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A central limit theorem for trigonometric series with bounded gaps

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Dedicated to Professor Norio Kôno on his 70th birthday

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Abstract In this paper it is proved that there exists a sequence $\{n_k\}$ of integers with $1 \le n_{k+1} - n_k \le 5$ such that the distribution of $(\cos 2\pi n_1 x + \cdots + \cos 2\pi n_N)/\sqrt{N}$ on $([0,1], \mathbf{B}, dx)$ converges to a gaussian distribution. It gives an affirmative answer to the long standing problem on lacunary trigonometric series which ask the existence of series with bounded gaps satisfying a central limit theorem.

1 Introduction

It is well known that the sequence $\{n_k x\}$ behaves like a sequence of independent random variables if n_k grows very fast. For example, the central limit theorem for lacunary trigonometric series

$$\left| \left\{ x \in [0,1] \;\middle|\; \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \cos 2\pi n_k x \le t \right\} \right| \to \mathfrak{N}_{0,1/2}(-\infty, t] \qquad (1)$$

where $\mathfrak{N}_{m,v}$ denotes the gaussian distribution with mean m and variance v, was first proved by Kac [7] under the large gap condition $n_{k+1}/n_k \to \infty$. The condition was relaxed to the Hadamard's gap condition $n_{k+1}/n_k > q > 1$ by Salem-Zygmund [9]. Erdős [5] proved (1) under weaker condition $n_{k+1}/n_k \ge 1 + c_k/\sqrt{k}$ with $c_k \to \infty$. Here divergence of c_k is necessary. This fact was stated by Erdős [5] without proof and was proved by Takahashi [12]. Cases of weaker gap conditions were studied by Berkes [1], Murai [8] and others [6].

Above sequences satisfy $n_k \ge \exp(k^\alpha)$ for some $\alpha > 0$, and it is natural to ask if there exists a sequence which diverges more slowly and obeys the central limit theorem.

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By using the central limit theorem for random trigonometric series by Salem-Zygmund [10], we can prove the existence of $\{n_k\}$ satisfying

$$k/n_k \to 1/2, \qquad \limsup_{k\to\infty}(n_{k+1}-n_k)/\log_2 k = 1, \tag{2}$$

and

$$\left|\left\{ x \in [\,0,1\,] \;\middle|\; \frac{1}{\sqrt{N}}\sum_{k=1}^{N}\cos 2\pi n_k x \le t \right\}\right| \to \mathfrak{N}_{0,1/4}(-\infty, t\,]. \tag{3}$$

In this result, the variance of the limit distribution is a half of that in (1), and this is the maximal variance under the condition (2). Actually Bobkov-Götze [4] proved that, if the law of $(\cos 2\pi n_1 x + \cdots + \cos 2\pi n_N)/\sqrt{N}$ converges weakly, then the second moment of the limit distribution is bounded from above by $(1 - \limsup k/n_k)/2$.

The study of this directions, however, was almost completed by the result by Berkes [2]: for any sequence $0 < L_k \to \infty$, there exists a sequence of positive integers satisfying $1 \le n_{k+1} - n_k = O(L_k)$ and obeying the central limit theorem (1).

Recently, Bobkov-Götze [4] constructed sequences with $1 \le n_{k+1} - n_k \le L < \infty$ such that the limit distribution is a variance mixture of centered gaussian distribution:

$$\left|\left\{ x \in [\,0,1\,] \;\middle|\; \frac{1}{\sqrt{N}}\sum_{k=1}^{N}\cos 2\pi n_k x \le t \right\}\right| \to \mathfrak{N}_{0,\rho^2}(-\infty, t\,], \tag{4}$$

where $\rho^2(x) = 1/2 - 1/2d - (1/d^2)\sum_{n=1}^{d-1}(d-n)\cos 2\pi nx$ $(d = 2,\ 3,\ \ldots)$ and the measure $\mathfrak{N}_{0,\rho^2}$ is given by

$$\mathfrak{N}_{0,\rho^2}(A) = \int_0^1 \mathfrak{N}_{0,\rho^2(x)}(A)\,dx, \quad A \in \mathcal{B}(\mathbf{R}).$$

There still remains the problem asking if a pure gaussian distribution can be a limit distribution in bounded gap case. There is also another problem to characterize the class of distributions which can be a limit distribution.

