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THE CONCRETE UPPER BOUND IN THE UNIFORM LAW OF THE ITERATED LOGARITHM

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Abstract. Kaufman and Philipp [6] and Dhompongsa [3] proved a uniform law of the iterated logarithm for $\sum f(n_k t)$. In this paper, we give the concrete upper bound for these results. Our bound is best possible in some cases.

1. Introduction

Let L_0^2 be the class of measurable functions f satisfying

$$f(x+1) = f(x), \quad \int_0^1 f(x)^2\,dx < \infty, \quad \text{and} \quad \int_0^1 f(x)\,dx = 0.$$

In this paper, we give the concrete upper bound for

$$\Psi[X; \{n_k\}](t) = \varlimsup_{K\to\infty} \sup_{f\in X} \frac{\sum_{k=1}^K f(n_k t)}{\sqrt{K \log\log K}},$$

where X is a subclass of L_0^2 and $\{n_k\}$ is an increasing sequence of integers.

We give here a simple survey of studies of this type. When $f \in \operatorname{Lip}\alpha \cap L_0^2$ $(\alpha > 0)$ where $\operatorname{Lip}\alpha$ is the class of functions satisfying Lipschitz condition of order α, Takahashi [7, 8] proved $\Psi[\{f\}; \{n_k\}] = \sqrt{2}\,\|f\|_2$ a.e. when $\{n_k\}$ satisfies the large gap condition:

$$n_{k+1}/n_k \to \infty \qquad (k \to \infty),$$

and $\Psi[\{f\}; \{n_k\}] \le C < \infty$ a.e. when $\{n_k\}$ satisfies Hadamard's gap condition:

$$n_{k+1}/n_k \ge 1 + \rho \qquad (\rho > 0).$$

For $\Lambda_\alpha = \{f \in L_0^2 \mid |f(t+h) - f(t)| \le |h|^\alpha\} \subset \operatorname{Lip}\alpha$ $(\alpha > 1/2)$, the estimate $\Psi[\Lambda_\alpha; \{n_k\}] \le C < \infty$ a.e. was proved by Kaufman and Philipp [6] when $\{n_k\}$ satisfies Hadamard's gap condition, and by Dhompongsa [3] when $\{n_k\}$ satisfies Takahashi's gap condition:

$$n_{k+1}/n_k \ge 1 + c/k^\beta \qquad (c > 0,\ \beta < 1/2).$$

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Assuming the same gap condition and $f \in \operatorname{Lip} \alpha$ $(\alpha > 1/2)$, Takahashi [11] gave the concrete upper bound $\Psi[\{f\}; \{n_k\}] \le \|f\|_A$ a.e. where $\|f\|_A$ denotes $\sum \sqrt{a_\nu^2 + b_\nu^2}$ for functions $f(t) \sim \sum (a_\nu \cos 2\pi\nu t + b_\nu \sin 2\pi\nu t) \in L_0^2$. The author [4] proved that this bound is best possible when $\{a_\nu, b_\nu\}$ is '*parallel*' in the following sense:

$$\text{There exists} \quad (a, b) \quad \text{such that} \quad (a_\nu, b_\nu) = \sqrt{a_\nu^2 + b_\nu^2}\,(a, b) \quad \text{for all} \quad \nu \in \mathbf{N}.$$

In view of these results, it is very natural to expect that we can give the concrete bound for $\Psi[X; \{n_k\}]$ under Takahashi's gap condition. In this context, we prove the following. Here we emphasize that the bound is independent of c and β in Takahashi's gap condition. Let $\psi = \{\psi(\nu)\}$ be a sequence of positive numbers satisfying $\sum 1/\psi(\nu) < \infty$, and denote

$$X_{\psi,M} = \left\{ \sum (a_\nu \cos 2\pi\nu t + b_\nu \sin 2\pi\nu t) \in L_0^2 \;\middle|\; \sum (a_\nu^2 + b_\nu^2)\psi(\nu) \le M \right\}.$$

