Kobe University Repository : Kernel

PDF issue: 2025-12-05

The central limit theorem for Rademacher system

Fukuyama, Katusi

(Citation)
Proceedings of The Japan Academy Series A-Mathematical Sciences,70(7):243-246

(Issue Date)
1994-09

(Resource Type)
journal article

(Version)
Version of Record

(URL)
https://hdl.handle.net/20.500.14094/90003861

56. The Central Limit Theorem for Rademacher System

By Katusi FUKUYAMA

Department of Mathematics, Kobe University

(Communicated by Kiyosi ITÔ, M. J. A., Sept. 12, 1994)

1. Introduction. Let (Ω, P) be the Lebesgue probability space, i.e., $\Omega := [0,1)$ and P is Lebesgue measure. In this note, we regard any function on Ω as a function on $\boldsymbol{R}$ with period 1. The Rademacher system is a system $\{r_i\}$ of random variables on (Ω, P) defined by

$$r_1(\omega) := -1_{[0,1/2)}(\omega) + 1_{[1/2,1)}(\omega) \text{ and } r_i(\omega) := r_1(2^{i-1}\omega), \quad (i \geq 2).$$

Note that $(r_i + 1)/2$ gives the i-th digit of the dyadic expansion of a real number ω. Since $\{r_i\}$ is i.i.d., the De Moivre-Laplace theorem claims that the law of

$$X^{(m)}(\omega) := \frac{1}{\sqrt{m}} \sum_{i=1}^{m} r_i(\omega)$$

converges weakly to the standard normal distribution, as $m \to \infty$.

We put $X_n^{(m,\alpha)}(\omega) := X^{(m)}(\omega + n\alpha)$ for $n \in \boldsymbol{Z}$ and $\alpha \in \boldsymbol{R}$, and study the limit behaviour of the sequence $\{X_n^{(m,\alpha)}\}_{n\in\boldsymbol{Z}}$ as $m \to \infty$. Since this sequence is given by iterating the Weyl automorphism, it is stationary and, in most cases, dependent. Having studied the quasi-Monte Carlo method, Sugita [6] conjectured that the dependence disappears as $m \to \infty$, for almost all α. He proved that, for almost all α,

$$R^{(m,\alpha)}(k) := E(X_n^{(m,\alpha)} X_{n+k}^{(m,\alpha)}) = o(m^{-\beta}) \text{ as } m \to \infty, \ (k \in \boldsymbol{N},\ 0 < \beta < 1/2).$$

We prove the following results related to the conjecture.

Theorem 1. *For almost all α with respect to Lebesgue measure, any finite dimensional distribution of $\{X_n^{(m,\alpha)}\}_{n\in\boldsymbol{Z}}$ converges weakly to the multi-dimensional standard normal law as $m \to \infty$; i.e., for all $n \in \boldsymbol{Z}$ and $k \in \boldsymbol{N}$,*

$$(1.1) \qquad (X_n^{(m,\alpha)}, \ldots, X_{n+k-1}^{(m,\alpha)}) \xrightarrow{\mathcal{D}} N(0, I_k) \text{ as } m \to \infty.$$

Here $\xrightarrow{\mathcal{D}}$ denotes convergence in law, and I_k the k-dimensional unit matrix.

Theorem 2. *For any α, the correlation $R^{(m,\alpha)}(k) := E(X_n^{(m,\alpha)} X_{n+k}^{(m,\alpha)})$ is given by*

$$(1.2) \quad R^{(m,\alpha)}(k) = \frac{1}{m} \sum_{i=1}^{m} \varphi(2^{i-1}k\alpha), \text{ where } \varphi(x) := |4x - 2| - 1, \ (x \in \Omega).$$

Moreover, for any $k \in \boldsymbol{N}$ and for almost all $\alpha \in \boldsymbol{R}$ with respect to Lebesgue measure, it holds that

$$(1.3) \qquad \limsup_{m\to\infty} \sqrt{\frac{m}{\log\log m}}\, R^{(m,\alpha)}(k) = \sqrt{\frac{2}{3}}.$$

Theorem 3. *The Hausdorff dimension of the set of α for which* (1.1) *does not hold is* 1.

