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Fukuyama, Katusi

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A metric discrepancy result for a lacunary sequence with small gaps

KATUSI FUKUYAMA

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Abstract For any $G(k) \uparrow \infty$, there exists a sequence $\{n_k\}$ of integers with $1 \leq n_{k+1} - n_k \leq G(k)$ such that the discrepancies of $\{n_k x\}$ obey the law of the iterated logarithm in the same way as uniform distributed i.i.d.

1 Introduction

In this paper we are concerned with the asymptotic behavior of the following discrepancies D_N and D_N^* of a sequence $\{x_k\}$ of real numbers:

$$D_N\{x_k\} = \sup_{0 \leq a' < a < 1} \left| \frac{1}{N} \sum_{k=1}^N \tilde{\mathbf{1}}_{[a', a)}(x_k) \right|; \quad D_N^*\{x_k\} = \sup_{0 \leq a < 1} \left| \frac{1}{N} \sum_{k=1}^N \tilde{\mathbf{1}}_{[0, a)}(x_k) \right|,$$

where $\tilde{\mathbf{1}}_{[a', a)}(x) = \mathbf{1}_{[a', a)}(\langle x \rangle) - (a - a')$, $\mathbf{1}_{[a', a)}$ denotes the indicator function of $[a', a)$, and $\langle x \rangle$ denotes the fractional part $x - [x]$ of real number x .

Philipp [17] proved the bounded law of the iterated logarithm

$$\frac{1}{4\sqrt{2}} < \overline{\lim}_{N \rightarrow \infty} \frac{ND_N\{n_k x\}}{\sqrt{2N \log \log N}} \leq C < \infty, \quad \text{a.e.} \quad (1)$$

under the Hadamard's gap condition $n_{k+1}/n_k > q > 1$, and many extensions and refinements were made on this result.

The first result proving that the limsup is constant a.e., was by Philipp [16] in case $n_k = 2^k$, although the concrete value of the constant was not calculated. The first

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K. Fukuyama
Department of mathematics,
Kobe University,
Rokko, Kobe, 657-8501, Japan
E-mail: fukuyama@math.kobe-u.ac.jp

result giving the concrete value was proved by Dhompongsa [9] as below: under the rather strong gap condition $\log(n_{k+1}/n_k)/\log \log k \rightarrow \infty$, one has

$$\overline{\lim}_{N \rightarrow \infty} \frac{ND_N\{n_k x\}}{\sqrt{2N \log \log N}} = \overline{\lim}_{N \rightarrow \infty} \frac{ND_N^*\{n_k x\}}{\sqrt{2N \log \log N}} = \frac{1}{2}, \quad \text{a.e.} \quad (2)$$

Note that this constant $1/2$ is the same as that in the Chung-Smirnov theorem, which proves the law of the iterated logarithm for discrepancies of uniform distributed i.i.d.

The above bounded law of the iterated logarithm (1) was extended to the case of so-called Hardy-Littlewood-Pólya sequences by Philipp [19], and to the case with some Diophantine conditions by Berkes-Philipp-Tichy [7, 8]. We can also find other extensions by Aistleitner [1] and Aistleitner-Berkes [5].

Recently, it became possible to calculate the exact value of limsup concretely.

It was proved in [11] that the limsup for $\{\theta^k x\}$ ($\theta > 1$) is constant a.e., and the constant was evaluated for every $\theta > 1$ except for countable many exceptions. For example, limsup equals to $\sqrt{42}/9$ when $\theta = 2$, the case Philipp treated before. In [13], for a Hardy-Littlewood-Pólya sequence, the limsup was again proved to be constant a.e., and the exact value was explicitly given when the sequence consists of odd numbers. Aistleitner gave a sufficient Diophantine condition to have (2) for sequences with Hadamard's gaps [2] and for sub-Hadamard sequences [4]. It was proved in [12] that the value of limsup may vary if we permute the order of the sequence $\{n_k\}$. Aistleitner [3] constructed a sequence with Hadamard's gap with non-constant limsup. In [14], one can also find the proof of existence of bounded gap sequences with non-constant limsup.

In this paper, we consider the problem whether there exists a sequence with small gaps satisfying (2). As to this problem, Berkes-Philipp-Tichy [7, 8] proved the existence of $n_k \geq e^{k^\beta}$ satisfying (1) by the method of randomization. Aistleitner [4] remarked that such sequence can be taken to satisfy (2).

