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A METRIC DISCREPANCY RESULT FOR LACUNARY SEQUENCES

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ABSTRACT. We prove that every value greater than or equal to 1/2 can be a constant appearing in the law of the iterated logarithm for discrepancies of a lacunary sequence satisfying the Hadamard's gap condition.

1. INTRODUCTION

In the theory of the uniform distribution, we use the following discrepancies of a sequence $\{a_k\}$:

$$D_N\{a_k\} = \sup_{0 \le a' < a < 1} \left| \frac{1}{N} \# \{k \le N \mid \langle a_k \rangle \in [\, a', a)\} - (a - a') \right|,$$

$$D_N^*\{a_k\} = \sup_{0 \le a < 1} \left| \frac{1}{N} \# \{k \le N \mid \langle a_k \rangle \in [\, 0, a)\} - a \right|,$$

where $\langle x \rangle$ denotes the fractional part $x - [\, x\,]$ of x.

One of the most typical result on asymptotic behavior of discrepancies is celebrated Chung-Smirnov theorem which asserts the law of the iterated logarithm for discrepancies of uniformly distributed i.i.d. $\{U_k\}$:

$$\overline{\lim_{N\to\infty}} \frac{N D_N^*\{U_k\}}{\sqrt{2N \log\log N}} = \overline{\lim_{N\to\infty}} \frac{N D_N\{U_k\}}{\sqrt{2N \log\log N}} = \frac{1}{2}, \quad \text{a.s.}$$

We have similar phenomena without assuming independence of sequence of random variables. For a sequence $\{n_k\}$ satisfying the Hadamard's gap condition

$$\inf_{k \in \mathbf{N}} n_{k+1}/n_k > 1, \tag{1.1}$$

Philipp [9, 10] proved the bounded law of the iterated logarithm

$$\frac{1}{4\sqrt{2}} \le \Sigma^*\{n_k x\} := \overline{\lim_{N\to\infty}} \frac{N D_N^*\{n_k x\}}{\sqrt{2N \log\log N}} \le \Sigma\{n_k x\} := \overline{\lim_{N\to\infty}} \frac{N D_N\{n_k x\}}{\sqrt{2N \log\log N}} \le C$$

for almost every x, where C is a constant depending only on the infimum in (1.1). Recently it became possible to calculate concrete values of $\Sigma\{n_k x\}$ and $\Sigma^*\{n_k x\}$.

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It is proved in [5] that, for all real number $\theta > 1$ there exists a constant Σ_θ such that

$$\Sigma^*\{\theta^k x\} = \Sigma\{\theta^k x\} = \Sigma_\theta, \quad \text{a.e.} \tag{1.2}$$

We have $\Sigma_\theta = 1/2$ if θ satisfies the condition $\theta^r \notin \mathbf{Q}$ for all $r \in \mathbf{N}$. Otherwise let us express θ by $\theta = \sqrt[r]{p/q}$ where $r = \min\{n \in \mathbf{N} \mid \theta^n \in \mathbf{Q}\}$, and $\gcd(p,q) = 1$. In this case Σ_θ does not depend on r and is evaluated in the following cases:

$$\Sigma_\theta = \begin{cases} \sqrt{(pq+1)/(pq-1)}/2 & \text{if } p \text{ and } q \text{ are odd,} \\ \sqrt{(p+1)p(p-2)/(p-1)^3}/2 & \text{if } p \geq 4 \text{ is even and } q = 1, \\ \sqrt{42}/9 & \text{if } p = 2 \text{ and } q = 1, \\ \sqrt{22}/9 & \text{if } p = 5 \text{ and } q = 2. \end{cases}$$

It is also proved that $\max_{\theta>1} \Sigma_\theta = \Sigma_2 = \sqrt{42}/9$ (Cf. [7]). Aistleitner [1] gave a nearly optimal Diophantine condition on the sequence $\{n_k\}$ to have $\Sigma^*\{n_k x\} = \Sigma\{n_k x\} = 1/2$ a.e., which coincides with the case of uniformly distributed i.i.d.

