsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N^*\{n_k x\}}{\sqrt{2N\log\log N}} = \sigma(x) \quad a.e. \quad x,$$
$$\varlimsup_{N\to\infty} \frac{ND_N\{n_k^\circ x\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N^*\{n_k^\circ x\}}{\sqrt{2N\log\log N}} = \sigma(x) \quad a.e. \quad x,$$

where

$$\sigma^2(x) = (4x\mathbf{1}_{[0,1/4)}(x) + \mathbf{1}_{[1/4,3/4)}(x) + 4(1-x)\mathbf{1}_{[3/4,1)}(x))/9 + 1/24. \quad (3)$$

We use a method of random series which originated with Salem-Zygmund [16] and Bobkov-Götze [5].

2. Proof

Let $J = \{-1,1\}^3 \setminus \{(1,1,1),(-1,-1,-1)\}$ and $\{\xi_n\}_{n=1}^\infty$ be a J-valued i.i.d. such that ξ_n is uniformly distributed over J. We define the sequence $\{X_n\}$ of random variables by $(X_{3n-2}, X_{3n-1}, X_{3n}) = \xi_n$ ($n \in \mathbf{N}$). Clearly $\{X_{3n-2}\}_{n=1}^\infty$, $\{X_{3n-1}\}_{n=1}^\infty$, and $\{X_{3n}\}_{n=1}^\infty$ are fair $\{-1,1\}$-valued i.i.d.

Denote $\mathbf{1}_{[a',a)}(\langle kx\rangle)$ and $\widetilde{\mathbf{1}}_{[a',a)}(\langle kx\rangle)$ simply by $\mathbf{1}_{[a',a)}\langle kx\rangle$ and $\widetilde{\mathbf{1}}_{[a',a)}\langle kx\rangle$. Put $i_k = \widetilde{\mathbf{1}}_{[a',a)}\langle kx\rangle$, $Y_k = i_{3k-2}X_{3k-2} + i_{3k-1}X_{3k-1} + i_{3k}X_{3k}$, $r(x,a) = \int_0^1 \widetilde{\mathbf{1}}_{[0,a)}\langle y\rangle \widetilde{\mathbf{1}}_{[0,a)}\langle y-x\rangle\,dy$, and $\varphi(x) = \sqrt{2x\log\log x}$.

Lemma 2. *For $x \notin \mathbf{Q}$, we have*

$$\frac{1}{3N}\sum_{k=1}^N EY_k^2 \to \sigma^2(x, a-a'), \quad (4)$$

where $\sigma^2(x,a) = r(0,a) - 4r(x,a)/9 - 2r(2x,a)/9$. The function $\sigma^2(x,a)$ is continuous in (x,a) and satisfies

$$\sigma^2(x,a) = \sigma^2(1-x,a). \quad (5)$$

Proof: By $EX_n = 0$, $EX_n^2 = 1$, and $EX_{3n-2}X_{3n-1} = EX_{3n-2}X_{3n} = EX_{3n-1}X_{3n} = -1/3$, we have

$$EY_k^2 = i_{3k-2}^2 + i_{3k-1}^2 + i_{3k}^2 - \frac{2}{3}i_{3k-2}i_{3k-1} - \frac{2}{3}i_{3k-1}i_{3k} - \frac{2}{3}i_{3k-2}i_{3k}.$$

Since the sequence $\{3kx\}$ is uniformly distributed mod 1, we have

$$\frac{1}{N}\sum_{k=1}^{N} i_{3k-2}i_{3k} = \frac{1}{N}\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a',a)}\langle(3k-2)x\rangle\widetilde{\mathbf{1}}_{[a',a)}\langle 3kx\rangle$$

$$\to \int_0^1 \widetilde{\mathbf{1}}_{[a',a)}\langle y-2x\rangle\widetilde{\mathbf{1}}_{[a',a)}\langle y\rangle\, dy = \int_0^1 \widetilde{\mathbf{1}}_{[0,a-a')}\langle y-2x\rangle\widetilde{\mathbf{1}}_{[0,a-a')}\langle y\rangle\, dy.$$

In the same way, we have

$$\frac{1}{N}\sum_{k=1}^{N} i_{3k-j}i_{3k-j-1} \to r(x,a-a') \quad \text{and} \quad \frac{1}{N}\sum_{k=1}^{N} i_{3k-j}^2 \to r(0,a-a')$$

for $j=0,\,1,\,2$. Therefore we have (4).

