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Metric discrepancy results for Erdős-Fortet sequence

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Abstract For the classical Erdős-Fortet sequence $n_k = 2^k - 1$ we show that the law of the iterated logarithm for star discrepancy of $\{n_k x\}$ has non-constant limsup, while the law for discrepancy has constant limsup.

Keywords discrepancy · lacunary sequence · law of the iterated logarithm

Mathematics Subject Classification (2000) 11K38 · 42A55 · 60F15

1 Introduction

After Kac [9] proved the central limit theorem

$$\left| \left\{ x \in [0,1] \;\middle|\; \frac{1}{\sqrt{N}} \sum_{k=1}^{N} f(2^k x) \le a \right\} \right| \to \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{a} e^{-u^2/2\sigma^2} \, du, \tag{1}$$

$$\sigma^2 = \int_0^1 f^2(t)\, dt + 2 \sum_{k=1}^{\infty} \int_0^1 f(t) f(2^k t)\, dt \tag{2}$$

for an f of bounded variation with period 1 satisfying $\int_0^1 f = 0$, a natural question arose asking if it is possible to replace the sequence $\{2^k\}$ by a general sequence $\{n_k\}$ diverging rapidly. Although Kac [11] proved that it is valid with $\sigma^2 = \int_0^1 f^2$ under the large

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gap condition $n_{k+1}/n_k \to \infty$, Erdős-Fortet [10,16,11] presented a counterexample satisfying the Hadamard's gap condition $n_{k+1}/n_k > q > 1$.

The example is very simple. For $n_k = 2^k - 1$ and $f(x) = \cos 2\pi x + \cos 4\pi x$, it holds that

$$\left|\left\{ x \in [\,0,1\,] \;\middle|\; \frac{1}{\sqrt{N}} \sum_{k=1}^{N} f(n_k x) \le a \right\}\right| \to \int_0^1 \frac{dx}{\sqrt{2\pi}} \int_{-\infty}^{a/\sqrt{2}\cos \pi x} e^{-u^2/2}\, du. \tag{3}$$

The contrast among the binary sequence $\{2^k\}$ and the Erdős-Fortet sequence $\{2^k - 1\}$ revealed a very delicate nature of sequences from the probabilistic point of view.

The aim of this paper is to give metric discrepancy results for the Erdős-Fortet sequence and its family. We first recall the definitions of discrepancies $D_N\{x_k\}$ and $D_N^*\{x_k\}$ of sequence $\{x_k\}$ of real numbers:

$$D_N\{x_k\} = \sup_{0 \le a < b < 1} \left| \frac{1}{N} \sum_{k=1}^{N} \widetilde{\mathbf{I}}_{a,b}(x_k) \right|; \quad D_N^*\{a_k\} = \sup_{0 \le a < 1} \left| \frac{1}{N} \sum_{k=1}^{N} \widetilde{\mathbf{I}}_{0,a}(a_k) \right|;$$

where $\widetilde{\mathbf{I}}_{a,b}$ is the centered periodic extension

$$\widetilde{\mathbf{I}}_{a,b}(x) = \left(\sum_{n \in \mathbf{Z}} \mathbf{1}_{[a,b)}(x+n) \right) - (b-a) \tag{4}$$

of the indicator function $\mathbf{1}_{[a,b)}$ of $[\,a,b)$.

For $\{n_k\}$ satisfying Hadamard's gap condition, Philipp [14,15] proved the bounded law of the iterated logarithm $1/4\sqrt{2} < \Sigma^*\{n_k x\} \le \Sigma\{n_k x\} \le C < \infty$, a.e., where

$$\Sigma\{n_k x\} = \varlimsup_{N\to\infty} \frac{N D_N\{n_k x\}}{\sqrt{2N \log\log N}} \quad \text{and} \quad \Sigma^*\{n_k x\} = \varlimsup_{N\to\infty} \frac{N D_N^*\{n_k x\}}{\sqrt{2N \log\log N}}.$$

Recently it became possible to compute the exact value of $\Sigma\{n_k x\}$. Actually we can find the following results in [6].

For any real number $\theta > 1$, there exists a constant Σ_θ such that

$$\Sigma^*\{\theta^k x\} = \Sigma\{\theta^k x\} = \Sigma_\theta, \quad \text{a.e.} \tag{5}$$

We have $\Sigma_\theta = 1/2$ if θ satisfies the condition

$$\theta^r \notin \mathbf{Q} \quad \text{for all} \quad r \in \mathbf{N}. \tag{6}$$

Otherwise let us express θ by

$$\theta = \sqrt[r]{p/q} \quad \text{where} \quad r = \min\{n \in \mathbf{N} \mid \theta^n \in \mathbf{Q}\}, \quad \text{and} \quad \gcd(p,q) = 1. \tag{7}$$

In this case Σ_θ does not depend on r. It is evaluated in the following cases:

$$\Sigma_\theta = \begin{cases} \sqrt{(pq+1)/(pq-1)}/2 & \text{if } p \text{ and } q \text{ are odd,} \\ \sqrt{(p+1)p(p-2)/(p-1)^3}/2 & \text{if } p \ge 4 \text{ is even and } q = 1, \\ \sqrt{42}/9 & \text{if } p = 2 \text{ and } q = 1, \\ \sqrt{22}/9 & \text{if } p = 5 \text{ and } q = 2, \end{cases}$$

Aistleitner [1] gave a nearly optimal Diophantine condition on the sequence $\{n_k\}$ to have $\Sigma^*\{n_kx\} = \Sigma\{n_kx\} = 1/2$, a.e., the result which corresponds to Chung-Smirnov law of the iterated logarithm for the uniformly distributed i.i.d.

It was a long standing problem whether $\Sigma^*\{n_kx\}$ and $\Sigma\{n_kx\}$ are always constant a.e. or not. The problem was negatively solved by a randomly constructed example with bounded gaps $n_{k+1} - n_k \le 5$ (Cf. [7]) and celebrated examples by Aistleitner [2, 3] below. Aistleitner's examples can be considered as a modification of Erdős-Fortet sequence. Aistleitner [2] introduced the example $n_{2k-1} = 2^{k^2}$ and $n_{2k} = 2^{k^2+1} - 1$ and proved that $\Sigma^*\{n_kx\}$ is not constant a.e. and that $\Sigma\{n_kx\} = 1/2$, a.e. This is the first example of the Hadamard's gap sequence such that $\Sigma^*\{n_kx\}$ is not a.e. constant as well as the deviation $\Sigma^*\{n_kx\} < \Sigma\{n_kx\}$ occurs with positive measure. In [3], Aistleitner gave another example such that $\Sigma\{n_kx\}$ is not constant a.e.

There is strong dependence among n_{2k-1} and n_{2k} which is a nature of original Erdős-Fortet sequence. The next block n_{2k+1}, n_{2k+2} is very far from the block n_{2k-1}, n_{2k} and hence they are almost independent. This is a reason why the Erdős-Fortet nature of this sequence can be extracted easily and the concrete evaluation is possible.

Although the original Erdős-Fortet sequence has more complicated dependence structure, it is still possible to study because it is almost stationary. Now we are in a position to state our results.

Theorem 1 *For $\theta > 1$, we have*

$$\Sigma\{(\theta^k - 1)x\} = \Sigma_\theta, \quad \text{a.e., and} \tag{8}$$

$$\Sigma^*\{(\theta^k - 1)x\} = \Sigma^*_\theta(x), \ \text{a.e.,} \tag{9}$$

where Σ_θ is a constant given by (5) and $\Sigma^*_\theta(x)$ is a continuous function on torus.

If θ satisfies (6), then $\Sigma^*_\theta(x) = 1/2 = \Sigma_\theta$. If θ is given by (7) and satisfies one of the following conditions, then $\Sigma^*_\theta(x)$ is not constant and $\Sigma^*_\theta(x) < \Sigma_\theta$ except for finitely many x: (i) p and q are odd; (ii) $q = 1$; (iii) $p = 5$ and $q = 2$.

As the graph of $\Sigma^*_2(x)$ drawn in figure 1 shows, limsup functions appearing here seem to be irregular when θ is a power root of integers.

Before closing this section, we mention the central limit theorem and the law of the iterated logarithm for Erdős-Fortet sequence.

Theorem 2 *Let f be a real valued function with period 1 which is locally square integrable and satisfies $\int_0^1 f = 0$. Suppose that f satisfies the L^2-Dini condition below:*

$$\int_0^1 \frac{\omega_2(y)}{y}\,dy < \infty \quad \text{where} \quad \omega_2(\delta) = \sup_{|h|\le\delta}\left(\int_0^1 \bigl|f(x+h) - f(x)\bigr|^2\,dx\right)^{1/2}.$$

Then for any $E \subset [0,1]$ with positive measure, we have

$$\frac{1}{|E|}\left|\left\{x \in E \;\Big|\; \frac{1}{\sqrt{N}}\sum_{k=1}^{N} f((\theta^k - 1)x) \le a\right\}\right| \to \frac{1}{|E|}\int_E N_{0,\varsigma^2(y)}(-\infty, a]\,dy,$$

where $N_{0,v}$ denotes the gaussian distribution with mean 0 and variance v, and

$$\varsigma^2(x) = \begin{cases} \int_0^1 f^2(y)\,dy & \text{if } \theta \text{ satisfies (6),} \\ \int_0^1 f^2(y)\,dy + 2\sum_{k=1}^\infty \int_0^1 f(p^k y - x) f(q^k y - x)\,dy & \text{if } \theta \text{ is given by (7).} \end{cases}$$

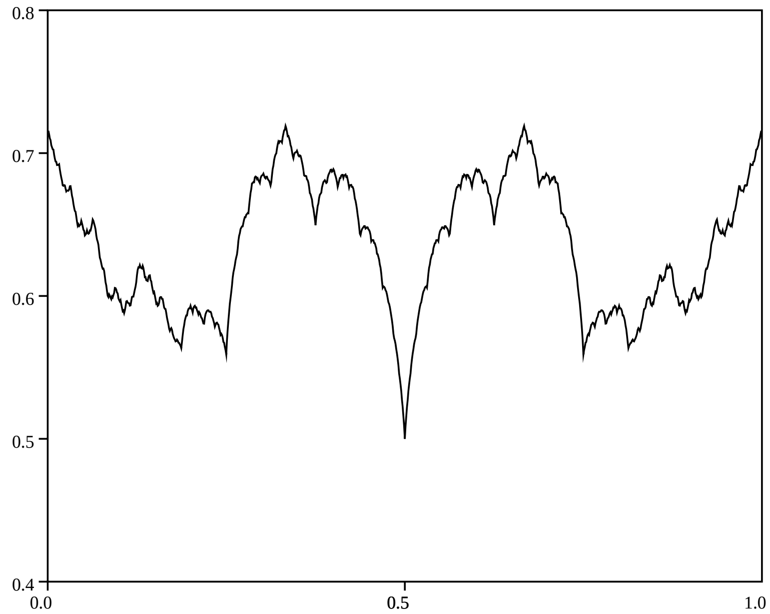

Fig. 1 Graph of $\Sigma_2^*(x)$.

