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Metric discrepancy results for subsequences of $\{\theta^k x\}$

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Abstract We determine the set of possible constants appearing in the law of the iterated logarithm for discrepancies of subsequences of $\{\theta^k x\}$. We prove that the set coincides with the interval $[1/2, \Sigma_\theta]$, where Σ_θ is the constant for $\{\theta^k x\}$. It implies that the discrepancies decrease if we take a subsequence of $\{\theta^k x\}$.

Keywords discrepancy · lacunary sequence · law of the iterated logarithm

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1 Introduction

Let us recall the definition of discrepancies of a sequence $\{a_k\}$ of real numbers:

$$D_N\{a_k\} = \sup_{0 \leq a' < a < 1} \left| \frac{1}{N} \sum_{k=1}^N \tilde{\mathbf{1}}_{a',a}(a_k) \right|; \quad D_N^*\{a_k\} = \sup_{0 \leq a < 1} \left| \frac{1}{N} \sum_{k=1}^N \tilde{\mathbf{1}}_{0,a}(a_k) \right|;$$

where $\tilde{\mathbf{1}}_{a',a}(x) = \mathbf{1}_{[a',a)}(\langle x \rangle) - (a - a')$, $\langle x \rangle$ denotes the fractional part $x - [x]$ of x , and $\mathbf{1}_{[a',a)}$ denotes the indicator function of $[a', a)$. We study the value distribution of

$$\Sigma\{n_k x\} = \overline{\lim}_{N \rightarrow \infty} \frac{ND_N\{n_k x\}}{\sqrt{2N \log \log N}} \quad \text{and} \quad \Sigma^*\{n_k x\} = \overline{\lim}_{N \rightarrow \infty} \frac{ND_N^*\{n_k x\}}{\sqrt{2N \log \log N}}.$$

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Philipp [12, 13] proved the bounded law of the iterated logarithm

$$1/(4\sqrt{2}) < \Sigma^*\{n_k x\} \leq \Sigma\{n_k x\} \leq C(q) < \infty, \quad \text{a.e.}$$

for $\{n_k\}$ satisfying Hadamard's gap condition

$$n_{k+1}/n_k > q > 1. \quad (1)$$

Recently it became possible to compute the exact values of $\Sigma^*\{n_k x\}$ and $\Sigma\{n_k x\}$. Actually the following results were proved in [6, 8].

For all real numbers $\theta > 1$, there exists a constant Σ_θ such that

$$\Sigma^*\{\theta^k x\} = \Sigma\{\theta^k x\} = \Sigma_\theta, \quad \text{a.e.} \quad (2)$$

We have $\Sigma_\theta = 1/2$ if and only if θ satisfies the condition

$$\theta^r \notin \mathbf{Q} \quad \text{for all } r \in \mathbf{N}. \quad (3)$$

Otherwise express θ in the following way:

$$\theta = \sqrt[r]{p/q} \quad \text{where } r = \min\{n \in \mathbf{N} \mid \theta^n \in \mathbf{Q}\}, \quad p, q \in \mathbf{N}, \quad \text{and } \gcd(p, q) = 1. \quad (4)$$

In this case Σ_θ does not depend on r . It is evaluated in the following cases:

$$\Sigma_\theta = \begin{cases} \sqrt{(pq+1)/(pq-1)}/2 & \text{if } p \text{ and } q \text{ are odd,} \\ \sqrt{(p+1)p(p-2)/(p-1)^3}/2 & \text{if } p \geq 4 \text{ is even and } q = 1, \\ \sqrt{42}/9 & \text{if } p = 2 \text{ and } q = 1, \\ \sqrt{22}/9 & \text{if } p = 5 \text{ and } q = 2. \end{cases}$$

In this note we determine the set of possible limsup values of $\Sigma^*\{\theta^{m(k)} x\}$ and $\Sigma\{\theta^{m(k)} x\}$ where $\{\theta^{m(k)} x\}$ is a subsequence of $\{\theta^k x\}$.

Denote by $\mathcal{N}$ the set of all strictly increasing sequences of natural numbers.

Theorem 1 *For any $\theta > 1$ and $\{m(k)\} \in \mathcal{N}$, there exists a constant $\Sigma_{\theta, \{m(k)\}} \geq 1/2$ such that*

$$\Sigma^*\{\theta^{m(k)} x\} = \Sigma\{\theta^{m(k)} x\} = \Sigma_{\theta, \{m(k)\}}, \quad \text{a.e.} \quad (5)$$

We have

$$\{\Sigma_{\theta, \{m(k)\}} \mid \{m(k)\} \in \mathcal{N}\} = [1/2, \Sigma_\theta].$$

By applying the same idea used in [7], we can prove the following corollary.

Corollary 1 *For any $\sigma \in [1/2, \Sigma_\theta]$, there exists a permutation π of $\mathbf{N}$ such that*

$$\Sigma^*\{\theta^{\pi(k)} x\} = \Sigma\{\theta^{\pi(k)} x\} = \sigma, \quad \text{a.e.}$$

Actually, we can prove a stronger result that the set of all possible values of $\Sigma\{\theta^{\pi(k)} x\}$ for permutations π of $\mathbf{N}$ coincides with $[1/2, \Sigma_\theta]$. We will visit this topic in a separate paper. As to the comprehensive studies on effects of permutation on limit theorems for lacunary series, we refer the readers to the survey [2] by Aistleitner-Berkes-Tichy.

2 Preliminary

For $x, y, \xi, \eta \in [0, 1]$, put $V(x, \xi) = x \wedge \xi - x\xi$ and $\tilde{V}(x, y, \xi, \eta) = V(x, \xi) + V(y, \eta) - V(x, \eta) - V(y, \xi)$. Note $V(x, \xi) = \tilde{V}(0, x, 0, \xi)$.

Lemma 1 *Let $a, b, c \in [0, 1]$. For any natural numbers P and Q ,*

$$V(\langle Pa \rangle, \langle Qb \rangle) \leq PQV(a, b). \quad (6)$$

If $\gcd(P, Q) = 1$,

$$\int_0^1 \tilde{\mathbf{1}}_{a,b}(Px) \tilde{\mathbf{1}}_{a,b}(Qx) dx = \frac{1}{PQ} \tilde{V}(\langle Pa \rangle, \langle Pb \rangle, \langle Qa \rangle, \langle Qb \rangle), \quad (7)$$

$$\tilde{V}(\langle Pa \rangle, \langle Pb \rangle, \langle Qa \rangle, \langle Qb \rangle) \leq V(\langle P(b-a) \rangle, \langle Q(b-a) \rangle), \quad (8)$$

If $\gcd(PP', QQ') = 1$,

$$\int_0^1 \tilde{\mathbf{1}}_{0,c}(PP'x) \tilde{\mathbf{1}}_{0,c}(QQ'x) dx \leq \int_0^1 \tilde{\mathbf{1}}_{0,c}(Px) \tilde{\mathbf{1}}_{0,c}(Qx) dx. \quad (9)$$

Proof Let us prove (6). We trivially have $\langle Pa \rangle \leq Pa$. By putting $a = 1 - b$, we have $1 - \langle Qb \rangle = \langle Q(1 - b) \rangle \leq Q(1 - b)$. By noting $V(x, \xi) = x(1 - \xi) \wedge (1 - x)\xi$, we have

$$\begin{aligned} V(\langle Pa \rangle, \langle Qb \rangle) &= \langle Pa \rangle(1 - \langle Qb \rangle) \wedge (1 - \langle Pa \rangle)\langle Qb \rangle \\ &\leq PaQ(1 - b) \wedge P(1 - a)Qb = PQV(a, b). \end{aligned}$$

