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Riesz-Raikov sums and Weyl transform

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Abstract — We investigate the dependence of stationary sequence which is obtained from the Riesz-Raikov sum by applying the Weyl transform. One of the results solves Sugita's conjecture on quasi - Monte Carlo methods.

1. Introduction

The Rademacher functions $\{r_i\}$ can be regarded as i.i.d. on the Lebesgue probability space $\big(\Omega := [0,1), P(d\omega) := d\omega\big)$. Let us recall that r_i is a function on $\mathbf{R}$ with period 1 defined by

$$r_i(\omega) := r_1(2^{i-1}\omega) \quad \text{and} \quad r_1(\omega) := \mathbf{1}_{[0,1/2)}(\omega) - \mathbf{1}_{[1/2,1)}(\omega) \quad (\omega \in \Omega).$$

By putting $S_m := \sum_{i=1}^{m} r_i$, we have the simple random walk. Thanks to the quasi-Monte Carlo method, or by virtue of the equi-distribution theorem of Weyl, we can evaluate the probability $P(S_m = a)$ in the following way: for all $\omega_0 \in \Omega$ and for any choice of $\alpha \notin \mathbf{Q}$, it holds that

$$\begin{aligned} P(S_m = a) &= \int_0^1 \mathbf{1}_{\{S_m = a\}}(\omega) d\omega \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{\{S_m=a\}}(\omega_0 + n\alpha). \end{aligned} \tag{1.1}$$

It is well known that if α is an algebraic irrational, the speed of convergence in (1.1) behaves like $O(N^{-1+\varepsilon})$ for all $\varepsilon > 0$ (Cf. Theorem 3.2 and Example 3.1 of [14]), and it is faster than the rate $O(N^{-1/2+\varepsilon})$ of Monte Carlo method.

According to Sugita [17], numerical experiments do not confirm this advantage when m is large. He conjectured that the sequence

$$S_m(\omega),\ S_m(\omega+\alpha),\ S_m(\omega+2\alpha),\ldots$$

is nearly independent when m is large, and thereby the observed rate of convergence is low. Related to the conjecture, he proved that the correlation $R_\alpha^{(m)}(n)$ of the stationary sequence

$$\mathbf{X}_\alpha^{(m)} = \{S_m(\omega)/\sqrt{m}, S_m(\omega+\alpha)/\sqrt{m}, S_m(\omega+2\alpha)/\sqrt{m},\ldots\}$$

goes to 0 and its order is of $O(m^{-1/2+\varepsilon})$ for every n, $\varepsilon > 0$, and for almost all α. As to this conjecture the author [8] proved the following results:

1. For almost every α the stationary sequence $\mathbf{X}_\alpha^{(m)}$ converges to a Gaussian i.i.d. as $m \to \infty$, in the sense of convergence of every finite dimensional distributions;
2. For almost all α and every n, it holds that
$$\limsup_{m\to\infty} \sqrt{\frac{m}{\log\log m}} R_\alpha^{(m)}(n) = \sqrt{\frac{2}{3}};$$
3. The Hausdorff dimension of the set of α for which the result (1) does not hold is 1.

Although the result (1) solves the conjecture affirmatively for almost all α, the result (3) claims that the exceptional set is also large. It is one of the aim of this paper to study on the limit behavior of the stationary sequence for exceptional α. Typical results are as follows:

4. For every α, the sequence $\mathbf{X}^{(m)}$ is relatively compact and the cluster point is a stationary gaussian sequence;
5. The Hausdorff dimension of the set of α for which the limit of $\mathbf{X}^{(m)}$ is dependent is equal to 1.

To prove these results, we regard $\mathbf{X}^{(m)}$ as a special case of the Riesz-Raikov sum $\sum f(\theta^k\omega)$ and use the methods of gap theorems.

2. Main results

Let $\theta > 1$, $\alpha \in \mathbf{R}$, and let f be a function on $\mathbf{R}$ with period 1 such that

$$\int_0^1 f(t)\,dt = 0 \quad \text{and} \quad 0 < \int_0^1 \left|f(t)\right|^2 dt < \infty. \tag{2.1}$$

Let us put

$$X^{(m)}(\omega) := \frac{1}{\sqrt{m}} \sum_{k=1}^{m} f(\theta^{k-1}\omega),$$

and define a sequence $\mathbf{X}_\alpha^{(m)}$ by applying the Weyl transform to $X^{(m)}$:

$$\mathbf{X}_\alpha^{(m)} := \left\{X_{\alpha;n}^{(m)}\right\}_{n\in\mathbf{Z}} \quad \text{where} \quad X_{\alpha;n}^{(m)}(\omega) := X^{(m)}(\omega + n\alpha)\,.$$

If we put $\theta := 2$ and $f = r_1$, $X^{(m)}$ is a normalized sum of Rademacher functions and the results listed in the Introduction follow from the results formulated below (Theorems 1 – 5).

Let us recall the next central limit theorem for $X^{(m)}$. We denote by $\|\cdot\|_2$ the $L^2[0,1]$-norm. For any $f \in L^2[0,1]$, let us denote its Fourier coefficients by $\widehat{f}(n)$ and an h-th subsum of the Fourier series by s_h, i.e.,

$$f(t) \sim \sum_{j\in\mathbf{Z}} \widehat{f}(j) e^{2\pi\sqrt{-1}jt} \quad \text{and} \quad s_h(t) = \sum_{|j|\le h} \widehat{f}(j) e^{2\pi\sqrt{-1}jt}.$$

Theorem A. *Let $\theta > 1$ and assume that f satisfies* (2.1) *and*

$$\sum_{k=0}^{\infty} \left\| s_{2^{k+1}} - s_{2^k} \right\|_2 < \infty. \tag{2.2}$$

Then for any measurable $\Omega \subset [0,1)$ with $|\Omega| > 0$,

$$X^{(m)} \xrightarrow{\mathcal{D}} N(0,v)$$

holds on $\left(\Omega, d\omega/|\Omega|\right)$. The limiting variance $v = v_{f,\theta}$ is determined as below:

If

$$\theta^r \notin \mathbf{Q} \qquad (r \in \mathbf{N}) \tag{2.3}$$

then $v = \|f\|_2^2$; Otherwise, by using p and q given by

$$\theta^r = \frac{p}{q} \text{ where } r = \min\{ n \in \mathbf{N} \mid \theta^n \in \mathbf{Q} \},\ p, q \in \mathbf{N} \text{ and } \gcd(p,q) = 1, \tag{2.4}$$

we put

$$v = \|f\|_2^2 + 2\sum_{k=1}^{\infty} \int_0^1 f(p^k t) f(q^k t)\, dt < \infty.$$

Now we are in a position to state our results.