Moreover, if f is of bounded variation or Hölder continuous, we have

$$\varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \sum_{k=1}^{N} f((\theta^k - 1)x) = \varsigma(x), \quad a.e.$$

The special case when $\theta = 2$ was stated in the famous survey by Kac [11] without any proof. The form of $\varsigma^2(x)$ is given there by means of the Fourier expansion when f is an even function.

The central theorem can be proved for a trigonometric polynomial f by martingale approximation given in the next section. The proof for general case is completed by approximating f by a subsum of the Fourier series of f, which is done in the same way as the proof given in [5].

The proof of the law of the iterated logarithm is completly same as that is given in the next section.

It is, however, worth giving a heuristic derivation of $\varsigma(x)$ for the original Erdős-Fortet sequence. The proof of (1) by Kac based on the property that $2^k x$ and x are asymptotically independent. Let us try to apply this property to

$$F(a) = \left|\left\{ x \in [0,1) \;\middle|\; \frac{1}{\sqrt{N}} \sum_{k=1}^{N} f(2^k x - x) \in (-\infty, a) \right\}\right|.$$

Since $2^k x$ is almost independent of x, it is asymptotically identical if we replace $2^k x$ by $2^k y$ using another independent variable as below:

$$F(a) \sim \left|\left\{ (x,y) \in [0,1)^2 \;\middle|\; \frac{1}{\sqrt{N}} \sum_{k=1}^{N} f(2^k y - x) \in (-\infty, a) \right\}\right|.$$

By applying Fubini's theorem and the central limit theorem (1) for $f(\,\cdot\,-x)$, we have

$$F(a) \sim \int_0^1 \left| \left\{ y \in [\,0,1) \;\middle|\; \frac{1}{\sqrt{N}} \sum_{k=1}^{N} f(2^k y - x) \in (-\infty, a) \right\} \right| dx$$
$$\to \int_0^1 N_{0,\sigma^2(f(\cdot - x))}(-\infty, a)\, dx.$$

2 Proof of Theorem 1

It is proved in [10], [12], and [5] that the law of $\frac{1}{\sqrt{N}} \sum_{k=1}^{N} f(\theta^k x)$ converges weakly to the centered normal distribution with variance $\sigma^2(f,\theta)$ for a function f of bounded variation with period 1 satisfying $\int_0^1 f = 0$. Here $\sigma(f,\theta)^2 \geq 0$ equals to $\int_0^1 f^2(y)\, dy$ if θ satisfies (6). If θ is given by (7),

$$\sigma^2(f,\theta) = \int_0^1 f^2(y)\, dy + 2 \sum_{k=1}^{\infty} \int_0^1 f(p^k y) f(q^k y)\, dy \tag{10}$$

$$= \begin{cases} 2 \sum\limits_{j \in P_{pq}} \sum\limits_{i=0}^{\infty} \left| \sum\limits_{s=0}^{i} \widehat{f}(jp^s q^{i-s}) \right|^2 & \text{if } q > 1, \\ 2 \sum\limits_{j \in P_p} \left| \sum\limits_{s=0}^{\infty} \widehat{f}(jp^s) \right|^2 & \text{if } q = 1, \end{cases} \tag{11}$$

where $P_\nu = \{ n \in \mathbf{N} \mid n \neq 0, \gcd(n,\nu) = 1 \}$.

Since $\widetilde{\mathbf{I}}_{a,b}$ is defined by (4), it is naturally extended for a and b satisfying $0 \leq b - a < 1$. By this extension, we naturally have

$$\widetilde{\mathbf{I}}_{a,b}(y-x) = \widetilde{\mathbf{I}}_{a+x,b+x}(y) \quad \text{and} \quad \widetilde{\mathbf{I}}_{a,b;d}(y-x) = \widetilde{\mathbf{I}}_{a+x,b+x;d}(y) \tag{12}$$

where $\widetilde{\mathbf{I}}_{a,b;d}(y)$ denotes the d-th subsum of the Fourier series of $\widetilde{\mathbf{I}}_{a,b}(y)$. Note that

$$\widetilde{\mathbf{I}}_{a,b}(x) = \begin{cases} \widetilde{\mathbf{I}}_{\langle a \rangle, \langle b \rangle}(x) & \text{if } \langle a \rangle \leq \langle b \rangle, \\ -\widetilde{\mathbf{I}}_{\langle b \rangle, \langle a \rangle}(x) & \text{if } \langle a \rangle > \langle b \rangle. \end{cases} \tag{13}$$

Because of $0 \leq b - a < 1$, there exists an integer m satisfying $m \leq a \leq b < m+1$ or $m-1 \leq a < m \leq b < m+1$. In the first case we have $\langle a \rangle \leq \langle b \rangle$ and $\widetilde{\mathbf{I}}_{a,b}(x) = \widetilde{\mathbf{I}}_{\langle a \rangle, \langle b \rangle}(x)$. In the second case we have $\langle a \rangle > \langle b \rangle$ and $\sum_{n \in \mathbf{Z}} \mathbf{1}_{[a,b)}(x+n) = 1 - \sum_{n \in \mathbf{Z}} \mathbf{1}_{[b,a+1)}(x+n)$ which yields $\widetilde{\mathbf{I}}_{a,b}(x) = -\widetilde{\mathbf{I}}_{b,a+1}(x) = -\widetilde{\mathbf{I}}_{\langle b \rangle, \langle a \rangle}(x)$ by $m \leq b < a+1 < m+1$.

The next lemma proves that the limit distribution of the central limit theorem for $\sum \widetilde{\mathbf{I}}_{a,b}$ never degenerates. This result has its own interest.

Lemma 1 *For any* $0 \leq a < b < 1$ *we have* $\sigma(\widetilde{\mathbf{I}}_{a,b}, \theta) > 0$.

Proof Denote $\sigma(\widetilde{\mathbf{I}}_{a,b}, \theta)$ by $\sigma_{a,b,\theta}$, and $\widetilde{\mathbf{I}}_{a,b}$ by f. By putting $e(x) = e^{2\pi\sqrt{-1}\,x}$, we have $\widehat{f}(n) = -(e(-nb) - e(-na))/(2\pi\sqrt{-1}\,n)$ for $n \neq 0$.

When θ satisfies (6), we have $\sigma_{a,b,\theta}^2 = (b-a)(1-(b-a)) > 0$.

We consider the case (7). When $q > 1$, if we have $\sigma_{a,b,\theta} = 0$, by (11) we have $\sum_{s=0}^{i} \widehat{f}(jp^s q^{i-s})$ for all $i = 0, 1, \ldots,$ and $j \in P_{pq}$. If we put $i = 0$ and $j = 1$, we have $\widehat{f}(1) = 0$, i.e., $e(-a) = e(-b)$ which is a contradiction.

When $q=1$, if we have $\sigma_{a,b,\theta}=0$, by (11) we have $\sum_{s=0}^{\infty}\widehat{f}(jp^s)=0$ for all $j\in P_p$. It proves

$$\sum_{s=0}^{\infty}\frac{e(-jp^sa)}{p^s}=\sum_{s=0}^{\infty}\frac{e(-jp^sb)}{p^s} \tag{14}$$

for all $j\in P_p$.

Consider the case $p>2$. Since $\left|\sum_{s=0}^{\infty}e(-jp^sx)/p^s-e(-jx)\right|\le 1/(p-1)\le 1/2$, the relation (14) can not hold if $|e(-ja)-e(-jb)|>1$. If there exists $j\in P_p$ such that $\langle j(b-a)\rangle\in(1/6,5/6)$, then we have $|e(-ja)-e(-jb)|=2|\sin\pi(b-a)|>1$ and have $\sigma_{a,b,\theta}>0$. Suppose the contrary case, i.e. there is no non-negative integer l such that $\langle(pl+1)(b-a)\rangle\in(1/6,5/6)$. It proves $p(b-a)=0 \bmod 1$, otherwise the sequence $\{\langle(pl+1)(b-a)\rangle\}_l$ is uniformly or periodically distributed on the torus and cannot be contained in $(1/6,5/6)^c$. Thus we have $b=a+n/p$ $(n=1,\ldots,p-1)$. In this case it holds $e(-p^sb)=e(-p^sa)$ for $s>0$ and $e(-b)\ne e(-a)$. Hence (14) does not hold for $j=1$, and $\sigma_{a,b,\theta}>0$ holds.

Let us consider the case $p=2$. First we consider special cases.

Case 1: The case $b-a=1/2$. We have $e(-2^sb)=e(-2^sa)$ for $s\ge1$, and $e(-b)\ne e(-a)$. Hence (14) does not hold for $j=1$, and $\sigma_{a,b,\theta}>0$ holds.