(7) and (8) are proved in [6]. By (6), we have $V(\langle PP'c \rangle, \langle QQ'c \rangle) \leq P'Q'V(\langle Pc \rangle, \langle Qc \rangle)$. By noting (7), we have (9). $\square$

For a function f of bounded variation with period 1 satisfying $\int_0^1 f = 0$, put

$$\sigma^2(f, \theta) = \begin{cases} \int_0^1 f^2(y) dy & \text{if } \theta \text{ satisfies (3),} \\ \int_0^1 f^2(y) dy + 2 \sum_{k=1}^{\infty} \int_0^1 f(p^k y) f(q^k y) dy & \text{if } \theta \text{ is given by (4).} \end{cases}$$

For a bounded measurable function g , we define the mean value $\int_{\mathbf{R}} g(x) \mu_R(dx)$ by

$$\int_{\mathbf{R}} g(x) \mu_R(dx) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T g(x) dx$$

if the limit on the right hand side exists. For a trigonometric polynomial g with period 1 satisfying $\int_0^1 g = 0$, we have $\int_{\mathbf{R}} g(\Theta x) g(x) \mu_R(dx) = 0$ for $\Theta \notin \mathbf{Q}$, and $\int_{\mathbf{R}} g((P/Q)x) g(x) \mu_R(dx) = \int_{\mathbf{R}} g(Px) g(Qx) \mu_R(dx) = \int_0^1 g(Px) g(Qx) dx$ for non-zero integers P and Q .

Suppose $0 \leq a < b < 1$. Denote by $\tilde{\mathbf{1}}_{a,b;d}$ the d -th subsum of the Fourier series of $\tilde{\mathbf{1}}_{a,b}$. Put

$$f_{a,b;d}(x) = \tilde{\mathbf{1}}_{a,b;d}(x) + \frac{1}{pd+1} \cos 2\pi(pd+1)x,$$

where p is that in (4) if θ is given by (4), and $p = 1$ if θ satisfies (3). Here we added the term $\frac{1}{pd+1} \cos 2\pi(pd+1)x$ additionally to have the lower bound (11) for the variance of summation. It makes easier to apply the martingale approximation machinery.

Lemma 2 *We have*

$$\begin{aligned} \int_{\mathbf{R}} \left(\sum_{k=M+1}^{M+N} f_{a,b;d}(\theta^{m(k)}x) \right)^2 \mu_R(dx) \\ \leq \int_{\mathbf{R}} \left(\sum_{k=M+1}^{M+N} f_{0,b-a;d}(\theta^{m(k)}x) \right)^2 \mu_R(dx) + NC_\theta \frac{\log d}{d}, \end{aligned} \quad (10)$$

$$\frac{N}{2(pd+1)^2} \leq \int_{\mathbf{R}} \left(\sum_{k=M+1}^{M+N} f_{a,b;d}(\theta^{m(k)}x) \right)^2 \mu_R(dx) \leq N \left(\sigma^2(\tilde{\mathbf{1}}_{0,b-a}, \theta) + C_\theta \frac{\log d}{d} \right), \quad (11)$$

$$\int_{\mathbf{R}} \left(\sum_{k=M+1}^{M+N} f_{0,1/2;d}(\theta^{m(k)}x) \right)^2 \mu_R(dx) \geq N \left(\frac{1}{4} - C_\theta \frac{\log d}{d} \right), \quad (12)$$

where C_θ is a constant depending only on θ .

Proof Let f be a trigonometric polynomial with period 1 satisfying $\int_0^1 f = 0$.

Assume that θ is given by (4), and put $I = \{m(k) \mid k = M+1, \dots, M+N\}$, $I_j = I \cap (r\mathbf{Z} + j)$, and $\kappa(j) = \#I_j$ ($j = 0, \dots, r-1$). Let us take $i(j; 1) < \dots < i(j; \kappa(j))$ such that $I_j = \{ri(j; 1) + j, \dots, ri(j; \kappa(j)) + j\}$. By noting $I = I_0 \cup \dots \cup I_{r-1}$ and $\int_{\mathbf{R}} f(\theta^{ri(j;s)+j}x) f(\theta^{ri(j';s')+j'}x) \mu_R(dx) = 0$ for $0 \leq j < j' \leq r-1$, we have

$$\int_{\mathbf{R}} \left(\sum_{k=M+1}^{M+N} f(\theta^{m(k)}x) \right)^2 \mu_R(dx) = \sum_{j=0}^{r-1} \int_{\mathbf{R}} \left(\sum_{s=1}^{\kappa(j)} f(\theta^{ri(j;s)+j}x) \right)^2 \mu_R(dx).$$

By noting

$$\int_{\mathbf{R}} f(\theta^{ri(j;s)+j}x) f(\theta^{ri(j';s')+j'}x) \mu_R(dx) = \int_0^1 f(p^{i(j;s')-i(j;s)}x) f(q^{i(j;s')-i(j;s)}x) dx$$

for $s' \geq s$, and by $N = \kappa(0) + \dots + \kappa(r-1)$, we have

$$\begin{aligned} \int_{\mathbf{R}} \left(\sum_{k=M+1}^{M+N} f(\theta^{m(k)}x) \right)^2 \mu_R(dx) = \\ N \int_0^1 f^2(x) dx + 2 \sum_{j=0}^{r-1} \sum_{1 \leq s < s' \leq \kappa(j)} \int_0^1 f(p^{i(j;s')-i(j;s)}x) f(q^{i(j;s')-i(j;s)}x) dx. \end{aligned} \quad (13)$$

Denote the right hand side of (13) by $\Upsilon(f)$. We clearly have $\Upsilon(\tilde{\mathbf{1}}_{a,b;d}) \geq 0$. To prove (10), (11) and (12), it is sufficient to prove

$$\Upsilon(\tilde{\mathbf{1}}_{a,b}) \leq \Upsilon(\tilde{\mathbf{1}}_{0,b-a}) \leq N\sigma^2(\tilde{\mathbf{1}}_{0,b-a}, \theta), \quad (14)$$

$$\Upsilon(\tilde{\mathbf{1}}_{0,1/2}) \geq N/4, \quad (15)$$

$$\Upsilon(f_{a,b;d}) = \Upsilon(\tilde{\mathbf{1}}_{a,b;d}) + \frac{N}{2(pd+1)^2}, \quad (16)$$

$$|\Upsilon(\tilde{\mathbf{1}}_{a,b}) - \Upsilon(\tilde{\mathbf{1}}_{a,b;d})| \leq 2N \left(\frac{\log d}{d \log p} + \frac{1}{(1-1/p)d} \right). \quad (17)$$

First let us prove (14). $\mathcal{R}(\tilde{\mathbf{I}}_{a,b}) \leq \mathcal{R}(\tilde{\mathbf{I}}_{0,b-a})$ is clear from (7) and (8). Let $s' \geq s$. Denoting $i(j; s') - i(j; s) - (s' - s) = A \geq 0$, we have $p^{i(j; s') - i(j; s)} = p^A p^{s' - s}$ and $q^{i(j; s') - i(j; s)} = q^A q^{s' - s}$. Hence by applying (9), we have

$$\int_0^1 \tilde{\mathbf{I}}_{0,c}(p^{i(j; s') - i(j; s)} x) \tilde{\mathbf{I}}_{0,c}(q^{i(j; s') - i(j; s)} x) dx \leq \int_0^1 \tilde{\mathbf{I}}_{0,c}(p^{s' - s} x) \tilde{\mathbf{I}}_{0,c}(q^{s' - s} x) dx,$$

where $c = b - a$. Therefore we have

$$\begin{aligned} \mathcal{R}(\tilde{\mathbf{I}}_{0,c}) &\leq N \int_0^1 \tilde{\mathbf{I}}_{0,c}^2(x) dx + 2 \sum_{j=0}^{r-1} \sum_{1 \leq s < s' \leq \kappa(j)} \int_0^1 \tilde{\mathbf{I}}_{0,c}(p^{s' - s} x) \tilde{\mathbf{I}}_{0,c}(q^{s' - s} x) dx \\ &\leq N \int_0^1 \tilde{\mathbf{I}}_{0,c}^2(x) dx + 2 \sum_{j=0}^{r-1} \kappa(j) \sum_{k=1}^{\infty} \int_0^1 \tilde{\mathbf{I}}_{0,c}(p^k x) \tilde{\mathbf{I}}_{0,c}(q^k x) dx \\ &= N \sigma^2(\tilde{\mathbf{I}}_{0,c}, \theta). \end{aligned}$$