Theorem. *Let $X \subset X_{\psi,M}$ for some $M > 0$ and $\psi = \{\psi(\nu)\}$ with $\sum 1/\psi(\nu) < \infty$. Then, for any sequence $\{n_k\}$ of positive integers satisfying Takahashi's gap condition, we have the uniform law of the iterated logarithm below:*

$$\Psi[X; \{n_k\}] \le \sup_{f \in X} \|f\|_A \quad \text{a.e.}$$

Let us suppose that for all $\sum (a_\nu \cos 2\pi\nu t + b_\nu \sin 2\pi\nu t) \in X$, there exists a parallel $\{a'_\nu, b'_\nu\}$ such that $a_\nu^2 + b_\nu^2 = (a'_\nu)^2 + (b'_\nu)^2$ and $\sum (a'_\nu \cos 2\pi\nu t + b'_\nu \sin 2\pi\nu t) \in X$. Then the above upper bound is best possible in the following sense:
(1) For all $\varepsilon > 0$, there exists a sequence $\{n_k\}$ of positive integers satisfying Hadamard's gap condition such that $\Psi[X; \{n_k\}] > \sup_{f \in X} \|f\|_A - \varepsilon$ a.e.
(2) For all $0 < \rho_k \to 0$, there exists a sequence $\{n_k\}$ of positive integers such that $n_{k+1}/n_k \ge 1 + \rho_k$ and $\Psi[X; \{n_k\}] = \sup_{f \in X} \|f\|_A$ a.e.

In Section 3, we discuss applications of our theorem to some examples. We will see that for $\alpha > 1/2$ we have $\Lambda_\alpha \subset X_{\psi,M}$ for some $M > 0$ and $\psi = \{\psi(\nu)\}$ with $\sum 1/\psi(\nu) < \infty$, and apply our Theorem. However, we can not prove the best possibility of this bound. On the other hand if we measure the modulus of continuity by L^2-norm, we can verify the conditions in our theorem for $\Lambda_\alpha^{L^2} = \{ f \in L_0^2 \mid \|f(\,\cdot + h) - f(\,\cdot\,)\|_2 \le |h|^\alpha \}$, and can prove the best possibility.

Remark. When we assume a stronger gap condition, say, the large gap condition, the situation changes drastically. Actually, we can prove

$$\Psi[X; \{n_k\}] = \sqrt{2} \sup_{f \in X} \|f\|_2 \quad \text{a.e.}$$

for $X \subset X_{\psi,M}$. We easily have

$$\Psi[X_{\psi,M}; \{n_k\}] = \sqrt{2M / \min_\nu \psi(\nu)} \quad \text{a.e.}$$

2. Proof of the Theorem

Set $\widehat{X} = \left\{\{a_\mu, b_\mu\} \mid \sum_\nu (a_\nu \cos 2\pi\nu t + b_\nu \sin 2\pi\nu t) \in X\right\}$, $e_1 = (1,0)$, $e_2 = (0,1)$,

$$S_K^{(a,b)}(t) = \sum_{k \le K} (a \cos 2\pi n_k t + b \sin 2\pi n_k t) \quad \text{and} \quad T_{(a,b)} = \sup_{K \in \mathbf{N}} \frac{|S_K^{(a,b)}|}{\sqrt{K \log\log K}}.$$

Lemma 4 of Berkes [2] claims that

$$\left|\left\{t \in [\,0,1\,] \;\middle|\; \sup_{K \ge K_p} \frac{|S_K^{e_r}(t)|}{\sqrt{K \log\log K}} > c\right\}\right| \le \frac{1}{c^p} \quad (c \ge c_p,\ r = 1,2,\ p > 1),$$

where $|A|$ denotes the Lebesgue measure of A, and K_p, c_p are positive absolute constants. In [2], it is claimed only for $p = 2$, but the proof is valid for all $p > 1$. From this, we can derive

$$C_{q,e_r} = \int_0^1 \left(T_{e_r}(t)\right)^q dt < \infty \quad (r = 1,2,\ q > 1).$$

Lemma 1. *For any N, we have*

$$U_N(t) = \varlimsup_{K\to\infty} \sup_{\{a_\mu,b_\mu\}\in\widehat{X}} \sum_{\nu=1}^N \frac{S_K^{(a_\nu,b_\nu)}(\nu t)}{\sqrt{K \log\log K}} \le \sup_{\{a_\mu,b_\mu\}\in\widehat{X}} \sum_{\nu=1}^N \sqrt{a_\nu^2 + b_\nu^2} \quad \text{a.e. } t.$$

