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Fukuyama, Katusi

Noda, Yutaro

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ON PERMUTATIONAL INVARIANCE OF THE METRIC DISCREPANCY RESULTS

Katusi Fukuyama — Yutaro Noda

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ABSTRACT. Let $\{n_k\}$ be a sequence of non-zero real numbers. We prove that the law of the iterated logarithm for discrepancies of the sequence $\{n_k x\}$ is permutational invariant if $|n_{k+1}/n_k| \to \infty$ is satisfied.

1. Introduction

A sequence $\{x_k\}$ of real numbers is said to be uniformly distributed modulo one (u. d. mod 1) if ${}^\#\{k \le N \mid \langle x_k \rangle \in [\,a, b)\}/N \to b - a$ for all $[\,a, b) \subset [\,0, 1)$, where $\langle x \rangle$ denotes the fractional part $x - [\,x\,]$ of real number x. To measure the speed of convergence, we use the discrepancy

$$D_N\{x_k\} = \sup_{0 \le a < b \le 1} \left| \frac{1}{N} {}^\#\{k \le N \mid \langle x_k \rangle \in [\,a, b)\} - (b - a) \right|.$$

General theory of discrepancies can be found in [13]. It is known that an arithmetic progression $\{kx\}$ is u. d. mod 1 if and only if x is irrational, and a geometric progression $\{\theta^k x\}$ is u. d. mod 1 for almost every x if and only if $|\theta| > 1$.

Manifestly the sequence $\{U_k\}$ of independent random variables with uniform distribution $P(a \le U_k < b) = b - a$ $(0 \le a < b \le 1)$ is u. d. mod 1, almost surely. The speed is described by Chung-Smirnov law of the iterated logarithm

$$\varlimsup_{N \to \infty} \frac{N D_N\{U_k\}}{\sqrt{2N \log\log N}} = \frac{1}{2} \quad \text{a.s.} \tag{1}$$

In Nijenrode Lecture, Erdős [6] made a conjecture: If a sequence $\{n_k\}$ of positive integers satisfies the Hadamard gap condition $n_{k+1}/n_k \ge q > 1$, then $\{n_k x\}$ imitates the behaviour of $\{U_k\}$ and obeys the law of Chung-Smirnov type (1). After Takahashi's method [16], Philipp [14] solved the conjecture by showing

$$\frac{1}{4\sqrt{2}} \le \varlimsup_{N \to \infty} \frac{N D_N\{n_k x\}}{\sqrt{2N \log\log N}} \le \frac{1}{\sqrt{2}} \left(166 + \frac{664}{q^{1/2} - 1} \right) \qquad \text{a.e.} \quad x.$$

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By martingale approximation, Philipp [15] removed the assumption that n_k's are integers. Dhompongsa [5] proved the exact result of Chung-Smirnov type

$$\varlimsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} = \frac{1}{2} \qquad \text{a.e.} \quad x,$$

assuming $\log(n_{k+1}/n_k)/\log\log k \to \infty$. It was relaxed in [8] to $n_{k+1}/n_k \to \infty$.

Results for geometric progressions $\{\theta^k x\}$ are complicated. If $|\theta| \le 1$, it is not u. d. mod 1. When $|\theta| > 1$, it is proved that there exists a real number Σ_θ such that

$$\varlimsup_{N\to\infty} \frac{ND_N\{\theta^k x\}}{\sqrt{2N\log\log N}} = \Sigma_\theta \qquad \text{a.e.} \quad x.$$

(Cf. [7, 9–11]). If $\theta^r \notin \mathbf{Q}$ for any $r \in \mathbf{N}$, then $\Sigma_\theta = 1/2$, and otherwise $\Sigma_\theta > 1/2$. When θ satisfies $\theta^r \in \mathbf{Q}$ for some $r \in \mathbf{N}$, take the smallest $r \in \mathbf{N}$ with $\theta^r \in \mathbf{Q}$ and write $\theta^r = p/q$ by $p \in \mathbf{Z}$ and $q \in \mathbf{N}$ with $\gcd(p, q) = 1$.

If p and q are odd, $\Sigma_\theta = \frac{1}{2}\sqrt{(|p|q+1)/(|p|q-1)}$. If $q = 1$, then Σ_θ equals to $\frac{1}{2}\sqrt{(|p|+1)/(|p|-1)}$, $\frac{1}{2}\sqrt{(|p|+1)|p|(|p|-2)/(|p|-1)^3}$, $\frac{1}{9}\sqrt{42}$, or $\frac{1}{49}\sqrt{910}$, according as p is odd, $|p| \ge 4$ is even, $p = 2$, or $p = -2$. If $p = \pm 5$ and $q = 2$, we have $\Sigma_\theta = \frac{1}{9}\sqrt{22}$. The values of Σ_θ for other cases are so far unknown.

Although (1) is permutational invariant, we can prove the contrary.