Continuity of $\sigma^2(x,a)$ is clear. By changing variable, we have

$$\int_0^1 \widetilde{\mathbf{1}}_{[0,a)}\langle y+x\rangle\widetilde{\mathbf{1}}_{[0,a)}\langle y\rangle\, dy = \int_0^1 \widetilde{\mathbf{1}}_{[0,a)}\langle y\rangle\widetilde{\mathbf{1}}_{[0,a)}\langle y-x\rangle\, dy,$$

or $r(-x,a)=r(x,a)$. Hence $\sigma^2(x,a)$ is a even function of x with period one, and thereby we have (5). □

Lemma 3. *For $x\notin \mathbf{Q}$, we have*

$$\sigma^2(x) = \max_{0\le a<1}\sigma^2(x,a)/2 = \max_{0\le a\le 1/2}\sigma^2(x,a)/2. \tag{6}$$

$\sigma^2(x,a)=0$ *implies* $a=0$.

Proof: Put $m(x)=x\wedge(1-x)$ and $M(x)=x\vee(1-x)$. $m(x)\le 1/2\le M(x)$ is clear. For $a,\,x\in[\,0,1)$, we have

$$r(x,a)=\begin{cases}-a^2 & 0\le a\le m(x)\\ a-a^2-m(x) & m(x)\le a\le M(x)\\ -(1-a)^2 & M(x)\le a<1.\end{cases} \tag{7}$$

We prove this in the case $x\le 1/2$ or $m(x)=x$. The other case can be proved by using (5). Note that $r(x,a)+a^2=\int_0^1 \mathbf{1}_{[0,a)}\langle y-x\rangle\mathbf{1}_{[0,a)}\langle y\rangle\, dy$ equals to the measure of $A_x=[\,0,a)\cap([\,0,a)+x \mod 1)$. We can easily see that $A_x=\emptyset$ if $0\le a\le x$, $A_x=[\,x,a)$ if $x\le a\le 1-x$, and $A_x=[\,0,a+x-1)\cup[\,x,a)$ if $1-x\le a<1$. (7) is verified by calculating the measure of A_x. Especially, we have $r(0,a)=a-a^2$.

By (5), it is sufficient to prove (6) for $x\in[\,0,1/2\,]$. The second equality of (6) follows from $r(x,a)=r(x,1-a)$ which is clear from (7).

By using (7), we calculate $9\sigma^2(x,a)$ for $a,\,x\in[\,0,1/2\,]$.

If $0\le x\le 1/4$, by noting $0\le m(x)=x\le m(2x)=2x\le 1/2$, we have $9\sigma^2(x,a)=9a-3a^2$ on $[\,0,x)$, $9\sigma^2(x,a)=5a-3a^2+4x$ on $[\,x,2x)$, and $9\sigma^2(x,a)=3a-3a^2+8x$ on $[\,2x,1/2)$.

If $1/4 \le x \le 1/3$, by noting $0 \le m(x) = x \le m(2x) = 1-2x \le 1/2$, we have $9\sigma^2(x,a) = 9a - 3a^2$ on $[\,0, x)$, $9\sigma^2(x,a) = 5a - 3a^2 + 4x$ on $[\,x, 1-2x)$, and $9\sigma^2(x,a) = 3a - 3a^2 + 2$ on $[\,1-2x, 1/2)$.

If $1/3 \le x \le 1/2$, by noting $0 \le m(2x) = 1-2x \le m(x) = x \le 1/2$, we have $9\sigma^2(x,a) = 9a - 3a^2$ on $[\,0, 1-2x)$, $9\sigma^2(x,a) = 7a - 3a^2 - 4x + 2$ on $[\,1-2x, x)$, and $9\sigma^2(x,a) = 3a - 3a^2 + 2$ on $[\,x, 1/2)$.