In [8], it is proved that $\Sigma^*\{n_k x\}$ and $\Sigma\{n_k x\}$ are equal to a constant if $\{n_k\}$ is a subsequence of $\{\theta^k\}$, and the set of possible values of constants coincides with the interval $[\,1/2, \Sigma_\theta\,]$. Therefore every value in $[\,1/2, \sqrt{42}/9\,]$ is proved to be a possible value of $\Sigma^*\{n_k x\}$ and $\Sigma\{n_k x\}$ for some $\{n_k\}$ satisfying the Hadamard's gap condition.

It is natural to ask whether values greater than $\sqrt{42}/9$ are also possible values of $\Sigma^*\{n_k x\}$ and $\Sigma\{n_k x\}$.

Relating to this question, we can find the following result by Berkes-Philipp [3]. For any real number L, there exists a sequence $\{n_k\}$ satisfying the Hadamard's gap condition (1.1) such that $\Sigma\{n_k x\} \geq L$ a.e. Unfortunately, we cannot know by this result, if $\Sigma\{n_k x\}$ is constant a.e. or not.

So far, only known greater constants are those for so-called Hardy-Littlewood-Pólya sequences defined as below: Let $q_1, \ldots, q_\tau \geq 2$ are relatively prime integers, and $\{n_k\}$ be an arrangement in increasing order of $\{q_1^{i_1} \ldots q_\tau^{i_\tau} \mid i_1, \ldots, i_\tau = 0, 1, 2, \ldots\}$. Then $\{n_k\}$ is said to be the Hardy-Littlewood-Pólya sequence generated by $q_1, \ldots, q_\tau$. As to this sequence $\Sigma^*\{n_k x\}$ and $\Sigma\{n_k x\}$ equal to a constant a.e. (Cf. [6]), and when the set of generators consists of odd integers, then

$$\Sigma^*\{n_k x\} = \Sigma\{n_k x\} = \frac{1}{2}\left(\prod_{i=1}^{\tau} \frac{q_i+1}{q_i-1}\right)^{1/2}, \quad \text{a.e.}$$

See also [2] to find a detailed study for permutations of these sequences.

The last value becomes arbitrarily large if we choose generators properly. Unfortunately, when $\tau \geq 2$, Hardy-Littlewood-Pólya sequences do not satisfy the Hadamard's gap condition. But the method of approximating the Hardy-Littlewood-Pólya sequence by subsequences satisfying the Hadamard's gap condition can be found in [4]. We adopt this method together with randomization technique to solve the above problem.

Now we are in a position to state our result.

Theorem 1.1. *For all $\sigma \geq 1/2$, there exists a sequence $\{n_k\}$ of positive integers satisfying the Hadamard's gap condition (1.1) such that*

$$\Sigma\{n_k x\} = \Sigma^*\{n_k x\} = \sigma, \quad \textit{a.e.} \tag{1.3}$$

For $0 < \sigma < 1/2$, there exists a sequence $\{n_k\}$ with bounded gaps $n_{k+1} - n_k = O(1)$ such that (1.3) holds (Cf. [7]). It is open whether we can take such $\{n_k\}$ satisfying the Hadamard's gap condition.

2. Proof

We first note that discrepancies are written by

$$D_N\{a_k\} = \sup_{0\le a<b<1} \left| \frac{1}{N} \sum_{k=1}^N \widetilde{\mathbf{1}}_{a,b}(a_k) \right|; \quad D_N^*\{a_k\} = \sup_{0\le a<1} \left| \frac{1}{N} \sum_{k=1}^N \widetilde{\mathbf{1}}_{0,a}(a_k) \right|;$$

where $\widetilde{\mathbf{1}}_{a,b}(x) = \mathbf{1}_{[a,b)}(\langle x\rangle) - (b-a)$, and $\mathbf{1}_{[a,b)}$ denotes the indicator function of $[\,a,b)$.