On the other hand, Berkes [6] proved that, for any sequence with $G(k) \uparrow \infty$, there exists a sequence $\{n_k\}$ satisfying $n_{k+1} - n_k \leq G(k)$ and $\{\cos 2\pi n_k x\}$ obeys the central limit theorem on the unit interval. By showing this result Berkes solved the long standing problem asking the possibility of the central limit theorem for lacunary trigonometric series with small gaps.

Our motivation is to investigate the metric discrepancy counterpart of this Berkes' result. Actually, we could prove a desirable result below.

Theorem 1 *For every sequence $1 \leq G(k) \uparrow \infty$ of real numbers, there exists a sequence $\{n_k\}$ of integers satisfying $1 \leq n_{k+1} - n_k \leq G(k)$ and (2).*

2 Preliminary Lemmas

Lemma 1 *Let us take a finite set $\Pi = \{q_1, \dots, q_k\} \subset [0, 1]$ and define a set of independent random variables $\{X_{q_i, n}\}_{i=1, \dots, k, n \in \mathbf{N}}$ satisfying $P(X_{q_i, n} = 1) = 1 - P(X_{q_i, n} = 0) = q_i$. Then*

$$C(\Pi) = \inf \left\{ P \left(\varepsilon \sum_{i=1}^k \sum_{j=1}^{n_i} (X_{q_i, j} - EX_{q_i, j}) \geq 0 \right) \middle| \varepsilon = \pm 1, n_1, \dots, n_k = 0, 1, \dots \right\} > 0.$$

Proof Denote $X_{q_i,j} - EX_{q_i,j}$ by $\tilde{X}_{q_i,j}$. Because of

$$P\left(\varepsilon \sum_{i=1}^k \sum_{j=1}^{n_i} \tilde{X}_{q_i,j} \geq 0\right) \geq \prod_{i=1}^k P\left(\varepsilon \sum_{j=1}^{n_i} \tilde{X}_{q_i,j} \geq 0\right) \geq C(\{q_1\}) \dots C(\{q_k\})$$

and the well known fact that $C(\{q\}) > 0$ for all $q \in [0, 1]$ (Cf. Feller [10] 205pp) we have $C(\Pi) \geq C(\{q_1\}) \dots C(\{q_k\}) > 0$. $\square$

Remark 1 The value of $C(\{q_1, \dots, q_k\})$ is determined by the set $\{q_1, \dots, q_k\}$ and is not effected by the multiplicity among $q_1, \dots, q_k$. For example, $C(\{q_1, \dots, q_k, q_{k+1}\}) = C(\{q_1, \dots, q_k\})$ if $q_k = q_{k+1}$.

We denote $\mathbf{1}_{[A,B]}(\langle x \rangle)$ simply by $\mathbf{1}_{[A,B]}(x)$.

Lemma 2 Let $\{\xi_k\}$ be a sequence of independent random variables satisfying

$$c = \inf_A P\left(\sum_{k \in A} \xi_k \geq 0\right) > 0,$$

where the infimum is taken over all non-empty finite subsets A of $\mathbf{N}$. For $A < B < 1$, $x \in [0, 1]$, and $\lambda > 0$, we have

$$P\left(\sup_{0 \leq a < B-A} \max_{n=1, \dots, N} \sum_{k=1}^n \mathbf{1}_{[A, A+a]}(kx) \xi_k \geq \lambda\right) \leq \frac{1}{c^2} P\left(\sum_{k=1}^N \mathbf{1}_{[A, B]}(kx) \xi_k \geq \lambda\right).$$

Proof The proof is a modification of that of Lemma 4 in [14]. For $N \in \mathbf{N}$ and $\lambda > 0$, denote

$$\tilde{S}_N = \sup_{a < B-A} \max_{n \leq N} \sum_{k=1}^n \mathbf{1}_{[A, A+a]}(kx) \xi_k.$$