Proof: Let Δ be a countable dense set in $\mathbf{R}^2$ containing e_1 and e_2. For $(a,b) \in \mathbf{R}^2$, Takahashi's law of the iterated logarithm [9, 10] claims that

$$\varlimsup_{K\to\infty} \frac{|S_K^{(a,b)}(t)|}{\sqrt{K \log\log K}} = \sqrt{a^2+b^2} \quad \text{a.e. } t.$$

Thus we can see that there exists $E_0 \subset [\,0,1\,]$ such that $|E_0| = 1$ and

$$\varlimsup_{K\to\infty} \frac{|S_K^{(a,b)}(\nu t)|}{\sqrt{K \log\log K}} = \sqrt{a^2+b^2} \quad \text{for all } t \in E_0,\ (a,b) \in \Delta, \text{ and } \nu \in \mathbf{N}.$$

Let us take $\varepsilon > 0$ arbitrary. Since $\widehat{X}_N = \{(a_1, b_1, \ldots, a_N, b_N) \mid \{a_\mu, b_\mu\} \in \widehat{X}\}$ is bounded, we can take finitely many points $(a_1^{(\lambda)}, b_1^{(\lambda)}, \ldots, a_N^{(\lambda)}, b_N^{(\lambda)}) \in \Delta^N$ ($\lambda = 1, \ldots, \Lambda$) such that the union of ε-neighborhoods of these points covers $\widehat{X}_N$. By using $S_K^{(a_\nu,b_\nu)} = S_K^{(a_\nu^{(\lambda)},b_\nu^{(\lambda)})} + (a_\nu - a_\nu^{(\lambda)})S_K^{e_1} + (b_\nu - b_\nu^{(\lambda)})S_K^{e_2}$, we have

$$\sup_{\{a_\mu,b_\mu\}\in\widehat{X}} \sum_{\nu=1}^N \frac{S_K^{(a_\nu,b_\nu)}(\nu t)}{\sqrt{K \log\log K}} \le \max_{\lambda=1}^{\Lambda} \sum_{\nu=1}^N \left(\frac{S_K^{(a_\nu^{(\lambda)},b_\nu^{(\lambda)})}}{\sqrt{K \log\log K}} + \varepsilon \frac{|S_K^{e_1}| + |S_K^{e_2}|}{\sqrt{K \log\log K}} \right)(\nu t).$$

Hence for $t \in E_0$, we have

$$U_N(t) \le \max_{\lambda=1}^{\Lambda} \sum_{\nu=1}^{N} \sqrt{(a_\nu^{(\lambda)})^2 + (b_\nu^{(\lambda)})^2} + 2N\varepsilon \le \sup_{\{a_\mu, b_\mu\} \in \widehat{X}} \sum_{\nu=1}^{N} \sqrt{a_\nu^2 + b_\nu^2} + 3N\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have the conclusion. ∎

Lemma 2. *As $N \to \infty$, we have*

$$V_N(t) = \sup_{K \in \mathbf{N}} \sup_{\{a_\mu, b_\mu\} \in \widehat{X}} \sum_{\nu > N} \frac{S_K^{(a_\nu, b_\nu)}(\nu t)}{\sqrt{K \log\log K}} \to 0 \quad \text{a.e. } t.$$

Proof: Clearly $|V_N(t)|$ is bounded from above by

$$W_N(t) = \sup_{\{a_\mu, b_\mu\} \in \widehat{X}} \sum_{\nu > N} \big(|a_\nu| T_{e_1}(\nu t) + |b_\nu| T_{e_2}(\nu t)\big).$$

By noting

$$\sum |a_\nu| T_{e_1}(\nu t) \le \Big(\sum a_\nu^2 \psi(\nu)\Big)^{1/2} \Big(\sum T_{e_1}^2(\nu t)/\psi(\nu)\Big)^{1/2}$$

and analogous inequality for T_{e_2}, we have the estimate

$$W_N^2 \le M \bigg[\sum_{r=1,2} \bigg(\sum_{\nu > N} \frac{T_{e_r}^2(\nu t)}{\psi(\nu)}\bigg)^{1/2}\bigg]^2 \le 2M \sum_{r=1,2} \sum_{\nu > N} \frac{T_{e_r}^2(\nu t)}{\psi(\nu)}.$$

