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A LAW OF LARGE NUMBERS FOR ARITHMETIC FUNCTIONS

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ABSTRACT. We prove the weighted strong law of the large numbers for every integrable i.i.d. sequence where the weights are given by a positive strongly additive function satisfying the Lindeberg condition. This result solves one of the open problems raised in the paper by Berkes-Weber [1].

1. MAIN RESULT

Let f be a strongly additive arithmetic function, i.e.,

$$\begin{aligned} f(mn) &= f(m) + f(n) \text{ if } \gcd(m, n) = 1, \quad \text{and} \\ f(p^n) &= f(p) \quad \text{for all primes } p \text{ and positive integers } n. \end{aligned}$$

Erdős-Kac [2] proved that if $f(p) = O(1)$ and $B_p \rightarrow \infty$ where p varies along primes, then the sequence $\{f(n)\}$ obeys the central limit theorem, i.e.,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{n \leq N \mid f(n) \leq A_N + xB_N^{1/2}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du,$$

where

$$A_n = \sum_{p < n} \frac{f(p)}{p}, \quad B_n = \sum_{p < n} \frac{f^2(p)}{p}.$$

Here and in the sequel, we follow the usual convention and denote the summation along the primes by \sum_p . Kubilius [4] and Shapiro [5] relaxed the condition $f(p) = O(1)$ to the Lindeberg condition below:

$$(1.1) \quad \lim_{N \rightarrow \infty} \frac{1}{B_N} \sum_{\{p < N : |f(p)| \geq \varepsilon B_N^{1/2}\}} \frac{f^2(p)}{p} = 0 \quad \text{for all } \varepsilon > 0.$$

The purpose of this paper is to prove the following theorem and show that the irregularity of positive strongly additive function f does not have an effect on the weighted law of large numbers.

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Theorem 1.1. *Suppose that a positive strongly additive function f satisfies the Lindeberg condition (1.1). Then for any sequence $\{X_n\}$ of independent and identically distributed integrable random variables, we have*

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N f(n)X_n \Big/ \sum_{n=1}^N f(n) = EX_1 \quad a.s.$$

Berkes-Weber [1] proved the same conclusion by assuming

$$f(p) = o(B_p^{1/2}) \quad \text{and} \quad B_p \rightarrow \infty \quad \text{as} \quad p \rightarrow \infty,$$

which is stronger than the Lindeberg condition (1.1). They also proved it by assuming the Lindeberg condition and the following smoothness condition

$$\sup_{n \leq p, p' \leq n^2} \frac{f(p)}{f(p')} = O(1),$$

and posed the question whether the Lindeberg condition alone is sufficient to have the same conclusion. Our result, which is proved by simple calculations without using randomization technique as is used in [1], gives an affirmative answer to this question.

2. PROOF

We use the following asymptotics which are proved in [1] under the Lindeberg condition (1.1):

$$(2.1) \quad \sum_{n=1}^N f(n) \sim NA_N,$$

$$(2.2) \quad \sum_{n=1}^N f^2(n) \sim NA_N^2.$$

By (2.2), we can take a constant $C > 0$ such that

$$(2.3) \quad \sum_{n=1}^N f^2(n) \leq \frac{CNA_N^2}{2}, \quad (N \geq 1).$$

To prove our theorem, we appeal to the characterization by Jamison-Orey-Pruitt [3]:

Lemma 2.1. *Let $\{w_k\}$ be a sequence of positive numbers and put $W_N = \sum_{n=1}^N w_n$. Then*

$$\lim_{N \rightarrow \infty} \frac{1}{W_N} \sum_{n=1}^N w_n X_n = EX_1 \quad a.s.$$

holds for any sequence $\{X_n\}$ of independent and identically distributed integrable random variables if and only if

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \#\{n : W_n \leq tw_n\} < \infty.$$