Define $k_1, \dots, k_M$ by $\{k_1, k_2, \dots, k_M\} = \{k \leq N \mid \langle kx \rangle \in [A, B]\}$ and $\langle k_1 x \rangle < \langle k_2 x \rangle < \dots < \langle k_M x \rangle$. By putting $J_{m,n} = \{j \leq m \mid k_j \leq n\}$ and $S_{m,n} = \sum_{j \in J_{m,n}} \xi_{k_j}$, we have $\tilde{S}_N = \max_{n \leq N} \max_{m \leq M} S_{m,n}$. Define n_0 and m_0 by $n_0 = \min\{n; \max_{m \leq M} S_{m,n} > \lambda\}$ and $m_0 = \min\{m; S_{m,n_0} > \lambda\}$, and put $C_{n,m} = \{n_0 = n, m_0 = m\}$. Then we have a disjoint decomposition $\{\tilde{S}_N > \lambda\} = \bigcup_{n \leq N} \bigcup_{m \leq M} C_{n,m}$. Since $C_{n,m}$ is defined by means of ξ_{k_j} ($k_j \leq n$), it is independent of $S_{m,N} - S_{m,n}$ which is a function of ξ_{k_j} ($k_j > n$). Hence by noting $P(S_{m,N} - S_{m,n} \geq 0) \geq c$, we have

$$\begin{aligned} c^2 P(C_{n,m}) &\leq c P(C_{n,m} \cap \{S_{m,n} > \lambda\}) P(S_{m,N} - S_{m,n} \geq 0) \\ &\leq c P(C_{n,m} \cap \{S_{m,N} > \lambda\}) \leq c P(C_{n,m} \cap \{\max_{m' \leq M} S_{m',N} > \lambda\}). \end{aligned}$$

By summing for $n \leq N$ and $m \leq M$, we have

$$c^2 P(\tilde{S}_N > \lambda) \leq c P(\tilde{S}_N > \lambda, \max_{m' \leq M} S_{m',N} > \lambda) = c P(\max_{m' \leq M} S_{m',N} > \lambda).$$

Defining $m'_0 = \min\{m'; S_{m',N} > \lambda\}$ and $D_m = \{m'_0 = m\}$, we have in the same way,

$$c P(D_m) \leq P(D_m \cap \{S_{m,N} > \lambda\}) P(S_{M,N} - S_{m,N} \geq 0) \leq P(D_m \cap \{S_{M,N} > \lambda\})$$

and $c P(\max_{m' \leq M} S_{m',N} > \lambda) \leq P(S_{M,N} > \lambda)$. $\square$

Lemma 3 Let $\{a_0, a_1, a_2, \dots\}$ be a sequence of numbers belonging to $[0, 1]$ and $\{i_0 = 1, i_1, i_2, \dots\}$ be a sequence of positive integers. Suppose that $a_L \sim a_{L+1}$ ($L \rightarrow \infty$), and that

$$i_1 + \dots + i_{L-1} \leq i_L^{1/3}, \quad a_L i_L^{1/3} \geq 1$$

hold for large L . If we define $b_N = a_L$ for $i_0 + \dots + i_{L-1} \leq N < i_0 + \dots + i_L$, we have

$$\sum_{j=1}^N b_j \sim N b_N, \quad (N \rightarrow \infty).$$

Proof Let $N \in \mathbf{N}$ and let L satisfy $i_0 + \dots + i_{L-1} \leq N < i_0 + \dots + i_L$. Note that

$$\sum_{j=1}^N b_j = \sum_{j=1}^{i_0 + \dots + i_{L-1} - 1} b_j + \sum_{j=i_0 + \dots + i_{L-1}}^N b_j = \sum_{j=1}^{L-1} a_j i_j + a_L (N - i_0 - \dots - i_{L-1} + 1).$$

Here we have

$$\begin{aligned} \sum_{j=1}^{L-1} a_j i_j &= \theta \sum_{j=1}^{L-2} i_j + a_{L-1} i_{L-1} = \theta' i_{L-1}^{1/3} + a_{L-1} i_{L-1} = a_{L-1} (\theta'' i_{L-1}^{2/3} + i_{L-1}) \\ &= a_{L-1} i_{L-1} (1 + o(1)) = a_L (i_1 + \dots + i_{L-1}) (1 + o(1)), \end{aligned}$$

where $0 \leq \theta, \theta', \theta'' \leq 1$. Thereby $\sum_{j=1}^N b_j = a_L N + a_L N o(1) = b_N N (1 + o(1))$. $\square$

3 Proof of the Theorem

For simplicity, we only prove for D_N^* . The proof for D_N is given in the same way.