Therefore, by $\int_0^1 T_{e_r}^2(\nu t)\,dt = \int_0^1 T_{e_r}^2(t)\,dt = C_{2,e_r}$, we have

$$\int_0^1 W_N^2(t)\,dt \le 2M(C_{2,e_1} + C_{2,e_2}) \sum_{\nu > N} \frac{1}{\psi(\nu)} \to 0 \quad (N \to \infty).$$

Since W_N is decreasing in N, by monotone convergence theorem, we have

$$\int_0^1 \lim_{N\to\infty} W_N^2(t)\,dt = \lim_{N\to\infty} \int_0^1 W_N^2(t)\,dt = 0$$

and hence $W_N \to 0$ a.e. ∎

Proof of the Theorem: The Fourier series of any $f \in X$ converges absolutely, and hence we can change the order of summation as

$$\sum_{k=1}^{K} f(n_k t) = \sum_{\nu=1}^{N} S_K^{(a_\nu, b_\nu)}(\nu t) + \sum_{\nu > N} S_K^{(a_\nu, b_\nu)}(\nu t),$$

and therefore we have

$$\Psi[X;\{n_k\}](t) \le U_N(t)+V_N(t) \le \sup_{\{a_\mu,b_\mu\}\in\widehat{X}} \sum_{\nu=1}^{N} \sqrt{a_\nu^2+b_\nu^2}+V_N(t) \to \sup_{f\in X}\|f\|_A$$

as $N\to\infty$ for almost every t. Thus the upper bound estimate is verified.

We here prove the best possibility. Let P be a subclass of L_0^2 consisting of all functions with parallel Fourier coefficients. By the assumption of Theorem, we have

$$\sup_{f\in X}\|f\|_A = \sup_{f\in X\cap P}\|f\|_A.$$

Thus the assertion of our Theorem derived from the following results [4, 5]:

(1) Suppose that $f\in L_0^2\cap P$ and $\|f\|_A<\infty$. Then, for all $\varepsilon>0$, there exists $\{n_k\}$ satisfying Hadamard's gap condition such that $\Psi[\{f\};\{n_k\}]\ge \|f\|_A-\varepsilon$ a.e.

(2) For all $0<\rho_k\to 0$, there exists a sequence $\{n_k\}$ of positive integers such that $n_{k+1}/n_k\ge 1+\rho_k$ and $\Psi[\{f\};\{n_k\}]=\|f\|_A$ a.e. for all $f\in L_0^2\cap P$ with $\|f\|_A<\infty$.

3. Estimate of the upper bound

In the book of Zygmund [(3.2) of Ch. 6; 12], we can find

$$\|f(\,\cdot\,+h)-f(\,\cdot\,)\|_2^2 = \|f(\,\cdot\,+h/2)-f(\,\cdot\,-h/2)\|_2^2 = 2\sum_{\nu=1}^{\infty}(a_\nu^2+b_\nu^2)\sin^2\pi\nu h.$$

We see that $\Lambda_\alpha^{L^2}$ is characterized only by $\{\sqrt{a_\nu^2+b_\nu^2}\,\}$ and it satisfies the conditions for best possibility in our theorem.

Let us suppose that $f\in\Lambda_\alpha^{L^2}$ and let $\gamma\in(0,2\alpha-1)$. By putting $h=1/2^{n+1}$ in the above formula, we have

$$\sum_{\nu=2^{n-1}}^{2^n-1}(a_\nu^2+b_\nu^2) \le 2\sum_{\nu=1}^{\infty}(a_\nu^2+b_\nu^2)\sin^2\frac{\pi\nu}{2^{n+1}} \le \|f(\,\cdot\,+h)-f(\,\cdot\,)\|_2^2 \le 2^{-2\alpha(n+1)}.$$

Therefore, we can verify

$$\sum_{\nu=1}^{\infty}\nu^{2\alpha-\gamma}(a_\nu^2+b_\nu^2) \le \sum_{n=1}^{\infty}(2^n)^{2\alpha-\gamma}\sum_{\nu=2^{n-1}}^{2^n-1}(a_\nu^2+b_\nu^2) \le \sum_{n=1}^{\infty}\frac{2^{-\gamma n}}{2^{2\alpha}} = \frac{1}{2^{2\alpha}(2^\gamma-1)},$$

and hence $\Lambda_\alpha^{L^2}\subset X_{\psi,M}$ ($\alpha>1/2$) for $\psi=\{\nu^{2\alpha-\gamma}\}$ and $M=1/2^{2\alpha}(2^\gamma-1)$. Clearly, we have $\sum 1/\nu^{2\alpha-\gamma}<\infty$. Therefore we have the best possible upper bound estimate for $\Lambda_\alpha^{L^2}$.