Theorem 2.1. *Suppose that a function f with period 1 satisfies* (2.1) *and* (2.2). *Then for all $\Omega \subset [0,1)$ with $|\Omega| > 0$, $\theta > 1$ and $\alpha \in \mathbf{R}$, on probability space $(\Omega, d\omega/|\Omega|)$, the sequence $\{\mathbf{X}_\alpha^{(m)}\}_{m\in\mathbf{N}}$ is relatively compact in the sense of convergence in distribution on $\mathbf{R}^{\mathbf{Z}}$, and all the possible limit is stationary Gaussian sequence, i.e., for any subsequence of $\{\mathbf{X}_\alpha^{(m)}\}_{m\in\mathbf{N}}$ there exists some subsequence which converges to stationary gaussian sequence. This subsequence can be taken independently of Ω and f, and the limit distribution does not depend on Ω.*

Corollary 2.1. *Let the conditions of Theorem 1 are satisfied and $\{m_j\}_{j\in\mathbf{N}}$ be a subsequence of $\mathbf{N}$. If $\{\mathbf{X}_\alpha^{(m_j)}\}_{j\in\mathbf{N}}$ converges to some gaussian sequence on some $\Omega \subset [0,1)$ with $|\Omega| > 0$, then converges to the same limit on any Ω with $|\Omega| > 0$.*

Next we give the equivalence between convergence in distribution and convergence of correlation.

Theorem 2.2. *Suppose that f satisfies* (2.1) *and* (2.2). *Let $\{m_j\}_{j\in\mathbf{N}}$ be a subsequence of $\mathbf{N}$ and $\Omega \subset [0,1)$ satisfy $|\Omega| > 0$. For any n and $l \in \mathbf{Z}$, the convergence of correlation*

$$\lim_{j\to\infty} E\big(X_{\alpha;n}^{(m_j)} X_{\alpha;l}^{(m_j)}\big) = r_{n,l} \tag{2.5}$$

holds if and only if $\mathbf{X}^{(m_j)}$ converges as $j \to \infty$ to the stationary gaussian sequence with mean vector $\mathbf{0}$ and correlation matrix $\{r_{n,l}\}_{n,l\in\mathbf{Z}}$.

Theorem 2.3. *Suppose that f satisfies* (2.1) *and* (2.2). *For almost all α with respect to the Lebesgue measure, the sequence $\mathbf{X}_\alpha^{(m)}$ converges in distribution to gaussian i. i. d. Exceptional set of α can be taken to be independent of Ω and f.*

Now let us confine ourselves to the case when $\theta > 1$ is an integer. As to the fluctuation of correlation, we have the following result.

Theorem 2.4. *Let $\Omega = [0,1)$, $\theta > 1$ be an integer and f satisfies* (2.1), *$v = v_{f,\theta} > 0$ and*

$$\sum_{k=0}^{\infty} k \big\| s_{2^{k+1}} - s_{2^k} \big\|_2 < \infty. \tag{2.6}$$

Then the correlation on $(\Omega, d|\Omega|)$ obeys

$$\limsup_{m\to\infty} \sqrt{\frac{m}{\log\log m}} E\big(X_{\alpha;n}^{(m_j)} X_{\alpha;l}^{(m_j)}\big) = \beta \quad a.e. \quad \alpha$$

for some positive constant β.

The next result claim that the exceptional set of α is not small.

Theorem 2.5. *Let $\theta > 1$ be an integer and assume that f satisfies* (2.1), *$v = v_{f,\theta} > 0$ and*

$$\sum_{k=0}^{\infty} 2^{k\gamma} \big\| s_{2^{k+1}} - s_{2^k} \big\|_2 < \infty \quad \text{for some} \quad \gamma > 0. \tag{2.7}$$

The set of α for which the limit distribution of $\mathbf{X}_\alpha^{(m)}$ is dependent has the Hausdorff dimension equal to 1.

Before closing this section, we explain the regularity conditions (2.2), (2.6) and (2.7). Let us define the L^2-modulus of continuity $\omega_2(\delta)$ of a function f by

$$\omega_2(\delta) := \sup_{|h|\le\delta} \left(\int_0^1 \big|f(t+h) - f(t)\big|^2 \, dt \right)^{1/2}.$$

The conditions (2.2), (2.6) or (2.7) are derived from

$$\int_0^1 \frac{\omega_2(y)}{y} \, dy < \infty, \quad \int_0^1 \frac{\omega_2(y) \log 1/y}{y} \, dy < \infty \quad \text{or} \quad \int_0^1 \frac{\omega_2(y)}{y^{1-\gamma}} \, dy < \infty, \tag{2.8}$$

respectively. (Cf. Zygmund [20], (3.3) of pp. 241). It is clear that these conditions are satisfied for Hölder continuous functions and functions of bounded variation.

3. Preliminaries

Let us introduce some notation. For a function satisfying (2.2), let us put

$$\Delta(x,y) := \|s_{[x]} - s_{[y]}\|_2, \Delta_\theta(k) := \Delta(\theta^k, \theta^{k+1}), \quad \text{and} \quad D_\theta(f) := \sum_{k=0}^{\infty} \Delta_\theta(k).$$

Let us define a function $\widetilde{\text{sgn}}(s)$ by

$$\widetilde{\text{sgn}}(s) := \begin{cases} 1 & s \geq 0 \\ -1 & s < 0. \end{cases}$$

Let B^2 be a class of B^2-almost periodic functions. As to the definition of this notion, we refer the reader to Besicovich [2]. Here, we summarize the properties of B^2 which will be used in this paper. The proof of these are also found in [2].

All locally square integrable periodic function belong to B^2 and the Fourier series as a function of B^2 coincides with the ordinary Fourier series as a periodic function. The set B^2 is a linear space and the Fourier series of linear combination of functions of B^2 is a linear combination of the Fourier series. For arbitrary functions f and g of B^2, the inner product

$$M_t\big(f(t)g(t)\big) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)g(t)\,dt$$

exists and is finite. The Parseval relation holds, i.e., if

$$f(t) \sim \sum c_j \exp\big(\sqrt{-1}\lambda_j t\big) \quad \text{and} \quad g(t) \sim \sum d_j \exp\big(\sqrt{-1}\lambda_j t\big)$$

where λ_j's are different from each other, then

$$M_t\big(f(t)g(t)\big) = \sum c_j \bar{d}_j.$$

The following theorem due to Salem and Zygmund plays an important role in this paper.

Theorem B. *Suppose that a sequence $\{\lambda_k\}$ of positive numbers satisfies the Hadamard gap condition:*

$$\lambda_{k+1}/\lambda_k > q > 1 \quad \textit{for all} \quad k \in \mathbf{N},$$

and assume that an array $\{c_{k,m}\}_{|k|\leq k_m, m\in\mathbf{N}}$ of complex numbers satisfies

$$\bar{c}_{k,m} = c_{-k,m}, C_m = \Big(\sum_{|k|\le k_m} |c_{k,m}|^2\Big)^{1/2} \to \infty \quad \textit{and} \quad \max_{|k|\le k_m} |c_{k,m}| = o(C_m).$$

If we put

$$S_m := \sum_{|k|\le k_m} c_{k,m} \exp(2\pi\sqrt{-1}\lambda_k \omega),$$

then for any $\Omega \subset [0,1)$ *with* $|\Omega| > 0$, *the law of* S_m/C_m *on* $(\Omega, d\omega/|\Omega|)$ *converges weakly to the standard normal distribution.* We note that this normalizing sequence C_m satisfies

$$C_m = M_t\big(S_m^2(t)\big)^{1/2}. \tag{3.1}$$

The proof of the next lemma can be found in Fukuyama [9].

