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On the central limit theorem and the law of the iterated logarithm

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Abstract: This paper investigates conditions under which the law of the iterated logarithm holds.

Key words: law of the iterated logarithm, sequence of random variables

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Introduction.

It has been a long standing problem to study the relation between the central limit theorem and the law of the iterated logarithm. As to this problem, Petrov (1971) proved the following result. Let $\{X_n\}$ be a sequence of independent random variables and put $S_n = X_1 + \cdots + X_n$. Let B_n be a sequence of real numbers satisfying $B_n \to \infty$ and $B_{n+1}/B_n \to 1$. If

$$\sup_x \bigl|\mathbf{P}\{S_n < B_n^{1/2} x\} - \Phi(x)\bigr| = O\bigl((\log B_n)^{-1-\delta}\bigr) \tag{1}$$

for some $\delta > 0$ where $\Phi(x) = \int_{-\infty}^{x} e^{-t^2/2}\, dt/\sqrt{2\pi}$, then

$$\limsup_{n\to\infty} S_n \big/ \sqrt{2B_n \log\log B_n} = 1 \quad \text{a.s.} \tag{2}$$

Without assuming the condition $B_{n+1}/B_n \to 1$, one still has the upper bound part. Actually, Petrov (2003) dropped that condition and proved

$$\limsup_{n\to\infty} S_n \big/ \sqrt{2B_n \log\log B_n} \le 1 \quad \text{a.s.}$$

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In the above results, $\delta > 0$ is necessary. Indeed, Egorov (1969) relaxed the condition to $\delta = 0$ and constructed an example such that

$$\limsup_{n\to\infty} S_n/\sqrt{2B_n \log\log B_n} = \infty.$$

Prior to these results, Petrov (1966, 2002) proved the same conclusions by assuming extra conditions $EX_n = 0$ and $B_n = EX_1^2 + \cdots + EX_n^2$. Assuming further that $\{X_n\}$ is an independent centered gaussian sequence with $B_n = EX_1^2 + \cdots + EX_n^2 \to \infty$, Kôno (1974) proved the generalised law of the iterated logarithm below: Let $n_1 < n_2 < \cdots$ be a sequence of integers defined by

$$B_{n_1} > 0 = B_{n_1 - 1} \quad \text{and} \quad B_{n_k} \geq eB_{n_{k-1}} > B_{n_k - 1}. \tag{3}$$

Then for any increasing function φ satisfying

$$\varphi(B_{n_k}) = \log k \tag{4}$$

one has the generalised law of the iterated logarithm:

$$\limsup_{n\to\infty} S_n/\sqrt{2B_n\varphi(B_n)} = 1 \quad \text{a.s.} \tag{5}$$

By (4), one can prove that φ satisfies the condition

$$\varphi^{-1}(\log(n+1))/\varphi^{-1}(\log n) \geq e \tag{6}$$

and the asymptotics $\limsup_{x\to\infty} \varphi(x)/\log\log x \leq 1$.

Conversely, for any φ satisfying (6) we can construct the sequence $\{B_n\}$ by $B_n = \varphi^{-1}(\log n)$, for which $B_{n+1}/B_n \geq e$ holds. Hence by taking an independent centered gaussian sequence with $EX_1^2 + \cdots + EX_n^2 = B_n$, we can apply the result of Kôno to have (5). Therefore we see that any increasing function φ with (6) can appear in the generalised law of the iterated logarithm (5).

Here, we mention that the condition (6) is mild to include the typical example $\varphi(x) = \alpha \log\log x$ $(0 < \alpha \le 1)$, $\varphi(x) = \log\log\log x$, $\varphi(x) = \log\log\log\log x$, and so on.

The other important point is that the condition $\sup B_{n+1}/B_n < \infty$ implies $\varphi(x) \sim \log\log x$ as $x \to \infty$, and we have the ordinary law of the iterated logarithm in this case.

As the extension of above results, we prove the following theorem.