Case 2: The case $b-a=1/4,\ 3/4$. We prove $e(-b)+e(-2b)/2\ne e(-a)+e(-2a)/2$. Actually, if the equality holds, by denoting $A=e(-a)$, we have $e(-2a)=A^2$, $e(-b)=\pm\sqrt{-1}\,A$, and $e(-2b)=-A^2$, and therefore $A+A^2/2=\pm\sqrt{-1}\,A-A^2/2$ which yields $A=0,\ \pm\sqrt{-1}-1$. It contradicts with $|A|=1$. On the other hand we have $e(-2^sb)=e(-2^sa)$ for $s\ge2$. Hence (14) does not hold for $j=1$, and $\sigma_{a,b,\theta}>0$ holds.

Case 3: The case $b-a=1/6,\ 5/6$. We have $e(-3\cdot2^sb)=e(-3\cdot2^sa)$ for $s\ge1$, and $e(-3b)\ne e(-3a)$. Hence (14) does not hold for $j=3$, and $\sigma_{a,b,\theta}>0$ holds.

Case 4: The case $a=1/3$ and $b=2/3$. Since $2=-1 \bmod 3$, we have $e(-2^s/3)=e(-(-1)^s/3)$ and $\sum_{s=0}^{\infty}e(-2^s/3)/2^s=\left(2e(\frac{1}{3})+4e(\frac{-1}{3})\right)/3\ne\left(2e(\frac{-1}{3})+4e(\frac{1}{3})\right)/3=\sum_{s=0}^{\infty}e(-2^s2/3)/2^s$. Hence (14) does not hold for $j=1$, and $\sigma_{a,b,\theta}>0$ holds.

Since we have $\left|\sum_{s=0}^{\infty}e(-j2^sx)/2^s-\left(e(-jx)+e(-2jx)/2\right)\right|\le 1/2$, if there exists an odd j such that

$$\Psi_{j,a,b}=\left|\left(e(-ja)+\frac{1}{2}e(-2ja)\right)-\left(e(-jb)+\frac{1}{2}e(-2jb)\right)\right|>1, \tag{15}$$

then (14) does not hold, and $\sigma_{a,b,\theta}>0$ holds. Note that we have a convenient formula $\Psi_{j,a,b}=|e(-ja)-e(-jb)|\left|1+\frac{1}{2}(e(-ja)+e(-jb))\right|$.

We here prove $\sigma_{a,b,\theta}>0$ if there exists an odd j such that $\langle j(b-a)\rangle\in[3/8,5/8]$. We see $|e(-ja)-e(-jb)|=|e(-j(a-b)/2)-e(j(a-b)/2)|=2|\sin2\pi j(b-a)/2|\ge 2\sin2\pi\frac{3}{16}$ and $|e(-ja)+e(-jb)|=2|\cos2\pi j(b-a)/2|\le2\cos2\pi\frac{3}{16}$. Hence $\Psi_{j,a,b}\ge 2\sin2\pi\frac{3}{16}(1-\cos2\pi\frac{3}{16})=\sqrt{2+\sqrt{2}}-\sqrt{2}/2>1$, and $\sigma_{a,b,\theta}>0$ holds.

We have $\sigma_{a,b,\theta}>0$ if there exists an l such that $\langle2(b-a)l+(b-a)\rangle\in[3/8,5/8]$. This sequence distributed uniformly or periodically on the torus. Since the length of $[3/8,5/8]$ is $1/4$, such l exists except for the case when $2(b-a)$ is an integer multiple of 1, 1/2, or 1/3. i.e., $b-a=1/2$ (Case 1), 1/4, 3/4 (Case 2), 1/6, 5/6 (Case 3), 1/3 and 2/3.

We only have to consider the case $b-a=1/3,\ 2/3$. We can take φ such that $a=\varphi+\frac{1}{6}$ and $b=\varphi-\frac{1}{6}$ mod 1. For $j=6l+1$, we have $|e(-ja)-e(-jb)|=|e(-\frac{1}{6})-e(\frac{1}{6})|=2\sin2\pi\frac{1}{6}=\sqrt{3}$ and $e(-ja)+e(-jb)=e(-j\varphi)(e(-\frac{1}{6})+e(\frac{1}{6}))=e(-j\varphi)$. Hence we have $\Psi_{j,a,b}^2-1=3\left(\frac{11}{12}+\cos2\pi j\varphi\right)$ and see that $\sigma_{a,b,\theta}>0$ if $\cos2\pi j\varphi>-\frac{11}{12}$. In the case when $\cos2\pi j\varphi\le-\frac{11}{12}$ is satisfied, we have $|\sin2\pi j\varphi|=$

$\sqrt{1-\cos^2 2\pi j\varphi} \le \sqrt{23}/12$ and $\Re(e(-ja))$, $\Re(e(-jb))$ equals to $\cos 2\pi(j\varphi \pm \frac{1}{6}) = \frac{1}{2}\bigl(\cos 2\pi j\varphi \mp \sqrt{3}\sin 2\pi j\varphi\bigr) \le (-11+\sqrt{69})/24 < 0$. In other words, if $\Re(e(-ja)) \ge 0$ or $\Re(e(-jb)) \ge 0$ then $\cos 2\pi j\varphi > -\frac{11}{12}$ and $\sigma_{a,b,\theta} > 0$. Therefore we can conclude $\sigma_{a,b,\theta} > 0$ if $b-a = 1/3,\ 2/3$ and there exists l such that $\Re(e(-(6l+1)a)) \ge 0$ or $\Re(e(-(6l+1)b)) \ge 0$.

Since the sequence $\{e(-(6l+1)a)\}_l$ is uniformly or periodically distributed on the torus, such l exists except for the case when a is an integer multiple of $1/6$. Such a satisfying $\Re(e(-a)) < 0$ are $a = 1/3,\ 1/2,\ 2/3$. In the same way, we have $b = 1/3,\ 1/2,\ 2/3$. By $b-a = 1/3,\ 2/3$, we have $a = 1/3$ and $b = 2/3$, which is the Case 4. □

The next lemma is proved in [6] (just after the proof of Lemma 1 in [6].)

Lemma 2 *As a function of (a,b) defined on 2-dimensional torus, $\sigma(\widetilde{\mathbf{I}}_{a,b},\theta)$ is uniformly continuous.*

Lemma 3 *If $0 \le a < b < 1$, we have $\min_x \sigma(\widetilde{\mathbf{I}}_{a+x,b+x;d},\theta) > 0$ for large enough d.*

Proof We prove that there exists C_θ depending only on θ such that

$$|\sigma^2(\widetilde{\mathbf{I}}_{a,b},\theta) - \sigma^2(\widetilde{\mathbf{I}}_{a,b;d},\theta)| \le C_\theta(\log d)/d. \tag{16}$$

Because of Lemma 1 and Lemma 2, we have $\min_x \sigma^2(\widetilde{\mathbf{I}}_{a+x,b+x},\theta) > 0$, and by (16), we have the conclusion of Lemma.

Denote $c(k) = \widehat{\widetilde{\mathbf{I}}}_{a,b}(k)$. We have $|c(k)| \le 1/|k|\pi$. When θ satisfies (6), it holds that $|\sigma^2(\widetilde{\mathbf{I}}_{a,b},\theta) - \sigma^2(\widetilde{\mathbf{I}}_{a,b;d},\theta)| = \sum_{|k|>d} |c(k)|^2 \le \sum_{|k|>d} 1/k^2\pi^2 \le 1/d$.

Let θ be given by (7). Note that $\int_0^1 \widetilde{\mathbf{I}}_{a,b}(p^k y)\widetilde{\mathbf{I}}_{a,b}(q^k y)\,dy = \sum_{l\ne 0} c(lq^k)\bar{c}(lp^k)$. By noting $p > q$, we have $\int_0^1 \widetilde{\mathbf{I}}_{a,b;d}(p^k y)\widetilde{\mathbf{I}}_{a,b;d}(q^k y)\,dy = \sum_{|l|\le d/p^k} c(lq^k)\bar{c}(lp^k)$. Thereby $\bigl|\int_0^1 \widetilde{\mathbf{I}}_{a,b}(p^k y)\widetilde{\mathbf{I}}_{a,b}(q^k y)\,dy - \int_0^1 \widetilde{\mathbf{I}}_{a,b;d}(p^k y)\widetilde{\mathbf{I}}_{a,b;d}(q^k y)\,dy\bigr| \le \sum_{|l|>d/p^k} 1/p^k q^k l^2\pi^2$. Hence the left hand side of (16) is bounded by $\sum_{k=0}^{\infty}\sum_{l>d/p^k} 4/p^k q^k l^2\pi^2$. We divide the sum into $\sum_{k<\log d/\log p}\sum_{l>d/p^k}$ and $\sum_{k\ge \log d/\log p}\sum_{l\ge 1}$. The first sum is bounded by $\sum_{k<\log d/\log p} 1/q^k d \le (\log d)/d\log p$. The second is bounded by $1/(1-1/p)d$. □

For a bounded measurable function g, we define *the mean value* $\int_{\mathbf{R}} g(x)\,\mu_R(dx)$ by

$$\int_{\mathbf{R}} g(x)\,\mu_R(dx) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x)\,dx$$

if the limit on the right hand side exists. For a trigonometric polynomial g with period 1 satisfying $\int_0^1 g = 0$, we have $\int_{\mathbf{R}} g(\Theta x)g(x)\,\mu_R(dx) = 0$ for $\Theta \notin \mathbf{Q}$, and $\int_{\mathbf{R}} g((P/Q)x)g(x)\,\mu_R(dx) = \int_{\mathbf{R}} g(Px)g(Qx)\,\mu_R(dx) = \int_0^1 g(Px)g(Qx)\,dx$ for non-zero integers P and Q.