By differentiating by a, we have $(9a-3a^2)' = 9 - 6a > 0$, $(5a - 3a^2 + 4x)' = 5 - 6a > 0$, $(7a - 3a^2 - 4x + 2)' = 7 - 6a > 0$, $(3a - 3a^2 + 4x)' = (3a - 3a^2 + 2)' = 3 - 6a > 0$ for $a \in [\,0, 1/2)$. Hence $9\sigma^2(x,a)$ strictly increases in $a \in [\,0, 1/2\,]$ and takes its maximum at $a = 1/2$. Hence (6) is proved. Clearly $\sigma^2(x,a) = 0$ only for $a = 0$. $\square$

Lemma 4. *For $l \in \mathbf{N}$, $i < 2^l$, and $x \notin \mathbf{Q}$, we have*

$$\varlimsup_{N\to\infty} \frac{1}{\varphi(N)} \sup_{a<2^{-l}} \left| \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[2^{-l}i, 2^{-l}i+a)} \langle kx \rangle X_k \right| \le 6 \cdot 2^{-l/2} \quad a.s.$$

Proof: Take an integer N and $\lambda > 0$ arbitrarily and denote

$$\widetilde{S}_N = \max_{n\le N} \sup_{a<2^{-l}} \sum_{k=1}^{n} \mathbf{1}_{[\,2^{-l}i, 2^{-l}i+a)} \langle 3kx \rangle X_{3k}, \quad b_N = \sum_{k=1}^{N} \mathbf{1}_{[\,2^{-l}i, 2^{-l}(i+1))} \langle 3kx \rangle.$$

Define $k_1, \ldots, k_M$ by $\{3k_1, 3k_2, \ldots, 3k_M\} = \{3k \le N \mid \langle 3kx \rangle \in [\,2^{-l}i, 2^{-l}(i+1))\}$ and $\langle 3k_1 x \rangle < \langle 3k_2 x \rangle < \cdots < \langle 3k_M x \rangle$. By putting $J_{m,n} = \{j \le m \mid 3k_j \le n\}$ and $S_{m,n} = \sum_{j\in J_{m,n}} X_{3k_j}$, we have $\widetilde{S}_N = \max_{n\le N} \max_{m\le M} S_{m,n}$. Defining random variables n_0 and m_0 by $n_0 = \min\{n; \max_{m\le M} S_{m,n} > \lambda\}$ and $m_0 = \min\{m; S_{m,n_0} > \lambda\}$, we have a disjoint decomposition $\{\widetilde{S}_N > \lambda\} = \bigcup_{n\le N} \bigcup_{m\le M} C_{n,m}$ where $C_{n,m} = \{n_0 = n, m_0 = m\}$. Since $C_{n,m}$ belongs to the sigma field generated by X_{3k_j} $(3k_j \le n)$, it is independent of $S_{m,N} - S_{m,n}$ which is a function of X_{3k_j} $(3k_j > n)$. Hence by noting $P(S_{m,N} - S_{m,n} \ge 0) \ge 1/2$, we have

$$\begin{aligned} P(C_{n,m}) &\le 2P\big(C_{n,m} \cap \{S_{m,n} > \lambda\}\big) P\big(S_{m,N} - S_{m,n} \ge 0\big) \\ &\le 2P\big(C_{n,m} \cap \{S_{m,N} > \lambda\}\big) \le 2P\big(C_{n,m} \cap \{\max_{m'\le M} S_{m',N} > \lambda\}\big). \end{aligned}$$

By summing for $n \le N$ and $m \le M$, we have

$$\begin{aligned} P\big(\widetilde{S}_N > \lambda\big) &\le 2P\big(\widetilde{S}_N > \lambda, \max_{m'\le M} S_{m',N} > \lambda\big) = 2P\big(\max_{m'\le M} S_{m',N} > \lambda\big) \\ &\le 4P\big(S_{M,N} > \lambda\big) = 4P\left(\sum_{k=1}^{N} \mathbf{1}_{[\,2^{-l}i, 2^{-l}(i+1))} \langle 3kx \rangle X_{3k} > \lambda\right), \end{aligned}$$

where the last inequality is by reflection principle. For fair ± 1 valued i.i.d. $\{Z_n\}$, we have $E\exp(t(Z_1+\cdots+Z_n)) \le e^{nt^2/2}$ (Lemma 4.2.1 of [16]), which implies $P(|Z_1+\cdots+Z_n| \ge \mu) \le 2\exp(-\mu^2/2n)$. By applying this we have $P\big(|\widetilde{S}_N| > \lambda\big) \le 8\exp(-\lambda^2/2b_N)$.

