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THE LAW OF THE ITERATED LOGARITHM FOR SUBSEQUENCES: A SIMPLE PROOF

Abstract. We give a simple proof for the law of the iterated logarithm for subsequence of sums of i.i.d.

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1. Result

There are various studies [2, 3] on the asymptotic behavior of subsequences of sums of i.i.d. The following result given by Weber [4] determined the speed of divergence of every subsequences.

Theorem 1. *Let $\nu_1 < \nu_2 < \cdots$ be an increasing sequence of natural numbers and $\{X_n\}$ be an i.i.d. with $EX_n = 0$ and $EX_n = 1$. We have*

$$\limsup_{j\to\infty} (X_1 + \cdots + X_{\nu_j}) \Big/ \sqrt{2\nu_j \Lambda(\nu_j)} = 1 \quad a.s.$$

where $p_n = {}^{\#}\{m \le n \mid \{\nu_j\} \cap (2^{m-1}, 2^m\,] \ne \emptyset\}$ and $\Lambda(k) = \log p_n$ if $k \in (2^{n-1}, 2^n\,]$.

Although Weber's proof consists of intricate and delicate long chaining arguments, we show that it can be proved very easily by modifying the short proof for Hartman-Wintner theorem given by de Acosta [1].

2. Proof

Put $S_n = X_1 + \cdots + X_n$. For $k \in (2^{n-1}, 2^n\,]$, put $\Phi(k) = ((2^n - k)\Lambda(2^{n-1}) + (k - 2^{n-1})\Lambda(2^n))/2^{n-1}$. Then $\Phi(2^n) = \Lambda(2^n)$ and $|\Phi(k) - \Lambda(k)| \le |\Lambda(2^n) - \Lambda(2^{n-1})|$, and hence by $\Lambda(2^n)/\Lambda(2^{n-1}) \to 1$, we have

$$1 \ge \Phi(k)/\Lambda(2^n) = \Phi(k)/\Lambda(k) \ge \Lambda(2^{n-1})/\Lambda(2^n) \to 1.$$

Put $a_n = \sqrt{2n\Phi(n)}$, $b_n = \sqrt{n/\Phi(n)}$. We prove $\limsup_{j\to\infty} S_{\nu_j}/a_{\nu_j} = 1$ a.s.

$\{b_n\}$ is increasing for large n. Actually, for $k \in (2^{n-1}, 2^n\,]$, we have $0 \le k(\Phi(k) - \Phi(k-1)) \le 2^n(\log p_n - \log p_{n-1})/2^{n-1} \le 2/\log p_{n-1}$, and hence

$$(b_k - b_{k-1})\Phi(k)\Phi(k-1) = \Phi(k) - k(\Phi(k) - \Phi(k-1))$$
$$\geq \log p_n - 2/p_{n-1} > 0.$$

We can prove the next two lemmas by replacing $\mathrm{LL}\, j$ by $\Phi(j)$ and rewriting the proofs of lemmas 2.2 and 2.3 of de Acosta [1]. The original proof uses the fact that $\sqrt{j/\mathrm{LL}\, j}$ is increasing, and in our case b_j plays its role.

Lemma 2. *Let $\{Y_n\}$ be a sequence of independent random variables with $EY_n = 0$, $V = \sup EY_n^2 < \infty$, $|Y_n| \leq \tau b_n$ for all n and some $\tau > 0$. Put $T_n = Y_1 + \cdots + Y_n$. Then for all $a \geq \sqrt{V}$, $t > 0$ and $n \in \mathbf{N}$,*

$$P(T_n/a_n > t) \leq \exp\bigl(-(t/a)^2\bigl(2 - \exp(\sqrt{t}a^{-2}\tau)\bigr)\Phi(n)\bigr).$$

Lemma 3. *Let $\{X_n\}$ be an i.i.d. with $EX_n = 0$ and $EX_n = 1$. Let $\tau > 0$ and put $Z_j = X_j \mathbf{1}_{|X_j| \geq \tau b_j}$. Then $\sum E|Z_j|/a_j < \infty$ and $(Z_1 + \cdots + Z_n)/a_n \to 0$, a.s.*

In the proof of Lemma 2, the estimate $1/a_1 + \cdots + 1/a_n = O(b_n)$ is used. In our case, we can verify it as follows. First take $m(n)$ as $p_{m(n)-1} < \frac{1}{2}p_n \leq p_{m(n)}$. For $k \in (2^{n-1}, 2^n\,]$, by $p_{n+1} - p_n = 0,\, 1$, we have $n - m(n) \geq p_n - p_{m(n)-1} - 1 \geq p_n/2 - 1$ and hence $m(n) \leq n - p_n/2 + 1$. We divide $\sum_{j=1}^{k} 1/a_j$ into two parts and estimate as follows:

$$\sum_{j=1}^{2^{m(k)-1}} \frac{1}{a_j} \leq \sum_{j=1}^{2^{m(k)-1}} \frac{1}{\sqrt{2j}} = O(2^{m(k)/2}) = O(2^{k/2}2^{-p_k/4}) = O(b_k),$$

$$\sum_{j=2^{m(n)-1}+1}^{k} \frac{1}{a_j} \leq \frac{1}{\sqrt{2\log(p_k/2)}} \sum_{j=2^{m(n)-1}+1}^{k} \frac{1}{\sqrt{j}} \leq \frac{O\bigl(\sqrt{k}\bigr)}{\sqrt{2\log(p_k/2)}} = O(b_k).$$