Theorem 1. *Let $\{X_n\}$be a sequence of independent random variables and put $S_n = X_1 + \cdots + X_n$. Let $0 = B_0 \le B_1 \le B_2 \le \cdots$ satisfy $B_n \to \infty$. Define a sequence $\{n_k\}$ and increasing function φ by (3) and (4). If*

$$\sup_x \left| \mathbf{P}\{S_n < B_n^{1/2} x\} - \Phi(x) \right| = O(e^{-(1+\delta)\varphi(B_n)}) \tag{7}$$

for some $\delta > 0$, then the generalised law of the iterated logarithm (5) holds.

By $\limsup \varphi(x)/\log\log x \le 1$, the condition (7) appears to be weaker than the condition (1). This asymptotic also explains why one always has the upper bound estimate.

Since $\sup B_{n+1}/B_n < \infty$ implies $\varphi(x) \sim \log\log x$, we have the following extension of the Petrov's law of the iterated logarithm.

Corollary. *Let $\{X_n\}$be a sequence of independent random variables and put $S_n = X_1 + \cdots + X_n$. Let $0 = B_0 \le B_1 \le B_2 \le \cdots$ satisfy $B_n \to \infty$ and $\sup B_{n+1}/B_n < \infty$. If (1) is satisfied for some $\delta > 0$, then the law of the iterated logarithm (2) holds.*

We therefore succeeded in weakening the condition $B_{n+1}/B_n \to 1$ of Petrov's result. As to the necessity of the condition on δ in Theorem 1, we have the result below:

Theorem 2. *Let φ be a non-negative increasing function satisfying (6) and $\varphi(x^2) = \varphi(x) + O(1)$ as $x \to \infty$. Then there exist a sequence of independent random variables $\{X_n\}$, an increasing sequence of real numbers $0 = B_0 \le B_1 \le B_2 \le \ldots$, and a increasing sequence of integers $\{n_k\}$, satisfying conditions (3), (4), and (7) with $\delta = 0$ such that*

$$\limsup_{n\to\infty} S_n \big/ \sqrt{2B_n\varphi(B_n)} = \infty, \quad \text{a.s.}$$

Proof of the law of the iterated logarithm

We use the following lemma due to Petrov (1971).

Lemma. *Let Y_1, Y_2, ... be a sequence of independent random variables and $0 = B_0 \le B_1 \le B_2 \le \cdots$ be a sequence of real numbers.*

(1) *If $\mathbf{P}\{Y_k + \cdots + Y_n \ge -C\} \ge 1/2$ $(k < n)$ holds for some $C \ge 0$, then we have*

$$\mathbf{P}\Big\{\max_{1\le k\le n}(Y_1 + \cdots + Y_k) \ge x\Big\} \le 2\mathbf{P}\{Y_1 + \cdots + Y_n \ge x - C\}$$

for every x.

(2) *If $\mathbf{P}\big\{Y_1 + \cdots + Y_n < xB_n^{1/2}\big\} \to \Phi(x)$ for any x, then*

$$\mathbf{P}\big\{Y_k + \cdots + Y_n \ge -b\sqrt{B_n}\,\big\} \ge 1/2 \quad (k < n)$$

for large enough b and n.

Put $M_n = \max\{S_1, \ldots, S_n\}$ and $C_n = \sqrt{2B_n\varphi(B_n)}$. Let p and q be positive integers. Take $\{m_k\}$ as below:

$$m_1 = n_1 \quad \text{and} \quad B_{m_k} \ge e^{1/p}B_{m_{k-1}} > B_{m_k-1}.$$

Then we have $m_{qp+1} \geq n_{q+1}$ and $m_q \leq n_q$ for any q. Put $\alpha_k = e^{1/p}C_{m_k-1}$ and $\beta_k = e^{1/2p}C_{m_k-1}$. By applying Lemma and $\alpha_k - b\sqrt{B_{m_k-1}} \geq \beta_k$ for large k, we have

$$a_k = \mathbf{P}\big\{M_{m_k-1} \geq \alpha_k\big\} \leq 2\mathbf{P}\big\{S_{m_k-1} \geq \beta_k\big\}.$$

By this and (7), we have

$$a_k = O\big(\exp[(-1-\delta)\varphi(B_{m_k-1})]\big) + 1 - \Phi\Big(\sqrt{2e^{1/p}\varphi(B_{m_k-1})}\Big).$$