Lemma 3.1. (1) *Suppose that a sequence* $\{\lambda_j\}_{j\in\mathbf{N}}$ *of real numbers greater than* 1 *satisfies the following Hadamard's gap condition*

$$\lambda_{j+1}/\lambda_j \ge q > 1 \quad \textit{for all} \quad j \in \mathbf{N}.$$

Let us put $\lambda_0 = 0$, $\lambda_{-j} = -\lambda_j$ *for* $j \in \mathbf{N}$. *Then there exists an absolute constant* $K_{n,q}$ *such that, for all* $n \in \mathbf{N}$ *and* $\{c_j\}_{j\in\mathbf{Z}} \in \ell^2$ *with* $c_{-j} = \bar{c}_j$, *it holds that*

$$\int_0^1 \Big(\sum_{j\in\mathbf{Z}} c_j e^{2\pi\sqrt{-1}\lambda_j t}\Big)^{2n} dt \le K_{n,q}\Big(\sum_{j\in\mathbf{Z}} |c_j|^2\Big)^n. \tag{3.2}$$

(2) *Suppose that* f *satisfies* (2.2). *Then for any* $\theta > 1$ *and* $m \in \mathbf{N}$,

$$\int_0^1 \Big(\sum_{k=1}^m f(\theta^k t)\Big)^2 dt \le CmD_\theta^2(f) \tag{3.3}$$

for some $C > 0$ *which depends only on* θ.

Note that the right hand side of (3.2) coincides with

$$M_t\bigg(\Big(\sum_{j\in\mathbf{Z}} c_j e^{2\pi\sqrt{-1}\lambda_j t}\Big)^2\bigg)^n.$$

4. The central limit theorem

We first prove

Lemma 4.1. *Let $\{m_j\}$ be a subsequence of* $\mathbf{N}$ *and μ_i be the limit distribution of $\{\langle\theta^{k-1}i\alpha\rangle\}_{k\in\mathbf{N}}$ along the sequence $\{m_j\}$, i.e., μ_i is a probability measure on $[0,1)$ such that*

$$\frac{1}{m_j}\sum_{k=1}^{m_j}\delta_{\langle\theta^{k-1}i\alpha\rangle}\xrightarrow{w}\mu_i \quad as \quad j\to\infty, \tag{4.1}$$

where δ_x denotes the delta measure with the mass at x and $\xrightarrow{w}$ denotes the weak convergence of measures. Then for all n and l with $l=n+i$, it holds that

$$\begin{aligned}R^{(m_\infty)}_{\alpha;n,l} &:= \lim_{j\to\infty}M_t\Big(X^{(m_j)}_{\alpha;n}(t)X^{(m_j)}_{\alpha;l}(t)\Big)\\ &= \sum_{s\in\mathbf{Z}}\int_0^1 M_t\Big(f(\theta^s t)f(t+\alpha)\Big)\,\mu_i(d\alpha) \qquad (4.2)\\ &= \begin{cases}\displaystyle\int_0^1\mu_i(d\alpha)\int_0^1 f(t)f(t+\alpha)dt & \text{if (2.3) holds,}\\ \displaystyle\sum_{s\in\mathbf{Z}}\int_0^1\mu_i(d\alpha)\int_0^1 f\Big(p^{|s|}t\Big)f\Big(q^{|s|}(t+\widetilde{\mathrm{sgn}}(s)\alpha)\Big)dt & \text{if (2.4) holds.}\end{cases}\end{aligned}$$

The terms in the last series obey the estimate

$$\left|\int_0^1\mu_i(d\alpha)\int_0^1 f\left(p^{|s|}t\right)f\left(q^{|s|}(t+\widetilde{\mathrm{sgn}}(s)\alpha)\right)dt\right|\le\sum_{u=0}^{\infty}\Delta_\theta(u)\Delta_\theta(u+s) \tag{4.3}$$

where the right-hand side is summable in s.

Proof. To simplify the notation, let us consider the case $\{m_j\}=\mathbf{N}$. There is no loss of generality if we put $i=1$ since the next relation is clear:

$$M_t\big(X^{(m_j)}_{\alpha;n}(t)X^{(m_j)}_{\alpha;l}(t)\big)=M_t\big(X^{(m_j)}_{i\alpha;n}(t)X^{(m_j)}_{i\alpha;n+1}(t)\big).$$

If $\theta^s \notin \mathbf{Q}$, the Fourier series of $f(\theta^s t)$ and $f(t+\alpha)$ have no common frequencies. Hence, by the Parseval relation for B^2-almost periodic functions, we have

$$M_t\Big(f\big(\theta^{s+k}(t+n\alpha)\big)f\big(\theta^k(t+(n+1)\alpha)\big)\Big) = M_t\big(f(\theta^s t)f(t+\alpha)\big) = 0.$$

In the case when $\theta^s = p/q$, we have

$$\begin{aligned} M_t\Big(f\big(\theta^{s+k}(t+n\alpha)\big)f\big(\theta^k(t+(n+1)\alpha)\big)\Big) &= M_t\Big(f\big(pt\big)f\big(q(t+\alpha)\big)\Big) \\ &= \int_0^1 f\big(pt\big)f\big(q(t+\alpha)\big)\,dt \end{aligned}$$

Therefore, the second equality in (4.2) is proved.

Let us prove the first equality. By simple calculation we find

$$\begin{aligned} M_t\big(X^{(m)}_{\alpha;n}(t)X^{(m)}_{\alpha;n+1}(t)\big) &= \frac{1}{m}\sum_{\substack{1\le k\le m\\ 1\le j\le m}} M_t\Big(f\big(\theta^{k-1}t\big)f\big(\theta^{j-1}(t+\alpha)\big)\Big) \\ &= \frac{1}{m}\sum_{\substack{1\le k\le m\\ 1\le j\le m}} M_t\Big(f\big(\theta^{k-j}t\big)f\big(t+\theta^{j-1}\alpha\big)\Big) \\ &= \sum_{|s|\le m-1}\frac{1}{m}\sum_{j=1\vee(1-s)}^{m\wedge(m-s)} M_t\big(f(\theta^s t)f(t+\theta^{j-1}\alpha)\big). \end{aligned}$$

If (2.3) is satisfied, the summand equals to 0 except for $s=0$. Thereby we get

$$\begin{aligned} M_t\big(X^{(m)}_{\alpha;n}(t)X^{(m)}_{\alpha;n+1}(t)\big) &= \frac{1}{m}\sum_{j=1}^{m} M_t\big(f(t)f(t+\theta^{j-1}\alpha)\big) \\ &= \frac{1}{m}\sum_{j=1}^{m}\int_0^1 f(t)f(t+\theta^{j-1}\alpha)\,dt. \end{aligned}$$

Since the last summand is continuous in α, by the definition of μ we have

$$M_t\big(X^{(m)}_{\alpha;n}(t)X^{(m)}_{\alpha;n+1}(t)\big) \to \int_0^1 \mu_1(d\alpha)\int_0^1 f(t)f(t+\alpha)\,dt \qquad (m\to\infty).$$

This proves the first equality.