Put $Y_j = X_j - Z_j - E(X_j - Z_j)$ and $T_n = Y_1 + \cdots + Y_n$. By $\sum E|Z_j|/a_j < \infty$ and $EX_n = 0$, we have $\sum |E(X_j - Z_j)|/a_j < \infty$ and $\sum_{j=1}^{k} E(X_j - Z_j) = o(a_k)$, and hence we have $S_n - T_n = o(a_n)$.

By the central limit theorem, T_n/a_n converges to 0 in probability, and hence $\min_{k<n} P\bigl(|T_n - T_k| \leq \varepsilon a_n\bigr) \geq 1/2 > 0$ for large n. By Ottaviani's inequality we have $P\bigl(\max_{k=1}^{n} |T_k| \geq (1 + 2\varepsilon)a_n\bigr) \leq 2P\bigl(|T_n| \geq (1 + \varepsilon)a_n\bigr)$. By Lemma 1, the right hand side is bounded from above by $2\exp\bigl(-(1+\varepsilon)^2\bigl(2 - \exp(\sqrt{1+\varepsilon}2\tau)\bigr)\Phi(n)\bigr)$. If we take τ small enough, it is less than $2e^{-\theta\Lambda(n)}$ where some $\theta > 1$. By this esitmate, denoting by

$\sum_n^*$ the summation for all n satisfying $p_n = p_{n-1} + 1$, we have

$$\sum_n{}^* \sum_{j=0}^{l-1} P\Big(\max_{k=1}^{[2^{n+j/l}]} |T_k| > (1+2\varepsilon) a_{[2^{n+j/l}]} \Big) \le Cl \sum_n{}^* p_n^{-\theta} = Cl \sum_n^{\infty} n^{-\theta} < \infty.$$

By Borel-Cantelli Lemma, for large n and for $k \in ([2^{n+(j-1)/l}], [2^{n+j/l}]\,]$, we have $|T_k| \le (1+2\varepsilon) a_{[2^{n+j/l}]} \le (1+2\varepsilon)(1+o(1)) 2^{1/l} a_k$. Hence by letting $l \to \infty$ and $\varepsilon \downarrow 0$, we have $\limsup_{j\to\infty} T_{\nu_j}/a_{\nu_j} \le 1$ a.s. This together with $S_n - T_n = o(a_n)$, we have the upper bound part of conclusion of our Theorem.

To have the lower bound part, we use the next lemma of [1].

Lemma 4. *Let $\{X_n\}$ be an i.i.d. with $EX_n = 0$ and $EX_n = 1$. Put $S_n = X_1 + \cdots + X_n$. Let $m_k \in \mathbf{N}$, $\alpha_k > 0$, $\alpha_k/m_k \to 0$, and $\alpha_k^2/m_k \to \infty$. Then for every $b \in \mathbf{R}$, $\varepsilon > 0$,*

$$\liminf_{k\to\infty} (m_k/\alpha_k^2) \log P\big(|S_{m_k}/\alpha_k - b| < \varepsilon \big) \ge -b^2/2.$$

For n satisfying $p_n = p_{n-1}+1$, denote by μ_{p_n} the largest $\nu_j \in (2^{n-1}, 2^n\,]$. If $\mu_k \in (2^{i-1}, 2^i\,]$, then $p_i = k$ and $\Phi(\mu_k) \le \Lambda(p_i) = \log k$. We see that $\mu_m/\mu_n \ge 2^{m-n-1}$. Putting

$$m_k = \mu_{lk} - \mu_{l(k-1)}, \quad \alpha_k = \sqrt{2(\mu_{lk} - \mu_{l(k-1)})\Phi(\mu_{lk})}, \quad \text{and} \quad b = 1 - 2\varepsilon,$$

and applying the above lemma, we have

$$P\big(S_{\mu_{lk}} - S_{\mu_{l(k-1)}} \ge (1-2\varepsilon)\alpha_k \big) \ge \exp\big(-(1-\varepsilon)\Phi(\mu_{lk}) \big) \ge (lk)^{-\theta},$$

where $\theta < 1$. Because of $\sum_k (lk)^{-\theta} = \infty$, by Borel-Cantelli Lemma and by upper bound estimate $|S_{\mu_{l(k-1)}}|/a_{\mu_{lk}} \le (1+\varepsilon)\sqrt{1/2^{l-1}}$ f.e., we have

$$S_{\mu_{lk}}/a_{\mu_{lk}} \ge (1-2\varepsilon)\sqrt{1 - 1/2^{l-1}} - (1+\varepsilon)\sqrt{1/2^{l-1}} \quad \text{i.o.}$$

By letting $l \to \infty$ and $\varepsilon \downarrow 0$, we have the lower bound part.

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