Due to Khintchine [15], it holds that $N D_N^* \{kx\} = O((\log N)(\log \log N)^{1+\varepsilon})$ a.e. x . Hence, for all $L \in \mathbf{N}$, there exists $\eta(L) \in \mathbf{N}$ and $H_L \subset [0, 1]$ such that $|H_L^c| \leq 1/2^L$ and

$$N D_N^* (kx) \leq (\log N)^2 / 2^L, \quad (x \in H_L, N \geq 2^{\eta(L)}). \quad (3)$$

We may take $\eta(L)$ large enough to satisfy

$$\eta(0) = 0, \quad \eta(1) \geq 3, \quad 3\eta(L) + 2L \leq \eta(L+1), \quad (4)$$

$$2^L \leq G(2^{\eta(L)+L} / 8 \log \log 2^{\eta(L)+L}). \quad (5)$$

We have hence $\eta(L) \geq 3^L$ and $\eta(L) + L \leq (\eta(L+1) + L)/3$.

We can construct a sequence $1 = p_0 \geq p_1 \geq p_2 \geq \dots$ satisfying the following conditions: $p_L \in \{1/k \mid k \in \mathbf{N}\}$; if $p_L = 1/k$ then $p_{L+1} = 1/k$ or $1/(k+1)$; $C(\{p_0, \dots, p_L\}) \geq 1/\log \log 2^{\eta(L)+L+2}$, and $p_L \downarrow 0$. (Note that $C(\{1\}) = 1$.) By the second condition, we have $p_L \geq 1/L$.

Put $I_L = [2^{\eta(L)+L}, 2^{\eta(L+1)+L+1})$ for $L \geq 0$. Let $\{X_k\}$ be a sequence of independent random variables defined as below. For $k \in I_L$, $P(X_k = 1) = 1$ if $2^L \mid k$, and $P(X_k = 1) = 1 - P(X_k = 0) = p_L$ otherwise. For $N \in I_L$, by Lemma 1 we have

$$\begin{aligned} \inf_{\varepsilon = \pm 1, A \subset \{1, \dots, N\}} P\left(\varepsilon \sum_{k \in A} \tilde{X}_k \geq 0\right) &\geq C(\{p_1, \dots, p_L\}) \geq 1/\log \log (2^{\eta(L)+L+2}) \\ &\geq 1/\log \log 4N, \end{aligned} \quad (6)$$

where $\tilde{X}_k = X_k - EX_k$.

Denote $\Xi_N = X_1 + \dots + X_N$, $e_N = E\Xi_N$, and $v_N = V\Xi_N$. For $k \in I_L$, put

$$\begin{aligned} c_k &= (EX_{2^{\eta(L)+L}} + \dots + EX_{2^{\eta(L)+L+2^L-1}})/2^L = p_L(2^L - 1)/2^L + 1/2^L, \\ d_k &= (VX_{2^{\eta(L)+L}} + \dots + VX_{2^{\eta(L)+L+2^L-1}})/2^L = p_L(1 - p_L)(2^L - 1)/2^L. \end{aligned}$$

Lemma 4 For $N \in \mathbb{N}$, take L satisfying $N \in I_L$ and denote it by L_N . We have

$$v_N \sim Nd_N \sim Np_{L_N}, \quad e_N \sim Nc_N, \quad v_N \sim e_N, \quad (N \rightarrow \infty). \quad (7)$$

Proof It is easily seen that

$$e_N = \sum_{k=1}^N c_k + O(1/2^{L_N}) \quad \text{and} \quad v_N = \sum_{k=1}^N d_k + O(1/2^{L_N}).$$

First we prove $\sum_{k=1}^N d_k \sim Nd_N$. We apply Lemma 3 for $i_L = 2^{\eta(L+1)+L+1} - 2^{\eta(L)+L}$, $a_L = p_L(1 - p_L)(2^L - 1)/2^L$, and $b_N = d_N$. Note that $\#I_L = i_L$. We have $i_L \geq 2^{\eta(L+1)+L}$. The conditions of Lemma 3 can be verified as below. By $p_L \sim p_{L+1}$ and $p_L \rightarrow 0$, we have

$$\frac{a_{L+1}}{a_L} = \frac{p_{L+1}(1 - p_{L+1})(2^{L+1} - 1)/2^{L+1}}{p_L(1 - p_L)(2^L - 1)/2^L} \sim 1,$$

$$i_1 + \dots + i_{L-1} = 2^{\eta(L)+L} - 1 \leq 2^{(\eta(L+1)+L)/3} \leq i_L^{1/3},$$

and

$$a_L i_L^{1/3} \geq p_L 2^{L/3}/2 \geq 2^{L/3}/L \rightarrow \infty.$$

By $L_N \rightarrow \infty$ and $p_L \rightarrow 0$, we have $d_N \sim p_{L_N}$ and hence $v_N \sim Np_{L_N}$.