Remark. We can improve Theorem 3 as follows: The Hausdorff dimension of the set of α such that finite dimensional distribution of $\{X_n^{(m,\alpha)}\}_{n\in\boldsymbol{Z}}$ converges to that of some stationary dependent gaussian sequence is 1. The proof will be given in a forthcoming paper.

2. Proof of Theorem 1. First we prove two lemmas.

Lemma 1. *For any sequence* $\{\alpha_i\}$ *of real numbers, the sequence* $\{r'_i(\omega) := r_i(\omega+\alpha_i)\}$ *is an i.i.d. on* (Ω, P).

Proof. Clearly, we have $P(r'_i = -1) = P(r'_i = 1) = 1/2$. To prove the independence, it is sufficient to prove

(2.1) $$P(r'_1 = \varepsilon_1, \dots, r'_n = \varepsilon_n) \le \frac{1}{2^n} \quad (\varepsilon_i = \pm 1,\ i = 1, \dots, n,\ n \in \boldsymbol{N}).$$

Actually, from (2.1), we have

$$1 = \sum_{\varepsilon_i = \pm 1, (i=1,\dots,n)} P(r'_1 = \varepsilon_1, \dots, r'_n = \varepsilon_n) \le \sum \frac{1}{2^n} = 1,$$

and thereby we see that the equality in (2.1) holds.

Take $\varepsilon_i = \pm 1$ arbitrarily and put $A_i := \{\omega \in \Omega : r'_i(\omega) = \varepsilon_i\}$. The next property of A_i is easily verified: If $\omega, \omega' \in \Omega$ satisfy $\omega \equiv \omega' + (2j+1)/2^i \pmod 1$ for some $j \in \boldsymbol{Z}$, then either $\omega \in A_i$ or $\omega' \in A_i$.

Now, we define a mapping $T_n : \Omega \to [0, 1/2^n)$ by $T_n(\omega) := (2^n\omega - \lfloor 2^n\omega \rfloor)/2^n$. If A is a measurable subset of Ω such that the restriction $T_n|_A$ of T_n to A is injective, then $T_n|_A$ is obviously measure-preserving and thereby the inequality $P(A) = P(T_n(A)) \le 1/2^n$ holds.

Here, we prove that $T_n|_{A_1 \cap \dots \cap A_n}$ is injective, from which (2.1) follows. The proof is by contradiction. Suppose that $\omega, \omega' \in A_1 \cap \dots \cap A_n$ satisfy $\omega < \omega'$ and $T_n(\omega) = T_n(\omega')$. Then there exists a $k \in [1, 2^n) \cap \boldsymbol{Z}$ such that $\omega' = \omega + k/2^n$. Factoring k, we have $k = 2^h(2j+1)$ ($h \in [0, n) \cap \boldsymbol{Z}$ and $j \in \boldsymbol{Z}$), from which $\omega' = \omega + (2j+1)/2^{n-h}$ follows. As we mentioned before, either $\omega \in A_{n-h}$ or $\omega' \in A_{n-h}$. This contradicts the assumption $\omega, \omega' \in A_1 \cap \dots \cap A_n$.

Lemma 2. *Suppose that* $p, q \in \boldsymbol{Z}$ *satisfy* $p < q$. *Then, for almost all* α *with respect to Lebesgue measure,*

(2.2) $$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} r_i(\omega + p\alpha) r_i(\omega + q\alpha) = 0, \quad P\text{-}a.e.\ \omega.$$

Proof. Since the sequence $\{r_i(\omega_1) r_i(\omega_2)\}$ is an i.i.d. on $(\Omega \times \Omega, P \times P)$, by the law of large numbers, we have

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} r_i(\omega_1) r_i(\omega_2) = 0, \quad P \times P\text{-}a.e.\ (\omega_1, \omega_2).$$

Since r_i is periodic, the Lebesgue measure of $\boldsymbol{R}^2 \setminus M$ is 0, where M is given by

$$M := \left\{ (\omega_1, \omega_2) \in \boldsymbol{R}^2 \,\middle|\, \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} r_i(\omega_1) r_i(\omega_2) = 0 \right\}.$$

Since the linear transformation $T : (\omega, \alpha) \mapsto (\omega + p\alpha, \omega + q\alpha)$ is regular, the Lebesgue measure of $T^{-1}(\boldsymbol{R}^2 \setminus M)$ is 0. Therefore, for almost all (ω, α),

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} r_i(\omega + p\alpha) r_i(\omega + q\alpha) = 0.$$

By Fubini's theorem, we have the conclusion.