By noting (7) we have $\int_0^1 \tilde{\mathbf{I}}_{0,1/2}(Px) \tilde{\mathbf{I}}_{0,1/2}(Qx) dx = V(\langle P/2 \rangle, \langle Q/2 \rangle) / PQ \geq 0$ and $\int_0^1 \tilde{\mathbf{I}}_{0,1/2}^2(x) dx = V(1/2, 1/2) = 1/4$, which prove (15).

Suppose $A \geq 1$. We prove $\widehat{f}_{a,b;d}(p^A \nu) \overline{\widehat{f}_{a,b;d}(q^A \nu)} = \widehat{\mathbf{I}}_{a,b;d}(p^A \nu) \overline{\widehat{\mathbf{I}}_{a,b;d}(q^A \nu)}$. Since $\widehat{f}_{a,b;d}(n) = \widehat{\mathbf{I}}_{a,b;d}(n)$ for all $n \neq \pm(pd + 1)$, we see $\widehat{f}_{a,b;d}(p^A \nu) = \widehat{\mathbf{I}}_{a,b;d}(p^A \nu)$. If $\widehat{f}_{a,b;d}(q^A \nu) \neq \widehat{\mathbf{I}}_{a,b;d}(q^A \nu)$, then $|q^A \nu| = pd + 1$. Hence we have $|p^A \nu| > |q^A \nu| = pd + 1$ and thereby $\widehat{f}_{a,b;d}(p^A \nu) = \widehat{\mathbf{I}}_{a,b;d}(p^A \nu) = 0$, which implies the conclusion. By $\int_0^1 f(p^A x) f(q^A x) dx = \sum_{\nu} \widehat{f}(p^A \nu) \overline{\widehat{f}(q^A \nu)}$, we have

$$\int_0^1 f_{a,b;d}(p^A x) f_{a,b;d}(q^A x) dx = \int_0^1 \tilde{\mathbf{I}}_{a,b;d}(p^A x) \tilde{\mathbf{I}}_{a,b;d}(q^A x) dx \quad (18)$$

for $A \geq 1$ and

$$\int_0^1 f_{a,b;d}^2(x) dx = \frac{1}{2(pd + 1)^2} + \int_0^1 \tilde{\mathbf{I}}_{a,b;d}^2(x) dx. \quad (19)$$

Therefore we have (16).

To prove (17), put

$$G(A) = \left| \int_0^1 \tilde{\mathbf{I}}_{a,b}(p^A x) \tilde{\mathbf{I}}_{a,b}(q^A x) dx - \int_0^1 \tilde{\mathbf{I}}_{a,b;d}(p^A x) \tilde{\mathbf{I}}_{a,b;d}(q^A x) dx \right|. \quad (20)$$

We have $G(A) \leq \sum_{|\nu| > d/p^A} |\widehat{\mathbf{I}}_{a,b}(q^A \nu) \widehat{\mathbf{I}}_{a,b}(p^A \nu)| \leq (2/\pi^2) \sum_{|\nu| > d/p^A} 1/p^A \nu^2$. By putting

$$\tilde{G}(A) = \begin{cases} 1/d & A < \log d / \log p, \\ 1/p^A & A \geq \log d / \log p, \end{cases}$$

we have $G(A) \leq \tilde{G}(A)$. Therefore we have

$$\begin{aligned} |\mathcal{R}(\tilde{\mathbf{I}}_{a,b}) - \mathcal{R}(\tilde{\mathbf{I}}_{a,b;d})| &\leq N \tilde{G}(0) + 2 \sum_{j=0}^{r-1} \sum_{1 \leq s < s' \leq \kappa(j)} \tilde{G}(i(j; s') - i(j; s)) \\ &\leq N \tilde{G}(0) + 2 \sum_{j=0}^{r-1} \sum_{1 \leq s < s' \leq \kappa(j)} \tilde{G}(s' - s) \leq 2N \sum_{A=0}^{\infty} \tilde{G}(A) \leq 2N \left(\frac{\log d}{d \log p} + \frac{1}{(1 - 1/p)d} \right). \end{aligned}$$

When θ satisfies (3), by noting $\int_{\mathbf{R}} f(\theta^{m(k)}x) f(\theta^{m(k')}x) \mu_R(dx) = 0$ for $k \neq k'$, and $\int_{\mathbf{R}} f^2(\theta^{m(k)}x) \mu_R(dx) = \int_0^1 f^2(x) dx$, we have

$$\begin{aligned} \int_{\mathbf{R}} \left(\sum_{k=M+1}^{M+N} f_{a,b;d}(\theta^{m(k)}x) \right)^2 \mu_R(dx) &= N \int_0^1 f_{a,b;d}^2(x) dx \\ &= N \int_0^1 \tilde{\mathbf{I}}_{a,b;d}^2(x) dx + \frac{N}{2(pd+1)^2}. \end{aligned}$$

By $G(0) \leq 1/d$, $\sigma(\tilde{\mathbf{I}}_{a,b}, \theta) = \sigma(\tilde{\mathbf{I}}_{0,b-a}, \theta)$, and $\sigma(\tilde{\mathbf{I}}_{0,1/2}, \theta) = 1/2$, we have (10), (11), and (12). $\square$

Lemma 3 *If g is a bounded measurable function with period 1 satisfying $\int_0^1 g = 0$, then for all $a < b$ and $\lambda > 0$, we have*

$$\left| \int_a^b g(\lambda x) dx \right| \leq \frac{\|g\|_{\infty}}{\lambda}.$$

Proof By changing variable $y = \lambda x$, the integral equals to

$$\frac{1}{\lambda} \int_{\lambda a}^{\lambda b} g(y) dy = \frac{1}{\lambda} \sum_{k=0}^{[\lambda(b-a)]-1} \int_{\lambda a+k}^{\lambda a+k+1} g(y) dy + \frac{1}{\lambda} \int_{\lambda a+[\lambda(b-a)]}^{\lambda b} g(y) dy.$$

We have $|\int_{\lambda a+[\lambda(b-a)]}^{\lambda b} g(y) dy| \leq (\lambda(b-a) - [\lambda(b-a)])\|g\|_{\infty} \leq \|g\|_{\infty}$, which completes the proof. $\square$

Lemma 4 *Let g be a trigonometric polynomial with period 1 and degree d satisfying $\int_0^1 g = 0$. There exists a constant C_{θ} depending only on θ such that, for a sequence $\{\lambda_k\}$ of real numbers satisfying Hadamard's gap condition $\lambda_{k+1}/\lambda_k \geq \theta > 1$ and $\lambda_1 \geq 1$,*

$$\int_0^1 \left(\sum_{k=M+1}^{M+N} g(\lambda_k x) \right)^4 dx \leq C_{\theta} \left(\sum_{|\nu| \leq d} |\hat{g}(\nu)| \right)^4 N^2.$$