Next we prove $\sum_{k=1}^N c_k \sim Nc_N$ by applying Lemma 3 for $a_L = p_L(2^L - 1)/2^L + 1/2^L$. The second condition of Lemma 3 is already verified and the third is verified as above by noting $a_L \geq p_L(2^L - 1)/2^L$. The first condition is verified as below.

$$\begin{aligned} 1 \sim \frac{p_{L+1}}{p_L} \frac{1}{1 + (L-1)/2^L} &\leq \frac{p_{L+1}}{p_L + (1 - p_L)/2^L} \leq \frac{p_{L+1} + (1 - p_{L+1})/2^{L+1}}{p_L + (1 - p_L)/2^L} = \frac{a_{L+1}}{a_L} \\ &\leq \frac{p_{L+1} + 1/2^{L+1}}{p_L} \leq \frac{p_{L+1}}{p_L} + \frac{L}{2^{L+1}} \sim 1. \end{aligned}$$

The third relation $v_N \sim e_N$ follows from first two as below.

$$\frac{e_N}{v_N} \sim \frac{c_N}{d_N} \sim \frac{1}{1 - p_{L_N}} + \frac{1}{p_{L_N}(1 - p_{L_N})(2^{L_N} - 1)}.$$

Here first term tends to 1, and the second term is bounded from above by $2L_N/(2^{L_N} - 1) \rightarrow 0$. Hence we have $e_N/v_N \rightarrow 1$. $\square$

By $2^{3^{L_N}} \leq 2^{\eta(L_N)+L_N} \leq N$, we have $L_N \leq \log \log N$. By above lemma, we have for large N ,

$$v_N \geq Nd_N/2 \geq Np_{L_N}/4 \geq N/4L_N \geq N/4 \log \log N. \quad (8)$$

Because of (3), for $x \in H_L$ and $N \geq N' \geq 2^{\eta(L)}$ we have

$$\sup_{0 \leq a < b < 1} \left| \sum_{k=N'+1}^N \tilde{\mathbf{I}}_{[a,b]}(kx) \right| \leq N' D_{N'}^*(kx) + N D_N^*(kx) \leq 2(\log N)^2 / 2^L.$$

Hence for $N \in I_L$,

$$\begin{aligned} & \sup_{0 \leq a < b < 1} \left| \sum_{k=2^{\eta(L)+L}}^N \tilde{\mathbf{I}}_{[a,b]}(kx) EX_k \right| \\ & \leq (1 - p_L) \sup_{0 \leq a < b < 1} \left| \sum_{k=2^{\eta(L)}}^{[N/2^L]} \tilde{\mathbf{I}}_{[a,b]}(k2^L x) \right| + p_L \sup_{0 \leq a < b < 1} \left| \sum_{k=2^{\eta(L)+L}}^N \tilde{\mathbf{I}}_{[a,b]}(kx) \right| \\ & \leq (\log N)^2 / 2^L \end{aligned}$$

if $x \in H_L$ and $2^L x \in H_L$.

Because of $\sum |H_L^c| < \infty$, by Borel-Cantelli Lemma, we see that $x \in H_L$ and $2^L x \in H_L$ for large L a.e. x . Suppose that $x \in H_L$ and $2^L x \in H_L$ for $L \geq L_0$. Then by putting $\zeta_k = \tilde{\mathbf{I}}_{[a,b]}(kx) EX_k$, we have

$$\begin{aligned} \left| \sum_{k=1}^N \tilde{\mathbf{I}}_{[a,b]}(kx) EX_k \right| & \leq \left| \sum_{k=1}^{2^{\eta(L_0)+L_0-1}} \zeta_k \right| + \sum_{L'=L_0}^{L-1} \left| \sum_{k \in I_{L'}} \zeta_k \right| + \left| \sum_{k=2^{\eta(L)+L}}^N \zeta_k \right| \\ & \leq \text{1st sum} + \sum_{L'=L_0}^{L-1} (\log 2^{\eta(L'+1)+L'})^2 / 2^{L'} + (\log N)^2 / 2^L \leq \text{1st sum} + (\log N)^2 \end{aligned}$$

since $L' \leq L-1$ implies $2^{\eta(L'+1)+L'} \leq 2^{\eta(L)+L} \leq N$. Hence we have