Lemma 4 *Let g be a trigonometric polynomial with period 1 and degree d satisfying $\int_0^1 g = 0$. There exists a constant $C_{\theta,d}$ depending only on θ and d such that*

$$\left|\int_{\mathbf{R}}\Bigl(\sum_{k=M+1}^{M+N} g(\theta^k y)\Bigr)^2 \mu_R(dy) - N\sigma^2(g,\theta)\right| \le C_{\theta,d}\|g\|_2^2.$$

Proof Put $\Gamma = \int_{\mathbf{R}} \bigl(\sum_{k=M+1}^{M+N} g(\theta^k y)\bigr)^2 \mu_R(dy) = \int_{\mathbf{R}} \bigl(\sum_{k=1}^{N} g(\theta^k y)\bigr)^2 \mu_R(dy)$. If θ satisfies (6). we have $\Gamma = N\|g\|_2^2 = N\sigma^2(g,\theta)$. Assume that θ is given by (7). We have $\Gamma = N\int_{\mathbf{R}} g^2(y)\,\mu_R(dy) + 2\sum_j (N-j)^+ \int_{\mathbf{R}} g(\theta^j y)g(y)\,\mu_R(dy)$. Note $\int_{\mathbf{R}} g(\theta^j y)g(y)\,\mu_R(dy) = 0$ if $j > G = \log_\theta d$ or $k \notin r\mathbf{N}$, and $\int_{\mathbf{R}} g(\theta^{rk} y)g(y)\,\mu_R(dy) = \int_0^1 g(p^k y)g(q^k y)\,dy$. Thereby $\int_0^1 g(p^k y)g(q^k y)\,dy = 0$ if $k > \frac{G}{r}$. Hence $\Gamma = N\int_0^1 g^2(y)\,dy + 2\sum_{k=1}^{G/r}(N - rk)^+ \int_0^1 g(p^k y)g(q^k y)\,dy$ and $\sigma^2(g,\theta) = \int_0^1 g^2 + 2\sum_{k=1}^{G/r}\int_0^1 g(p^k y)g(q^k y)\,dy$. These imply $|\Gamma - N\sigma^2(g,\theta)| \le 2\sum_{k=1}^{G/r} rk\|g\|_2^2$ by $\bigl|\int_0^1 g(p^k y)g(q^k y)\,dy\bigr| \le \|g\|_2^2$. □

Lemma 5 *If g is a bounded measurable function with period 1 satisfying $\int_0^1 g = 0$, then for all $a < b$ and $\lambda > 0$, we have*

$$\left|\int_a^b g(\lambda x)\,dx\right| \le \frac{\|g\|_\infty}{\lambda}.$$

Proof By changing variable $y = \lambda x$, the integral equals to

$$\frac{1}{\lambda}\int_{\lambda a}^{\lambda b} g(y)\,dy = \frac{1}{\lambda}\sum_{k=0}^{[\lambda(b-a)]}\int_{\lambda a+k}^{\lambda a+k+1} g(y)\,dy + \frac{1}{\lambda}\int_{\lambda a+[\lambda(b-a)]}^{\lambda b} g(y)\,dy.$$

The proof ends by $|\int_{\lambda a+[\lambda(b-a)]}^{\lambda b} g(y)\,dy| \le (\lambda(b-a) - [\lambda(b-a)])\|g\|_\infty \le \|g\|_\infty$. □

Lemma 6 *For $0 \le a < b < 1$, we have $\|\widetilde{\mathbf{I}}_{a,b;d}\|_\infty \le 2$.*

Proof Let σ_d be the d-th Cesaro sum of the Fourier series of $\widetilde{\mathbf{I}}_{a,b}$. We have $\|\sigma_d\|_\infty \le \|\widetilde{\mathbf{I}}_{a,b}\|_\infty \le 1$. Hence $|\widetilde{\mathbf{I}}_{a,b;d}| \le |\sigma_d| + \sum_{|j|\le d}(j/d)|\widehat{\widetilde{\mathbf{I}}}_{a,b}(j)e(jx)| \le 2$ by $|\widehat{\widetilde{\mathbf{I}}}_{a,b}(j)| \le 1/\pi|j|$. □

Lemma 7 *Let g be a trigonometric polynomial with period 1 and degree d satisfying $\int_0^1 g = 0$. There exists a constant C_θ depending only on θ such that, for a sequence $\{\lambda_k\}$ of real numbers satisfying the Hadamard's gap condition $\lambda_{k+1}/\lambda_k > \theta > 1$ and $\lambda_1 \ge 1$,*

$$\int_0^1 \Bigl(\sum_{k=M+1}^{M+N} g(\lambda_k x)\Bigr)^4 dx \le C_\theta \Bigl(\sum_{|\nu|\le d} |\widehat{g}(\nu)|\Bigr)^4 N^2.$$

Proof The left hand side is bounded by $\bigl(\sum_{|\nu|\le d}|\widehat{g}(\nu)|\bigr)^4 \int_0^1 \bigl(\sum_{k=M+1}^{M+N} e(\lambda_k \nu x)\bigr)^4 dx$. Hence by applying the following inequality (Lemma 1 (1) of [5]), we have the conclusion:

$$\int_0^1 \Bigl(\sum_{j=1}^{\infty}(c_j\cos 2\pi\lambda_j x + d_j \sin 2\pi\lambda_j x)\Bigr)^4 dx \le C_\theta \Bigl(\sum_{j=1}^{\infty}(c_j^2 + d_j^2)\Bigr)^2. \square$$

Let $\theta > 1$ and denote $n_k = \theta^k - 1$. Since $(\theta x - 1)/(x-1) > 1$ holds for $x > 1$, $\{n_k\}$ satisfies the Hadamard's gap condition $n_{k+1}/n_k > \theta$.

Let $0 \le a < b < 1$ and take $d \in \mathbf{N}$ such that the conclusion of Lemma 3 holds. Put $f = \widetilde{\mathbf{I}}_{a,b;d}$. Then $\int_0^1 f = 0$ is clear and $\|f\|_\infty \le 2$ by Lemma 6.

We follow the method of the martingale approximation given in [2], which originated with Berkes and Philipp [4,13].

Let us divide $\mathbf{N}$ into consecutive blocks $\Delta_1', \Delta_1, \Delta_2', \Delta_2, \dots$ satisfying ${}^\#\Delta_i' = [\,1 + 9\log_\theta i\,]$ and ${}^\#\Delta_i = i$. Denote $i^- = \min \Delta_i$ and $i^+ = \max \Delta_i$. We have

$$n_{i^-}/n_{(i-1)^+} \ge \theta^{9\log_\theta i} \ge i^9$$

Put $\mu(i) = [\,\log_2 i^4 n_{i^+}\,] + 1$ and $\mathcal{F}_i = \sigma\{[\,j2^{-\mu(i)}, (j+1)2^{-\mu(i)}) \mid j = 0, \dots, 2^{\mu(i)} - 1\}$. Note that $i^4 n_{i^+} \le 2^{\mu(i)} \le 2i^4 n_{i^+}$. Put

$$T_i(x) = \sum_{k\in\Delta_i} f(n_k x), \quad T_i'(x) = \sum_{k\in\Delta_i'} f(n_k x), \quad Y_i = E(T_i \mid \mathcal{F}_i) - E(T_i \mid \mathcal{F}_{i-1}).$$

Then $\{Y_i, \mathcal{F}_i\}$ forms a martingale difference sequence. Here let us prove

$$\|Y_i - T_i\|_\infty \le (\|f'\|_\infty + 2\|f\|_\infty)/i^3, \tag{17}$$

$$\|Y_i^2 - T_i^2\|_\infty \le 3\|f\|_\infty(\|f'\|_\infty + 2\|f\|_\infty)/i^2, \tag{18}$$

$$\|Y_i^4 - T_i^4\|_\infty \le 15\|f\|_\infty^3(\|f'\|_\infty + 2\|f\|_\infty). \tag{19}$$

If $k \in \Delta_i$ and $x \in I = [\,j2^{-\mu(i)}, (j+1)2^{-\mu(i)}) \in \mathcal{F}_i$ then we have $|f(n_k x) - E(f(n_k\,\cdot\ \mid \mathcal{F}_i)| = \left||I|^{-1}\int_I (f(n_k x) - f(n_k y))\,dy\right| \le \max_{y\in I}|f(n_k x) - f(n_k y))| \le \|f'\|_\infty n_k 2^{-\mu(i)} \le \|f'\|_\infty n_k/i^4 n_{i^+} \le \|f'\|_\infty/i^4$. Hence we have $|T_i - E(T_i \mid \mathcal{F}_i)| \le \|f'\|_\infty {}^\#\Delta_i/i^4 = \|f'\|_\infty/i^3$. Put $J = [\,j2^{-\mu(i-1)}, (j+1)2^{-\mu(i-1)}) \in \mathcal{F}_{i-1}$. Then by applying Lemma 5, $|E(f(n_k\,\cdot\,) \mid \mathcal{F}_{i-1})| = \left||J|^{-1}\int_J f(n_k y)\,dy\right| \le \|f\|_\infty 2^{\mu(i-1)}/n_k \le \|f\|_\infty 2(i-1)^4 n_{(i-1)^+}/n_{i^-} \le 2\|f\|_\infty/i^5$. Hence we have $|E(T_i \mid \mathcal{F}_{i-1})| \le 2\|f\|_\infty {}^\#\Delta_i/i^5 = 2\|f\|_\infty/i^4$ and (17).

By $\|T_i\|_\infty \le i\|f\|_\infty$, we have $\|E(T_i \mid \mathcal{F}_i)\|_\infty, \|E(T_i \mid \mathcal{F}_{i-1})\|_\infty \le i\|f\|_\infty$. Hence we have $\|Y_i\|_\infty \le 2i\|f\|_\infty$, $\|Y_i + T_i\|_\infty \le 3i\|f\|_\infty$, $\|Y_i^2 + T_i^2\|_\infty \le 5i^2\|f\|_\infty^2$. Applying these to $\|Y_i^2 - T_i^2\|_\infty \le \|Y_i - T_i\|_\infty\|Y_i + T_i\|_\infty$ and $\|Y_i^4 - T_i^4\|_\infty \le \|Y_i^2 - T_i^2\|_\infty\|Y_i^2 + T_i^2\|_\infty$, we have (18) and (19).