Proof Applying the triangle inequality for L^4 -norm, we see that the left hand side is bounded by $(\sum_{|\nu| \leq d} |\hat{g}(\nu)|)^4 \int_0^1 (\sum_{k=M+1}^{M+N} \exp(2\pi\sqrt{-1}\lambda_k \nu x))^4 dx$. Hence by applying the following inequality (Lemma 1 (1) of [5]), we have the conclusion:

$$\int_0^1 \left(\sum_{j=1}^{\infty} (c_j \cos 2\pi\lambda_j x + d_j \sin 2\pi\lambda_j x) \right)^4 dx \leq C_{\theta} \left(\sum_{j=1}^{\infty} (c_j^2 + d_j^2) \right)^2. \square$$

Lemma 5 *Let $\{Z_k\}$ and $\{Z'_k\}$ be standard normal i.i.d. Suppose that $\{v_k\}$ and $\{v'_k\}$ are sequences of positive numbers satisfying $c_1 i \leq v_i \leq c_2 i$, $d_1 i \leq v'_i \leq d_2 i$, and $v_i \leq v'_i + \gamma i$ for some $0 < c_1 < c_2 < \infty$, $0 < d_1 < d_2 < \infty$, and $0 < \gamma < \infty$. Put $l_M = M(M+1)/2$, $\beta_M = v_1 + \dots + v_M$, $\beta'_M = v'_1 + \dots + v'_M$, and $\phi(x) = \sqrt{2x \log \log x}$. Then we have*

$$\sqrt{c_1} \leq \lim_{M \rightarrow \infty} \frac{1}{\phi(l_M)} \left| \sum_{k \leq \beta_M} Z_k \right| \leq \lim_{M \rightarrow \infty} \frac{1}{\phi(l_M)} \left| \sum_{k \leq \beta'_M} Z'_k \right| + \sqrt{\gamma} \leq \sqrt{d_2} + \sqrt{\gamma}, \quad a.s.,$$

where all limsups above are constants a.s.

Proof By 0-1 law, we see that all limsups are constants a.s. Hence it is sufficient to prove the conclusion by putting $Z'_k = Z_k$.

We use the following result ((1.2.4) in Theorem 1.2.1 of [4]). Let $W(t)$ be a standard brownian motion. For non-decreasing A_T such that $0 < A_T \leq T$ and T/A_T is non-decreasing, we have

$$\overline{\lim}_{T \rightarrow \infty} \sup_{0 \leq t \leq T - A_T} \sup_{0 \leq s \leq A_T} B_T |W(t+s) - W(t)| = 1, \quad \text{a.s.}$$

where $B_T^2 = 1/(2A_T(\log(T/A_T) + \log \log T))$. By putting $A_T = 1$, $A_T = C\sqrt{T}$, and $A_T = DT$ ($0 < D < 1$), we have

$$\sup_{0 \leq s \leq 1} |W(t+s) - W(t)| = O((\log t)^{1/2}), \quad (t \rightarrow \infty) \quad \text{a.s.} \quad (21)$$

$$\sup_{0 \leq s \leq C\sqrt{t}} |W(t+s) - W(t)| = O((\sqrt{t} \log t)^{1/2}), \quad (t \rightarrow \infty) \quad \text{a.s.} \quad (22)$$

$$\sup_{0 \leq u \leq (1-D)s} |W(u+Ds) - W(u)| \leq (1+o(1))\sqrt{2Ds \log \log s}, \quad (s \rightarrow \infty) \quad \text{a.s.} \quad (23)$$

It is sufficient to prove

$$\sqrt{c_1} \leq \overline{\lim}_{M \rightarrow \infty} \frac{|W(\lfloor \beta_M \rfloor)|}{\phi(l_M)} \leq \overline{\lim}_{M \rightarrow \infty} \frac{|W(\lfloor \beta'_M \rfloor)|}{\phi(l_M)} + \sqrt{\gamma} \leq \sqrt{d_2} + \sqrt{\gamma}, \quad \text{a.s.}$$

Note that $c_1 l_M < \beta_M < c_2 l_M$ and $d_1 l_M < \beta'_M < d_2 l_M$. By applying (21), we have $|W(\beta_M) - W(\lfloor \beta_M \rfloor)| = O((\log l_M)^{1/2}) = o(\phi(l_M))$ and thereby

$$\overline{\lim}_{M \rightarrow \infty} \frac{|W(\beta_M)|}{\phi(l_M)} = \overline{\lim}_{M \rightarrow \infty} \frac{|W(\lfloor \beta_M \rfloor)|}{\phi(l_M)} \quad \text{and} \quad \overline{\lim}_{M \rightarrow \infty} \frac{|W(\beta'_M)|}{\phi(l_M)} = \overline{\lim}_{M \rightarrow \infty} \frac{|W(\lfloor \beta'_M \rfloor)|}{\phi(l_M)}, \quad \text{a.s.}$$

Let us define functions $\beta(t)$, $\beta'(t)$, and $l(t)$ by putting $\beta(M) = \beta_M$, $\beta'(M) = \beta'_M$, and $l(M) = l_M$ for $M \in \mathbb{N}$ and linearly interpolating on each interval $[M, M+1]$. Clearly they are strictly increasing functions satisfying $c_1 l(t) < \beta(t) < c_2 l(t)$, $d_1 l(t) < \beta'(t) < d_2 l(t)$, and $\beta(t) < \beta'(t) + \gamma l(t)$

Thanks to $\beta_M - \beta_{M-1} = O(l_M^{1/2})$, by applying (22), we have

$$\sup_{\beta_{M-1} \leq t \leq \beta_M} |W(t) - W(\beta_M)| = O((l_M^{1/2} \log l_M)^{1/2}) = o(\phi(l_M)), \quad \text{a.s.}$$

and hence

$$\overline{\lim}_{M \rightarrow \infty} \frac{|W(\beta_M)|}{\phi(l_M)} = \overline{\lim}_{t \rightarrow \infty} \frac{|W(\beta(t))|}{\phi(l(t))} \quad \text{and} \quad \overline{\lim}_{M \rightarrow \infty} \frac{|W(\beta'_M)|}{\phi(l_M)} = \overline{\lim}_{t \rightarrow \infty} \frac{|W(\beta'(t))|}{\phi(l(t))}, \quad \text{a.s.}$$

By $\beta(t) \geq c_1 l(t)$, we have

$$\overline{\lim}_{t \rightarrow \infty} \frac{|W(\beta(t))|}{\phi(l(t))} = \overline{\lim}_{t \rightarrow \infty} \frac{|W(\beta(t))|}{\phi(\beta(t))} \frac{\phi(\beta(t))}{\phi(l(t))} \geq \sqrt{c_1} \overline{\lim}_{t \rightarrow \infty} \frac{|W(\beta(t))|}{\phi(\beta(t))} = \sqrt{c_1}, \quad \text{a.s.}$$

By $\beta'(t) \leq d_2 l(t)$, in the same way we have

$$\overline{\lim}_{t \rightarrow \infty} \frac{|W(\beta'(t))|}{\phi(l(t))} \leq \sqrt{d_2}, \quad \text{a.s.}$$

Put $\beta''(t) = \beta'(t) + \gamma l(t)$ and denote the inverse functions of $\beta(t)$ and $\beta''(t)$ by $\beta^{-1}(s)$ and $\beta''^{-1}(s)$. By $\beta(t) \leq \beta''(t)$ we have $\beta^{-1}(s) \geq \beta''^{-1}(s)$. Therefore we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{|W(\beta(t))|}{\phi(l(t))} &= \lim_{t \rightarrow \infty} \sup_{s \geq t} \frac{|W(\beta(s))|}{\phi(l(s))} = \lim_{t \rightarrow \infty} \sup_{u \geq t} \frac{|W(u)|}{\phi(l(\beta^{-1}(u)))} \\ &\leq \lim_{t \rightarrow \infty} \sup_{u \geq t} \frac{|W(u)|}{\phi(l(\beta''^{-1}(u)))} = \lim_{t \rightarrow \infty} \frac{|W(\beta''(t))|}{\phi(l(t))}, \quad \text{a.s.} \end{aligned}$$