Let (2.4) be satisfied. Then, since the summand equals to 0 except for s is a multiple of r, we have

$$\begin{aligned}
&M_t\big(X^{(m)}_{\alpha;n}(t)X^{(m)}_{\alpha;n+1}(t)\big)\\
&\quad= \sum_{|s|\le(m-1)/r}\frac{1}{m}\sum_{j=1\vee(1-sr)}^{m\wedge(m-sr)} M_t\big(f(\theta^{sr}t)f(t+\theta^{j-1}\alpha)\big)\\
&\quad= \sum_{|s|\le(m-1)/r}\frac{1}{m}\sum_{j=1\vee(1-sr)}^{m\wedge(m-sr)} \int_0^1 f(p^{|s|}t)f\big(q^{|s|}(t+\widetilde{\mathrm{sgn}}(s)\theta^{j-1}\alpha)\big)\,dt\\
&\quad= \sum_{|s|\le(m-1)/r} C_{m,s}, \qquad \text{(say)}.
\end{aligned}$$

By the definition of μ_1 again, we have

$$C_{m,s}\to\sum_{s\in\mathbf{Z}}\int_0^1\mu_1(d\alpha)\int_0^1 f(p^{|s|r}t)f\big(q^{|s|r}(t+\widetilde{\mathrm{sgn}}(s)\alpha)\big)\,dt, \qquad (m\to\infty).$$

Each summand in the definition of $C_{m,s}$ is bounded

$$\begin{aligned}
&\Big|\int_0^1 f(p^{|s|}t)f\big(q^{|s|}(t+\widetilde{\mathrm{sgn}}(s)\theta^{j-1}\alpha)\big)\,dt\Big|\\
&\quad\le \sum_{j\in q^{|s|}\mathbf{Z}}\Big|\widehat{f}(j)\widehat{f}(-jp^{|s|}/q^{|s|})\Big|\\
&\quad\le \sum_{u=0}^{\infty}\Big(\sum_{\theta^u<j\le\theta^{u+1}}|\widehat{f}(j)|^2\Big)^{1/2}\Big(\sum_{\theta^u<j\le\theta^{u+1}}|\widehat{f}(j\theta^{r|s|})|^2\Big)^{1/2} \qquad (4.4)\\
&\quad= \sum_{u=0}^{\infty}\Delta_\theta(u)\Delta_\theta(u+r|s|).
\end{aligned}$$

Therefore the arithmetic mean $C_{m,s}$ is also estimated by the same bound. Since this bound is summable in s

$$\begin{aligned}\sum_{s\in\mathbf{Z}}\sum_{u=0}^{\infty}\Delta_\theta(u)\Delta_\theta(u+|s|) &\leq 2\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\Delta_\theta(u)\Delta_\theta(u+s)\\ &= 2\sum_{u=0}^{\infty}\Delta_\theta(u)\sum_{s=0}^{\infty}\Delta_\theta(u+s)\\ &\leq D_\theta^2(f)<\infty.\end{aligned}$$

we conclude by Lebesgue's convergence theorem.

□

Let us prove Theorem 2.1. We first show that we can take measures μ_i on $[0,1)$ and a subsequence $\{m_j\}$ of given sequence such that (4.1) holds for all $i\in\mathbf{N}$. Since the sequence of probability measures on torus is tight, we can take a subsequence $\{m_j^{(1)}\}$ of the given sequence such that (4.1) holds for $i=1$. By the same reason, we can take $\{m_j^{(2)}\}$ out of $\{m_j^{(1)}\}$ such that (4.1) holds for $i=2$. Inductively, we can take $\{m_j^{(k)}\}$ out of $\{m_j^{(k-1)}\}$ such that (4.1) holds for $i=k$. Let us put $m_j=m_j^{(j)}$. Since $\{m_j\}$ is a subsequence of $\{m_j^{(i)}\}$ for all $i\in\mathbf{N}$, (4.1) holds for all i.

We prove that $\mathbf{X}_\alpha^{(m_j)}$ converges in law to the stationary gaussian sequence with mean 0 and correlation $R_{\alpha;n,l}^{(m_\infty)}$ given in (4.2).

Let us take $d\in\mathbf{N}$ and $\beta_n\in\mathbf{R}$ $(|n|\leq d)$ arbitrary and put

$$T_m=\sum_{|n|\leq d}\beta_n X_{\alpha;m}^{f,\theta;m}\quad\text{and}\quad v=\sum_{|n|,|l|\leq d}\beta_n\beta_l R_{\alpha;n,l}^{f,\theta;m_\infty}.$$

To show the convergence of a finite dimensional distribution, by the Cramér-Wold theorem (Theorem 7.7 of Billingsley [5]), it is sufficient to show

$$T_{m_j}\xrightarrow{\mathcal{D}}N(0,v),\qquad(j\to\infty).\tag{4.5}$$

By Lemma 4.1, we have

$$\lim_{j\to\infty}M_t\big(T_{m_j}^2(t)\big)=v.\tag{4.6}$$

We first prove (4.5) when f is a trigonometric polynomial. In this case the frequencies appearing in $S_m:=\sqrt{m}\,T_m$ have the Hadamard gaps, and coefficients are bounded by the same bound for all m. Let us put

$$C_m^2 := M_t(S_m^2(t)) = mM_t(T_m^2(t)).$$

In the case $v > 0$, by (4.6), we have $C_{m_j} \to \infty$ and we can use Theorem B to get

$$T_{m_j} = \frac{C_{m_j}}{\sqrt{m_j}} \cdot \frac{S_{m_j}}{C_{m_j}} \xrightarrow{\mathcal{D}} N(0,v), \qquad (j \to \infty).$$

In the case when $v = 0$, we have $C_{m_j} = o(\sqrt{m_j})$. By using (3.2) with $n = 1$, we find

$$\|T_{m_j}\|_2^2 = O(M_t(T_{m_j}^2)) = O(C_{m_j}^2/m_j) = o(1)$$

which implies (4.5).

Next let us prove (4.5) for f with (2.2). From now on, denoting f explicitly, we use $X_{\alpha;n}^{(f;m)}$, $\mathbf{X}_{\alpha;n}^{(f;m)}$ and $R_{\alpha;n,l}^{(f;m_\infty)}$ to denote $X_{\alpha;n}^{(m)}$, $\mathbf{X}_{\alpha;n}^{(m)}$ and $R_{\alpha;n,l}^{(m_\infty)}$, respectively.