Put $C = \min\{[\log_\theta \nu - \log_\theta \nu'\,]^* \mid \nu, \nu' = 1, \dots, d, \log_\theta \nu - \log_\theta \nu' \notin \mathbf{Z}\} \in (0,1)$ where $[\,x\,]^* = \min_{n\in\mathbf{Z}}|x - n|$. By denoting $D = (\theta^C - 1) \wedge 1 > 0$, we prove

$$|\theta^k\nu + \theta^l\nu'| \ge D\theta^L \quad \text{if} \quad k, l \ge L,\ |\nu|, |\nu'| \le d,\ \theta^k\nu + \theta^l\nu' \ne 0. \tag{20}$$

If $\nu\nu' \ge 0$, then it is trivial. Assume $\nu > 0$ an $\nu' < 0$ and put $\nu'' = -\nu'$. If $\log_\theta \nu - \log_\theta \nu'' \notin \mathbf{Z}$, then we have $|\log_\theta(\theta^k\nu) - \log_\theta(\theta^l\nu'')| = |(k - l) + (\log_\theta \nu - \log_\theta \nu'')| \ge [\log_\theta \nu - \log_\theta \nu''\,]^* \ge C$. If $\log_\theta \nu - \log_\theta \nu'' \in \mathbf{Z}$, then by $\theta^k\nu \ne \theta^l\nu''$, $\log_\theta(\theta^k\nu) - \log_\theta(\theta^l\nu'')$ is a non-zero integer and hence $|\log_\theta(\theta^k\nu) - \log_\theta(\theta^l\nu'')| \ge 1 \ge C$. Hence $\theta^k\nu/\theta^l\nu'' \ge \theta^C$ if $\theta^k\nu > \theta^l\nu''$, and $\theta^l\nu''/\theta^k\nu \ge \theta^C$ otherwise. By $\theta^k\nu/\theta^l\nu'' \ge \theta^C$ we have $\theta^k\nu - \theta^l\nu'' \ge (\theta^C - 1)\theta^l\nu'' \ge D\theta^L$, which is (20). The rest is similar.

Put

$$\zeta_i(x,y) = \Bigl(\sum_{k\in\Delta_i} f(\theta^k y - x)\Bigr)^2 \quad \text{and} \quad \xi_i(x) = \int_{\mathbf{R}} \zeta_i(x,y)\,\mu_R(dy).$$

Clearly we have $T_i^2(x) = \zeta_i(x,x)$. Expanding $\zeta_i(x,y)$, we have

$$\zeta_i(x,y) = \sum_k\sum_l\sum_\nu\sum_{\nu'} \widehat{f}(\nu)\widehat{f}(\nu')e(-(\nu+\nu')x)e((\theta^k\nu + \theta^l\nu')y).$$

Since $\xi_i(x)$ is a sum of above summands with $\theta^k\nu + \theta^l\nu' = 0$, $\zeta_i(x,y) - \xi_i(x)$ is a sum of summands with $\theta^k\nu + \theta^l\nu' \ne 0$, and hence $|\theta^k\nu + \theta^l\nu'| \ge D\theta^{i^-}$ by (20). By regarding x

as a constant, $\zeta_i(x,y)-\xi_i(x)$ is a sum of trigonometric functions of y with frequencies greater than $D\theta^{i^-}$. Therefore the frequencies in $W_i(x)=\zeta_i(x,x)-\xi_i(x)$ are greater than $D\theta^{i^-}-2d \geq D\theta^{i^-}/2 = n_{i^-}D/2$.

Since f is a trigonometric polynomial with degree d, we see that T_i can be expressed by a sum of at most $2di$ many trigonometric functions. Hence T_i^2 as well as W_i is a sum of at most $8d^2i^2$ many trigonometric functions. Therefore, by Lemma 5 again, we have $|E(W_i \mid \mathcal{F}_{i-1})| \leq 8d^2i^2(2/Dn_{i^-})2^{\mu(i-1)} \leq i^2(i-1)^4(32d^2/D)n_{(i-1)^+}/n_{i^-} \leq 32d^2/Di^3$. By noting $T_i^2-\xi_i=W_i$, we have

$$\left\|\sum_{i=1}^{M} E(T_i^2 \mid \mathcal{F}_{i-1}) - \sum_{i=1}^{M} E(\xi_i \mid \mathcal{F}_{i-1})\right\|_\infty = O(1). \tag{21}$$

By applying Lemma 4 to $g(y)=f_x(y)=f(y-x)$ and denoting $V_f(x)=\sigma^2(f_x,\theta)$, we have $|\xi_i(x)-iV_f(x)| \leq C_{\theta,d}\|f_x\|_2^2 = C_{\theta,d}\|f\|_2^2$ and hence

$$|E(\xi_i \mid \mathcal{F}_{i-1}) - iE(V_f \mid \mathcal{F}_{i-1})| \leq C_{\theta,d}\|f\|_2^2.$$

Since f is a trigonometric polynomial, $\int_0^1 f_x(p^ky)f_x(q^ky)\,dy=0$ for large k, and $V_f(x)$ is also a trigonometric polynomial. Hence $V_f(x)$ has continuous derivative and

$$|iE(V_f \mid \mathcal{F}_{i-1}) - iV_f| \leq \|V_f'\|_\infty i2^{-\mu(i-1)} \leq \|V_f'\|_\infty i/(i-1)^4 n_{(i-1)^+} = o(1/i^3).$$

By these we have $\left\|\sum_{i=1}^M E(\xi_i \mid \mathcal{F}_{i-1}) - \sum_{i=1}^M iV_f\right\|_\infty = O(M)$. By this, (21), and $\left\|\sum_{i=1}^M E(Y_i^2 \mid \mathcal{F}_{i-1}) - \sum_{i=1}^M E(T_i^2 \mid \mathcal{F}_{i-1})\right\|_\infty \leq \sum_{i=1}^M \|Y_i^2-T_i^2\|_\infty = O(1)$, we have

$$\left\|\sum_{i=1}^{M} E(Y_i^2 \mid \mathcal{F}_{i-1}) - V_f\frac{M(M+1)}{2}\right\|_\infty = O(M). \tag{22}$$

Now we use the next Lemma which can be found in [2], which is a modification of corollary in [17].

Lemma 8 *Let $\{Y_i,\mathcal{F}_i\}$ be a martingale difference sequence with finite 4-th moment and $\mathcal{F}_0=\{\emptyset,\Omega\}$. Suppose that*

$$V_M = \sum_{i=1}^{M} E(Y_i^2 \mid \mathcal{F}_{i-1}) \to \infty, \quad a.s., \tag{23}$$

and that there exists a sequence $\{r_M\}$ of positive numbers satisfying $r_M\to\infty$ and

$$\liminf_{M\to\infty}\frac{V_M}{r_M} \geq 1, \quad a.s., \tag{24}$$

$$\sum_{M=1}^{\infty}\frac{(\log r_M)^{10}}{r_M^2}EY_M^4 < \infty. \tag{25}$$

Then

$$\varlimsup_{M\to\infty}\frac{1}{\sqrt{2V_M\log\log V_M}}\left|\sum_{i=1}^{M}Y_i\right| = 1, \quad a.s.$$

Denote $\phi(N)=\sqrt{2\log\log N}$. Note $f_x(y)=\widetilde{\mathbf{I}}_{a,b;d}(y-x)=\widetilde{\mathbf{I}}_{a+x,b+x;d}(y)$ and $V_f(x)=\sigma^2(\widetilde{\mathbf{I}}_{a+x,b+x;d},\theta)$. Thanks to Lemma 3, $\rho=\min_x V_f(x)$ is positive. Putting $l_M=M(M+1)/2$ and $r_M=\rho l_M$, by (22) we have $\lim V_M/r_M=V_f(x)/\rho\geq 1$, which proves (24) and (23). (25) is clear from $EY_M^4=ET_M^4+O(1)=O(M^2)=O(l_M)$ which follows from (19) and Lemma 7. By applying the Lemma 8 we have

$$\overline{\lim_{M\to\infty}}\,\phi^{-1}(l_M)\Big|\sum_{i=1}^{M}Y_i\Big|=\sigma(\widetilde{\mathbf{I}}_{a+x,b+x;d},\theta),\quad \text{a.e.} \tag{26}$$

where $V_M\sim\sigma^2(\widetilde{\mathbf{I}}_{a+x,b+x;d},\theta)l_M$ is used. By (17), we can replace Y_i in (26) by T_i.

Now we use the following Lemma which is Lemma 3.1 of [1]. In [1], it is proved by assuming that λ_k is an integer and applying method by Takahashi [18] and Philipp [14]. We can remove this extra condition by applying modification of proof which is used in [6].

Lemma 9 *Let $r(x)$ be a function of the form $r(x)=\sum_{j=d}^{\infty}(a_j\cos 2\pi jx+b_j\sin 2\pi jx)$, where $|a_j|,\ |b_j|\leq 1/j$ for $j\geq d$. If a sequence $\{\lambda_k\}$ of real numbers satisfies the Hadamard's condition $\lambda_{k+1}/\lambda_k>\theta>1$, then*

$$\overline{\lim_{N\to\infty}}\,\frac{1}{\sqrt{2N\log\log N}}\Big|\sum_{k=1}^{N}r(\lambda_k x)\Big|\leq C'_\theta d^{-1/4},\quad a.e.,$$

where C'_θ is a positive constant depending on θ.