$$\sup_{a < b < 1} \left| \sum_{k=1}^N \tilde{\mathbf{I}}_{[a,b]}(kx) EX_k \right| \leq 2(\log N)^2 \leq 3(\log v_N)^2 \quad (9)$$

for large N a.e. x by (8). By noting $\mathbf{1}_{[a,b]}^2 = \mathbf{1}_{[a,b]}$, we have

$$\begin{aligned} & V \left(\sum_{k=2^{\eta(L)+L}}^N \mathbf{1}_{[a,b]} \langle kx \rangle X_k \right) \\ & = \sum_{k=2^{\eta(L)+L}}^N \mathbf{1}_{[a,b]} \langle kx \rangle V X_k = \sum_{k=2^{\eta(L)+L}}^N \tilde{\mathbf{I}}_{[a,b]}(kx) V X_k + (b-a) \sum_{k=2^{\eta(L)+L}}^N V X_k \\ & = (b-a) \sum_{k=2^{\eta(L)+L}}^N V X_k + p_L (1 - p_L) \left(\sum_{k=2^{\eta(L)+L}}^N \tilde{\mathbf{I}}_{[a,b]}(kx) - \sum_{k=2^{\eta(L)}}^{[N/2^L]} \tilde{\mathbf{I}}_{[a,b]}(k2^L x) \right). \end{aligned}$$

Hence if $x \in H_L$ and $2^L x \in H_L$, we have

$$\left| V \left(\sum_{k=2^{\eta(L)+L}}^N \mathbf{1}_{[a,b]} \langle kx \rangle X_k \right) - (b-a) \sum_{k=2^{\eta(L)+L}}^N V X_k \right| \leq 2(\log N)^2 / 2^L.$$

In the same way as above, we have $|V_N^{a,b} - (b-a)v_N| \leq 4(\log N)^2 \leq 12(\log v_N)^2$ for large N a.e. x , where

$$V_N^{a,b} = V\left(\sum_{k=1}^N \mathbf{1}_{[a,b)}\langle kx \rangle X_k\right).$$

Hence we have

$$V_N^{a,b} \sim (b-a)v_N. \quad (10)$$

By noting $\tilde{\mathbf{1}}_{[a,b)}^2 = (1-2(b-a))\mathbf{1}_{[a,b)} + (b-a)^2$, we have

$$\begin{aligned} \tilde{V}_N^{a,b} &= V\left(\sum_{k=1}^N \tilde{\mathbf{1}}_{[a,b)}(kx)X_k\right) = (1-2(b-a))V_N^{a,b} + (b-a)^2v_N \\ &\sim (b-a)(1-(b-a))v_N. \end{aligned} \quad (11)$$

Recall the Esseen's inequality below. For a sequence $\{Y_n\}$ of independent random variables with $EY_j = 0$ and $E|Y_j|^3 < \infty$, we have

$$\sup_{-\infty < t < \infty} \left| P\left(B_n^{-1/2} \sum_{j=1}^n Y_j \leq t\right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-u^2/2} du \right| \leq \gamma B_n^{-3/2} \sum_{j=1}^n E|Y_j|^3,$$

where $B_n = \sum_{j=1}^n VY_j$ and γ is an absolute constant.

We apply this inequality for $Y_k = \mathbf{1}_{[A,B)}\langle kx \rangle \tilde{X}_k$ where $0 \leq A < B < 1$. It is clear that $B_n = V_n^{a,b}$. Because of $|Y_j| \leq 1$, we have $E|Y_j|^3 \leq EY_j^2$ and $B_n^{-3/2} \sum_{j=1}^n E|Y_j|^3 \leq B_n^{-1/2} = (V_n^{a,b})^{-1/2} = O(v_n^{-1/2})$. By (8) and (10), we have $V_N^{A,B} \geq N^{1/2}$ for large N . Put $\varphi(x) = \sqrt{2x \log \log x}$. By noting Lemma 2, the bound (6), and Esseen's inequality we have

$$\begin{aligned} &P\left(\sup_{0 \leq a < B-A} \max_{n=1, \dots, N} \left| \sum_{k=1}^n \mathbf{1}_{[A, A+a)}\langle kx \rangle \tilde{X}_k \right| \geq \sqrt{2}\varphi(V_N^{A,B})\right) \\ &\leq \sum_{\varepsilon=\pm 1} P\left(\sup_{0 \leq a < B-A} \max_{n=1, \dots, N} \varepsilon \sum_{k=1}^n \mathbf{1}_{[A, A+a)}\langle kx \rangle \tilde{X}_k \geq \sqrt{2}\varphi(V_N^{A,B})\right) \\ &\leq (\log \log 4N)^2 \sum_{\varepsilon=\pm 1} P\left(\sum_{k=1}^N \mathbf{1}_{[A, B)}\langle kx \rangle \varepsilon \tilde{X}_k \geq \sqrt{2}\varphi(V_N^{A,B})\right) \\ &\leq 2(\log \log 4N)^2 \left((\log v_N)^{-2} + O(v_N^{-1/2}) \right) \leq 8(\log \log 4N)^2 (\log N)^{-2} \end{aligned}$$