Since $X_{\alpha;n}^{(f;m)}$ is linear in f, by (3.3), we have

$$\|X_{\alpha;n}^{(f;m_j)} - X_{\alpha;n}^{(s_h;m_j)}\|_2 = \|X_{\alpha;n}^{(f-s_h;m_j)}\|_2 \leq C^{1/2}D_\theta(f - s_h) \to 0 \quad (h \to \infty).$$

Thus, when h is large, any finite dimensional distribution of $\mathbf{X}_\alpha^{(f;m_j)}$ is approximated by $\mathbf{X}_\alpha^{(s_h;m_j)}$ uniformly in j. Therefore the limit distribution is also approximated. Thus the proof ends if we prove

$$\lim_{h\to\infty} R_{\alpha;n,l}^{(s_h;m_\infty)} = R_{\alpha;n,l}^{(f;m_\infty)}. \tag{4.7}$$

Let us recall the formula (4.2). If (2.3) holds, (4.7) follows from $\|f - s_h\|_2 \to 0$. If (2.4) is satisfied, by using (4.2) we expand $R_{\alpha;n,l}^{(s_h;m_\infty)}$ and $R_{\alpha;n,l}^{(f;m_\infty)}$ into series of integrals. We can easily see the termwise convergence of these series. Note that each term of the series for $R_{\alpha;n,l}^{(s_h;m_\infty)}$ is bounded by the left hand side of (4.3) which is summable in s and independent of h. Therefore we can use the Lebesgue convergence theorem and prove (4.7). □

Next we prove the Corollary 2.1. From Theorem 2.1, we can take a subsequence $\{m'_j\}_{j\in\mathbf{N}}$ of $\{m_j\}_{j\in\mathbf{N}}$ such that $\{\mathbf{X}_\alpha^{(m'_j)}\}_{j\in\mathbf{N}}$ converges on all $\Omega \subset [0,1)$ with $|\Omega| > 0$. Since this limit does not depend on Ω, our conclusion follows. □

Let us proceed with the proof of Theorem 2.2. Let us take $\Omega \subset [0,1)$ with $|\Omega| > 0$. On this space we have generally, $E|Y|^2 \leq \|Y\|_2^2/|\Omega|$.

First we prove that the convergence in law implies the convergence of correlation. If f is a trigonometric polynomial, by (3.2), we have

$$E(X_{\alpha;n}^{(m)})^4 \leq K\Big(M_t(X_{\alpha;n}^{(m)})^2\Big)^2/|\Omega| = O(1).$$

Therefore the second moment of $\{X_{\alpha;n}^{(m)}X_{\alpha;l}^{(m)}\}_{m\in\mathrm{N}}$ is bounded, and hence it is uniformly integrable. This together with the convergence in law implies (Cf. Theorem 5.4 in Billingsley [5].)

$$\lim_{m\to\infty} E(X_{\alpha;n}^{(f;m)}X_{\alpha;l}^{(f;m)}) = R_{\alpha;n,l}^{(f;\infty)} := \lim_{m\to\infty} M_t(X_{\alpha;n}^{(f;m)}(t)X_{\alpha;l}^{(f;m)}(t)). \quad (4.8)$$

For f with (2.2), we prove this by showing that from any subsequence $\{m_j\}_{j\in\mathrm{N}}$ we can choose a subsequence $\{m_j'\}_{j\in\mathrm{N}}$ such that the convergence of correlation holds. By Theorem 2.1, we can take $\{m_j'\}_{j\in\mathrm{N}}$ such that $\{\mathbf{X}_\alpha^{(f;m_j')}\}_{j\in\mathrm{N}}$ converges in law for all f. Applying

$$\begin{aligned}&|EY_1Z_1 - EY_2Z_2| \\ &\quad\leq (\|Y_1 - Y_2\|_2\|Z_1 - Z_2\|_2 + \|Y_1\|_2\|Z_1 - Z_2\|_2 + \|Y_1 - Y_2\|_2\|Z_1\|_2)/|\Omega|,\end{aligned}$$

and (3.3) we find

$$\begin{aligned}&|EX_{\alpha;n}^{(f;m_j')}X_{\alpha;l}^{(f;m_j')} - EX_{\alpha;n}^{(s_h;m_j')}X_{\alpha;l}^{(s_h;m_j')}| \\ &\quad\leq C\Big(D_\theta^2(f - s_h) + 2D_\theta(f - s_n)D_\theta(f)\Big) \to 0, \qquad (h\to\infty).\end{aligned}$$

From this, combining (4.7), (4.8) we see that (4.7) holds for our f and the subsequence $\{m_j'\}$.

Let us prove the converse part of the Corollary 2.1. Suppose that $\{m_j\}$ is an arbitrary sequence. Thanks to Theorem 2.1 we can choose a subsequence $\{m_j'\}$ for which $\{\mathbf{X}_\alpha^{(f;m_j')}\}_{j\in\mathrm{N}}$ converges in law. By using the above result, we see that the correlation $EX_{\alpha;n}^{(f;m_j')}X_{\alpha;l}^{(f;m_j')}$ converges to the correlation of the limit distribution. By (2.5), the limit distribution of this subsequence is stationary gaussian with correlation $r_{n,l}$. This proves that $\{\mathbf{X}_\alpha^{(f;m)}\}_{m\in\mathrm{N}}$ converges in law to this distribution. □

Finally, we prove Theorem 2.3. By equidistribution theorem of Weyl, (Cf. Erdős-Taylor [6]), the sequence $\{\langle\theta^{k-1}i\alpha\rangle\}_{k\in\mathrm{N}}$ is distributed over $[0,1)$ for almost all α. Thus for almost all α, μ_i of (4.1) is a Lebesgue measure and (4.2) shows that the limiting correlation is 0. □

5. Exceptional set

Let $p \in [0,1]$ and $\{Y_i\}_{i\in\mathbf{N}}$ be a Bernoulli i.i.d. such that

$$P\{Y_i = 0\} = 1 - p \quad \text{and} \quad P\{Y_i = 1\} = p.$$

Let us denote by ν_p the law of $\sum_{i=1}^{\infty} Y_i/2^i$. Since the binary transform is ergodic under this measure, the limit distribution in a sense of Lemma 4.1 of $\{2^k\alpha\}_{k\in\mathbf{N}}$ is ν_p for almost all α with respect to ν_p. By the proof of Theorem 2.1, we see that the correlation of limit distribution is given by (4.2). Let us denote the correlation with $i = 1$ by V_p. By (4.2) and from the above consideration, we have

$$V_p = \sum_{s\in\mathbf{Z}} \int_0^1 \nu_p(d\alpha) \int_0^1 f(2^{|s|}t) f(t + \widetilde{\operatorname{sgn}}(s)\alpha)\, dt$$

$$= \sum_{s\in\mathbf{Z}} \sum_{j\in\mathbf{Z}} \overline{\widehat{f}(2^{|s|}j)} \widehat{f}(j) \widehat{\nu}_p(\widetilde{\operatorname{sgn}}(s) 2^{|s|} j)$$

where $\widehat{\nu}$ denotes the characteristic function of ν. Since $\nu_{1/2}$ is a Lebesgue measure, by the proof of Theorem 3, we have $V_{1/2} = 0$. Since ν_0 and ν_1 are delta measures, we can easily prove that $V_0 = V_1 = 1$. We prove that V_p is continuous in p on $[0,1]$ and it is a real analytic function on $(0,1)$. Since V_p is not a constant, there is no zero in some neighbourhood of $1/2$. Thus for $p \neq 1/2$ of this neighbourhood, the limit correlation is not zero for almost every α with respect to ν_p. In other words, the exceptional set L has ν_p-full measure for any $p \neq 1/2$ sufficiently close to $1/2$. By using the following theorem due to Billingsley (Billingsley [4], p. 141 onwards), we have $\dim L \geq e_p$. Noting $e_p \to 1$ as $p \to 1/2$, we have the conclusion.