Theorem 1 ([8]). *For an unbounded sequence $\{n_k\}$ of positive real numbers, there exists a permutation σ over $\mathbf{N}$ (i.e., a bijection $\mathbf{N} \to \mathbf{N}$) such that*

$$\varlimsup_{N\to\infty} \frac{ND_N\{n_{\sigma(k)} x\}}{\sqrt{2N\log\log N}} = \frac{1}{2} \qquad a.e. \quad x. \tag{2}$$

For $a = 2, 3, \ldots$, there exists a permutation σ such that

$$\varlimsup_{N\to\infty} \frac{ND_N\{2^{\sigma(k)} x\}}{\sqrt{2N\log\log N}} = \frac{1}{2}\sqrt{\frac{(2^a+1)2^a(2^a-2)}{(2^a-1)^3}} \qquad a.e. \quad x. \tag{3}$$

Aistleitner-Berkes-Tichy [1–3] studied the effect of permutation on metric discrepancy results. As a corollary, they derived the beautiful result below.

Theorem 2 ([3: Aistleitner-Berkes-Tichy]). *If a sequence $\{n_k\}$ of positive integers satisfies $n_{k+1}/n_k \to \infty$, then (2) holds for any permutation σ over $\mathbf{N}$.*

It can be applied only for integers $\{n_k\}$, and we here prove it for real numbers.

Theorem 3. *If a sequence $\{n_k\}$ of non-zero real numbers satisfies the condition $|n_{k+1}/n_k| \to \infty$, then (2) holds for any permutation σ over $\mathbf{N}$.*

2. Proof

We first introduce the notion of weakly multiplicative system. It is said that a sequence $\{X_k\}$ of random variables is a weakly multiplicative system if it satisfies

$$\sum_{r=1}^{\infty} \sum_{k(1),\dots,k(r):\ k(1)<\dots<k(r)} \Big| E X_{k(1)} \dots X_{k(r)} \Big| < \infty. \tag{4}$$

We use the next law of the iterated logarithm for weakly multiplicative systems.

Theorem 4 ([4: Berkes]). *Let $\{X_k\}$ be a sequence of uniformly bounded random variables. If both of $\{X_k\}$ and $\{X_k^2 - 1\}$ are weakly multiplicative systems, then*

$$\varlimsup_{N\to\infty} \frac{1}{\sqrt{2N \log\log N}} \sum_{k=1}^{N} X_k = 1 \quad a.s. \tag{5}$$

Since (4) is permutational invariant, (5) remains valid if we replace X_k by $X_{\sigma(k)}$. Moreover, if $\{X_k\}$ satisfies (4), then $\{-X_k\}$ also. Therefore we have

$$\varlimsup_{N\to\infty} \frac{1}{\sqrt{2N \log\log N}} \left| \sum_{k=1}^{N} X_{\sigma(k)} \right| = 1 \quad \text{a.s.} \tag{6}$$

The next key lemma states a result on the probability space $\big([0,1), \mathcal{B}[0,1), \mathrm{d}x\big)$.

Lemma 1. *Let f be a trigonometric polynomial satisfying*

$$f(x+1) = f(x), \qquad \int_0^1 f(x)\mathrm{d}x = 0 \quad \textit{and} \quad \|f\|_2^2 = \int_0^1 f^2(x)\,\mathrm{d}x = 1.$$

For $\{n_k\}$ in Theorem 3*, both of $\{f(n_k x)\}$ and $\{f^2(n_k x) - 1\}$ are uniformly bounded weakly multiplicative systems.*

Proof. Denote

$$f(x) = \sum_{j=1}^{J} a_j \cos(2\pi j x + \gamma_j) \quad \text{and} \quad L = 1 + \sum_{j=1}^{J} |a_j| \ge 1.$$

Since $\{|n_k|\}$ is eventually increasing, and since $|n_k|$ diverges to infinity, we see that $|n_1|, \dots, |n_k| \le |n_k|$ holds for large k. Take K_0 such that

$$\left|\frac{n_{k+1}}{n_k}\right| \ge 6JL \quad (k \ge K_0) \quad \text{and} \quad \max\{|n_1|, \dots, |n_{K_0}|\} = |n_{K_0}|. \tag{7}$$