In this paper we prove that $\mathfrak{N}_{0,1/4}$ belongs to that class, and that the class includes measures defined by various ρ^2 functions.

Theorem 1 *Let $\{a_n\}$ be a sequence of real numbers satisfying*

$$\sum_{n=1}^{\infty}|a_n| \le \frac{1}{12}. \tag{5}$$

There exists a sequence $\{n_k\}$ of positive integers satisfying $1 \le n_{k+1} - n_k \le 9$ and a central limit theorem (4) for

$$\rho^2(x) = \frac{1}{4} + \sum_{n=1}^{\infty} a_n \cos 2\pi nx.$$

Also, there exists a sequence satisfying $1 \le n_{k+1} - n_k \le 5$ and a central limit theorem (3).

For this last sequence with pure gaussian limit, we can also prove the exact law of the iterated logarithm for discrepancies of $\{n_k x\}$. We study this phenomenon in a separate paper.

2 Proof

Put $a_0 = 1/4$ and let $\varepsilon_n = \pm 1$ satisfies $a_n = \varepsilon_n |a_n|$. Note that $\varepsilon_0 = 1$. Let us define $l(\nu, \varepsilon)$ and $g(\nu, \varepsilon)$ by

$$\begin{aligned} &(l(0,+1), g(0,+1)) = (4,0),\\ &(l(1,+1), g(1,+1)) = (6,1), \quad (l(1,-1), g(1,-1)) = (2,1),\\ &(l(2,+1), g(2,+1)) = (8,2), \quad (l(2,-1), g(2,-1)) = (4,2),\\ &(l(\nu,\varepsilon), g(\nu,\varepsilon)) = \begin{cases} (6m, m), & \text{if } (\nu,\varepsilon) = (3m, \pm 1) \text{ for } m \geq 1,\\ (6m+2, m+1), & \text{if } (\nu,\varepsilon) = (3m+1, \pm 1) \text{ for } m \geq 1,\\ (6m+4, m+2), & \text{if } (\nu,\varepsilon) = (3m+2, \pm 1) \text{ for } m \geq 1. \end{cases} \end{aligned}$$

Note that $g(n, \pm 1) \leq l(n, \pm 1)$ for all n, $0 < g(n, \pm 1)$ and $l(n, \pm 1) \leq 6g(n, \pm 1) \leq 6n$ for $n \in \mathbf{N}$.

We prove Theorem 1 by assuming

$$\sum_{n=1}^{\infty} \frac{2|a_n| l(n, \varepsilon_n)}{g(n, \varepsilon_n)} \leq 1, \tag{6}$$

instead of (5). It is clear that (5) implies (6).

Lemma 1 *Assume (6) and put*

$$\mu = l(0,+1) \Bigg/ \left(1 - \sum_{n=1}^{\infty} \frac{2|a_n| l(n,\varepsilon_n)}{g(n,\varepsilon_n)} + l(0,+1) \sum_{n=1}^{\infty} \frac{2|a_n|}{g(n,\varepsilon_n)}\right).$$

Then there exists a sequence $\{\nu_k\}$ of non-negative integers such that

$$\nu_k = O(\log k), \tag{7}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} l(\nu_k, \varepsilon_{\nu_k}) = \mu, \tag{8}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \varepsilon_{\nu_k} g(\nu_k, \varepsilon_{\nu_k}) \cos 2\pi \nu_k x = 2\mu\Big(\rho^2(x) - \frac{1}{4}\Big) \quad a.e.\ x. \tag{9}$$

Proof Let $\{\xi_k\} = \{\xi_k(\omega)\}$ be an i.i.d. whose distribution is given by $P(\xi_k = n) = 2\mu|a_n|/g(n,\varepsilon_n)$ for $n \in \mathbf{N}$ and

$$P(\xi_k = 0) = \frac{\mu}{l(0,+1)}\left(1 - \sum_{n=1}^{\infty} \frac{2|a_n| l(n,\varepsilon_n)}{g(n,\varepsilon_n)}\right).$$