Theorem C. *The Hausdorff dimension of $A \subset [0,1)$ is equal to or greater than $e_p := -(p\log_2 p + (1-p)\log_2(1-p))$ if $\nu_p(A) > 0$.*

Let us prove the analyticity. Let us put $\widetilde{V}_z := V_{(z+1)/2}$ and $\widetilde{\nu}_z := \nu_{(z+1)/2}$, where $z \in [-1,1]$. Then,

$$\widetilde{V}_z = \sum_{s\in\mathbf{Z}} \sum_{j\in\mathbf{Z}} \overline{\widehat{f}(2^{|s|}j)} \widehat{f}(j) \widehat{\widetilde{\nu}}_z(\widetilde{\operatorname{sgn}}(s) 2^{|s|} j). \tag{5.1}$$

By the definition of ν_p we have

$$\begin{aligned}
\widehat{\nu}_p(2^k(2l+1)) &= E\Big[\exp\Big(2\pi\sqrt{-1}\,2^k(2l+1)\sum_{i=1}^{\infty}\frac{Y_i}{2^i}\Big)\Big] \\
&= E\Big[\exp\Big(2\pi\sqrt{-1}\,(2l+1)\sum_{i=1}^{\infty}\frac{Y_i}{2^i}\Big)\Big] \\
&= \exp\Big(2\pi\sqrt{-1}\,(2l+1)\sum_{i=1}^{\infty}\frac{1}{2^{i+1}}\Big) \\
&\quad\times\prod_{i=1}^{\infty}E\Big[\exp\Big(2\pi\sqrt{-1}\,(2l+1)\frac{Y_i-1/2}{2^i}\Big)\Big].
\end{aligned}$$

The last expectation can be evaluated as follows:

$$\begin{aligned}
E\quad&\Big[\exp\Big(2\pi\sqrt{-1}\,(2l+1)\frac{Y_i-1/2}{2^i}\Big)\Big] \\
&= (1-p)\exp\Big(-\frac{\pi\sqrt{-1}(2l+1)}{2^i}\Big)+p\exp\Big(\frac{\pi\sqrt{-1}(2l+1)}{2^i}\Big) \\
&= (2p-1)\sqrt{-1}\sin\frac{\pi(2l+1)}{2^i}+\cos\frac{\pi(2l+1)}{2^i}.
\end{aligned}$$

Note that when $i=1$, it reduces to $\sqrt{-1}^{2l+1}(2p-1)$. Combining these results, we get an infinite product expression for $\widehat{\nu}_z(j)$:

$$\begin{aligned}
\widetilde{\nu}_z(2^k(2l+1)) &= \sqrt{-1}^{2l+3}z\prod_{i=2}^{\infty}\Big(z\sqrt{-1}\sin\frac{\pi(2l+1)}{2^i}+\cos\frac{\pi(2l+1)}{2^i}\Big) \\
&= \sqrt{-1}^{2l+3}z\prod_{i=2}^{\infty}\cos\frac{\pi(2l+1)}{2^i}\prod_{i=2}^{\infty}\Big(1+z\sqrt{-1}\tan\frac{\pi(2l+1)}{2^i}\Big) \\
&= \sqrt{-1}^{2l+3}z\frac{\sin\pi(l+1/2)}{\pi(l+1/2)}\prod_{i=2}^{\infty}\Big(1+z\sqrt{-1}\tan\frac{\pi(2l+1)}{2^i}\Big) \\
&= -\sqrt{-1}\,z\frac{2}{\pi(2l+1)}\prod_{i=2}^{\infty}\Big(1+z\sqrt{-1}\tan\frac{\pi(2l+1)}{2^i}\Big).
\end{aligned}$$

Here we used the Viète formula $\prod_{i=1}^{\infty} \cos x/2^i = (\sin x)/x$. Since we have

$$\sum_{i=2}^{\infty} \left| \tan \frac{\pi(2l+1)}{2^i} \right| \ll \sum_{i=2}^{\infty} \left| \frac{\pi(2l+1)}{2^i} \right| < \infty,$$

the infinite product converges for all $z \in \mathbf{C}$ and hence an analytic function is defined. In this way we extend $\widetilde{\nu}_z(j)$ to the whole plane.

From now on, we prove that the series (5.1) converges uniformly on

$$D := \{ z \in \mathbf{C} \mid |z| \le 1, |\mathrm{Im} z| \le H \},$$

where $H = (2^\gamma - 1)/2$. From this our conclusion follows.

First let us put $L(x) = \big(\log_2(2x+1)\big) / 2$ and prove that

$$\left|\widetilde{\nu}_z(j)\right| \le 2e^2 j^{\gamma/2}, \quad (z \in D). \tag{5.2}$$

Let us put $n_l = \left[\log_2(2l+1)\right] + 2$ and divide $\widetilde{\nu}_z\big(2^k(2l+1)\big)$ into three parts:

$$\begin{aligned} \widetilde{\nu}_z\big(2^k(2l+1)\big) &= \sqrt{-1}^{2l+3} z \prod_{i=2}^{n_l} \left(z\sqrt{-1} \sin \frac{\pi(2l+1)}{2^i} + \cos \frac{\pi(2l+1)}{2^i} \right) \\ &\quad \times \prod_{i=n_l+1}^{\infty} \cos \frac{\pi(2l+1)}{2^i} \prod_{i=n_l+1}^{\infty} \left(1 + z\sqrt{-1} \tan \frac{\pi(2l+1)}{2^i} \right) \\ &= \sqrt{-1}^{2l+3} z \prod\nolimits_1 \times \prod\nolimits_2 \times \prod\nolimits_3 \qquad \text{(say)}. \end{aligned}$$

Clearly we have $\left|\prod_2\right| \le 1$. If $i \ge n_l + 1$, we have $\pi(2l+1)/2^i \le \pi/4$ and thereby

$$0 < \tan \frac{\pi(2l+1)}{2^i} \le \frac{2l+1}{2^{i-2}}.$$

Thus, by $|z| \le 1$ we have

$$\left|\prod\nolimits_3\right| \le \prod_{i=n_l+1}^{\infty} \left(1 + \frac{2l+1}{2^{i-2}} \right) \le \exp\left(\sum_{i=n_l+1}^{\infty} \frac{2l+1}{2^{i-2}} \right) \le \exp\left(\frac{2l+1}{2^{n_l-2}} \right) \le e^2.$$

In the case when $n_l \ge \log_2(2l+1) + 1$, denoting $z = \xi + i\eta$, we write

$$\begin{aligned}\left|\sqrt{-1}z\sin\theta+\cos\theta\right|^2 &= \left|\sqrt{-1}\zeta\sin\theta+(\cos\theta-\eta\sin\theta)\right|^2\\ &= |z|^2\sin^2\theta+\cos^2\theta-2\eta\sin\theta\cos\theta\\ &\leq 1+\eta.\end{aligned}$$