Applying Lemma 9 by putting $r=\widetilde{\mathbf{I}}_{a,b;d}$, by $\sum_{i=1}^{M}[\,1+9\log_\theta i\,]=O(M\log M)$, we have $\big|\sum_{k=1}^{M}T'_k\big|=O\big(\sqrt{M\log M\log\log(M\log M)}\big)=o(\sqrt{l_M})$. Thereby (26) is valid if we replace Y_i by T'_i and $\sigma(\widetilde{\mathbf{I}}_{a+x,b+x;d},\theta)$ by 0. Hence by noting $M^+=l_M+\sum_{i=1}^{M}[\,1+9\log_\theta i\,]\sim l_M$, we have $\overline{\lim}_{M\to\infty}\phi^{-1}(M^+)\big|\sum_{i=1}^{M}\sum_{k\in\Delta'_i\cup\Delta_i}f((\theta^k-1)x)\big|=\sigma(\widetilde{\mathbf{I}}_{a+x,b+x;d},\theta)$ a.e. By $\sum_{k\in\Delta'_M\cup\Delta_M}\|f((\theta^k-1)x)\|_\infty=o(\phi(M^+))$, we have

$$\overline{\lim_{N\to\infty}}\,\phi^{-1}(N)\Big|\sum_{k=1}^{N}\widetilde{\mathbf{I}}_{a,b;d}((\theta^k-1)x)\Big|=\sigma(\widetilde{\mathbf{I}}_{a+x,b+x;d},\theta),\quad \text{a.e.} \tag{27}$$

By Lemma 9 we have $\overline{\lim}_{N\to\infty}\phi^{-1}(N)\big|\sum_{k=1}^{N}(\widetilde{\mathbf{I}}_{a,b}-\widetilde{\mathbf{I}}_{a,b;d})((\theta^k-1)x)\big|\leq C'_\theta d^{-1/4}$, a.e. This together with (16) and (27) implies $\overline{\lim}_{N\to\infty}\phi^{-1}(N)\big|\sum_{k=1}^{N}\widetilde{\mathbf{I}}_{a,b}((\theta^k-1)x)\big|\in[\,\eta_d^-,\eta_d^+\,]$, a.e., where $\eta_d^\pm=\pm C'_\theta d^{-1/4}+\big(\sigma^2(\widetilde{\mathbf{I}}_{a+x,b+x},\theta)\pm C_\theta(\log d)/d\big)^{1/2}$. Since $\eta_d^\pm\to\sigma(\widetilde{\mathbf{I}}_{a+x,b+x},\theta)$ as $d\to\infty$, we have

$$\overline{\lim_{N\to\infty}}\,\phi^{-1}(N)\Big|\sum_{k=1}^{N}\widetilde{\mathbf{I}}_{a,b}((\theta^k-1)x)\Big|=\sigma(\widetilde{\mathbf{I}}_{a+x,b+x},\theta),\quad \text{a.e.}$$

We use the fundamental relation below which can be found in [8].

Lemma 10 *For any countable dense subset S of $[\,0,1)$ and for any sequence $\{n_k\}$ of real numbers (not necessarily integers) satisfying the Hadamard's gap condition, we* *have*

$$\varlimsup_{N\to\infty}\frac{ND_N\{n_kx\}}{\sqrt{2N\log\log N}}=\sup_{S\ni a'<a\in S}\varlimsup_{N\to\infty}\frac{1}{\sqrt{2N\log\log N}}\left|\sum_{k=1}^N\widetilde{\mathbf{I}}_{a',a}(n_kx)\right|,\quad a.e.,$$

$$\varlimsup_{N\to\infty}\frac{ND_N^*\{n_kx\}}{\sqrt{2N\log\log N}}=\sup_{a\in S}\varlimsup_{N\to\infty}\frac{1}{\sqrt{2N\log\log N}}\left|\sum_{k=1}^N\widetilde{\mathbf{I}}_{0,a}(n_kx)\right|,\quad a.e.$$

Hence $\Sigma\{(\theta^k-1)x\}=\sup_{S\ni a<b\in S}\sigma(\widetilde{\mathbf{I}}_{a+x,b+x},\theta)=\sup_{0\le a<b<1}\sigma(\widetilde{\mathbf{I}}_{a+x,b+x},\theta)$, a.e., where the second equality is by continuity of $\sigma(\widetilde{\mathbf{I}}_{a,b},\theta)$ with respect to (a,b). In the same way, we have $\Sigma^*\{(\theta^k-1)x\}=\sup_{0\le a<1}\sigma(\widetilde{\mathbf{I}}_{x,a+x},\theta)$, a.e.

In [6], it is proved that $\Sigma_\theta=\sup_{0\le a<b<1}\sigma(\widetilde{\mathbf{I}}_{a,b},\theta)=\sup_{0\le a<1}\sigma(\widetilde{\mathbf{I}}_{0,a},\theta)$. By (13), we have

$$\sigma(\widetilde{\mathbf{I}}_{a+x,b+x},\theta)=\begin{cases}\sigma(\widetilde{\mathbf{I}}_{\langle a+x\rangle,\langle b+x\rangle},\theta) & \text{if } \langle a+x\rangle\le\langle b+x\rangle\\ \sigma(\widetilde{\mathbf{I}}_{\langle b+x\rangle,\langle a+x\rangle},\theta) & \text{if } \langle b+x\rangle<\langle a+x\rangle\end{cases}$$

Hence we have $\sup_{0\le a<b<1}\sigma(\widetilde{\mathbf{I}}_{a+x,b+x},\theta)\le\Sigma_\theta$. On the other hand, for given $0\le c<d<1$, we can find $0\le a<b<1$ such that $c=\langle b+x\rangle$ and $d=\langle a+x\rangle$ if $c<x\le d$, and $c=\langle a+x\rangle$ and $d=\langle b+x\rangle$ otherwise. Therefore we can conclude $\sup_{0\le a<b<1}\sigma(\widetilde{\mathbf{I}}_{a+x,b+x},\theta)=\Sigma_\theta$ and (8).

We have (9) by putting $\Sigma_\theta^*(x)=\sup_{0\le a<1}\sigma(\widetilde{\mathbf{I}}_{x,a+x},\theta)$. Clearly we have $\Sigma_\theta^*(x)\le\Sigma_\theta$. Continuity of $\Sigma_\theta^*(x)$ is proved by using Lemma 2. To investigate $\Sigma_\theta^*(x)$, we use some lemmas.

Lemma 11 *Let $\theta>1$, and let a, $b\in\mathbf{R}$ satisfy $0\le b-a<1$. If θ satisfies (6), then $\sigma^2(\widetilde{\mathbf{I}}_{a,b},\theta)$ equals to $|\langle b\rangle-\langle a\rangle|(1-|\langle b\rangle-\langle a\rangle|)$. If θ is given by (7), then*

$$\sigma^2(\widetilde{\mathbf{I}}_{a,b},\theta)=\widetilde{V}(\langle a\rangle,\langle b\rangle,\langle a\rangle,\langle b\rangle)+2\sum_{k=1}^\infty\frac{\widetilde{V}(\langle p^ka\rangle,\langle p^kb\rangle,\langle q^ka\rangle,\langle q^kb\rangle)}{p^kq^k},\tag{28}$$

where $\widetilde{V}(x,y,\xi,\eta)=V(x,\xi)+V(y,\eta)-V(x,\eta)-V(y,\xi)$ and $V(x,\xi)=x\wedge\xi-x\xi$.

Proof We prove for θ given by (7), since it is trivial otherwise. It is already proved (Lemma 1 of [6]) that,

$$\int_0^1\widetilde{\mathbf{I}}_{a,b}(\mu t)\widetilde{\mathbf{I}}_{a,b}(\nu t)\,dt=\frac{\widetilde{V}(\langle\mu a\rangle,\langle\mu b\rangle,\langle\nu a\rangle,\langle\nu b\rangle)}{\mu\nu}\tag{29}$$

for $0\le a<b<1$ and μ, ν with $\gcd(\mu,\nu)=1$. If $\langle a\rangle\le\langle b\rangle$, then (28) is trivial from (10), (13), and (29). If $\langle b\rangle<\langle a\rangle$, then we have $\widetilde{\mathbf{I}}_{a,b}(\mu t)\widetilde{\mathbf{I}}_{a,b}(\nu t)=\widetilde{\mathbf{I}}_{\langle b\rangle,\langle a\rangle}(\mu t)\widetilde{\mathbf{I}}_{\langle b\rangle,\langle a\rangle}(\nu t)$ by (13). Therefore we have (28) by noting $\widetilde{V}(x,y,\xi,\eta)=\widetilde{V}(y,x,\eta,\xi)$. □

If θ satisfies (6) and if $0\le a<1$, by noting

$$\langle a+x\rangle-\langle x\rangle=\begin{cases}a & \text{if } \langle x\rangle\le\langle a+x\rangle,\\ a-1 & \text{otherwise,}\end{cases}$$

we have $\sigma^2(\widetilde{\mathbf{I}}_{x,a+x},\theta)=a(1-a)$. By taking supremum for $0\le a<1$, we have $(\Sigma_\theta^*(x))^2=1/4$, i.e. $\Sigma_\theta^*(x)=1/2$.

Assume that θ is given by (7). Note again that we have $\Sigma_\theta=\sup_{0\le a<1}\sigma(\widetilde{\mathbf{I}}_{0,a},\theta)$.

Lemma 12 *When μ and ν are relatively prime each other and $0 \le a < b < 1$,*

$$\widetilde{V}(\langle \mu a\rangle, \langle \mu b\rangle, \langle \nu a\rangle, \langle \nu b\rangle) \le V(\langle \mu(b-a)\rangle, \langle \nu(b-a)\rangle). \tag{30}$$

We have $\sigma^2(\widetilde{\mathbf{I}}_{a,b}, \theta) \le \sigma^2(\widetilde{\mathbf{I}}_{0,b-a}, \theta)$ for a, b with $0 \le b-a < 1$. Denote $M_\theta = \{0 \le a < 1 \mid \sigma^2(\widetilde{\mathbf{I}}_{0,a}, \theta) = \Sigma_\theta\}$. We have $M_\theta = \{1/2\}$ if both of p and q are odd, $M_\theta = \{(p/2 \pm 1)/(p-1)\}$ if $p \ge 4$ is even and $q = 1$, $M_\theta = \{1/3, 2/3\}$ if $p = 2$ and $q = 1$, and $M_\theta = \{1/3, 2/3\}$ if $p = 5$ and $q = 2$.

The inequality (30) and the determination of M_θ are essentially proved in [6]. The inequality $\sigma^2(\widetilde{\mathbf{I}}_{a,b}, \theta) \le \sigma^2(\widetilde{\mathbf{I}}_{0,b-a}, \theta)$ under $0 \le b-a < 1$ is proved by Lemma 11 and (30).