By applying (23) for $s = (d_2 + \gamma)l(t)$, $u = \beta'(t)$, and $D = \gamma/(d_2 + \gamma)$, we have

$$|W(\beta'(t) + \gamma l(t)) - W(\beta'(t))| \leq (1 + o(1))\sqrt{2\gamma l(t) \log \log l(t)}, \quad \text{a.s.}$$

and hence

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{|W(\beta''(t))|}{\phi(l(t))} &\leq \lim_{t \rightarrow \infty} \frac{|W(\beta'(t))|}{\phi(l(t))} + \lim_{t \rightarrow \infty} \frac{|W(\beta'(t) + \gamma l(t)) - W(\beta'(t))|}{\phi(l(t))} \\ &\leq \lim_{t \rightarrow \infty} \frac{|W(\beta'(t))|}{\phi(l(t))} + \sqrt{\gamma}, \quad \text{a.s.} \quad \square \end{aligned}$$

3 Law of the iterated logarithm

In this section we prove (5) and $\Sigma_{\theta, \{m(k)\}} \in [1/2, \Sigma_\theta]$. Denote $f_{a,b;d}$ simply by f . Put $n_k = \theta^{m(k)}$ and note $n_{k+1}/n_k \geq \theta$. We follow the method of martingale approximation given in [1], which originated with Berkes [3] and Philipp [11].

Let us divide $\mathbf{N}$ into consecutive blocks $\Delta'_1, \Delta_1, \Delta'_2, \Delta_2, \dots$ satisfying $\# \Delta'_i = [1 + 9 \log_\theta i]$ and $\# \Delta_i = i$. Denote $i^- = \min \Delta_i$ and $i^+ = \max \Delta_i$. We have

$$n_{i^-}/n_{(i-1)^+} \geq \theta^{9 \log_\theta i} = i^9.$$

Put $\mu(i) = \lfloor \log_2 i^4 n_{i^+} \rfloor + 1$ and $\mathcal{F}_i = \sigma\{[j2^{-\mu(i)}, (j+1)2^{-\mu(i)}] \mid j = 0, \dots, 2^{\mu(i)} - 1\}$. Note that $i^4 n_{i^+} \leq 2^{\mu(i)} \leq 2i^4 n_{i^+}$. Put

$$T_i(x) = \sum_{k \in \Delta_i} f(n_k x), \quad T'_i(x) = \sum_{k \in \Delta'_i} f(n_k x), \quad Y_i = E(T_i \mid \mathcal{F}_i) - E(T_i \mid \mathcal{F}_{i-1}).$$

Then $\{Y_i, \mathcal{F}_i\}$ forms a martingale difference sequence. Here let us prove

$$\|Y_i - T_i\|_\infty \leq (\|f'\|_\infty + 2\|f\|_\infty)/i^3, \quad (24)$$

$$\|Y_i^2 - T_i^2\|_\infty \leq 3\|f\|_\infty(\|f'\|_\infty + 2\|f\|_\infty)/i^2, \quad (25)$$

$$\|Y_i^4 - T_i^4\|_\infty \leq 15\|f\|_\infty^3(\|f'\|_\infty + 2\|f\|_\infty). \quad (26)$$

If $k \in \Delta_i$ and $x \in I = [j2^{-\mu(i)}, (j+1)2^{-\mu(i)}] \in \mathcal{F}_i$, then we have $|f(n_k x) - E(f(n_k \cdot) \mid \mathcal{F}_i)| = |I|^{-1} \int_I (f(n_k x) - f(n_k y)) dy| \leq \max_{y \in I} |f(n_k x) - f(n_k y)| \leq \|f'\|_\infty n_k 2^{-\mu(i)} \leq \|f'\|_\infty n_k / i^4 n_{i^+} \leq \|f'\|_\infty / i^4$. Hence we obtain $|T_i - E(T_i \mid \mathcal{F}_i)| \leq \|f'\|_\infty \# \Delta_i / i^4 = \|f'\|_\infty / i^3$. Take $J = [j2^{-\mu(i-1)}, (j+1)2^{-\mu(i-1)}] \in \mathcal{F}_{i-1}$. Then by Lemma 3, we have $|E(f(n_k \cdot) \mid \mathcal{F}_{i-1})| = |J|^{-1} \int_J f(n_k y) dy| \leq \|f\|_\infty 2^{\mu(i-1)} / n_k \leq \|f\|_\infty 2(i-1)^4 n_{(i-1)^+} / n_{i^-} \leq 2\|f\|_\infty / i^5$. Thus $|E(T_i \mid \mathcal{F}_{i-1})| \leq 2\|f\|_\infty \# \Delta_i / i^5 = 2\|f\|_\infty / i^4$, and (24) is proved.

By $\|T_i\|_\infty \leq i\|f\|_\infty$, we have $\|E(T_i | \mathcal{F}_i)\|_\infty, \|E(T_i | \mathcal{F}_{i-1})\|_\infty \leq i\|f\|_\infty$. Hence we have $\|Y_i\|_\infty \leq 2i\|f\|_\infty$, $\|Y_i + T_i\|_\infty \leq 3i\|f\|_\infty$, $\|Y_i^2 + T_i^2\|_\infty \leq 5i^2\|f\|_\infty^2$. Apply these to $\|Y_i^2 - T_i^2\|_\infty \leq \|Y_i - T_i\|_\infty \|Y_i + T_i\|_\infty$ and $\|Y_i^4 - T_i^4\|_\infty \leq \|Y_i^2 - T_i^2\|_\infty \|Y_i^2 + T_i^2\|_\infty$. Then we have (25) and (26).

Put $C = \min\{\langle \log_\theta \nu - \log_\theta \nu' \rangle^* \mid \nu, \nu' = 1, \dots, pd+1, \log_\theta \nu - \log_\theta \nu' \notin \mathbf{Z}\} \in (0, 1)$ where $\langle x \rangle^* = \min_{n \in \mathbf{Z}} |x - n|$. Denote $D = \theta^C - 1 > 0$. We prove

$$|\theta^k \nu + \theta^l \nu'| \geq D\theta^L \quad \text{if } k, l \geq L, |\nu|, |\nu'| \leq pd+1, \theta^k \nu + \theta^l \nu' \neq 0. \quad (27)$$

If $\nu\nu' > 0$, then it is trivial. Assume $\nu > 0$ and $\nu' < 0$ and put $\nu'' = -\nu'$. If $\log_\theta \nu - \log_\theta \nu'' \notin \mathbf{Z}$, then we have $|\log_\theta(\theta^k \nu) - \log_\theta(\theta^l \nu'')| = |(k-l) + (\log_\theta \nu - \log_\theta \nu'')| \geq \langle \log_\theta \nu - \log_\theta \nu'' \rangle^* \geq C$. If $\log_\theta \nu - \log_\theta \nu'' \in \mathbf{Z}$, then by $\theta^k \nu \neq \theta^l \nu''$, $\log_\theta(\theta^k \nu) - \log_\theta(\theta^l \nu'')$ is a non-zero integer and hence $|\log_\theta(\theta^k \nu) - \log_\theta(\theta^l \nu'')| \geq 1 \geq C$. Hence $\theta^k \nu / \theta^l \nu'' \geq e^C$ if $\theta^k \nu > \theta^l \nu''$, and $\theta^l \nu'' / \theta^k \nu \geq e^C$ otherwise. By $\theta^k \nu / \theta^l \nu'' \geq e^C$ we have $\theta^k \nu - \theta^l \nu'' \geq (e^C - 1)\theta^l \nu'' \geq D\theta^L$, which is (27). The rest is similar.