4. Proof of the Remark

Lemma 3. *If $\{n_k\}$ has the large gap, then we have for any N,*

$$\varlimsup_{K\to\infty} \sup_{\{a_\mu,b_\mu\}\in\widehat{X}} \sum_{\nu=1}^{N} \frac{S_K^{(a_\nu,b_\nu)}(\nu t)}{\sqrt{K\log\log K}} \le \sqrt{2} \sup_{\{a_\mu,b_\mu\}\in\widehat{X}} \left(\sum_{\nu=1}^{N}(a_\nu^2+b_\nu^2)\right)^{1/2} \quad \text{a.e. } t.$$

Proof: Take $\varepsilon > 0$ arbitrary, and take $(a_1^{(\lambda)}, b_1^{(\lambda)}, \dots, a_N^{(\lambda)}, b_N^{(\lambda)})$ as in the proof of Lemma 1. Since the large gap condition is satisfied, we can apply $\Psi[\{f\};\{n_k\}] = \sqrt{2}\,\|f\|_2$ a.e. (Takahashi [8]) for the trigonometric polynomial $f(t) = \sum_{\nu=1}^{N}(a_\nu^{(\lambda)}\cos 2\pi\nu t + b_\nu^{(\lambda)}\sin 2\pi\nu t)$ and have

$$\varlimsup_{K\to\infty} \sum_{\nu=1}^{N} \frac{S_K^{(a_\nu^{(\lambda)},b_\nu^{(\lambda)})}(\nu t)}{\sqrt{K\log\log K}} = \sqrt{2}\left(\sum_{\nu=1}^{N}\left(a_\nu^{(\lambda)}\right)^2 + (b_\nu^{(\lambda)})^2\right)\Bigg)^{1/2} \quad \text{a.e. } t.$$

By applying this, we have the conclusion as the proof of Lemma 1. ■

By Lemma 2 and Lemma 3, we can prove $\Psi[X;\{n_k\}] \le \sqrt{2}\sup_{f\in X}\|f\|_2$ a.e. in the same way as before. Since $\Psi[\{f\};\{n_k\}] = \sqrt{2}\,\|f\|_2$ a.e holds for all $f \in X$, the reversed inequality is trivial.

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References

[1] N. K. Bari, Treatise of trigonometric series, vol I & II, Pergamon, Oxford, 1964.

[2] I. Berkes, On the convergence of $\sum c_n f(nx)$ and the Lip 1/2 class, Trans. Amer. Math. Soc. **349** (1997) 4143–4158

[3] S. Dhompongsa, Uniform laws of the iterated logarithm for Lipschitz classes of functions, Acta Sci. Math. **50** (1986) 105-124.

[4] K. Fukuyama, *An asymptotic property of gap series*, Acta Mathematica Hungarica, **97** (2002) 209–216.

[5] K. Fukuyama, *An asymptotic property of gap series III*, (preprint)

[6] R. Kaufman and W. Philipp, A uniform law of the iterated logarithm for classes of functions, Ann. Probab., **5** (1978) 930–952.

[7] S. Takahashi, An asymptotic property of a gap sequence, Proc. Japan Acad. **38** (1962) 101-104.

[8] S. Takahashi, The law of the iterated logarithm for a gap sequence with infinite gaps, Tôhoku Math. J., **15** (1963) 281–288.

[9] S. Takahashi, On the law of the iterated logarithm for lacunary trigonometric series, Tôhoku Math. J., **24** (1972) 319–329.

[10] S. Takahashi, On the law of the iterated logarithm for lacunary trigonometric series II Tôhoku Math. J., **27** (1975) 391–403.

[11] S. Takahashi, An asymptotic behavior of $\{f(n_k t)\}$, Sci. Rep. Kanazawa Univ., **33** (1988) 27-36.

[12] A. Zygmund, Trigonometric series, Vol I., Cambridge University Press, 1959.

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