By law of large numbers, we have

$$\frac{1}{N}\sum_{k=1}^{N} l(\xi_k, \varepsilon_{\xi_k}) \to El(\xi_1, \varepsilon_{\xi_1}) = \mu, \tag{10}$$

$$\frac{1}{N}\sum_{k=1}^{N} \varepsilon_{\xi_k} g(\xi_k, \varepsilon_{\xi_k}) \cos 2\pi\xi_k x \to E\varepsilon_{\xi_1} g(\xi_1, \varepsilon_{\xi_1}) \cos 2\pi \xi_1 x = 2\mu\Big(\rho^2(x) - \frac{1}{4}\Big). \tag{11}$$

for almost every ω and all x.

If $f \geq 0$ and $Ef(\xi_1) < \infty$, we have

$$\begin{aligned} Ef(\xi_1) &= \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(\xi_k) \geq \limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(\xi_k)\mathbf{1}_{\xi_k \leq \log k} \\ &\geq \liminf_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(\xi_k)\mathbf{1}_{\xi_k \leq \log k} \geq \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(\xi_k)\mathbf{1}_{\xi_k \leq L} = Ef(\xi_1)\mathbf{1}_{\xi_1 \leq L} \end{aligned}$$

for all L a.s. When $L \to \infty$ the right hand side tends to $Ef(\xi_1)$, and hence we have

$$\frac{1}{N}\sum_{k=1}^{N} f(\xi_k)\mathbf{1}_{\xi_k \leq \log k} \to Ef(\xi_1) \quad \text{or} \quad \frac{1}{N}\sum_{k=1}^{N} f(\xi_k)\mathbf{1}_{\xi_k > \log k} \to 0 \quad \text{a.s.}$$

Therefore we have

$$\varXi_N = \sum_{k=1}^{N} \mathbf{1}_{\xi_k \leq \log k} \sim N \quad \text{and} \quad \frac{1}{N}\sum_{k=1}^{N} l(\xi_k, \varepsilon_{\xi_k})\mathbf{1}_{\xi_k > \log k} \to 0 \quad \text{a.s.} \tag{12}$$

By noting $|\varepsilon_{\xi_k} g(\xi_k, \varepsilon_{\xi_k})| \leq l(\xi_k, \varepsilon_{\xi_k})$, we have

$$\frac{1}{N}\sum_{k=1}^{N} \varepsilon_{\xi_k} g(\xi_k, \varepsilon_{\xi_k})\mathbf{1}_{\xi_k > \log k} \cos 2\pi\xi_k x \to 0 \quad \text{a.s.} \tag{13}$$

By combining (10), (11), (12), and (13), we have

$$\frac{1}{\varXi_N}\sum_{k=1}^{N} \mathbf{1}_{\xi_k \leq \log k} l(\xi_k, \varepsilon_{\xi_k}) \to \mu \quad \text{a.s.}$$

$$\frac{1}{\varXi_N}\sum_{k=1}^{N} \varepsilon_{\xi_k} g(\xi_k, \varepsilon_{\xi_k})\mathbf{1}_{\xi_k \leq \log k} \cos 2\pi\xi_k x \to 2\mu\Big(\rho^2(x) - \frac{1}{4}\Big) \quad \text{a.s.}$$

Define the sequuence $\{\nu_j\}$ by $\nu_{\varXi_k} = \xi_k$ for k with $\xi_k \leq \log k$. We have $\nu_{\varXi_k} \leq \log k \sim \log \varXi_k$, and above two asymptotic formulas imply that (7), (8), and (9) are valid almost surely for all x. By virtue of Fubini's theorem, (7), (8), and (9) are valid for almost all x, almost surely, and hence we can choose one sample of $\{\nu_k\}$ such that (7), (8), and (9) are valid for almost all x. □

Let $\{Y_j\}$ be a fair ± 1-valued i.i.d. and we define another sequence $\{\widetilde{Y}_j\}$ by modifying $\{Y_j\}$ in the following way.