By $qp < k \leq (q+1)p$, we have $m_k - 1 \geq n_{q+1} - 1 \geq n_q$. Hence we have $B_{m_k-1} \geq B_{n_q}$ and $\varphi(B_{m_k-1}) \geq \varphi(B_{n_q}) = \log q$. Thus we have $a_k = O(q^{-1-\delta}) + Cq^{-e^{1/p}}$ and $\sum_k a_k < \infty$. By the Borel Cantelli Lemma, we have $\mathbf{P}\{M_{m_k-1} < \alpha_k$ for all large $k\} = 1$ and $\mathbf{P}\{S_n < e^{1/p}C_n$ for all large $n\} = 1$. By letting $p \to \infty$, we have the upperbound estimate.

Put $\{m'_k\}$ as below:

$$m'_1 = n_1 \quad \text{and} \quad B_{m'_k} \geq e^p B_{m'_{k-1}} > B_{m'_k - 1}.$$

We have $n_{qp+1} \geq m'_{q+1}$. Put $\rho_k = (1-e^{-p})C_{m'_k}$ and $\mu_k = (1+e^{-p})C_{m'_{k-1}}$. Then we have

$$\begin{aligned} b_k = \mathbf{P}\{S_{m'_k} - S_{m'_{k-1}} \geq \rho_k - \mu_k\} &\geq \mathbf{P}\{S_{m'_k} \geq \rho_k, S_{m'_{k-1}} < \mu_k\} \\ &\geq \mathbf{P}\{S_{m'_k} \geq \rho_k\} - \mathbf{P}\{S_{m'_{k-1}} \geq \mu_k\} = c_k - d_k \end{aligned}$$

As before, we have

$$\sum d_k < \infty \quad \text{and} \quad \sum c_k = \sum O((pk)^{-1-\delta}) + (pk)^{1-\varepsilon}/\log(pk) = \infty.$$

Thus we have $\sum b_k = \infty$. By Borel-Cantelli Lemma, we have

$$\mathbf{P}\{S_{m'_k} - S_{m'_{k-1}} \geq \rho_k - \mu_k \text{ i.o.}\} = 1.$$

On the other hand, we have $\mathbf{P}\{S_{m'_{k-1}} \geq -\mu_k \text{ for all large } k\} = 1$. Thus we have $\mathbf{P}\{S_{m'_k} \geq \rho_k - 2\mu_k \text{ i.o.}\} = 1$. Since $\rho_k - 2\mu_k \geq D_p C_{m'_k}$, where $D_p = \big((1-e^{-p}) - 2(1+e^{-p})/e^{p/2}\big)$, we have $\mathbf{P}\{S_n \geq D_p C_n \text{ i.o.}\} = 1$. By letting $p \to \infty$, we have the lower bound estimate.

Construction of the counter example

We modify the method of Egorov (1969).

Let $\lambda_k = \varphi^{-1}(\log k)$, $n_k = [\,\lambda_k + 1/2\,]$, and $\widetilde{\varphi}(x) = \log(k+1)$ on $[\,\lambda_k, \lambda_{k+1})$. By (6) we have $\lambda_{k+1}/\lambda_k \geq e$, by definition $e^{\widetilde{\varphi}(x)}/e^{\varphi(x)} \to 1$, and $\widetilde{\varphi}(x^2) = \widetilde{\varphi}(x) + O(1)$.

Put $\alpha_n = 1$ if $n \notin \{n_k\} \cup \{n_k + 1\}$, $\alpha_{n_k} = \sqrt{\lambda_k - n_k + 1}$, $\alpha_{n_k+1} = \sqrt{n_k + 1 - \lambda_k}$,

Let $\{Y_n\}$ be a sequence of independent random variables satisfying

$$P(Y_n = \alpha_n x_n) = P(Y_n = -\alpha_n x_n) = p_n,$$

$$P(Y_n = \alpha_n \sqrt{2}) = P(Y_n = -\alpha_n \sqrt{2}) = 1/4 - x_n^2 p_n/2,$$

$$P(Y_n = 0) = 1/2 - 2p_n + x_n^2 p_n,$$

where $x_n = \sqrt{n\widetilde{\varphi}(n) \log_2^+ \widetilde{\varphi}(n)}$, $p_n = (2ne^{\widetilde{\varphi}(n)} \log_{1,3}^+ e^{\widetilde{\varphi}(n)})^{-1}$, $\log_1^+ x = 1 \vee \log x$, $\log_n^+ x = \log_1^+(\log_{n-1}^+ x)$, and $\log_{1,3}^+ x = (\log_1^+ x)(\log_2^+ x)(\log_3^+ x)$.