Therefore, we have

$$\left|\prod\nolimits_1\right| \leq (1+\eta)^{n_l/2} \leq (1+\eta)^{1+(\log_2(2l+1))/2} \leq (1+\eta)(2l+1)^{L(\eta)}.$$

By combining these estimates, we have

$$\left|\widetilde{\nu}_z(j)\right| \leq \left|\widetilde{\nu}_z\big(2^k(2l+1)\big)\right| \leq 2e^2(2l+1)^{L(H)} \leq 2e^2 j^{\gamma/2}$$

By (5.2) we have the next estimate, which implies the uniform convergence:

$$\begin{aligned}&\sum_{s\in\mathbf{Z}}\ \sum_{j\in\mathbf{Z}}\left|\overline{\widehat{f}(2^{|s|}j)}\widehat{f}(j)\widetilde{\nu}_z(\widetilde{\mathrm{sgn}}(s)2^{|s|}j)\right|\\ &\leq 4e^2\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\sum_{2^u<|j|\leq 2^{u+1}}\left|\overline{\widehat{f}(2^sj)}\widehat{f}(j)j^{\gamma/2}\right|\\ &\leq 4e^2\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\Big(\sum_{2^u<|j|\leq 2^{u+1}}|\widehat{f}(2^sj)|^2\Big)^{1/2}\Big(\sum_{2^u<|j|\leq 2^{u+1}}|\widehat{f}(j)|^2\Big)^{1/2}2^{(u+1)\gamma/2}\\ &\leq 8e^2\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\Delta_2(s+u)\Delta_2(s)2^{(s+u)\gamma/2}\\ &\leq 8e^2\Big(\sum_{s=0}^{\infty}\Delta_2(s)\Big)\Big(\sum_{v=0}^{\infty}\Delta_2(v)2^{v\gamma/2}\Big)<\infty.\end{aligned}$$

6. The law of iterated logarithm

We use the next theorem due to Heyde & Scott [11].

Theorem D. *Let T be an bijective ergodic automorphism on a probability space $(\Omega, \mathcal{F}, P)$ and X_0 be a square integrable random variable. Suppose that a σ-field $\mathcal{M}_0$ satisfy $\mathcal{M}_0 \subset \mathcal{F}$ and $\mathcal{M}_0 \subset T^{-1}(\mathcal{M}_0)$. Let us put $\mathcal{M}_k := T^{-k}(\mathcal{M}_0)$, $X_k(\omega) := X_0(T^k\omega)$ and $S_n = X_1 + \ldots + X_n$. If the condition*

$$\sum_{m=0}^{\infty}\left(\left\|E(X_0 \mid \mathcal{M}_{-m})\right\|_2 + \left\|X_0 - E(X_0 \mid \mathcal{M}_m)\right\|_2\right) < \infty, \tag{6.1}$$

is satisfied, the limit

$$\sigma := \lim_{n\to\infty} \|S_n\|_2/\sqrt{n}$$

exists and is finite, and if σ is positive we have

$$\limsup_{n\to\infty} \frac{S_m}{\sqrt{2n\log\log n}} = \sigma \quad a.s.$$

From this, we derive the next corollary.

Corollary 6.1. *Let $(\Omega, \mathcal{F}, P)$ be a Lebesgue probability space. If f satisfies (2.1), the first condition of (2.8), and $\sigma > 0$, then*

$$\limsup_{m\to\infty} \frac{\sum_{k=1}^{m} f(2^{k-1}\omega)}{\sqrt{2m\log\log m}} = \sigma \quad a.s.$$

To obtain this corollary, we consider $(\widetilde{\Omega}, \widetilde{F}, \widetilde{P})$, the direct product of two Lebesgue probability spaces, and extend the binary transform to the bijective automorphism $\widetilde{T}(\omega_1, \omega_2) \mapsto (\langle 2\omega_1\rangle, (\omega_2 - [\omega_1]))$ and put $X_0(\omega_1, \omega_2) := f(\omega_1)$. Let $\mathcal{F}_n$ be a σ-field generated by $\{\, [(i-1)2^{-n}, i2^{-n}) \mid i = 1, \ldots, 2^n \,\}$. The condition (6.1) reduces to

$$\sum_{m=0}^{\infty} \|f - E(f \mid \mathcal{F}_m)\|_2 < \infty$$

and is equivalent to the first condition of (2.8). Thus we can apply Theorem D and obtain the conclusion. □

Remark 6.1. The above Corollary 6.1 holds also for the measure ν_p if we assume that

$$\int_0^1 f(\omega)\,\nu_p(d\omega) = 0 \quad \text{and} \quad \int \frac{\omega_u(h)}{h}\,dh < \infty,$$

where ω_u is the uniform modulus of continuity of f.

Now we proceed with the proof of Theorem 2.4. We use the notation

$$R^{(m)}(\alpha) := E\big(X^{(m)}_{\alpha;n}(t)X^{(m)}_{\alpha;n+1}(t)\big).$$

By the proof of Lemma 4.1, we have

$$R^{(m)}(\alpha) = \sum_{|s|\le(m-1)} \frac{1}{m} \sum_{j=1\vee(1-s)}^{m\wedge(m-s)} \int_0^1 f(2^{|s|}t)f(t+\widetilde{\operatorname{sgn}}(s)2^{j-1}\alpha)\,dt.$$

Let us put

$$R^{(m)}_*(\alpha) := \frac{1}{m}\sum_{j=1}^{m} \varphi(2^{j-1}\alpha)$$

where

$$\varphi(\alpha) := \sum_{s\in\mathbf{Z}} \int_0^1 f(2^{|s|}t)f(t+\widetilde{\operatorname{sgn}}(s)\alpha)\,dt.$$

First we prove

$$R^{(m)}(\alpha) = R^{(m)}_*(\alpha) + O\big(1/\sqrt{m}\big). \tag{6.2}$$

By (4.4), we have

$$|R^{(m)}(\alpha) - R^{(m)}_*(\alpha)| \le \frac{1}{m}\sum_{s\in\mathbf{Z}} \sum_{\substack{j=1,\dots,m\\ j+s\notin[1,m]}} \sum_{u=0}^{\infty} \Delta_2(u)\Delta_2(u+|s|). \tag{6.3}$$

Since the summand is independent of j, the sum in j reduces to the multiplication by $m \wedge s$. We divide this summation into sums over $|s| \le \sqrt{m}$ and over $|s| > \sqrt{m}$. As to the first part, we have

$$\sum_{|s|\le\sqrt{m}} \frac{s}{m} \sum_{u=0}^{\infty} \Delta_2(u)\Delta_2(u+s) \le \frac{1}{\sqrt{m}}\Big(\sum_{u=0}^{\infty}\Delta_2(u)\Big)^2 = O\big(1/\sqrt{m}\big).$$

The second part is estimated as follows:

$$\sum_{|s|>\sqrt{m}} \frac{s\wedge m}{m} \sum_{u=0}^{\infty} \Delta_2(u)\Delta_2(u+s) \le \sum_{u=0}^{\infty} \Delta_2(u) \sum_{|s|>\sqrt{m}} \Delta_2(s)$$

$$\le \frac{2}{\sqrt{m}} \sum_{u=0}^{\infty} \Delta_2(u) \sum_{|s|>\sqrt{m}} s\Delta_2(s)$$

$$= O\big(1/\sqrt{m}\big).$$

Combining these, we get (6.2).