for large N . By putting $N = 2^k$, the last term is summable in k , and we can conclude that

$$\sup_{0 \leq a < 2^{-M}} \max_{n=1, \dots, 2^k} \left| \sum_{k=1}^n \mathbf{1}_{[I2^{-M}, I2^{-M}+a)}\langle kx \rangle \tilde{X}_k \right| < \sqrt{2}\varphi(V_{2^k}^{I2^{-M}, (I+1)2^{-M}})$$

for large k , a.e. x , a.s. By (10), we have $\varphi(V_{2^k}^{I2^{-M}, (I+1)2^{-M}}) \sim \varphi(v_{2^k})2^{-M/2}$.

By $L_{2^k} = L_{2^{k-1}}$ or $L_{2^{k-1}} + 1$, and by (7), we see that $v_{2^k} \sim 2^k p_{L_{2^k}} \sim 2^k p_{L_{2^{k-1}}} \sim 2v_{2^{k-1}}$. Hence we have $\varphi(v_{2^k})2^{-M/2} \sim \sqrt{2}\varphi(v_{2^{k-1}})2^{-M/2}$ and

$$\sup_{0 \leq a < 2^{-M}} \max_{n=1, \dots, N} \left| \sum_{k=1}^n \mathbf{1}_{[I2^{-M}, I2^{-M}+a)}\langle kx \rangle \tilde{X}_k \right| < 2 \cdot 2^{-M/2} \varphi(v_N),$$

for large N , a.e. x , a.s. Therefore we have

$$\overline{\lim}_{N \rightarrow \infty} \sup_{0 < a < 2^{-M}} \left| \sum_{k=1}^N \mathbf{1}_{[I2^{-M}, I2^{-M}+a)}(kx) \tilde{X}_k \right| \varphi(v_N)^{-1} \leq 2 \cdot 2^{-M/2}, \quad \text{a.e. } x, \text{ a.s.}$$

On the other hand, the ordinary law of the iterated logarithm implies

$$\overline{\lim}_{N \rightarrow \infty} \sup_{0 < a < 2^{-M}} \left| \sum_{k=1}^N a \tilde{X}_k \right| \varphi(v_N)^{-1} = \overline{\lim}_{N \rightarrow \infty} 2^{-M} \left| \sum_{k=1}^N \tilde{X}_k \right| \varphi(v_N)^{-1} = 2^{-M}, \quad \text{a.e. } x, \text{ a.s.}$$

By (9), we have

$$\overline{\lim}_{N \rightarrow \infty} \sup_{0 < a < 2^{-M}} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{[I2^{-M}, I2^{-M}+a)}(kx) EX_k \right| \varphi(v_N)^{-1} = 0, \quad \text{a.e. } x, \text{ a.s.}$$

By applying these to $|\sum \tilde{\mathbf{1}}X| \leq |\sum \mathbf{1}\tilde{X}| + |\sum a\tilde{X}| + |\sum \tilde{\mathbf{1}}EX|$, we have

$$\overline{\lim}_{N \rightarrow \infty} \sup_{0 < a < 2^{-M}} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{[I2^{-M}, I2^{-M}+a)}(kx) X_k \right| \varphi(v_N)^{-1} \leq 3 \cdot 2^{-M/2}, \quad \text{a.e. } x, \text{ a.s.} \quad (12)$$

The law of the iterated logarithm for bounded centered random variables claims

$$\overline{\lim}_{N \rightarrow \infty} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{[0,a)}(kx) \tilde{X}_k \right| \varphi(\tilde{V}_N^{0,a})^{-1} = 1, \quad \text{a.e. } x, \text{ a.s.}$$

By (11), we have

$$\overline{\lim}_{N \rightarrow \infty} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{[0,a)}(kx) \tilde{X}_k \right| \varphi(v_N)^{-1} = \sqrt{a(1-a)} \quad \text{a.e. } x, \text{ a.s.}$$