Denote the sequence of all odd prime numbers by $3 = q_1 < q_2 < \cdots$. Put $\phi(x) = x \vee 0$ and $\psi(x) = (-x) \vee 0$. We have $\phi(x) + \psi(x) = |x|$. Denote

$$\Phi_{I;j_1,\dots,j_\tau} = \frac{\phi(I-|j_1|)}{I} \cdots \frac{\phi(I-|j_\tau|)}{I} \quad \text{and} \quad \Psi(x) = \sqrt{2x\log\log x}.$$

We use the next proposition.

Proposition 2.1. *Let $0 < p < 1$ and $\{X_n\}$ be a sequence of independent random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = p$. Let us regard the set $\{q_{\tau+1}^i \mid X_i(\omega) = 1\}$ as an increasing sequence and denote it by $\{n_k^\circ(\omega)\}$. Then*

$$\mathbb{P}\left(\#\{n_k^\circ\} = \infty,\ \varlimsup_{N\to\infty} \Psi^{-1}(N) \left| \sum_{k=1}^N f(n_k^\circ x) \right| = \sigma(f,p) \ \text{ a.e. } x, \text{ for all } f \in BV_0 \right) = 1,$$

where BV_0 denotes the class of functions f of bounded variation with period 1 satisfying $\int_0^1 f(t)\,dt = 0$, and $\sigma^2(f,p)$ is given by

$$\sigma^2(f,p) = \int_0^1 f^2(x)\,dx + 2p \sum_{k=1}^\infty \int_0^1 f(q_{\tau+1}^k x) f(x)\,dx.$$

In [8] we proved a similar statement given by replacing 'for all $f \in BV_0$' with 'for all $\widetilde{\mathbf{1}}_{a,b}$ with $0 \le a < b < 1$' and $q_{\tau+1}$ with 3. Proposition 2.1 can be proved completely in the same way. We apply it for

$$f_{a,b}(x) = \frac{1}{I^{\tau/2}} \sum_{i_1=0}^{I-1} \cdots \sum_{i_\tau=0}^{I-1} \widetilde{\mathbf{1}}_{a,b}(q_1^{i_1} \dots q_\tau^{i_\tau} x).$$

Let us evaluate $\sigma^2(f_{a,b}, p)$. By changing variable $y = q^{i\wedge i'}x$ and by noting $i - i\wedge i' = \phi(i-i')$ and $i' - i\wedge i' = \psi(i-i')$, we have

$$\frac{1}{I} \sum_{i=0}^{I-1} \sum_{i'=0}^{I-1} \int_0^1 g(q'q^i x) g(q''q^{i'} x)\,dx = \sum_{j\in\mathbf{Z}} \frac{\phi(I-|j|)}{I} \int_0^1 g(q'q^{\phi(j)}y) g(q''q^{\psi(j)}y)\,dy.$$

Here we used the fact that the number of $(i,i') \in \{0,1,\dots,I-1\}^2$ such that $j = i - i'$ equals to $\phi(I-|j|)$. Hence we have