We apply this characterization by putting $w_n = f(n)$. Because of (2.1), it is sufficient to prove

$$(2.4) \quad \#\{n : nA_n \leq mf(n)\} \leq (1+C)^2 m.$$

To begin with, we have

$$(2.5) \quad \#\{n : nA_n \leq mf(n)\} \leq m + m^2 \sum_{n>m} \frac{f^2(n)}{n^2 A_n^2}.$$

To bound the second term, we first prove

$$(2.6) \quad \sum_{m< n \leq M} \frac{f^2(n)}{n^2 A_n^2} \leq \frac{C}{m} + C \sum_{m< p < M} \frac{f(p)}{p^2 A_p}, \quad (m < M).$$

By using the partial summation method, we have

$$\begin{aligned} & \sum_{m< n \leq M} \frac{f^2(n)}{n^2 A_n^2} \\ &= \sum_{m< n < M} \left(\sum_{k=m+1}^n f^2(k) \right) \left(\frac{1}{n^2 A_n^2} - \frac{1}{(n+1)^2 A_{n+1}^2} \right) + \left(\sum_{k=m+1}^M f^2(k) \right) \frac{1}{M^2 A_M^2} \\ &= \sum_{m< n < M} \left(\sum_{k=m+1}^n f^2(k) \right) \left(\frac{1}{A_n^2} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right) + \frac{1}{(n+1)^2} \left(\frac{1}{A_n^2} - \frac{1}{A_{n+1}^2} \right) \right) \\ & \quad + \left(\sum_{k=m+1}^M f^2(k) \right) \frac{1}{M^2 A_M^2}. \end{aligned}$$

Thanks to (2.3), we have $\sum_{k=m+1}^n f^2(k) \leq CnA_n^2/2$ and hence

$$\begin{aligned} \sum_{m< n \leq M} \frac{f^2(n)}{n^2 A_n^2} &\leq C \sum_{m< n < M} \left(\frac{1}{n(n+1)} + \frac{A_{n+1}^2 - A_n^2}{2nA_{n+1}^2} \right) + \frac{C}{2M} \\ &= \frac{C}{m+1} - \frac{C}{M} + C \sum_{m< n < M} \frac{A_{n+1}^2 - A_n^2}{2nA_{n+1}^2} + \frac{C}{2M}. \end{aligned}$$

Since $A_{n+1}^2 - A_n^2$ vanishes if n is not prime, and

$$\frac{A_{p+1}^2 - A_p^2}{2pA_{p+1}^2} = \frac{(A_{p+1} + A_p)(A_{p+1} - A_p)}{2pA_{p+1}^2} \leq \frac{2A_{p+1}f(p)}{2p^2A_{p+1}^2} \leq \frac{f(p)}{p^2A_p}$$

for prime p , we have (2.6).

By applying (2.6), we have

$$\begin{aligned} \sum_{m< p < M} \frac{f(p)}{p^2 A_p} &\leq \sum_{m< n \leq M} \frac{f(n)}{n^2 A_n} \leq \left(\sum_{m< n \leq M} \frac{1}{n^2} \right)^{1/2} \left(\sum_{m< n \leq M} \frac{f^2(n)}{n^2 A_n^2} \right)^{1/2} \\ &\leq \frac{1}{\sqrt{m}} \left(\frac{C}{m} + C \sum_{m< p < M} \frac{f(p)}{p^2 A_p} \right)^{1/2}. \end{aligned}$$

Therefore

$$m^2 \left(\sum_{m< p < M} \frac{f(p)}{p^2 A_p} \right)^2 \leq C + Cm \sum_{m< p < M} \frac{f(p)}{p^2 A_p},$$

and thereby

$$(2.7) \quad \sum_{m< p < M} \frac{f(p)}{p^2 A_p} \leq \frac{Cm + \sqrt{C^2 m^2 + 4Cm^2}}{2m^2} \leq \frac{C+1}{m}.$$

By letting $M \rightarrow \infty$, we see that (2.6) and (2.7) are valid even in the case $M = \infty$. Combining these with (2.5), we have (2.4). \square

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