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Fukuyama, Katusi

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# THE CONCRETE UPPER BOUND IN THE UNIFORM LAW OF THE ITERATED LOGARITHM

K. FUKUYAMA (Kobe)

**Abstract.** Kaufman and Philipp [6] and Dhompongsa [3] proved a uniform law of the iterated logarithm for  $\sum f(n_k t)$ . In this paper, we give the concrete upper bound for these results. Our bound is best possible in some cases.

## 1. Introduction

Let  $L_0^2$  be the class of measurable functions  $f$  satisfying

$$f(x+1) = f(x), \quad \int_0^1 f(x)^2 dx < \infty, \quad \text{and} \quad \int_0^1 f(x) dx = 0.$$

In this paper, we give the concrete upper bound for

$$\Psi[X; \{n_k\}](t) = \overline{\lim}_{K \rightarrow \infty} \sup_{f \in X} \frac{\sum_{k=1}^K f(n_k t)}{\sqrt{K \log \log K}},$$

where  $X$  is a subclass of  $L_0^2$  and  $\{n_k\}$  is an increasing sequence of integers.

We give here a simple survey of studies of this type. When  $f \in \text{Lip } \alpha \cap L_0^2$  ( $\alpha > 0$ ) where  $\text{Lip } \alpha$  is the class of functions satisfying Lipschitz condition of order  $\alpha$ , Takahashi [7, 8] proved  $\Psi[\{f\}; \{n_k\}] = \sqrt{2} \|f\|_2$  a.e. when  $\{n_k\}$  satisfies the large gap condition:

$$n_{k+1}/n_k \rightarrow \infty \quad (k \rightarrow \infty),$$

and  $\Psi[\{f\}; \{n_k\}] \leq C < \infty$  a.e. when  $\{n_k\}$  satisfies Hadamard's gap condition:

$$n_{k+1}/n_k \geq 1 + \rho \quad (\rho > 0).$$

For  $\Lambda_\alpha = \{f \in L_0^2 \mid |f(t+h) - f(t)| \leq |h|^\alpha\} \subset \text{Lip } \alpha$  ( $\alpha > 1/2$ ), the estimate  $\Psi[\Lambda_\alpha; \{n_k\}] \leq C < \infty$  a.e. was proved by Kaufman and Philipp [6] when  $\{n_k\}$  satisfies Hadamard's gap condition, and by Dhompongsa [3] when  $\{n_k\}$  satisfies Takahashi's gap condition:

$$n_{k+1}/n_k \geq 1 + c/k^\beta \quad (c > 0, \beta < 1/2).$$

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Assuming the same gap condition and  $f \in \text{Lip } \alpha$  ( $\alpha > 1/2$ ), Takahashi [11] gave the concrete upper bound  $\Psi[\{f\}; \{n_k\}] \leq \|f\|_A$  a.e. where  $\|f\|_A$  denotes  $\sum \sqrt{a_\nu^2 + b_\nu^2}$  for functions  $f(t) \sim \sum (a_\nu \cos 2\pi\nu t + b_\nu \sin 2\pi\nu t) \in L_0^2$ . The author [4] proved that this bound is best possible when  $\{a_\nu, b_\nu\}$  is ‘parallel’ in the following sense:

There exists  $(a, b)$  such that  $(a_\nu, b_\nu) = \sqrt{a_\nu^2 + b_\nu^2} (a, b)$  for all  $\nu \in \mathbf{N}$ .

In view of these results, it is very natural to expect that we can give the concrete bound for  $\Psi[X; \{n_k\}]$  under Takahashi’s gap condition. In this context, we prove the following. Here we emphasize that the bound is independent of  $c$  and  $\beta$  in Takahashi’s gap condition. Let  $\psi = \{\psi(\nu)\}$  be a sequence of positive numbers satisfying  $\sum 1/\psi(\nu) < \infty$ , and denote

$$X_{\psi, M} = \left\{ \sum (a_\nu \cos 2\pi\nu t + b_\nu \sin 2\pi\nu t) \in L_0^2 \mid \sum (a_\nu^2 + b_\nu^2) \psi(\nu) \leq M \right\}.$$