Moreover we have

$$\overline{\lim}_{N \rightarrow \infty} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{[0,a)}(kx) EX_k \right| \varphi(v_N)^{-1} = 0, \quad \text{a.e. } x, \text{ a.s.}$$

because of (9). By these we have

$$\overline{\lim}_{N \rightarrow \infty} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{[0,a)}(kx) X_k \right| \varphi(v_N)^{-1} = \sqrt{a(1-a)}, \quad \text{a.e. } x, \text{ a.s.} \quad (13)$$

In general, for a function $g(N, a)$, we have $\overline{\lim}_N g(N, a') \leq \overline{\lim}_N \sup_a g(N, a)$ for all a' and hence $\sup_a \overline{\lim}_N g(N, a) \leq \overline{\lim}_N \sup_a g(N, a)$. Thereby, we have

$$\frac{1}{2} = \sup_{0 < a < 1} \sqrt{a(1-a)} \leq \overline{\lim}_{N \rightarrow \infty} \sup_{0 < a < 1} \left| \sum_{k=1}^N \tilde{\mathbf{1}}_{[0,a)}(kx) X_k \right| \varphi(v_N)^{-1}, \quad \text{a.e. } x, \text{ a.s.}$$

For all M and $a \in [0, 1)$, by taking I satisfying $I2^{-M} \leq a < (I+1)2^{-M}$ and noting $\tilde{\mathbf{I}}_{[0,a)} = \tilde{\mathbf{I}}_{[0,I2^{-M})} + \tilde{\mathbf{I}}_{[I2^{-M},a)}$, we have

$$\begin{aligned} & \sup_{0 < a < 1} \left| \sum_{k=1}^N \tilde{\mathbf{I}}_{[0,a)}(kx) X_k \right| \\ & \leq \max_{I=0, \dots, 2^M-1} \left(\left| \sum_{k=1}^N \tilde{\mathbf{I}}_{[0,I2^{-M})}(kx) X_k \right| + \sup_{a < 2^{-M}} \left| \sum_{k=1}^N \tilde{\mathbf{I}}_{[I2^{-M}, I2^{-M}+a)}(kx) X_k \right| \right). \end{aligned}$$

By (12) and (13), we have

$$\sup_{0 < a < 1} \left| \sum_{k=1}^N \tilde{\mathbf{I}}_{[0,a)}(kx) X_k \right| \varphi(v_N)^{-1} \leq \frac{1}{2} + 3 \cdot 2^{-M/2}, \quad \text{a.e. } x, \text{ a.s.}$$

By letting $M \rightarrow \infty$, we have

$$\overline{\lim}_{N \rightarrow \infty} \sup_{0 < a < 1} \left| \sum_{k=1}^N \tilde{\mathbf{I}}_{[0,a)}(kx) X_k \right| \varphi(v_N)^{-1} \leq \frac{1}{2}, \quad \text{a.e. } x, \text{ a.s.}$$

By the law of the iterated logarithm, we have $|\Xi_N - e_N| = O((v_N \log \log v_N)^{1/2})$. By $v_N \sim e_N$, we have $\Xi_N \sim v_N$, a.s. Hence

$$\overline{\lim}_{N \rightarrow \infty} \sup_{0 < a < 1} \left| \sum_{k=1}^N \tilde{\mathbf{I}}_{[0,a)}(kx) X_k \right| \varphi(\Xi_N)^{-1} = \frac{1}{2} \quad \text{a.e. } x, \text{ a.s.} \quad (14)$$

By arranging $\{k : X_k = 1\}$ in increasing order and denoting by $\{n_k\}$, we have (2).

If $n_k \in I_L$, then we have $n_{k+1} - n_k \leq 2^L \leq G(2^{\eta(L)+L}/8 \log \log 2^{\eta(L)+L})$ by (5). By $2^{\eta(L)+L} \leq n_k$, we have $k \geq \Xi(2^{\eta(L)+L}) \sim v_{2^{\eta(L)+L}} \geq 2^{\eta(L)+L}/4 \log \log 2^{\eta(L)+L}$. Hence we have $G(2^{\eta(L)+L}/8 \log \log 2^{\eta(L)+L}) \leq G(k)$ and $n_{k+1} - n_k \leq G(k)$ for large k . If we redefine first finitely many terms to satisfy $n_{k+1} - n_k \leq G(k)$, we have desired $\{n_k\}$.

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