Next we apply the Corollary 6.1 to φ and get the log log result for $R_*^{(m)}$. We verify the condition

$$\varphi(\alpha+h)-\varphi(\alpha) = o\big((\log 1/h)^{-2}\big). \tag{6.4}$$

Assume that h satisfies $2^{-m-1}/\pi \le |h| \le 2^{-m}/\pi$. Then,

$$|\varphi(\alpha+h)-\varphi(\alpha)|$$

$$\le \sum_{s\in\mathbf{Z}} \left| \int_0^1 f(2^{|s|}t)\big\{f\big(t+\widetilde{\operatorname{sgn}}(s)(\alpha+h)\big) - f\big(t+\widetilde{\operatorname{sgn}}(s)\alpha\big)\big\}\, dt \right|$$

$$\le \sum_{s\in\mathbf{Z}} \sum_{i\in\mathbf{Z}} \big|\widehat{f}(i)\widehat{f}(2^{|s|}i)\big| \big|\exp(2\pi\sqrt{-1}2^{|s|}i\widetilde{\operatorname{sgn}}(s)h) - 1\big|$$

$$\le 8 \sum_{s=0}^{\infty} \sum_{i\in\mathbf{Z}} \big|\widehat{f}(i)\widehat{f}(2^{|s|}i)\big| (1\wedge 2^{|s|-m})$$

$$\le 8 \sum_{s=0}^{\infty} \sum_{u=0}^{\infty} \Delta_2(u)\Delta_2(u+s)(1\wedge 2^{|s|-m})$$

$$\le 8\Bigg(\sum_{u=0}^{m/2} \Delta_2(u) \sum_{s=0}^{(m/2)-u} + \sum_{u>m/2} \Delta_2(u) \sum_{s>(m/2)-u}\Bigg) \Delta_2(u+s)(1\wedge 2^{|s|-m})$$

$$\le 8\Bigg(2^{-m/2}D_2^2(f) + \frac{4}{m^2} \sum_{u>m/2} u\Delta_2(u) \sum_{s>(m/2)-u} (u+s)\Delta_2(u+s)\Bigg)$$

$$= o(m^{-2}) = o\big((\log 1/h)^{-2}\big).$$

We have previously proved the following: If f satisfies (2.1) and (2.2), the limiting variance $v = v_{f,2}$ is given by

$$
\begin{aligned}
v &= \int_0^1 f^2(t)\,dt + 2\sum_{k=1}^{\infty}\int_0^1 f(t)f(2^k t)\,dt \\
&= 2\sum_{j\in P_2}\left|\sum_{s=0}^{\infty}\widehat{f}(j2^s)\right|^2,
\end{aligned}
\tag{6.5}
$$

and these series are absolutely convergent (see [9]) Here P_2 denotes the set of all positive odd numbers. By Lebesgue's convergence theorem we have

$$
\begin{aligned}
\frac{1}{m}\int_0^1\left(\sum_{k=1}^{m}(2^{k-1}t)\right)^2 dt &= \int_0^1 f^2(t)\,dt + 2\sum_{k=1}^{m-1}\frac{m-k}{m}\int_0^1 f(t)f(2^k t)\,dt \\
&\to \int_0^1 f^2(t)\,dt + 2\sum_{k=1}^{\infty}\int_0^1 f(t)f(2^k t)\,dt
\end{aligned}
$$

and thereby, σ^2 in the Corollary 6.1 above equals to the right-hand side of (6.5).

We now prove that

$$
\beta^2 = \lim_{m\to\infty}\frac{1}{m}\int_0^1\left(\sum_{k=1}^{m}\varphi(2^{k-1}t)\right)^2 dt = 2\sum_{j\in P_2}\left|\sum_{s=0}^{\infty}\widehat{f}(j2^s)\right|^4. \tag{6.6}
$$

Since we assumed that $v > 0$, by (6.5) and (6.6) we have $\beta > 0$. By this and (6.3), we can apply the Corollary 6.1 which yield

$$
\limsup_{m\to\infty}\sqrt{\frac{m}{\log\log m}}R_*^{(m)}(\alpha) = \beta \quad \text{a.e.} \quad \alpha.
$$

From this, we conclude by (6.2).

To prove (6.6) we evaluate the Fourier series of φ. By definition, we have the following expansion. As we have seen in the proof of (6.4), all the series below are absolutely convergent, and hence the order of summation can be changed arbitrarily:

$$\begin{aligned}
\varphi(\alpha) &= \sum_{s\in\mathbf{Z}}\sum_{i\in\mathbf{Z}} \overline{\widehat{f}(i)}\widehat{f}(2^{|s|}i)\exp(2\pi\sqrt{-1}2^{|s|}i\widetilde{\operatorname{sgn}}(s)\alpha)\\
&= 2\sum_{i=1}^{\infty}\Big(|\widehat{f}(i)|^2\cos(2\pi i\alpha) + 2\sum_{s=1}^{\infty}\operatorname{Re}\Big(\overline{\widehat{f}(i)}\widehat{f}(2^s i)\Big)\cos(2\pi 2^s i\alpha)\Big)\\
&= 2\sum_{l\in P_2}\sum_{u=0}^{\infty}\Big(|\widehat{f}(2^u l)|^2\cos(2\pi 2^u l\alpha)\\
&\quad + 2\sum_{s=1}^{\infty}\operatorname{Re}\Big(\overline{\widehat{f}(2^u l)}\widehat{f}(2^{u+s}l)\Big)\cos(2\pi 2^{u+s}l\alpha)\Big)\\
&= 2\sum_{l\in P_2}\sum_{v=0}^{\infty}\left(\left|\sum_{s=0}^{v}\widehat{f}(2^s l)\right|^2 - \left|\sum_{s=0}^{v-1}\widehat{f}(2^s l)\right|^2\right)\cos(2\pi 2^v l\alpha),
\end{aligned}$$

where $\sum_{s=0}^{v-1}\dots$ is 0 when $v=0$. Thus we have

$$\widehat{\varphi}(j2^v) = \left|\sum_{s=0}^{v}\widehat{f}(2^s j)\right|^2 - \left|\sum_{s=0}^{v-1}\widehat{f}(2^s j)\right|^2$$

for $s = 0, 1, \dots,$ and for $j \in \pm P_2$. By using (6.5), we have

$$\beta^2 = 2\sum_{j\in P_2}\left|\sum_{s=0}^{\infty}\widehat{\varphi}(j2^s)\right|^2 = 2\sum_{j\in P_2}\left|\sum_{s=0}^{\infty}\widehat{f}(j2^s)\right|^4.$$

□

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