**Theorem.** *Let  $X \subset X_{\psi, M}$  for some  $M > 0$  and  $\psi = \{\psi(\nu)\}$  with  $\sum 1/\psi(\nu) < \infty$ . Then, for any sequence  $\{n_k\}$  of positive integers satisfying Takahashi’s gap condition, we have the uniform law of the iterated logarithm below:*

$$\Psi[X; \{n_k\}] \leq \sup_{f \in X} \|f\|_A \quad \text{a.e.}$$

Let us suppose that for all  $\sum (a_\nu \cos 2\pi\nu t + b_\nu \sin 2\pi\nu t) \in X$ , there exists a parallel  $\{a'_\nu, b'_\nu\}$  such that  $a_\nu^2 + b_\nu^2 = (a'_\nu)^2 + (b'_\nu)^2$  and  $\sum (a'_\nu \cos 2\pi\nu t + b'_\nu \sin 2\pi\nu t) \in X$ . Then the above upper bound is best possible in the following sense:

- (1) For all  $\varepsilon > 0$ , there exists a sequence  $\{n_k\}$  of positive integers satisfying Hadamard’s gap condition such that  $\Psi[X; \{n_k\}] > \sup_{f \in X} \|f\|_A - \varepsilon$  a.e.
- (2) For all  $0 < \rho_k \rightarrow 0$ , there exists a sequence  $\{n_k\}$  of positive integers such that  $n_{k+1}/n_k \geq 1 + \rho_k$  and  $\Psi[X; \{n_k\}] = \sup_{f \in X} \|f\|_A$  a.e.

In Section 3, we discuss applications of our theorem to some examples. We will see that for  $\alpha > 1/2$  we have  $\Lambda_\alpha \subset X_{\psi, M}$  for some  $M > 0$  and  $\psi = \{\psi(\nu)\}$  with  $\sum 1/\psi(\nu) < \infty$ , and apply our Theorem. However, we can not prove the best possibility of this bound. On the other hand if we measure the modulus of continuity by  $L^2$ -norm, we can verify the conditions in our theorem for  $\Lambda_\alpha^{L^2} = \{f \in L_0^2 \mid \|f(\cdot + h) - f(\cdot)\|_2 \leq |h|^\alpha\}$ , and can prove the best possibility.

**Remark.** When we assume a stronger gap condition, say, the large gap condition, the situation changes drastically. Actually, we can prove

$$\Psi[X; \{n_k\}] = \sqrt{2} \sup_{f \in X} \|f\|_2 \quad \text{a.e.}$$

for  $X \subset X_{\psi, M}$ . We easily have

$$\Psi[X_{\psi, M}; \{n_k\}] = \sqrt{2M / \min_\nu \psi(\nu)} \quad \text{a.e.}$$

## 2. Proof of the Theorem

Set  $\widehat{X} = \{\{a_\mu, b_\mu\} \mid \sum_\nu (a_\nu \cos 2\pi\nu t + b_\nu \sin 2\pi\nu t) \in X\}$ ,  $e_1 = (1, 0)$ ,  $e_2 = (0, 1)$ ,

$$S_K^{(a,b)}(t) = \sum_{k \leq K} (a \cos 2\pi n_k t + b \sin 2\pi n_k t) \quad \text{and} \quad T_{(a,b)} = \sup_{K \in \mathbf{N}} \frac{|S_K^{(a,b)}|}{\sqrt{K \log \log K}}.$$

Lemma 4 of Berkes [2] claims that

$$\left| \left\{ t \in [0, 1] \mid \sup_{K \geq K_p} \frac{|S_K^{e_r}(t)|}{\sqrt{K \log \log K}} > c \right\} \right| \leq \frac{1}{c^p} \quad (c \geq c_p, \ r = 1, 2, \ p > 1),$$

where  $|A|$  denotes the Lebesgue measure of  $A$ , and  $K_p, c_p$  are positive absolute constants. In [2], it is claimed only for  $p = 2$ , but the proof is valid for all  $p > 1$ . From this, we can derive

$$C_{q,e_r} = \int_0^1 (T_{e_r}(t))^q dt < \infty \quad (r = 1, 2, \ q > 1).$$

**Lemma 1.** *For any  $N$ , we have*

$$U_N(t) = \overline{\lim}_{K \rightarrow \infty} \sup_{\{a_\mu, b_\mu\} \in \widehat{X}} \sum_{\nu=1}^N \frac{S_K^{(a_\nu, b_\nu)}(\nu t)}{\sqrt{K \log \log K}} \leq \sup_{\{a_\mu, b_\mu\} \in \widehat{X}} \sum_{\nu=1}^N \sqrt{a_\nu^2 + b_\nu^2} \quad \text{a.e. } t.$$