It is clear that $EY_n = 0$ and $EY_n^2 = \alpha_n^2$. By $|x_n| \leq n^2$ and $\widetilde{\varphi}(x_n) = \varphi(n) + C$, we can verify and $EY_n^2 g(Y_n) = O(1)$, where $g(x) = e^{\widetilde{\varphi}(x)} \log_1^+ \widetilde{\varphi}(x)$. Put $T_n = Y_1 + \cdots + Y_n$ and $C_n = ET_n^2$. We have $|C_n - n| \leq 1$ and $C_{n_k} = \lambda_k$.

We use Theorem 5.6 of Petrov (1995) below:

Theorem. *Let g be a non-negative even function such that $g(x)$ and $x/g(x)$ are both non-decreasing for $x > 0$. Let $\{Y_n\}$ be a sequence of independent random variables such that $EY_n = 0$ and $EY_n^2 g(Y_n) < \infty$ for all n. Put*

$T_n = Y_1 + \cdots + Y_n$ *and* $C_n = ET_n^2$. *Then there exists an absolute constant* A *such that*

$$\sup_x \Big| \mathbf{P}\{T_n < C_n^{1/2} x\} - \Phi(x) \Big| \le \frac{A}{C_n g(C_n^{1/2})} \sum_{j=1}^{n} EY_n^2 g(Y_n).$$

Clearly g is non-decreasing, and so is $x/g(x)$ by

$$\lambda_{k+1}/g(\lambda_{k+1}) - \lambda_k/g(\lambda_k) \ge e\lambda_k/(k+1)\log(k+1) - \lambda_k/k\log k > 0.$$

Thus we can apply the above theorem and have

$$\sup_x \Big| \mathbf{P}\{T_n < C_n^{1/2} x\} - \Phi(x) \Big| = O\big(e^{-\widetilde{\varphi}(C_n^{1/2})}/\log_1^+ \varphi(C_n^{1/2})\big) = o(e^{-\varphi(C_n)}). \quad (8)$$

Let us define a sequence $\{X_i\}$ of random variables by $X_i = Y_i$ if $n_j < i \le en_j/2$, $X_i = 0$ if $en_j/2 < i < n_{j+1}$, and $X_{n_{j+1}} = \sum_{en_j/2 < k \le n_{j+1}} Y_k$. Put $S_n = X_1 + \cdots + X_n$, and $B_n = ES_n^2$. We have $B_{n_i} < \cdots < B_{[en_j/2]} = \cdots = B_{n_{j+1}-1} = C_{[en_j/2]} = [en_j/2] < e\lambda_j$, and $B_{n_{j+1}} = \lambda_{j+1} \ge e\lambda_j = eB_{n_j}$. Hence $\{B_n\}$ and $\{n_j\}$ together with φ satisfy conditions (3)and (4). It is clear from (8) that condition (7) is satisfied with $\delta = 0$. Moreover we have

$$\sum_{j=1}^{\infty} \sum_{n_j < k \le en_j/2} p_k \ge \frac{e-2}{e} \sum_{j=1}^{\infty} \frac{1}{e^{\log(j+1)} \log_{1,3}^+ e^{\log(j+1)}} = \infty$$

by $\sum_{n_j < k \le en_j/2} 1/k \ge (en_j/2 - n_j)/(en_j/2) = (e-2)/e$. Thus by Borel-Cantelli Lemma, we see that $X_j \ge \sqrt{B_j \varphi(B_j) \log_2^+ \varphi(B_j)}$ i.o., and hence we have the conclusion of Theorem 2.

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