If we expand T_i^2 into trigonometric polynomial, the constant term is given by $v_{a,b;d;i} = \int_{\mathbf{R}} T_i^2(x) \mu_R(dx)$. The polynomial $T_i^2 - v_{a,b;d;i}$ has at most $8(d+1)^2 i^2$ terms and the absolute values of frequencies are greater than Dn_{i-} . Therefore, by Lemma 3 again, we have $|E(T_i^2 - v_{a,b;d;i} | \mathcal{F}_{i-1})| \leq 8(d+1)^2 i^2 (1/Dn_{i-}) 2^{\mu(i-1)} = O(1/i^3)$. Hence we have

$$\left\| \sum_{i=1}^M E(T_i^2 | \mathcal{F}_{i-1}) - \beta_{a,b;d;M} \right\|_\infty = O(1), \quad (28)$$

where $\beta_{a,b;d;M} = \sum_{i=1}^M v_{a,b;d;i}$.

Since (25) implies $\left\| \sum_{i=1}^M (E(Y_i^2 | \mathcal{F}_{i-1}) - E(T_i^2 | \mathcal{F}_{i-1})) \right\|_\infty = O(1)$, we have

$$\left\| \sum_{i=1}^M E(Y_i^2 | \mathcal{F}_{i-1}) - \beta_{a,b;d;M} \right\|_\infty = O(1). \quad (29)$$

Denote $l_M = M(M+1)/2$. By (10), (11) and (12), we have

$$\frac{i}{2(pd+1)^2} \leq v_{a,b;d;i} \leq v_{0,b-a;d;i} + iC_\theta \frac{\log d}{d}, \quad (30)$$

$$v_{0,b-a;d;i} \leq i \left(\sigma^2(\tilde{\mathbf{I}}_{0,b-a}, \theta) + C_\theta \frac{\log d}{d} \right) \quad (31)$$

$$v_{0,1/2;d;i} \geq i \left(\frac{1}{4} - C_\theta \frac{\log d}{d} \right), \quad (32)$$

$$l_M \frac{1}{2(pd+1)^2} \leq \beta_{a,b;d;M} \leq l_M \left(\sigma^2(\tilde{\mathbf{I}}_{0,b-a}, \theta) + 2C_\theta \frac{\log d}{d} \right) \quad (33)$$

Now we use the following theorem by Monrad-Philipp [10] which is a version of the theorem by Strassen [14].

Theorem 2 *Let $\{Y_i, \mathcal{F}_i\}$ be a real-valued square-integrable martingale difference sequence. Let ψ be a nondecreasing function with $\psi(x) \rightarrow \infty$ as $x \rightarrow \infty$ and such that $\psi(x)(\log x)^\alpha/x$ is non-increasing for some $\alpha > 50$. Suppose that*

$$V_M = \sum_{i=1}^M E(Y_i^2 | \mathcal{F}_{i-1}) \rightarrow \infty \quad \text{a.s.} \quad \text{and} \quad \sum_{i=1}^\infty E(Y_i^2 \mathbf{1}_{\{Y_i^2 \geq \psi(V_i)\}} / \psi(V_i)) < \infty.$$

Suppose that there exists a random variable which is uniformly distributed over $[0, 1]$ and independent of the sequence $\{Y_n\}$. Then there exists a standard normal i.i.d. $\{Z_i\}$ such that with probability 1,

$$\sum_{i \geq 1} Y_i \mathbf{1}_{\{V_i \leq t\}} = \sum_{i \leq t} Z_i + o(t^{1/2}(\psi(t)/t)^{1/50}), \quad (t \rightarrow \infty).$$

To have an independent uniformly distributed variable, we make a product space with another unit interval equipped with the Lebesgue measure. By Lemma 4 and by (26), we have $EY_i^4 = ET_i^4 + O(1) = O(i^2)$. By (29) and (33), we have $V_M \geq l_M/C''$ for some $C'' > 0$. Hence we can verify the condition in the above theorem by putting $\psi(x) = x/(\log x)^{51}$ as below

$$\sum_i E(Y_i^2 \mathbf{1}_{\{Y_i^2 \geq \psi(V_i)\}} / \psi(V_i)) \leq \sum_i \frac{EY_i^4}{\psi^2(l_i/C'')} = O\left(\sum_i \frac{i^2 (\log l_i)^{102}}{l_i^2}\right) = O(1).$$

Because of $\|Y_i\|_\infty = O(i)$ and (29), we have

$$\sum_{i \geq 1} Y_i \mathbf{1}_{\{V_i \leq \beta_{a,b;d;M}\}} = \sum_{k=1}^M Y_k + O(M) = \sum_{k=1}^M Y_k + o(\phi(l_M)).$$

By (33), we have $\beta_{a,b;d;M} = O(l_M)$. Hence by applying previous theorem for $t = \beta_{a,b;d;M}$ by noting the estimate (24), we have

$$\begin{aligned} \sum_{k=1}^M T_k &= \sum_k Y_k + O(1) = \sum_{i \geq 1} Y_i \mathbf{1}_{\{V_i \leq \beta_{a,b;d;M}\}} + o(\phi(l_M)) \\ &= \sum_{i \leq \beta_{a,b;d;M}} Z_i + o(\phi(l_M)) \quad \text{a.s.} \end{aligned}$$

Therefore we have

$$\overline{\lim}_{M \rightarrow \infty} \phi^{-1}(l_M) \left| \sum_{k=1}^M T_k \right| = \overline{\lim}_{M \rightarrow \infty} \phi^{-1}(l_M) \left| \sum_{i \leq \beta_{a,b;d;M}} Z_i \right| = C(a, b; d), \quad \text{a.s.} \quad (34)$$

By noting (30), (31) and (32), and by applying Lemma 5, we have

$$C(a, b; d) \leq C(0, b-a; d) + \sqrt{C_\theta \frac{\log d}{d}} \leq \sigma(\tilde{\mathbf{1}}_{0,b-a}, \theta) + 2\sqrt{C_\theta \frac{\log d}{d}}, \quad (35)$$

$$C(0, 1/2; d) \geq \left(\frac{1}{4} - C_\theta \frac{\log d}{d}\right)^{1/2}. \quad (36)$$

By the standard argument using Fubini's theorem, $\overline{\lim}_{M \rightarrow \infty} \phi^{-1}(l_M) \left| \sum_{k=1}^M T_k \right| = C(a, b; d)$ a.e. on the original unit interval.

Now we use the following Lemma which is Lemma 3.1 of [1]. In [1], it is proved by assuming that λ_k is an integer and applying method by Takahashi [15] and Philipp [12]. We can remove this extra condition by applying modification of proof which is used in [6].

Lemma 6 Let $r(x)$ be a function of the form $r(x) = \sum_{j=d}^{\infty} (a_j \cos 2\pi jx + b_j \sin 2\pi jx)$, where $|a_j|, |b_j| \leq 1/j$ for $j \geq d$. If a sequence $\{\lambda_k\}$ of real numbers satisfies the Hadamard's condition $\lambda_{k+1}/\lambda_k \geq \theta > 1$, then

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{\sqrt{2N \log \log N}} \left| \sum_{k=1}^N r(\lambda_k x) \right| \leq C'_\theta d^{-1/4}, \quad a.e.$$

where C'_θ is a positive constant depending on θ .