We investigate the condition on x to have $\Sigma^*_\theta(x) = \Sigma_\theta$. If $a \notin M_\theta$, we have $\sigma^2(\widetilde{\mathbf{I}}_{x,x+a}, \theta) \le \sigma^2(\widetilde{\mathbf{I}}_{0,a}, \theta) < \Sigma_\theta$. Let $a \in M_\theta$. We have

$$\widetilde{V}(\langle p^n x\rangle, \langle p^n(x+a)\rangle, \langle q^n x\rangle, \langle q^n(x+a)\rangle) \le V(\langle p^n a\rangle, \langle q^n a\rangle). \tag{31}$$

If there exists an n such that the equality does not hold, then $\sigma^2(\widetilde{\mathbf{I}}_{x,x+a}, \theta) < \sigma^2(\widetilde{\mathbf{I}}_{0,a}, \theta)$.

Lemma 13 *Let $0 \le \xi, \eta < 1$ and $0 < \alpha < 1$. Then*

$$\widetilde{V}(\xi, \langle \xi + \alpha\rangle, \eta, \langle \eta + \alpha\rangle) \le V(\alpha, \alpha) = \alpha(1-\alpha) \tag{32}$$

The equality holds if and only if $\xi = \eta$.

Proof Suppose that $0 < \alpha < 1$. Recall that $0 \le V(\alpha, \xi) \le V(\alpha, \alpha)$, and that $V(\alpha, \xi) = V(\alpha, \alpha)$ if and only if $\xi = \alpha$. By symmetry, we may assume $\xi \le \eta$. The left hand side equals to

$$\int_0^1 \widetilde{\mathbf{I}}_{\xi,\xi+\alpha}(t)\widetilde{\mathbf{I}}_{\eta,\eta+\alpha}(t)\,dt = \int_0^1 \widetilde{\mathbf{I}}_{0,\alpha}(u)\widetilde{\mathbf{I}}_{\eta-\xi,\eta-\xi+\alpha}(u)\,du.$$

Here we changed variable by $t = u + \xi$ and used (12). The right hand side equals to $V(\alpha, \eta-\xi+\alpha) - V(\alpha, \eta-\xi)$, which proves the conclusion. □

First consider the case p and q are odd, and the case $p \ge 4$ is even and $q = 1$. In these cases we have $p^n a = a = q^n a$ for all $a \in M_\theta$ and $n \in \mathbf{N}$. Thus (31) reduces to

$$\widetilde{V}(\langle p^n x\rangle, \langle p^n x + a\rangle, \langle q^n x\rangle, \langle q^n x + a\rangle) \le V(\langle p^n a\rangle, \langle q^n a\rangle). \tag{33}$$

Applying Lemma 13 by putting $\xi = \langle p^n x\rangle$ and $\eta = \langle q^n y\rangle$, we see that the equality in (33) holds if and only if $\langle p^n x\rangle = \langle q^n x\rangle$. By this we have $(p^n - q^n)x = 0 \bmod 1$, i.e., $x = i/(p^n - q^n)$. If we put $n = 1$, we have $x = i/(p-q)$ $(i = 0, \dots, p-q-1)$. These satisfy $(p^n - q^n)x = 0 \bmod 1$ for all n and hence satisfy the equality in (33) for all n. From this, we can conclude that $\sigma^2(\widetilde{\mathbf{I}}_{x,x+a}, \theta) = \sigma^2(\widetilde{\mathbf{I}}_{0,a}, \theta)$ if and only if $x = i/(p-q)$ $(i = 0, \dots, p-q-1)$. Therefore we have $\Sigma^*_\theta(x) = \Sigma_\theta$ if $x = i/(p-q)$ $(i = 0, \dots, p-q-1)$ and $\Sigma^*_\theta(x) < \Sigma_\theta$ otherwise. Since $\Sigma_\theta(x)$ is continuous, by $\Sigma^*\{(\theta^k - 1)x\} = \Sigma_\theta(x)$ a.e., we can conclude that $\Sigma^*\{(\theta^k - 1)x\}$ is not constant a.e.

In case $p = 5$ and $q = 2$, we have $5^{2n}\left(\frac{1}{3}\right) = \frac{1}{3} = 2^{2n}\left(\frac{1}{3}\right)$, $5^{2n+1}\left(\frac{1}{3}\right) = \frac{2}{3} = 2^{2n+1}\left(\frac{1}{3}\right)$, $5^{2n}\left(\frac{2}{3}\right) = \frac{2}{3} = 2^{2n}\left(\frac{2}{3}\right)$, and $5^{2n+1}\left(\frac{2}{3}\right) = \frac{1}{3} = 2^{2n+1}\left(\frac{2}{3}\right) \bmod 1$. For $a \in M_\theta$, (31) reduces to (33) again. It implies that $\sigma^2(\widetilde{\mathbf{I}}_{x,x+a}, \theta) = \sigma^2(\widetilde{\mathbf{I}}_{0,a}, \theta) = \Sigma_\theta$ if and only if $x = 0$, $1/3$, $2/3$. Hence $\Sigma^*_\theta(x) = \Sigma_\theta$ if and only if $x = 0$, $1/3$, $2/3$.

Lastly, consider the case $p = 2$ and $q = 1$. In this case we have $2^{2n}\left(\frac{1}{3}\right) = \frac{1}{3}$, $2^{2n+1}\left(\frac{1}{3}\right) = \frac{2}{3}$, $2^{2n}\left(\frac{2}{3}\right) = \frac{2}{3}$, $2^{2n+1}\left(\frac{2}{3}\right) = \frac{1}{3} \bmod 1$. Hence for $a \in M_\theta$ and even n, (31) reduces to (33) again. In the same way as above $x = 0$, $1/3$, $2/3$. If $a = 1/3$ and $x = 2/3$, we have $\widetilde{\mathbf{I}}_{2/3,1} = -\widetilde{\mathbf{I}}_{0,2/3}$ and hence $\sigma^2(\widetilde{\mathbf{I}}_{2/3,1}, \theta) = \sigma^2(\widetilde{\mathbf{I}}_{0,2/3}, \theta) = \Sigma_\theta$. If $a = 2/3$ and $x = 1/3$, we have $\widetilde{\mathbf{I}}_{1/3,1} = -\widetilde{\mathbf{I}}_{0,1/3}$ and hence $\sigma^2(\widetilde{\mathbf{I}}_{1/3,1}, \theta) = \sigma^2(\widetilde{\mathbf{I}}_{0,1/3}, \theta) = \Sigma_\theta$. Thus $\Sigma^*_\theta(x) = \Sigma_\theta$ if and only if $x = 0$, $1/3$, $2/3$.

3 Aistleitner's sequences and its variation

In both of Aistleitner's papers [2,3], the limiting variance functions in the law of the iterated logarithm was given by calculations of the Fourier series and it is difficult to know the reason why we have these functions.

In this section we will give a simple derivation of these functions and try to reveal the reason.

First let us consider the Aistleitner's first sequence $n_{2k-1} = 2^{k^2}$, $n_{2k} = 2^{k^2+1} - 1$, $(k = 1, 2, \dots)$. For this sequence, Aistleitner [2] proved that

$$\varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^{N} \widetilde{\mathbf{I}}_{a,b}(n_k x) \right| = \sigma(x), \quad \text{a.e.} \tag{34}$$

$$\sigma^2(x) = (b-a)(1-(b-a)) - \frac{1}{2}\int_0^1 \mathbf{I}_{0,1-(b-a)}(t)\mathbf{I}_{0,\langle 2(b-a)\rangle}(x-a-t)\,dt, \tag{35}$$

where $\mathbf{I}_{a,b}$ coincides with our $\widetilde{\mathbf{I}}_{a,b}$ when $0 \le a < b < 1$, and is defined by $\mathbf{I}_{a,b} = -\mathbf{I}_{b,a}$ when $0 \le b < a < 1$.

Denoting $F(x,y) = \widetilde{\mathbf{I}}_{a,b;d}(y) + \widetilde{\mathbf{I}}_{a,b;d}(2y-x)$, we have $\int_0^1 F(x,y)F(x,2^{k'^2-k^2}y)\,dy = 0$ if $k'^2 - k^2 > \log_2(2d)$. By $k'^2 - k^2 \ge k' + k$, it is valid except for finitely many (k,k'). Hence we have

$$\begin{aligned}
&\frac{1}{2N}\int_0^1 \Big(\sum_{k=M+1}^{M+N} \Big(\widetilde{\mathbf{I}}_{a,b;d}(2^{k^2}y) + \widetilde{\mathbf{I}}_{a,b;d}(2^{k^2+1}y - x)\Big)\Big)^2\,dy \\
&= \frac{1}{2N}\int_0^1 \Big(\sum_{k=M+1}^{M+N} F(x,2^{k^2}y)\Big)^2\,dy \\
&= \frac{1}{2}\int_0^1 F^2(x,y)\,dy + \frac{1}{N}\sum_{M<k<k'\le M+N}\int_0^1 F(x,y)F(x,2^{k'^2-k^2}y)\,dy \\
&= \int_0^1 \widetilde{\mathbf{I}}^2_{a,b;d}(y)\,dy + \int_0^1 \widetilde{\mathbf{I}}_{a,b;d}(y)\widetilde{\mathbf{I}}_{a,b;d}(2y-x)\,dy + O(1/N).
\end{aligned}$$

In the same way as the proof given in the last section, we can prove (34) for

$$\sigma^2(x) = \int_0^1 \widetilde{\mathbf{I}}^2_{a,b}(y)\,dy + \int_0^1 \widetilde{\mathbf{I}}_{a,b}(y)\widetilde{\mathbf{I}}_{a,b}(2y-x)\,dy.$$