*Proof:* Let  $\Delta$  be a countable dense set in  $\mathbf{R}^2$  containing  $e_1$  and  $e_2$ . For  $(a, b) \in \mathbf{R}^2$ , Takahashi's law of the iterated logarithm [9, 10] claims that

$$\overline{\lim}_{K \rightarrow \infty} \frac{|S_K^{(a,b)}(t)|}{\sqrt{K \log \log K}} = \sqrt{a^2 + b^2} \quad \text{a.e. } t.$$

Thus we can see that there exists  $E_0 \subset [0, 1]$  such that  $|E_0| = 1$  and

$$\overline{\lim}_{K \rightarrow \infty} \frac{|S_K^{(a,b)}(\nu t)|}{\sqrt{K \log \log K}} = \sqrt{a^2 + b^2} \quad \text{for all } t \in E_0, \ (a, b) \in \Delta, \text{ and } \nu \in \mathbf{N}.$$

Let us take  $\varepsilon > 0$  arbitrary. Since  $\widehat{X}_N = \{(a_1, b_1, \dots, a_N, b_N) \mid \{a_\mu, b_\mu\} \in \widehat{X}\}$  is bounded, we can take finitely many points  $(a_1^{(\lambda)}, b_1^{(\lambda)}, \dots, a_N^{(\lambda)}, b_N^{(\lambda)}) \in \Delta^N$  ( $\lambda = 1, \dots, \Lambda$ ) such that the union of  $\varepsilon$ -neighborhoods of these points covers  $\widehat{X}_N$ . By using  $S_K^{(a_\nu, b_\nu)} = S_K^{(a_\nu^{(\lambda)}, b_\nu^{(\lambda)})} + (a_\nu - a_\nu^{(\lambda)})S_K^{e_1} + (b_\nu - b_\nu^{(\lambda)})S_K^{e_2}$ , we have

$$\sup_{\{a_\mu, b_\mu\} \in \widehat{X}} \sum_{\nu=1}^N \frac{S_K^{(a_\nu, b_\nu)}(\nu t)}{\sqrt{K \log \log K}} \leq \max_{\lambda=1}^{\Lambda} \sum_{\nu=1}^N \left( \frac{S_K^{(a_\nu^{(\lambda)}, b_\nu^{(\lambda)})}}{\sqrt{K \log \log K}} + \varepsilon \frac{|S_K^{e_1}| + |S_K^{e_2}|}{\sqrt{K \log \log K}} \right) (\nu t).$$

Hence for  $t \in E_0$ , we have

$$U_N(t) \leq \max_{\lambda=1}^{\Lambda} \sum_{\nu=1}^N \sqrt{(a_{\nu}^{(\lambda)})^2 + (b_{\nu}^{(\lambda)})^2} + 2N\varepsilon \leq \sup_{\{a_{\mu}, b_{\mu}\} \in \widehat{X}} \sum_{\nu=1}^N \sqrt{a_{\nu}^2 + b_{\nu}^2} + 3N\varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, we have the conclusion. ■

**Lemma 2.** As  $N \rightarrow \infty$ , we have

$$V_N(t) = \sup_{K \in \mathbf{N}} \sup_{\{a_{\mu}, b_{\mu}\} \in \widehat{X}} \sum_{\nu > N} \frac{S_K^{(a_{\nu}, b_{\nu})}(\nu t)}{\sqrt{K \log \log K}} \rightarrow 0 \quad \text{a.e. } t.$$