Applying Lemma 6 by putting $r = f_{a,b;d}$, by $\sum_{i=1}^M [1 + 9 \log_\theta i] = O(M \log M)$, we have $|\sum_{k=1}^M T'_k| = O(\sqrt{M \log M \log \log(M \log M)}) = o(\sqrt{l_M})$, and hence we have $\overline{\lim}_{M \rightarrow \infty} \phi^{-1}(l_M) |\sum_{k=1}^M T'_k| = 0$. By noting $M^+ = l_M + \sum_{i=1}^M [1 + 9 \log_\theta i] \sim l_M$ and (34), we see that $\lim_{M \rightarrow \infty} \phi^{-1}(M^+) |\sum_{i=1}^M \sum_{k \in \Delta'_i \cup \Delta_i} f(\theta^{m(k)})| = C(a, b; d)$ a.e. By $\sum_{k \in \Delta'_M \cup \Delta_M} \|f((\theta^{m(k)})x)\|_\infty = o(\phi(M^+))$, we have

$$\overline{\lim}_{N \rightarrow \infty} \phi^{-1}(N) \left| \sum_{k=1}^N f_{a,b;d}(\theta^{m(k)}x) \right| = C(a, b; d), \quad a.e.$$

Noting $\tilde{\mathbf{I}}_{a,b} - f_{a,b;d} = (\tilde{\mathbf{I}}_{a,b} - \tilde{\mathbf{I}}_{a,b;d}) - \frac{1}{pd+1} \cos 2\pi(pd+1)x$, we see that $(\tilde{\mathbf{I}}_{a,b} - f_{a,b;d})/2$ satisfies the condition of Lemma 6. Hence we have

$$\overline{\lim}_{N \rightarrow \infty} \phi^{-1}(N) \left| \sum_{k=1}^N (\tilde{\mathbf{I}}_{a,b} - f_{a,b;d})(\theta^{m(k)}x) \right| \leq 2C'_\theta d^{-1/4}, \quad a.e.$$

Therefore we see that $\overline{\lim}_{N \rightarrow \infty} \phi^{-1}(N) |\sum_{k=1}^N \tilde{\mathbf{I}}_{a,b}(\theta^{m(k)}x)|$ belongs to the interval with center $C(a, b; d)$ and radius $2C'_\theta d^{-1/4}$, a.e. By letting $d \rightarrow \infty$, we have

$$\overline{\lim}_{N \rightarrow \infty} \phi^{-1}(N) \left| \sum_{k=1}^N \tilde{\mathbf{I}}_{a,b}(\theta^{m(k)}x) \right| = \lim_{d \rightarrow \infty} C(a, b; d) = \sigma(\tilde{\mathbf{I}}_{a,b}, \theta, \{m(k)\}), \quad a.e.$$

By letting $d \rightarrow \infty$ in (35) and (36), we have $\sigma(\tilde{\mathbf{I}}_{a,b}, \theta, \{m(k)\}) \leq \sigma(\tilde{\mathbf{I}}_{0,b-a}, \theta, \{m(k)\}) \leq \sigma(\tilde{\mathbf{I}}_{0,b-a}, \theta)$ and $\sigma(\tilde{\mathbf{I}}_{0,1/2}, \theta, \{m(k)\}) \geq 1/2$.

We use the fundamental relations below which can be found in [8].

Lemma 7 For any countable dense subset S of $[0, 1]$ and for any sequence $\{n_k\}$ of real numbers (not necessarily integers) satisfying Hadamard's gap condition, we have

$$\overline{\lim}_{N \rightarrow \infty} \frac{ND_N \{n_k x\}}{\sqrt{2N \log \log N}} = \sup_{S \ni a' < a \in S} \overline{\lim}_{N \rightarrow \infty} \frac{1}{\sqrt{2N \log \log N}} \left| \sum_{k=1}^N \tilde{\mathbf{I}}_{a',a}(n_k x) \right|, \quad a.e.,$$

$$\overline{\lim}_{N \rightarrow \infty} \frac{ND_N^* \{n_k x\}}{\sqrt{2N \log \log N}} = \sup_{a \in S} \overline{\lim}_{N \rightarrow \infty} \frac{1}{\sqrt{2N \log \log N}} \left| \sum_{k=1}^N \tilde{\mathbf{I}}_{0,a}(n_k x) \right|, \quad a.e.$$

Applying the lemma by putting $S = [0, 1] \cap \mathbf{Q}$, $\Sigma_{\theta, \{m(k)\}}^* = \sup_{a \in S} \sigma(\tilde{\mathbf{I}}_{0,a}, \theta, \{m(k)\})$, and $\Sigma_{\theta, \{m(k)\}} = \sup_{S \ni a < b \in S} \sigma(\tilde{\mathbf{I}}_{a,b}, \theta, \{m(k)\})$, we have $\Sigma^* \{\theta^{m(k)}x\} = \Sigma_{\theta, \{m(k)\}}^*$ and $\Sigma \{\theta^{m(k)}x\} = \Sigma_{\theta, \{m(k)\}}$ a.e. Although we have a trivial inequality $\Sigma_{\theta, \{m(k)\}}^* \leq \Sigma_{\theta, \{m(k)\}}$, by $\sigma(\tilde{\mathbf{I}}_{a,b}, \theta, \{m(k)\}) \leq \sigma(\tilde{\mathbf{I}}_{0,b-a}, \theta, \{m(k)\})$ we have the converse inequality and thereby we have $\Sigma_{\theta, \{m(k)\}}^* = \Sigma_{\theta, \{m(k)\}}$.

By $\sigma(\tilde{\mathbf{I}}_{a,b}, \theta, \{m(k)\}) \leq \sigma(\tilde{\mathbf{I}}_{0,b-a}, \theta)$ and $\Sigma_\theta = \sup_{0,a} \sigma(\tilde{\mathbf{I}}_{0,a}, \theta)$ (Cf. [6]), we have $\Sigma_{\theta, \{m(k)\}} \leq \Sigma_\theta$. By $\sigma(\tilde{\mathbf{I}}_{0,1/2}, \theta, \{m(k)\}) \geq 1/2$, we have $\Sigma_{\theta, \{m(k)\}} \geq 1/2$.

4 Randomization

In this section we prove the following: For any $\sigma \in [1/2, \Sigma_\theta]$, there exists a sequence $\{m(k)\} \in \mathcal{N}$ such that $\Sigma_{\theta, \{m(k)\}} = \sigma$. When θ satisfies (3), we have $\Sigma_\theta = 1/2$ and nothing is left to be proved. We assume that θ is given by (4). Let us first consider two extreme cases. For $\sigma = \Sigma_\theta$, $m(k) = k$ is sufficient since it is proved in [6], and for $\sigma = 1/2$, $m(k) = k^2$ is sufficient since we have $\theta^{m(k+1)}/\theta^{m(k)} \rightarrow \infty$ which is treated in [7]. Therefore it is enough to consider the case $\sigma \in (1/2, \Sigma_\theta)$. To construct such a sequence, we adopt the method of randomization.