We must prove that this $\sigma(x)$ coincides with $\sigma(x)$ given by (35). First we prove

$$\widetilde{V}(\langle\alpha\rangle,\langle\beta\rangle,\langle\gamma\rangle,\langle\delta\rangle) = \int_0^1 \widetilde{\mathbf{I}}_{\langle\alpha\rangle,\langle\alpha\rangle+\langle\beta-\alpha\rangle}(t)\widetilde{\mathbf{I}}_{\langle\gamma\rangle,\langle\gamma\rangle+\langle\delta-\gamma\rangle}(t)\,dt. \tag{36}$$

If $\langle\alpha\rangle \le \langle\beta\rangle$ and $\langle\gamma\rangle \le \langle\delta\rangle$, by (29) we have (36) thanks to $\langle\beta\rangle = \langle\alpha\rangle + \langle\beta-\alpha\rangle$, etc. If $\langle\alpha\rangle > \langle\beta\rangle$ and $\langle\gamma\rangle \le \langle\delta\rangle$, by the definition of $\widetilde{V}$, we have

$$\widetilde{V}(\langle\alpha\rangle,\langle\beta\rangle,\langle\gamma\rangle,\langle\delta\rangle) = -\widetilde{V}(\langle\beta\rangle,\langle\alpha\rangle,\langle\gamma\rangle,\langle\delta\rangle) = \int_0^1 -\widetilde{\mathbf{I}}_{\langle\beta\rangle,\langle\alpha\rangle}(t)\widetilde{\mathbf{I}}_{\langle\gamma\rangle,\langle\delta\rangle}(t)\,dt.$$

Here we have $-\widetilde{\mathbf{I}}_{\langle\beta\rangle,\langle\alpha\rangle} = \widetilde{\mathbf{I}}_{\langle\alpha\rangle,\langle\beta\rangle+1} = \widetilde{\mathbf{I}}_{\langle\alpha\rangle,\langle\alpha\rangle+\langle\beta-\alpha\rangle}$ and hence we have (36). The other cases can be proved in the same way.

By changing variable $y=z+x$ and $z-\langle a-x\rangle=-w$ in turn, we have

$$\begin{aligned}
&\int_0^1 \widetilde{\mathbf{I}}_{a,b}(y)\widetilde{\mathbf{I}}_{a,b}(2y-x)\,dy=\int_0^1 \widetilde{\mathbf{I}}_{a-x,b-x}(z)\widetilde{\mathbf{I}}_{a-x,b-x}(2z)\,dz\\
&=\frac{1}{2}\widetilde{V}(\langle a-x\rangle,\langle b-x\rangle,\langle 2a-2x\rangle,\langle 2b-2x\rangle) \qquad (37)\\
&=\frac{1}{2}\int_0^1 \widetilde{\mathbf{I}}_{\langle a-x\rangle,\langle a-x\rangle+\langle b-a\rangle}(z)\widetilde{\mathbf{I}}_{\langle 2a-2x\rangle,\langle 2a-2x\rangle+\langle 2(b-a)\rangle}(z)\,dz\\
&=\frac{1}{2}\int_0^1 \widetilde{\mathbf{I}}_{0,\langle b-a\rangle}(-w)\widetilde{\mathbf{I}}_{\langle a-x\rangle,\langle a-x\rangle+\langle 2(b-a)\rangle}(-w)\,dw \qquad (38)\\
&=-\frac{1}{2}\int_0^1 \widetilde{\mathbf{I}}_{0,1-\langle b-a\rangle}(w)\widetilde{\mathbf{I}}_{0,\langle 2(b-a)\rangle}(x-a-w)\,dw,
\end{aligned}$$

where we used $\widetilde{\mathbf{I}}_{0,c}(-w)=-\widetilde{\mathbf{I}}_{0,1-c}(w)$ a.e. Since $\int_0^1 \widetilde{\mathbf{I}}^2_{a,b}(y)\,dy=(b-a)(1-(b-a))$ is clear, we complete the proof.

Let us consider a modification of this example. That is the sequence given by $n_{2k-1}=2^{k^2}-1$ and $n_{2k}=2^{k^2+1}-1$, which is a subsequence of Erdős-Fortet sequence. By putting $F(x,y)=\widetilde{\mathbf{I}}_{a,b;d}(y-x)+\widetilde{\mathbf{I}}_{a,b;d}(2y-x)$, we have (34) with

$$\begin{aligned}
\sigma^2(x)&=\int_0^1 \widetilde{\mathbf{I}}^2_{a,b}(y)\,dy+\int_0^1 \widetilde{\mathbf{I}}_{a,b}(y-x)\widetilde{\mathbf{I}}_{a,b}(2y-x)\,dy\\
&=\int_0^1 \widetilde{\mathbf{I}}^2_{a,b}(y)\,dy+\int_0^1 \widetilde{\mathbf{I}}_{a,b}(y)\widetilde{\mathbf{I}}_{a,b}(2y+x)\,dy.
\end{aligned}$$

Hence we have the same $\Sigma^*(x)$ function as Aistleitner's case.

Let us consider the second example given by Aistleitner [3].

$$n_k=\begin{cases}2^{k^2} & k=1 \mod 4,\\ 2^{(k-1)^2+1}-1 & k=2 \mod 4,\\ 2^{k^2+k} & k=3 \mod 4,\\ 2^{(k-1)^2+(k-1)+1}-2 & k=0 \mod 4.\end{cases} \qquad (39)$$

For this sequence, Aistleitner [3] proved (34) for

$$\begin{aligned}
\sigma^2(x)=(b-a)(1-(b-a))&+\frac{1}{4}\int_0^1 \mathbf{I}_{1-b,1-a}(t)\mathbf{I}_{\langle 2a\rangle,\langle 2b\rangle}(x-t)\,dt\\
&+\frac{1}{4}\int_0^1 \mathbf{I}_{1-b,1-a}(t)\mathbf{I}_{\langle 2a\rangle,\langle 2b\rangle}(\langle 2x\rangle-t)\,dt.
\end{aligned}$$

In the same way as above, we can prove (34) for

$$\sigma^2(x)=\int_0^1 \widetilde{\mathbf{I}}^2_{a,b}(y)\,dy+\frac{1}{2}\int_0^1 \widetilde{\mathbf{I}}_{a,b}(y)\widetilde{\mathbf{I}}_{a,b}(2y-x)\,dy+\frac{1}{2}\int_0^1 \widetilde{\mathbf{I}}_{a,b}(y)\widetilde{\mathbf{I}}_{a,b}(2y-2x)\,dy.$$

We prove the coincidence of these two expressions. As to the second term, we see that it equals to the half of (38) which can be written as

$$\frac{1}{4}\int_0^1 \widetilde{\mathbf{I}}_{a,b}(-w)\widetilde{\mathbf{I}}_{\langle a-x\rangle+a,\langle a-x\rangle+\langle 2(b-a)\rangle+a}(-w)\,dw.$$

By $\widetilde{\mathbf{I}}_{a,b}(-w) = \widetilde{\mathbf{I}}_{1-b,1-a}(w) = \mathbf{I}_{1-b,1-a}(w)$ and $\widetilde{\mathbf{I}}_{\langle a-x\rangle+a,\langle a-x\rangle+\langle 2(b-a)\rangle+a}(-w) = \widetilde{\mathbf{I}}_{2a,2b}(x-w) = \mathbf{I}_{2a,2b}(x-w)$, we see that it equals to the second term of Aistleitner's expression. Replacing x by $2x$, we see that the third terms of both expressions are identical.

4 M. Kac's expression of limiting variance functions

As to Theorem 2, Kac [11] gave an expression of $\varsigma(x)$ by mean of the Fourier coefficient of f when $\theta = 2$ and f is an even function. Here we give a full general expression $\varsigma(x)$ in the same manner.

We can easily see that $\varsigma(x) = \sigma(f_x, \theta)$ where $f_x(t) = f(t-x)$. In case when (6) is satisfied, we trivially have

$$\varsigma^2(x) = \int_0^1 f^2(t)\,dt = \frac{1}{2}\sum_{\nu=1}^{\infty}(a_\nu^2 + b_\nu^2)$$

where a_ν and b_ν are the Fourier coefficients of f, i.e.,

$$f(t) = \sum_{\nu=1}^{\infty}(a_\nu \cos 2\pi\nu t + b_\nu \sin 2\pi\nu t).$$

Let us consider the case when θ is given by (7). Because of

$$\begin{aligned}\widehat{f_x}(\nu) &= \widehat{f}(\nu)\exp(-2\pi\sqrt{-1}\,\nu x)\\ &= \frac{1}{2}(a_\nu \cos 2\pi\nu x - b_\nu \sin 2\pi\nu x) - \frac{\sqrt{-1}}{2}(a_\nu \sin 2\pi\nu x + b_\nu \cos 2\pi\nu x),\end{aligned}$$

By applying (11), we have

$$\begin{aligned}\varsigma^2(x) = \frac{1}{2}\sum_{j\in P_{pq}}\sum_{i=0}^{\infty}\Bigg\{&\Bigg(\sum_{s=0}^{i}(a_{jp^sq^{i-s}}\cos 2\pi jp^sq^{i-s}x - b_{jp^sq^{i-s}}\sin 2\pi jp^sq^{i-s}x)\Bigg)^2\\ &+\Bigg(\sum_{s=0}^{i}(a_{jp^sq^{i-s}}\sin 2\pi jp^sq^{i-s}x + b_{jp^sq^{i-s}}\cos 2\pi jp^sq^{i-s}x)\Bigg)^2\Bigg\}\end{aligned}$$

if $q > 1$, and

$$\begin{aligned}\varsigma^2(x) = \frac{1}{2}\sum_{j\in P_p}\Bigg\{&\Bigg(\sum_{s=0}^{\infty}(a_{jp^s}\cos 2\pi jp^sx - b_{jp^sq^{i-s}}\sin 2\pi jp^sx)\Bigg)^2\\ &+\Bigg(\sum_{s=0}^{\infty}(a_{jp^s}\sin 2\pi jp^sx + b_{jp^s}\cos 2\pi jp^sx)\Bigg)^2\Bigg\}\end{aligned}$$

if $q = 1$. The last expression coincides with Kac's expression when $p = 2$ and $b_\nu = 0$.

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