$$\begin{aligned}
&\int_0^1 f_{a,b}(q_{\tau+1}^{\phi(k)}x) f_{a,b}(q_{\tau+1}^{\psi(k)}x)\,dx \\
&= \frac{1}{I^\tau}\sum_{i_1,i_1'=0}^{I-1}\cdots\sum_{i_\tau,i_\tau'=0}^{I-1}\int_0^1 \widetilde{\mathbf{1}}_{a,b}(q_1^{i_1}\dots q_\tau^{i_\tau} q_{\tau+1}^{\phi(k)}x)\widetilde{\mathbf{1}}_{a,b}(q_1^{i_1'}\dots q_\tau^{i_\tau'} q_{\tau+1}^{\psi(k)}x)\,dx \\
&= \sum_{j_1,\dots,j_\tau\in\mathbf{Z}} \Phi_{I;j_1,\dots,j_\tau}\int_0^1 \widetilde{\mathbf{1}}_{a,b}(q_1^{\phi(j_1)}\dots q_\tau^{\phi(j_\tau)} q_{\tau+1}^{\phi(k)}x)\widetilde{\mathbf{1}}_{a,b}(q_1^{\psi(j_1)}\dots q_\tau^{\psi(j_\tau)} q_{\tau+1}^{\psi(k)}x)\,dx.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\sigma^2(f_{a,b},0) &= \int_0^1 f_{a,b}^2(x)\,dx \\
&= \sum_{j_1,\dots,j_\tau\in\mathbf{Z}} \Phi_{I;j_1,\dots,j_\tau}\int_0^1 \widetilde{\mathbf{1}}_{a,b}(q_1^{\phi(j_1)}\dots q_\tau^{\phi(j_\tau)}x)\widetilde{\mathbf{1}}_{a,b}(q_1^{\psi(j_1)}\dots q_\tau^{\psi(j_\tau)}x)\,dx,
\end{aligned}$$

and by noting

$$\sigma^2(f_{a,b},1) = \sum_{k\in\mathbf{Z}}\int_0^1 f_{a,b}(q_{\tau+1}^{\phi(k)}x) f_{a,b}(q_{\tau+1}^{\psi(k)}x)\,dx,$$

we see that $\sigma^2(f_{a,b},1)$ equals to

$$\sum_{k,j_1,\dots,j_\tau\in\mathbf{Z}} \Phi_{I;j_1,\dots,j_\tau}\int_0^1 \widetilde{\mathbf{1}}_{a,b}(q_1^{\phi(j_1)}\dots q_\tau^{\phi(j_\tau)} q_{\tau+1}^{\phi(k)}x)\widetilde{\mathbf{1}}_{a,b}(q_1^{\psi(j_1)}\dots q_\tau^{\psi(j_\tau)} q_{\tau+1}^{\psi(k)}x)\,dx.$$

To evaluate the integrals above, we use the next lemma, which is proved in [5].

Lemma 2.2. *For x, y, ξ, $\eta \in [0,1]$, put $V(x,\xi) = x\wedge\xi - x\xi$ and $\widetilde{V}(x,y,\xi,\eta) = V(x,\xi)+V(y,\eta)-V(x,\eta)-V(y,\xi)$. For any positive integers P and Q with $\gcd(P,Q)=1$, we have*

$$\begin{aligned}
\int_0^1 \widetilde{\mathbf{1}}_{a,b}(Px)\widetilde{\mathbf{1}}_{a,b}(Qx)\,dx &= \frac{1}{PQ}\widetilde{V}(\langle Pa\rangle,\langle Pb\rangle,\langle Qa\rangle,\langle Qb\rangle), \\
\widetilde{V}(\langle Pa\rangle,\langle Pb\rangle,\langle Qa\rangle,\langle Qb\rangle) &\le V(\langle P(b-a)\rangle,\langle Q(b-a)\rangle) \le 1/4.
\end{aligned}$$

If P and Q are odd, and if $a=0$, $b=1/2$, then $\widetilde{V}(\langle Pa\rangle,\langle Pb\rangle,\langle Qa\rangle,\langle Qb\rangle) = 1/4$.