*Proof:* Clearly  $|V_N(t)|$  is bounded from above by

$$W_N(t) = \sup_{\{a_{\mu}, b_{\mu}\} \in \widehat{X}} \sum_{\nu > N} (|a_{\nu}|T_{e_1}(\nu t) + |b_{\nu}|T_{e_2}(\nu t)).$$

By noting

$$\sum |a_{\nu}|T_{e_1}(\nu t) \leq \left( \sum a_{\nu}^2 \psi(\nu) \right)^{1/2} \left( \sum T_{e_1}^2(\nu t) / \psi(\nu) \right)^{1/2}$$

and analogous inequality for  $T_{e_2}$ , we have the estimate

$$W_N^2 \leq M \left[ \sum_{r=1,2} \left( \sum_{\nu > N} \frac{T_{e_r}^2(\nu t)}{\psi(\nu)} \right)^{1/2} \right]^2 \leq 2M \sum_{r=1,2} \sum_{\nu > N} \frac{T_{e_r}^2(\nu t)}{\psi(\nu)}.$$

Therefore, by  $\int_0^1 T_{e_r}^2(\nu t) dt = \int_0^1 T_{e_r}^2(t) dt = C_{2,e_r}$ , we have

$$\int_0^1 W_N^2(t) dt \leq 2M(C_{2,e_1} + C_{2,e_2}) \sum_{\nu > N} \frac{1}{\psi(\nu)} \rightarrow 0 \quad (N \rightarrow \infty).$$

Since  $W_N$  is decreasing in  $N$ , by monotone convergence theorem, we have

$$\int_0^1 \lim_{N \rightarrow \infty} W_N^2(t) dt = \lim_{N \rightarrow \infty} \int_0^1 W_N^2(t) dt = 0$$

and hence  $W_N \rightarrow 0$  a.e. ■

*Proof of the Theorem:* The Fourier series of any  $f \in X$  converges absolutely, and hence we can change the order of summation as

$$\sum_{k=1}^K f(n_k t) = \sum_{\nu=1}^N S_K^{(a_{\nu}, b_{\nu})}(\nu t) + \sum_{\nu > N} S_K^{(a_{\nu}, b_{\nu})}(\nu t),$$

and therefore we have

$$\Psi[X; \{n_k\}](t) \leq U_N(t) + V_N(t) \leq \sup_{\{a_\mu, b_\mu\} \in \widehat{X}} \sum_{\nu=1}^N \sqrt{a_\nu^2 + b_\nu^2} + V_N(t) \rightarrow \sup_{f \in X} \|f\|_A$$

as  $N \rightarrow \infty$  for almost every  $t$ . Thus the upper bound estimate is verified.

We here prove the best possibility. Let  $P$  be a subclass of  $L_0^2$  consisting of all functions with parallel Fourier coefficients. By the assumption of Theorem, we have

$$\sup_{f \in X} \|f\|_A = \sup_{f \in X \cap P} \|f\|_A.$$

Thus the assertion of our Theorem derived from the following results [4, 5]:

(1) Suppose that  $f \in L_0^2 \cap P$  and  $\|f\|_A < \infty$ . Then, for all  $\varepsilon > 0$ , there exists  $\{n_k\}$  satisfying Hadamard's gap condition such that  $\Psi[\{f\}; \{n_k\}] \geq \|f\|_A - \varepsilon$  a.e.

(2) For all  $0 < \rho_k \rightarrow 0$ , there exists a sequence  $\{n_k\}$  of positive integers such that  $n_{k+1}/n_k \geq 1 + \rho_k$  and  $\Psi[\{f\}; \{n_k\}] = \|f\|_A$  a.e. for all  $f \in L_0^2 \cap P$  with  $\|f\|_A < \infty$ .

### 3. Estimate of the upper bound

In the book of Zygmund [(3.2) of Ch. 6; 12], we can find

$$\|f(\cdot + h) - f(\cdot)\|_2^2 = \|f(\cdot + h/2) - f(\cdot - h/2)\|_2^2 = 2 \sum_{\nu=1}^{\infty} (a_\nu^2 + b_\nu^2) \sin^2 \pi \nu h.$$

We see that  $\Lambda_\alpha^{L^2}$  is characterized only by  $\{\sqrt{a_\nu^2 + b_\nu^2}\}$  and it satisfies the conditions for best possibility in our theorem.