Thanks to $b_N/N \to 2^{-l}$ we have $b_N < 2N2^{-l}$ for large N, and hence $P\big(|\widetilde{S}_N| > \lambda\big) \le 8\exp\big(-\lambda^2/4N2^{-l}\big)$. By putting $N = 2^j$ and $\lambda = \sqrt{5\cdot 2^j 2^{-l}\log\log 2^j}$ we have

$$\sum_j P\big(|\widetilde{S}_{2^j}| > \sqrt{5\cdot 2^j 2^{-l}\log\log 2^j}\big) \le \sum_j 8(j\log 2)^{-5/4} < \infty.$$

By Borel-Cantelli Lemma, we see $|\widetilde{S}_{2^j}| \le \sqrt{5\cdot 2^j 2^{-l}\log\log 2^j}$ a.s. for large j, and hence

$$\varlimsup_{N\to\infty} \frac{1}{\varphi(N)} \sup_{a<2^{-l}} \Big| \sum_{k=1}^N \mathbf{1}_{[2^{-l}i, 2^{-l}i+a)}\langle 3kx\rangle X_{3k} \Big| \le \sqrt{10\cdot 2^{-l}} \quad \text{a.s.}$$

It remains valid if we replace $3k$ by $3k-1$ or $3k-2$, and combining these, we have

$$\varlimsup_{N\to\infty} \frac{1}{\varphi(N)} \sup_{a<2^{-l}} \Big| \sum_{k=1}^N \mathbf{1}_{[2^{-l}i, 2^{-l}i+a)}\langle kx\rangle X_{k} \Big| \le \sqrt{30\cdot 2^{-l}} \quad \text{a.s.}$$

By decomposing $\sum_{k=1}^N X_k$ into the sum of $\sum_{k\le (N+2)/3} X_{3k-2}$, $\sum_{k\le (N+1)/3} X_{3k-1}$, and $\sum_{k\le N/3} X_{3k}$, and by applying the law of the iterated logarithm for each, we have

$$\varlimsup_{N\to\infty} \sup_{a<2^{-l}} \frac{a}{\varphi(N)} \Big| \sum_{k=1}^N X_k \Big| \le \varlimsup_{N\to\infty} \frac{2^{-l}}{\varphi(N)} \Big| \sum_{k=1}^N X_k \Big| \le 2^{-l}\sqrt{3}.$$

By adding these we have the conclusion. □

Lemma 5. *For $x \notin \mathbf{Q}$ and $s = \pm 1$, we have*

$$\varlimsup_{N\to\infty} \sup_{a<1} \sum_{k=1}^N \frac{s\widetilde{\mathbf{1}}_{[0,a)}\langle kx\rangle X_k}{\varphi(N)} = \varlimsup_{N\to\infty} \sup_{a'<a<1} \sum_{k=1}^N \frac{s\widetilde{\mathbf{1}}_{[a',a)}\langle kx\rangle X_k}{\varphi(N)} = \sqrt{2}\sigma(x) \quad \textit{a.s.}$$

Proof: We prove for $s=1$. The other case is proved in the same way. Denote $\Sigma_{a',a}^{N,x} = \sum_{k=1}^N \widetilde{\mathbf{1}}_{[a',a)}\langle kx\rangle X_k/\varphi(N)$. Since $\{Y_k\}$ is an independent sequence satisfying $EY_k = 0$, $|Y_k| \le 3$, $V_N = EY_1^2+\cdots+EY_N^2 \sim 3N\sigma^2(x, a-a') \to \infty$ $(a' < a)$, by the law of the iterated logarithm [11], we have $\varlimsup_{N\to\infty}(Y_1+\cdots+Y_N)/\sqrt{2V_N\log\log V_N} = 1$ a.s. For