First, let $\{\nu_k\}$ be a sequence satisfying the properties in Lemma 1, and put

$$\varLambda_0 = 0 \quad \text{and} \quad \varLambda_n = \sum_{k=1}^{n} l(\nu_k, \varepsilon_{\nu_k}) \quad (n = 1, 2, \dots).$$

Secondly, we define a sequence $\{\widetilde{Y}_j\}$ in blocks

$$\widetilde{Y}_{\varLambda_{n-1}+1}, \dots, \widetilde{Y}_{\varLambda_{n-1}+l(\nu_n, \varepsilon_{\nu_n})} = \widetilde{Y}_{\varLambda_n}. \quad (n = 1, 2, \dots)$$

Define the value of each block according to the value of $(\nu_n, \varepsilon_{\nu_n})$ as below: Here we denote $\varLambda_{n-1}$, ν_n, and ε_{ν_n} simply by $\varLambda$, ν, and ε.

If $\nu = 0$, 1, 2, we let $(\widetilde{Y}_{\Lambda+1}, \ldots, \widetilde{Y}_{\Lambda+l(\nu,\varepsilon)})$ equal to

$$\begin{cases} (Y_{\Lambda+1}, Y_{\Lambda+1}, Y_{\Lambda+3}, -Y_{\Lambda+3}) & \text{if } (\nu,\varepsilon) = (0,+1), \\ (Y_{\Lambda+1}, Y_{\Lambda+1}, Y_{\Lambda+3}, -Y_{\Lambda+3}, Y_{\Lambda+5}, Y_{\Lambda+5}) & \text{if } (\nu,\varepsilon) = (1,+1), \\ (Y_{\Lambda+1}, -Y_{\Lambda+1}) & \text{if } (\nu,\varepsilon) = (1,-1), \\ (Y_{\Lambda+1}, Y_{\Lambda+2}, Y_{\Lambda+1}, Y_{\Lambda+2}, Y_{\Lambda+5}, -Y_{\Lambda+5}, Y_{\Lambda+7}, Y_{\Lambda+7}) & \text{if } (\nu,\varepsilon) = (2,+1), \\ (Y_{\Lambda+1}, Y_{\Lambda+2}, -Y_{\Lambda+1}, -Y_{\Lambda+2}) & \text{if } (\nu,\varepsilon) = (2,-1); \end{cases}$$

If $\nu = 3m$ $(m \in \mathbf{N})$, we define by

$$\begin{aligned} \widetilde{Y}_{\Lambda+3j+1} &= \varepsilon \widetilde{Y}_{\Lambda+3m+3j+1} = Y_{\Lambda+3j+1} \quad (j = 0, 1, \ldots, m-1), \\ \widetilde{Y}_{\Lambda+3j+2} &= (-1)^j \widetilde{Y}_{\Lambda+3j+3} = Y_{\Lambda+3j+2} \quad (j = 0, 1, \ldots, 2m-1); \end{aligned}$$

If $\nu = 3m+1$ $(m \in \mathbf{N})$, we define by

$$\begin{aligned} \widetilde{Y}_{\Lambda+3j+1} &= \varepsilon \widetilde{Y}_{\Lambda+3m+3j+2} = Y_{\Lambda+3j+1} \quad (j = 0, 1, \ldots, m), \\ \widetilde{Y}_{\Lambda+3j+2} &= (-1)^j \widetilde{Y}_{\Lambda+3j+3} = Y_{\Lambda+3j+2} \quad (j = 0, 1, \ldots, m-1), \\ \widetilde{Y}_{\Lambda+3j+3} &= (-1)^j \widetilde{Y}_{\Lambda+3j+4} = Y_{\Lambda+3j+3} \quad (j = m, m+1, \ldots, 2m-1); \end{aligned}$$

If $\nu = 3m+2$ $(m \in \mathbf{N})$, we define by

$$\begin{aligned} \widetilde{Y}_{\Lambda+3j+1} &= \varepsilon \widetilde{Y}_{\Lambda+3m+3j+3} = Y_{\Lambda+3j+1} \quad (j = 0, 1, \ldots, m), \\ \widetilde{Y}_{\Lambda+3m+2} &= \varepsilon \widetilde{Y}_{\Lambda+6m+4} = Y_{\Lambda+3m+2}, \\ \widetilde{Y}_{\Lambda+3j+2} &= (-1)^j \widetilde{Y}_{\Lambda+3j+3} = Y_{\Lambda+3j+2} \quad (j = 0, 1, \ldots, m-1), \\ \widetilde{Y}_{\Lambda+3j+4} &= (-1)^j \widetilde{Y}_{\Lambda+3j+5} = Y_{\Lambda+3j+4} \quad (j = m, m+1, \ldots, 2m-1). \end{aligned}$$