By applying this lemma and by noting $\phi(x)+\psi(x) = |x|$, we have

$$\int_0^1 \widetilde{\mathbf{1}}_{a,b}(q_1^{\phi(j_1)}\dots q_\tau^{\phi(j_\tau)} q_{\tau+1}^{\phi(k)}x)\widetilde{\mathbf{1}}_{a,b}(q_1^{\psi(j_1)}\dots q_\tau^{\psi(j_\tau)} q_{\tau+1}^{\psi(k)}x)\,dx \le \frac{1}{4q_1^{|j_1|}\dots q_\tau^{|j_\tau|}q_{\tau+1}^{|k|}},$$

and we see that the equality holds if $a=0$ and $b=1/2$. Put

$$Q_\tau = \frac{1}{2}\biggl(\prod_{i=1}^{\tau}\frac{q_i+1}{q_i-1}\biggr)^{1/2} \quad\text{and}\quad Q_{\tau,I} = \frac{1}{2}\biggl(\sum_{j_1,\dots,j_\tau\in\mathbf{Z}}\frac{\Phi_{I;j_1,\dots,j_\tau}}{q_1^{|j_1|}\dots q_\tau^{|j_\tau|}}\biggr)^{1/2}.$$

Hence we see

$$\sigma^2(f_{a,b},0) \le \frac{1}{4} \sum_{j_1,\dots,j_\tau \in \mathbf{Z}} \frac{\Phi_{I;j_1,\dots,j_\tau}}{q_1^{|j_1|}\dots q_\tau^{|j_\tau|}} = Q_{\tau,I}^2 = \sigma^2(f_{0,1/2},0),$$

$$\sigma^2(f_{a,b},1) \le \frac{1}{4} \sum_{k,j_1,\dots,j_\tau \in \mathbf{Z}} \frac{\Phi_{I;j_1,\dots,j_\tau}}{q_1^{|j_1|}\dots q_\tau^{|j_\tau|} q_{\tau+1}^{|k|}} = Q_{\tau,I}^2 \frac{q_{\tau+1}+1}{q_{\tau+1}-1} = \sigma^2(f_{0,1/2},1),$$

which imply

$$\sup_{S \ni a<b \in S} \sigma(f_{a,b},0) = \sup_{a\in S} \sigma(f_{0,a},0) = \sigma(f_{0,1/2},0) = Q_{\tau,I},$$

$$\sup_{S \ni a<b \in S} \sigma(f_{a,b},1) = \sup_{a\in S} \sigma(f_{0,a},1) = \sigma(f_{0,1/2},1) = Q_{\tau,I}\left(\frac{q_{\tau+1}+1}{q_{\tau+1}-1}\right)^{1/2}$$

where $S = [\,0,1) \cap \mathbf{Q}$. By $\sigma^2(f_{a,b},p) = (1-p)\sigma^2(f_{a,b},0) + p\sigma^2(f_{a,b},1)$, we also have

$$\sup_{S \ni a<b \in S} \sigma(f_{a,b},p) = \sup_{a\in S} \sigma(f_{0,a},p) = \sigma(f_{0,1/2},p) = \left(1-p+p\frac{q_{\tau+1}+1}{q_{\tau+1}-1}\right)^{1/2} Q_{\tau,I}.$$

Let $\{n_k\}$ be an arrangement in increasing order of

$$\{n_k^\circ q_1^{i_1}\dots q_\tau^{i_\tau} : k=0,1,\dots; i_1,\dots,i_\tau = 0,1,\dots,I-1\}.$$

It is clear that $\{n_k\}$ satisfies the Hadamard's gap condition. Let us take $\rho(k)$ such that $n_{\rho(k)} = n_k^\circ q_1^{I-1}\dots q_\tau^{I-1}$. If $n_{k'}^\circ q_1^{i_1'}\dots q_\tau^{i_\tau'} \le n_k^\circ q_1^{I-1}\dots q_\tau^{I-1}$, then $n_{k'}^\circ/n_k^\circ \le q_1^{I-1}\dots q_\tau^{I-1}$. In case $k' \ge k$, we have $3^{k'-k} \le n_{k'}^\circ/n_k^\circ$ and thereby $k' \le k + \log_3 q_1^{I-1}\dots q_\tau^{I-1}$. The last inequality is also valid when $k' < k$. Hence we see

$$\{n_k^\circ q_1^{i_1}\dots q_\tau^{i_\tau} \mid k \le K, i_1,\dots,i_\tau < I\} \subset \{n_k \mid k \le \rho(K)\}$$
$$\subset \{n_k^\circ q_1^{i_1}\dots q_\tau^{i_\tau} \mid k \le K + \log_3 q_1^{I-1}\dots q_\tau^{I-1}, i_1,\dots,i_\tau < I\}.$$