$a = a'$, we have $Y_k = 0$. Hence $\overline{\lim}_{N\to\infty} \Sigma^{N,x}_{a',a} = \sigma(x, a-a')$ a.s., and thereby

$$\overline{\lim_{N\to\infty}} \max_{j'<j<2^l} \Sigma^{N,x}_{2^{-l}j',2^{-l}j} = \overline{\lim_{N\to\infty}} \max_{j<2^l} \Sigma^{N,x}_{0,2^{-l}j} = \max_{j<2^l} \sigma(x, 2^{-l}j) \quad \text{a.s.}$$

By taking limsup in

$$\max_{j<2^l} \Sigma^{N,x}_{0,2^{-l}j} \leq \sup_{a<1} \Sigma^{N,x}_{0,a} \leq \sup_{a'<a<1} \Sigma^{N,x}_{a',a} \leq \max_{j'<j<2^l} \Sigma^{N,x}_{2^{-l}j',2^{-l}j} + 2 \max_{i<2^l} \sup_{a<2^{-l}} \left|\Sigma^{N,x}_{2^{-l}i,2^{-l}i+a}\right|,$$

we have

$$\max_{j<2^l} \sigma(x, 2^{-l}j) \leq \overline{\lim_{N\to\infty}} \sup_{a<1} \Sigma^{N,x}_{0,a} \leq \overline{\lim_{N\to\infty}} \sup_{a'<a<1} \Sigma^{N,x}_{a',a} \leq \max_{j<2^l} \sigma(x, 2^{-l}j) + 12\cdot 2^{-l/2} \quad \text{a.s.}$$

By letting $l \to \infty$, we have the conclusion. □

By applying the result $ND_N\{kx\} = O((\log N)(\log\log N)^{1+\varepsilon})$ a.e. x by Khintchine [10] together with Lemma 5 to

$$\inf_{a<1} \sum_{k=1}^N \widetilde{\mathbf{1}}_{[0,a)}\langle kx\rangle \frac{1}{2} + \sup_{a<1} \sum_{k=1}^N \widetilde{\mathbf{1}}_{[0,a)}\langle kx\rangle \frac{sX_k}{2} \leq \sup_{a<1} \sum_{k=1}^N \widetilde{\mathbf{1}}_{[0,a)}\langle kx\rangle \frac{1+sX_k}{2}$$
$$\leq \sup_{a<1} \sum_{k=1}^N \widetilde{\mathbf{1}}_{[0,a)}\langle kx\rangle \frac{1}{2} + \sup_{a<1} \sum_{k=1}^N \widetilde{\mathbf{1}}_{[0,a)}\langle kx\rangle \frac{sX_k}{2} \quad (s = \pm 1),$$

we have

$$\overline{\lim_{N\to\infty}} \frac{1}{\varphi(N)} \sup_{a<1} \sum_{k=1}^N \widetilde{\mathbf{1}}_{[0,a)}\langle kx\rangle \frac{1+sX_k(\omega)}{2} = \sigma(x)/\sqrt{2} \quad \text{a.e. } (x,\omega).$$

It remains valid if we replace $1 + sX_k(\omega)$ by $-1 - sX_k(\omega)$. Hence

$$\overline{\lim_{N\to\infty}} \frac{1}{\varphi(N)} \sup_{a<1} \left| \sum_{k=1}^N \widetilde{\mathbf{1}}_{[0,a)}\langle kx\rangle \frac{1+sX_k(\omega)}{2} \right| = \sigma(x)/\sqrt{2} \quad \text{a.e. } (x,\omega).$$

The law of large numbers $B_N = \sum_{k=1}^N (1 + sX_k(\omega))/2 \sim N/2$ a.s. implies

$$\overline{\lim_{N\to\infty}} \frac{1}{\varphi(B_N)} \sup_{a<1} \left| \sum_{k=1}^N \widetilde{\mathbf{1}}_{[0,a)}\langle kx\rangle \frac{1+sX_k(\omega)}{2} \right| = \sigma(x) \qquad (8)$$

for almost every (x,ω). By taking ω which satisfies the formula (8) for a.e. x, and by denoting $\{n_k\} = \{j \mid X_k(\omega) = 1\}$, (8) with $s = 1$ yields the law of the iterated logarithm for $D^*_N\{n_k\}$. The law for $D_N\{n_k\}$ can be proved in the same way. (8) with $s = -1$ yields the laws for $\{n^\circ_k\}$.

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