Let us suppose that  $f \in \Lambda_\alpha^{L^2}$  and let  $\gamma \in (0, 2\alpha - 1)$ . By putting  $h = 1/2^{n+1}$  in the above formula, we have

$$\sum_{\nu=2^{n-1}}^{2^n-1} (a_\nu^2 + b_\nu^2) \leq 2 \sum_{\nu=1}^{\infty} (a_\nu^2 + b_\nu^2) \sin^2 \frac{\pi \nu}{2^{n+1}} \leq \|f(\cdot + h) - f(\cdot)\|_2^2 \leq 2^{-2\alpha(n+1)}.$$

Therefore, we can verify

$$\sum_{\nu=1}^{\infty} \nu^{2\alpha-\gamma} (a_\nu^2 + b_\nu^2) \leq \sum_{n=1}^{\infty} (2^n)^{2\alpha-\gamma} \sum_{\nu=2^{n-1}}^{2^n-1} (a_\nu^2 + b_\nu^2) \leq \sum_{n=1}^{\infty} \frac{2^{-\gamma n}}{2^{2\alpha}} = \frac{1}{2^{2\alpha}(2^\gamma - 1)},$$

and hence  $\Lambda_\alpha^{L^2} \subset X_{\psi, M}$  ( $\alpha > 1/2$ ) for  $\psi = \{\nu^{2\alpha-\gamma}\}$  and  $M = 1/2^{2\alpha}(2^\gamma - 1)$ . Clearly, we have  $\sum 1/\nu^{2\alpha-\gamma} < \infty$ . Therefore we have the best possible upper bound estimate for  $\Lambda_\alpha^{L^2}$ .

#### 4. Proof of the Remark

**Lemma 3.** *If  $\{n_k\}$  has the large gap, then we have for any  $N$ ,*

$$\overline{\lim}_{K \rightarrow \infty} \sup_{\{a_\mu, b_\mu\} \in \widehat{X}} \sum_{\nu=1}^N \frac{S_K^{(a_\nu, b_\nu)}(\nu t)}{\sqrt{K \log \log K}} \leq \sqrt{2} \sup_{\{a_\mu, b_\mu\} \in \widehat{X}} \left( \sum_{\nu=1}^N (a_\nu^2 + b_\nu^2) \right)^{1/2} \quad \text{a.e. } t.$$

*Proof:* Take  $\varepsilon > 0$  arbitrary, and take  $(a_1^{(\lambda)}, b_1^{(\lambda)}, \dots, a_N^{(\lambda)}, b_N^{(\lambda)})$  as in the proof of Lemma 1. Since the large gap condition is satisfied, we can apply  $\Psi[\{f\}; \{n_k\}] = \sqrt{2} \|f\|_2$  a.e. (Takahashi [8]) for the trigonometric polynomial  $f(t) = \sum_{\nu=1}^N (a_\nu^{(\lambda)} \cos 2\pi\nu t + b_\nu^{(\lambda)} \sin 2\pi\nu t)$  and have

$$\overline{\lim}_{K \rightarrow \infty} \sum_{\nu=1}^N \frac{S_K^{(a_\nu^{(\lambda)}, b_\nu^{(\lambda)})}(\nu t)}{\sqrt{K \log \log K}} = \sqrt{2} \left( \sum_{\nu=1}^N (a_\nu^{(\lambda)})^2 + (b_\nu^{(\lambda)})^2 \right)^{1/2} \quad \text{a.e. } t.$$

By applying this, we have the conclusion as the proof of Lemma 1. ■

By Lemma 2 and Lemma 3, we can prove  $\Psi[X; \{n_k\}] \leq \sqrt{2} \sup_{f \in X} \|f\|_2$  a.e. in the same way as before. Since  $\Psi[\{f\}; \{n_k\}] = \sqrt{2} \|f\|_2$  a.e holds for all  $f \in X$ , the reversed inequality is trivial.

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DEPARTMENT OF MATHEMATICS  
KOBE UNIVERSITY  
ROKKO KOBE 657-8501 JAPAN  
E-MAIL: fukuyama@math.kobe-u.ac.jp