By definition, the sequence $\widetilde{Y}_{\Lambda+1}, \ldots, \widetilde{Y}_{\Lambda+l(\nu,\varepsilon)}$ contains both of $+1$ and -1, and thus cannot be a run of the same number. The maximum length of run of the same number is at most 8. And the length of run of the same number on the left end is at most 5, and on the right end it is 3. Hence, the maximum length of run of the same number in the sequence $\{\widetilde{Y}_j\}$ is at most 8.

To have the pure gaussian limit distribution $\mathfrak{N}_{0,1/4}$, we put $\nu_k \equiv 0$, $\varepsilon_{\nu_k} \equiv +1$. In this case the maximum length of run in $\{\widetilde{Y}_j\}$ is at most 4.

Having defined $\{\widetilde{Y}_j\}$, we define γ_n and φ_n by

$$\gamma_n = \sum_{j=\Lambda_{n-1}+1}^{\Lambda_n} \widetilde{Y}_j \quad \text{and} \quad \varphi_n = \sum_{j=\Lambda_{n-1}+1}^{\Lambda_n} \widetilde{Y}_j \cos 2\pi jx.$$

Clearly $\{\gamma_n\}$ and $\{\varphi_n\}$ are sequences of independent random variables.

Denoting $\cos 2\pi nx$ by c_n, we have

$$E\varphi_n^2 = \frac{l(\nu_n, \varepsilon_{\nu_n})}{2} + \varepsilon_{\nu_n} g(\nu_n, \varepsilon_{\nu_n}) c_{\nu_n} + H_1(\Lambda_{n-1}, \nu_n, \varepsilon_{\nu_n}) + H_2(\Lambda_{n-1}, \nu_n, \varepsilon_{\nu_n}), \quad (14)$$

where

$$
\begin{aligned}
H_1(\Lambda,\nu,\varepsilon) &= \frac{1}{2}\sum_{j=1}^{l(\nu,\varepsilon)} c_{2\Lambda+2j},\\
H_2(\Lambda,0,+1) &= c_{2\Lambda+3} - c_{2\Lambda+7},\\
H_2(\Lambda,1,+1) &= c_{2\Lambda+3} - c_{2\Lambda+7} + c_{2\Lambda+11},\\
H_2(\Lambda,1,-1) &= -c_{2\Lambda+3},\\
H_2(\Lambda,2,+1) &= c_{2\Lambda+4} + c_{2\Lambda+6} - c_{2\Lambda+11} + c_{2\Lambda+15},\\
H_2(\Lambda,2,-1) &= -c_{2\Lambda+4} - c_{2\Lambda+6},\\
H_2(\Lambda,3m,\varepsilon) &= \varepsilon\sum_{j=0}^{m-1} c_{2\Lambda+3m+6j+2} + \sum_{j=0}^{2m-1}(-1)^j c_{2\Lambda+6j+5},\\
H_2(\Lambda,3m+1,\varepsilon) &= \varepsilon\sum_{j=0}^{m} c_{2\Lambda+3m+6j+3} + \sum_{j=0}^{m-1}(-1)^j c_{2\Lambda+6j+5} + \sum_{j=m}^{2m-1}(-1)^j c_{2\Lambda+6j+7},\\
H_2(\Lambda,3m+2,\varepsilon) &= \varepsilon\sum_{j=0}^{m} c_{2\Lambda+3m+6j+4} + \varepsilon c_{2\Lambda+9m+6}\\
&\quad + \sum_{j=0}^{m-1}(-1)^j c_{2\Lambda+6j+5} + \sum_{j=m}^{2m-1}(-1)^j c_{2\Lambda+6j+9}.
\end{aligned}
$$