Let $w \in (0, 1)$ and $\{X_n\}$ be a sequence of independent random variables defined on the other probability space $(\Omega, \mathcal{E}, \mathbb{P})$ satisfying $\mathbb{P}(X_n = 1) = 1 - \mathbb{P}(X_n = 0) = w$. Put

$$H_N = \sum_{k=1}^N X_k \quad \text{and} \quad \Xi_{s,M} = \frac{1}{l_M} \sum_{i=1}^M \sum_{k \in \Delta_i \cap (\Delta_i - s)} X_k X_{k+s} \quad (s = 0, 1, 2, \dots),$$

where $\Delta_i - s = \{j - s \mid j \in \Delta_i\}$. Clearly the condition $k \in \Delta_i \cap (\Delta_i - s)$ is equivalent to that $k, k + s \in \Delta_i$. Since a subsequence of $\{X_k X_{k+s}\}_k$ forms an s -dependent sequence of identically distributed random variables, it obeys the law of large numbers. Since we have

$$\frac{1}{l_M} \sum_{i=1}^M \sum_{k \in \Delta_i \cap (\Delta_i - s)} 1 = \frac{1}{l_M} \sum_{i=1}^M \max\{i - s, 0\} \rightarrow 1,$$

by putting

$$\Omega_0 = \{\omega \in \Omega \mid H_N(\omega)/N \rightarrow w, \Xi_{0,M}(\omega) \rightarrow w \text{ and } \Xi_{s,M}(\omega) \rightarrow w^2 \ (s = 1, 2, \dots)\},$$

we have $\mathbb{P}(\Omega_0) = 1$. We assume that $\omega \in \Omega_0$. Denote

$$\sigma^2(f, \theta, w) = \int_0^1 f^2(x) dx + 2w \sum_{k=1}^{\infty} \int_0^1 f(p^k x) f(q^k x) dx.$$

By applying (18) and (19), we can take a constant $\tilde{C}_\theta$ depending only on θ such that

$$\begin{aligned} |\sigma^2(f_{a,b;d}, \theta, w) - \sigma^2(\tilde{\mathbf{f}}_{a,b}, \theta, w)| &\leq \frac{1}{2(pd+1)^2} + G(0) + 2w \sum_{k=1}^{\infty} G(k) \\ &\leq \frac{1}{2(pd+1)^2} + \frac{\log d}{d \log p} + \frac{1}{(1-1/p)d} \leq \tilde{C}_\theta \frac{\log d}{d}, \end{aligned} \quad (37)$$

where $G(k)$ above is that defined by (20).

By recalling that θ is given by (4), we put $m(k) = rk$, $n_k = \theta^{m(k)} = (p/q)^k$, $T_i(x) = \sum_{k \in \Delta_i} X_k f(n_k x)$, and $T'_i(x) = \sum_{k \in \Delta'_i} X_k f(n_k x)$. We follow the proof of the last section and have completely the same conclusions until (29).

We have

$$\begin{aligned} v_{a,b;d;i} &= \sum_{k \in \Delta_i} X_k \int_0^1 f_{a,b;d}^2(x) dx \\ &\quad + 2 \sum_{s=1}^{\infty} \sum_{k \in \Delta_i \cap (\Delta_i - s)} X_k X_{k+s} \int_0^1 f_{a,b;d}(p^s x) f_{a,b;d}(q^s x) dx. \end{aligned}$$

Note that $\int_0^1 f_{a,b;d}(p^k x) f_{a,b;d}(q^k x) dx = 0$ except for finitely many k because $f_{a,b;d}$ is a trigonometric polynomial. Hence we have

$$\begin{aligned} \beta_{a,b;d;M}/l_M &= \Xi_{0,M} \int_0^1 f_{a,b;d}^2(x) dx + 2 \sum_{s=1}^{\infty} \Xi_{s,M} \int_0^1 f_{a,b;d}(p^s x) f_{a,b;d}(q^s x) dx \\ &\rightarrow w\sigma^2(f_{a,b;d}, \theta, w). \end{aligned}$$

Hence we have (34) with $C(a, b, d) = \sqrt{w}\sigma(f_{a,b;d}, \theta, w)$. By following the same argument, we have

$$\overline{\lim}_{N \rightarrow \infty} \phi^{-1}(N) \left| \sum_{k=1}^N X_k f_{a,b;d}((p/q)^k x) \right| = \sqrt{w}\sigma(f_{a,b;d}, \theta, w), \quad \text{a.e.}$$

By applying Lemma 6 and (37), we conclude as before that

$$\overline{\lim}_{N \rightarrow \infty} \phi^{-1}(N) \left| \sum_{k=1}^N X_k \tilde{\mathbf{I}}_{a,b}((p/q)^k x) \right| = \sqrt{w}\sigma(\tilde{\mathbf{I}}_{a,b}, \theta, w), \quad \text{a.e.}$$

By $H_N \sim wN$, we have

$$\overline{\lim}_{N \rightarrow \infty} \phi^{-1}(H_N) \left| \sum_{k=1}^N X_k \tilde{\mathbf{I}}_{a,b}((p/q)^k x) \right| = \sigma(\tilde{\mathbf{I}}_{a,b}, \theta, w), \quad \text{a.e.}$$

Now we regard the set $\{(p/q)^k \mid X_k = 1\}$ as an increasing sequence and denote it by $\{n_k^\circ\}$. Then we can rewrite the law of the iterated logarithm above as below:

$$\overline{\lim}_{N \rightarrow \infty} \phi^{-1}(N) \left| \sum_{k=1}^N \tilde{\mathbf{I}}_{a,b}(n_k^\circ x) \right| = \sigma(\tilde{\mathbf{I}}_{a,b}, \theta, w), \quad \text{a.e.}$$

By applying Lemma 7, we have $\Sigma^*(n_k^\circ x) = \sup_{0 \leq a < 1} \sigma(\tilde{\mathbf{I}}_{0,a}, \theta, w)$ and $\Sigma(n_k^\circ x) = \sup_{0 \leq a < b < 1} \sigma(\tilde{\mathbf{I}}_{a,b}, \theta, w)$. By noting (8), we have

$$\sup_{0 \leq a < b < 1} \sigma(\tilde{\mathbf{I}}_{a,b}, \theta, w) = \sup_{0 \leq a < 1} \sigma(\tilde{\mathbf{I}}_{0,a}, \theta, w)$$

and hence $\Sigma^*(n_k^\circ x) = \Sigma(n_k^\circ x)$ a.e. as before. We here prove that $\sigma(\tilde{\mathbf{I}}_{0,a}, \theta, w)$ is a uniformly continuous function of $(a, w) \in \mathbf{T} \times [0, 1]$. Since $V(x, \xi)$ is continuous on $\mathbf{T}^2$, by expressing

$$\sigma(\tilde{\mathbf{I}}_{0,a}, \theta, w) = V(a, a) + 2w \sum_{k=1}^{\infty} \frac{V(\langle p^k x \rangle, \langle q^k x \rangle)}{p^k q^k}$$

using (7), we see that this is a uniformly convergent series of uniformly continuous functions on $\mathbf{T} \times [0, 1]$. Hence $\sigma(\tilde{\mathbf{I}}_{0,a}, \theta, w)$ is uniformly continuous on $\mathbf{T} \times [0, 1]$, and therefore $\sup_{0 \leq a < 1} \sigma(\tilde{\mathbf{I}}_{0,a}, \theta, w)$ is continuous in w . Thanks to

$$\sup_{0 \leq a < 1} \sigma(\tilde{\mathbf{I}}_{0,a}, \theta, 0) = \sup_{0 \leq a < 1} \left(\int_0^1 \tilde{\mathbf{I}}_{0,a}(x) dx \right)^{1/2} = \frac{1}{2}$$

and

$$\sup_{0 \leq a < 1} \sigma(\tilde{\mathbf{I}}_{0,a}, \theta, 1) = \sup_{0 \leq a < 1} \sigma(\tilde{\mathbf{I}}_{0,a}, \theta) = \Sigma_\theta,$$

for any given $\sigma \in (1/2, \Sigma_\theta)$ we can find a $w \in (0, 1)$ such that

$$\Sigma^*(n_k^\circ x) = \Sigma(n_k^\circ x) = \sup_{0 \leq a < 1} \sigma(\tilde{\mathbf{I}}_{0,a}, \theta, w) = \sigma, \quad \text{a.e.}$$

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