Therefore we have $KI^\tau \le \rho(K) \le KI^\tau + I^\tau \log_3 q_1^{I-1}\dots q_\tau^{I-1}$ and $\rho(K) \sim KI^\tau$.

For given N, take the largest K such that $KI^\tau \le N$. Then $N < KI^\tau + I^\tau$. It implies $|N-\rho(K)| \le I^\tau + I^\tau \log_3 q_1^{I-1}\dots q_\tau^{I-1}$. Therefore we have

$$\left|\sum_{k=1}^N \widetilde{\mathbf{1}}_{a,b}(n_k x) - \sum_{k=1}^K \sum_{i_1,\dots,i_\tau<I} \widetilde{\mathbf{1}}_{a,b}(n_k^\circ q_1^{i_1}\dots q_\tau^{i_\tau} x)\right| = O(1),$$

and thereby we have

$$\sigma(f_{a,b},p) = \varlimsup_{K\to\infty} \Psi(K)^{-1} \sum_{k=1}^K f_{a,b}(n_k^\circ x) = \varlimsup_{N\to\infty} \Psi(N)^{-1} \left|\sum_{k=1}^N \widetilde{\mathbf{1}}_{a,b}(n_k x)\right|, \quad \text{a.e.}$$

We use the fundamental relation below which can be found in [7].

Lemma 2.3. *For any countable dense subset S of $[\,0,1)$ and for any sequence $\{n_k\}$ of positive real numbers satisfying the Hadamard's gap condition, we have*

$$\varlimsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} = \sup_{S\ni a<b\in S} \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left|\sum_{k=1}^N \widetilde{\mathbf{1}}_{a,b}(n_k x)\right|, \quad \textit{a.e.},$$

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{n_k x\}}{\sqrt{2N\log\log N}} = \sup_{a\in S} \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left|\sum_{k=1}^N \widetilde{\mathbf{1}}_{0,a}(n_k x)\right|, \quad \textit{a.e.}$$

By applying the lemma, we have

$$\Sigma\{n_k x\} = \Sigma^*\{n_k x\} = \left(1 - p + p\frac{q_{\tau+1}+1}{q_{\tau+1}-1}\right)^{1/2} Q_{\tau,I}, \quad \text{a.e.}$$

Therefore, for every $\sigma \in (Q_{\tau,I}, \big(\frac{q_{\tau+1}+1}{q_{\tau+1}-1}\big)^{1/2} Q_{\tau,I})$, there exists a $p \in (0,1)$ such that (1.3) holds. Because of $Q_{\tau,I} < Q_\tau$, $Q_{\tau,I} \uparrow Q_\tau$ as $I \to \infty$, and $\big(\frac{q_{\tau+1}+1}{q_{\tau+1}-1}\big)^{1/2} Q_\tau = Q_{\tau+1}$, we see that for every $\sigma \in [\,Q_\tau, Q_{\tau+1})$ there exists a sequence $\{n_k\}$ satisfying (1.1) and (1.3). Because of $Q_1 = 1/\sqrt{2}$ and $Q_\tau \to \infty$ as $\tau \to \infty$, we can conclude that for every $\sigma > 1/\sqrt{2}$ there exists a sequence $\{n_k\}$ satisfying (1.1) and (1.3). On the other hand, for any $\sigma \in [\,1/2, 1/\sqrt{2}\,]$ there exists a sub-sequence $\{n_k\}$ of $\{3^k\}$ such that (1.1) and (1.3) holds. (Cf. [8]).

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