To prove Theorem 1, we use the next theorem due to McLeish [5].

Theorem A. *Let* $\{\zeta_{m,j}\,;\, 1 \le j \le k_m\}$ *be a triangular array of random vari-*

ables and put $L_m := \Pi_{j \le k_m}(1 + \sqrt{-1}\, t\zeta_{m,j})$. *The law of* $\sum_{j \le k_m} \zeta_{m,j}$ *converges to the standard normal distribution as* $m \to \infty$, *provided that the following four conditions are satisfied for all* $t \in \boldsymbol{R}$:

(1) $EL_m \to 1$ *as* $m \to \infty$;

(2) *The sequence* $\{L_m\}_{m \in \mathrm{N}}$ *is uniformly integrable*;

(3) $\sum_{j \le k_m} \zeta_{m,j}^2 \to 1$ *in probability as* $m \to \infty$;

(4) $\max_{j \le k_m} |\zeta_{m,j}| \to 0$ *in probability as* $m \to \infty$.

Because of Lemma 2, it is sufficient for us to prove (1.1) assuming (2.2) for all $p < q$. We put $n = 1$ in (1.1), since the sequence is stationary. We prove that, for any $a_1, \ldots, a_k$ satisfying $a_1^2 + \cdots + a_k^2 = 1$, the law of $a_1 X_1^{(m,\alpha)} + \cdots + a_k X_k^{(m,\alpha)}$ converges to the standard normal law. Because of Cramér-Wold's theorem (Theorem 7.7 in Billingsley [2]), this is equivalent to (1.1).

Putting $\eta_i(\omega) := \sum_{j=1}^k a_j r_i(\omega + j\alpha)$, we have

$$a_1 X_1^{(m,\alpha)} + \cdots + a_k X_k^{(m,\alpha)} = \frac{1}{\sqrt{m}} \sum_{i=1}^m \eta_i.$$

We note the following two properties of $\{\eta_i\}$:

(a) $|\eta_i| \le k$;

(b) $E(\eta_{i_1} \ldots \eta_{i_s}) = 0$, for any $i_1 < \cdots < i_s$ and $s \in \boldsymbol{N}$.

(a) is trivial. (b) is verified by expanding $\eta_{i_1} \ldots \eta_{i_s}$ into a linear combination of

$$r_{i_1}(\omega + j_1\alpha) \ldots r_{i_s}(\omega + j_s\alpha), \quad (j_t = 1, \ldots, k,\ t = 1, \ldots, s),$$

and by noting that the expectations of these are 0 because of Lemma 1.

Putting $k_m := m$ and $\zeta_{m,j} := \eta_j / \sqrt{m}$, we apply Theorem A to prove the convergence. The four conditions are easily verified: $EL_m = 1$ follows from (b); By (a), we have $|L_m| \le (1 + t^2k^2/m)^{m/2} \le e^{t^2k^2/2}$, which implies (2); Since $\sum_{j \le k_m} \zeta_{m,j}^2(\omega) - 1$ is expanded into a linear combination of the sums in (2.2), (3) follows from the assumption; (4) is clear from (a).

3. Proof of Theorem 2. By using Lemma 1, we have

$$R^{(m,\alpha)}(k) = EX_0^{(m,\alpha)} X_k^{(m,\alpha)} = \frac{1}{m} \sum_{i=1}^m E(r_i(\omega) r_i(\omega + k\alpha)).$$

An easy calculation gives $E(r_1(\omega) r_1(\omega + k\alpha)) = \varphi(k\alpha)$. Since the dyadic transformation is measure preserving, we have

$$\begin{aligned} E(r_i(\omega) r_i(\omega + k\alpha)) &= E(r_1(2^{i-1}\omega) r_1(2^{i-1}\omega + 2^{i-1}k\alpha)) \\ &= E(r_1(\omega) r_1(\omega + 2^{i-1}k\alpha)) \\ &= \varphi(2^{i-1}k\alpha). \end{aligned}$$