Actually, when $\nu = 3m$, φ_n equals to

$$
\sum_{j=0}^{m-1} Y_{\Lambda+3j+1}(c_{\Lambda+3j+1} + \varepsilon c_{\Lambda+3m+3j+1}) + \sum_{j=0}^{2m-1} Y_{\Lambda+3j+2}(c_{\Lambda+3j+2} + (-1)^j c_{\Lambda+3j+3}),
$$

and hence we have

$$
\begin{aligned}
E\varphi_n^2 &= \sum_{j=0}^{m-1}(c_{\Lambda+3j+1} + \varepsilon c_{\Lambda+3m+3j+1})^2 + \sum_{j=0}^{2m-1}(c_{\Lambda+3j+2} + (-1)^j c_{\Lambda+3j+3})^2\\
&= \sum_{j=1}^{6m} c_{\Lambda+j}^2 + 2\varepsilon\sum_{j=0}^{m-1} c_{\Lambda+3j+1}c_{\Lambda+3m+3j+1} + 2\sum_{j=0}^{2m-1}(-1)^j c_{\Lambda+3j+2}c_{\Lambda+3j+3}\\
&= 3m + \frac{1}{2}\sum_{j=1}^{6m} c_{2\Lambda+2j} + \varepsilon m c_{3m} + \varepsilon\sum_{j=0}^{m-1} c_{2\Lambda+3m+6j+2}\\
&\quad + c_1\sum_{j=0}^{2m-1}(-1)^j + \sum_{j=0}^{2m-1}(-1)^j c_{2\Lambda+6j+5}.
\end{aligned}
$$

The other cases can be proved in the same way. Put $\Theta_n = H_1(\Lambda_{n-1},\nu_n,\varepsilon_{\nu_n}) + H_2(\Lambda_{n-1},\nu_n,\varepsilon_{\nu_n})$.

Note that Θ_k is a sum of at most $l(\nu_k,\varepsilon_{\nu_k})$ cosines with bounded coefficients and frequencies among $[\,2\Lambda_{k-1}+2, 2\Lambda_k\,]$. Hence we have $\|\Theta_k\|_\infty = O(\log k)$ and see that $\{\Theta_k\}$ is an orthogonal sequence. Due to Rademacher-Menchoff theorem (Cf. [11]),

$\sum(\log k)^2 \int_0^1 \Theta_k^2\,dx/k^2 = O(\sum(\log k)^4/k^2) < \infty$ implies almost everywhere convergence of the series $\sum \Theta_k/k$. Hence by Kronecker's Lemma, we have

$$\frac{1}{N}\sum_{k=1}^{N}\Theta_k(x) \to 0 \quad \text{a.e. } x \tag{15}$$

By noting (14) and combining asymptotics (8), (9), and (15), we have

$$\frac{1}{N}\sum_{k=1}^{N} E\varphi_k^2 \to 2\mu\rho^2(x) \quad \text{a.e. } x. \tag{16}$$

On the other hand we can express φ_n by

$$\varphi_n = \sum_{j=\Lambda_{n-1}/2+1}^{\Lambda_n/2} Y_{\alpha_j}(c_{\beta_{2j-1}} \pm c_{\beta_{2j}}),$$

where $\{\alpha_j\}$ is an increasing sequence of integers, and $\{\beta_j\}$ is a permutation of $\mathbf{N}$. We see

$$E\varphi_n^2 = \sum_{j=\Lambda_{n-1}/2+1}^{\Lambda_n/2} (c_{\beta_{2j-1}} \pm c_{\beta_{2j}})^2.$$

By (8), we have $\Lambda_n/2 \sim \mu n/2$, and hence by (16) we have

$$\frac{1}{\Lambda_n/2}\sum_{j=1}^{\Lambda_n/2}(c_{\beta_{2j-1}} \pm c_{\beta_{2j}})^2 \to 4\rho^2(x) \quad \text{and thereby} \quad \frac{1}{M}\sum_{j=1}^{M}(c_{\beta_{2j-1}} \pm c_{\beta_{2j}})^2 \to 4\rho^2(x),$$

for a.e. x. We here apply the next Proposition by Bobkov [3]: Although the special case $\rho = 1$ is proved in [3], as Bobkov-Götze states in [4] (Cf. pp74), the following version can be proved in the same way.

Proposition 1 *Let $\{\xi_n\}$ be a sequence of measurable functions on $[0,1]$ satisfying $\int_0^1 \xi_n(x)\xi_m(x)\,dx = \delta_{nm}$. Suppose that we have ± 1-valued fair i.i.d. $\{Y_n\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$. If two conditions*

$$\left|\left\{x \in [0,1] \;\middle|\; \frac{|\xi_1(x)| \vee \cdots \vee |\xi_n(x)|}{\sqrt{n}} > \varepsilon\right\}\right| \to 0 \quad (n \to \infty, \varepsilon > 0) \tag{17}$$

$$\left|\left\{x \in [0,1] \;\middle|\; \left|\frac{\xi_1^2(x) + \cdots + \xi_n^2(x)}{n} - \rho^2(x)\right| > \varepsilon\right\}\right| \to 0 \quad (n \to \infty, \varepsilon > 0) \tag{18}$$

are satisfied, then we have

$$\left|\left\{x \in [0,1] \;\middle|\; \frac{1}{\sqrt{M}}\sum_{j=1}^{M} Y_j(\omega)\xi_j(x) \le t\right\}\right| \to \mathfrak{N}_{0,\rho^2}(-\infty, t\,] \quad (M \to \infty, t \in \mathbf{R}) \tag{19}$$

for P-almost every ω.

By applying this, we have

$$\left|\left\{x\in[\,0,1\,]\;\middle|\;\frac{1}{\sqrt{M}}\sum_{j=1}^{M}Y_{\alpha_j}(\omega)(c_{\beta_{2j-1}}(x)\pm c_{\beta_{2j}}(x))\le t\right\}\right|\to\mathfrak{N}_{0,4\rho^2}(-\infty,t\,],$$

for P-almost every ω. By putting $M=\Lambda_n/2$ and by noting $\sum_{j=1}^{\Lambda_n/2}Y_{\alpha_j}(c_{\beta_{2j-1}}\pm c_{\beta_{2j}})=\sum_{j=1}^{\Lambda_n}\widetilde{Y}_j c_j$, we have

$$\left|\left\{x\in[\,0,1\,]\;\middle|\;\frac{1}{\sqrt{\Lambda_n/2}}\sum_{j=1}^{\Lambda_n}\widetilde{Y}_j(\omega)c_j(x)\right\}\right|\to\mathfrak{N}_{0,4\rho^2}(-\infty,t\,],$$

for P-almost every ω, and thereby

$$\left|\left\{x\in[\,0,1\,]\;\middle|\;\frac{1}{\sqrt{M/2}}\sum_{j=1}^{M}\widetilde{Y}_j(\omega)c_j(x)\right\}\right|\to\mathfrak{N}_{0,4\rho^2}(-\infty,t\,],$$

for P-almost every ω. By taking the arithmetic mean with the normalized Dirichlet kernels $(1/\sqrt{M/2})\sum_{j=1}^{M}c_j(x)\to 0$, we have

$$\left|\left\{x\in[\,0,1\,]\;\middle|\;\frac{1}{\sqrt{M/2}}\sum_{j=1}^{M}\frac{\widetilde{Y}_j(\omega)+1}{2}c_j(x)\right\}\right|\to\mathfrak{N}_{0,\rho^2}(-\infty,t\,],$$

for P-almost every ω. By law of large numbers, one has $\sum_{j=1}^{M}(\widetilde{Y}_j(\omega)+1)/2\sim M/2$ for P-almost every ω, and hence

$$\left|\left\{x\in[\,0,1\,]\;\middle|\;\sum_{j=1}^{M}\frac{\widetilde{Y}_j(\omega)+1}{2}c_j(x)\bigg/\left(\sum_{j=1}^{M}\frac{\widetilde{Y}_j(\omega)+1}{2}\right)^{1/2}\right\}\right|\to\mathfrak{N}_{0,\rho^2}(-\infty,t\,]$$

as $M\to\infty$, for all $t\in\mathbf{R}$ and for P-almost every ω. If we denote $\{k\in\mathbf{N}\mid\widetilde{Y}_k=1\}$ by $\{n_j\}$, we have the conclusion.

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