These prove (1.2). To prove (1.3), we apply the following law of the iterated logarithm due to Maruyama [4]:

$$\limsup_{m \to \infty} \frac{1}{\sqrt{2m \log\log m}} \sum_{i=1}^m \varphi(2^{i-1}x) = \sigma, \quad \text{a.e. } x,$$

where

$$\sigma^2 = \int_0^1 \varphi^2(x)\,dx + 2 \sum_{j=1}^\infty \int_0^1 \varphi(x)\varphi(2^j x)\,dx = \frac{1}{3}.$$

The last evaluation of σ^2 is easily given by $\varphi(x) = 2r_1(x) \sum_{j=2}^\infty 2^{-j} r_j(x)$.

4. Proof of Theorem 3. We follow the method of Hawkes (Proof of

Theorem 4 of [3]). We introduce a Bernoulli i.i.d. $\{Y_n\}$ on another probability space (Ω_0, P_0) such that $P_0\{Y_n = 0\} = q$ and $P_0\{Y_n = 1\} = p$, $(p + q = 1)$. Let μ_p be the law of $\sum_{j=1}^{\infty} 2^{-j} Y_j$. We can easily verify that μ_p is a probability measure on Ω and that, on the probability space (Ω, μ_p), the dyadic transformation $\omega \mapsto 2\omega$ is measure-preserving and ergodic.

Because of (1.2) and the ergodicity, we have

$$R^{(m,\alpha)}(1) \to \int_\Omega \varphi(x)\mu_p(dx), \text{ as } m \to \infty, \ \mu_p\text{-a.e. } \alpha.$$

The last integral is evaluated as follows: By the definition of μ_p,

$$\begin{aligned}\int_\Omega \varphi(x)\mu_p(dx) &= E_{P_0}\varphi\Big(\sum_{j=1}^{\infty} 2^{-j} Y_j\Big) \\ &= \sum_{i=0}^{1} E_{P_0}\varphi\Big(\frac{i}{2} + \sum_{j=2}^{\infty} 2^{-j} Y_j\Big) P_0\{Y_1 = i\};\end{aligned}$$

Since $0 \le \sum_{j=2}^{\infty} 2^{-j} Y_j \le 1/2$, we have $\int_\Omega \varphi(x)\mu_p(dx) = (1-2p)^2$.

Now we recall the following result by Billingsley ([1] p. 141 onwards): Hausdorff dimension of $A \subset \Omega$ is equal to or greater than $e_p := -(p \log_2 p + q \log_2 q)$ if $\mu_p(A) > 0$. Then, we see that $R^{(m,\alpha)}(1) \to (1-2p)^2$ holds on the set of α whose dimension is at least e_p.

$E(X_n^{(m,\alpha)})^4 \le 4$ implies the uniform integrability of $\{X_0^{(m,\alpha)} X_1^{(m,\alpha)}\}_{m \in \mathbf{N}}$. Thus, $R^{(m,\alpha)}(1) \to 0$ follows from (1.1).

Therefore, the dimension of the set of α such that (1.1) does not hold is at least $e_p (p \ne 1/2)$. Taking the supremum, we have the conclusion.

References

[1] Billingsley, P.: Ergodic Theory and Information. Wiley, New York (1967).
[2] ——: Convergence of Probability Measures. Wiley, New York (1968).
[3] Hawkes, J.: Probabilistic behaviour of some lacunary series. Z. Wahr. verw. Geb., **53**, 21–33 (1980).
[4] Maruyama, G.: On an asymptotic property of a gap sequence. Kôdai Math. Sem. Rep., **1950**, 31–32 (1950) (Reproduced version is available in Selection of Gisiro Maruyama, pp. 63–66, Kaigai, Tokyo, 1988)
[5] McLeish, D. L.: Dependent central limit theorem and invariance principles. Annals of Prob., **2**, 620–628 (1974).
[6] Sugita, H.: A private communication (1993).