Put

$$K_1 = [\,K_0 + 2 + \log_{6JL} K_0\,] \ge K_0$$

and

$$C = (6JL)^{-K_0} \frac{|n_{K_0}|}{2} > 0.$$

We have

$$(6JL)^{k-K_0-1} \ge K_0 \quad (k \ge K_1). \tag{8}$$

By (7), we have

$$|n_k| = \big|(n_k/n_{k-1}) \dots (n_{K_0+1}/n_{K_0}) n_{K_0}\big| \ge 2C(6JL)^k$$

for $k \ge K_0$. Suppose that $k(1) < \cdots < k(r)$ and $k(r) \ge K_1$. By (7) and (8), we have

$$\begin{aligned}\sum_{i=1}^{r-1}|n_{k(i)}| &\le \sum_{i=1}^{k(r)-1}|n_i| \le \sum_{i=K_0+1}^{k(r)-1}|n_i| + K_0|n_{K_0}| \\ &\le \sum_{i=K_0+1}^{k(r)-1}\frac{|n_{k(r)}|}{(6JL)^{k(r)-i}} + \frac{K_0|n_{k(r)}|}{(6JL)^{k(r)-K_0}} \\ &\le |n_{k(r)}|\Big(\frac{2}{6JL}+\frac{1}{6JL}\Big) \le \frac{|n_{k(r)}|}{2J}.\end{aligned}$$

Hence, for $j_1, \dots, j_r \le J$, and $\varsigma_1, \dots, \varsigma_{r-1} = \pm 1$, $\varsigma_r = 1$, we have the estimate

$$\begin{aligned}\Big|\sum_{i=1}^{r}\varsigma_i j_i n_{k(i)}\Big| &\ge |n_{k(r)}| - J\sum_{i=1}^{r-1}|n_{k(i)}| \\ &\ge \frac{|n_{k(r)}|}{2} \ge C(6L)^{k(r)},\end{aligned}$$

and thereby

$$\Big|\int_0^1 \cos\sum_{i=1}^{r}(2\pi\varsigma_i j_i n_{k(i)}x + \varsigma_i\gamma_i)\,\mathrm{d}x\Big| \le \frac{1}{C(6L)^{k(r)}}.$$

Let $I_{k(1),\dots,k(r)}$ denotes

$$\int_0^1 f(n_{k(1)}x)\dots f(n_{k(r)}x)\,\mathrm{d}x = \sum_{j_1}^{J}\dots\sum_{j_r}^{J}\int_0^1\prod_{i=1}^{r}a_{j_i}\cos(2\pi j_i n_{k(i)}x+\gamma_i)\,\mathrm{d}x.$$

$$\prod_{i=1}^{r}\cos(2\pi j_i n_{k(i)}x+\gamma_i) = \frac{1}{2^{r-1}}\sum_{\varsigma_1,\dots,\varsigma_{r-1}=\pm1}\cos\sum_{i=1}^{r}(2\pi\varsigma_i j_i n_{k(i)}x+\varsigma_i\gamma_i)$$

gives

$$\begin{aligned}|I_{k(1),\dots,k(r)}| &\le \frac{1}{C(6L)^{k(r)}}\Big(\sum_{j=1}^{J}|a_j|\Big)^r \\ &\le \frac{L^r}{C(6L)^{k(r)}} \le \frac{1}{C6^{k(r)}}\end{aligned}$$

when $k(1) < \cdots < k(r)$ and $k(r) \ge K_1$. Note that we have

$$\sum_{r=1}^{\infty}\sum_{\substack{k(1),\dots,k(r):\\ k(1)<\cdots<k(r)}}|I_{k(1),\dots,k(r)}| = \sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{\substack{k(1),\dots,k(r-1):\\ k(1)<\cdots<k(r-1)<k}}|I_{k(1),\dots,k(r-1),k}|$$

We divide the summation as $\sum_{k=1}^{\infty} = \sum_{k=1}^{K_1-1} + \sum_{k=K_1}^{\infty}$. The first is finite and the second is bounded by

$$\sum_{k=K_1}^{\infty}\sum_{r=1}^{k}\binom{k-1}{r-1}\frac{1}{C6^k} \le \sum_k \frac{2^{k-1}}{C6^k} < \infty.$$

□

We prove Theorem 3. For a trigonometric polynomial f satisfying

$$f(x+1)=f(x),\qquad \int_0^1 f(x)\,\mathrm{d}x=0 \quad\text{and}\quad \int_0^1 f^2(x)\,\mathrm{d}x>0,$$

by applying Lemma 1, we see that $\{f(n_k x)/\|f\|_2\}$ and $\{(f(n_k x)/\|f\|_2)^2-1\}$ satisfy (4). By (6), we have

$$\varlimsup_{N\to\infty}\frac{1}{\sqrt{2N\log\log N}}\left|\sum_{k=1}^{N}f(n_{\sigma(k)}x)\right|=\|f\|_2 \qquad \text{a.e.}\quad x. \tag{9}$$

The next result is convenient to deal with the discrepancies.

Theorem 5 ([12])**.** *Let $\{n_k\}$ be a sequence of non-zero real numbers satisfying $|n_{k+1}/n_k|\ge q>1$, σ be a permutation over $\mathbf{N}$, and S denotes $\mathbf{Q}\cap[0,1)$. Then*

$$\varlimsup_{N\to\infty}\frac{ND_N\{n_{\sigma(k)}x\}}{\sqrt{2N\log\log N}}=\sup_{S\ni a<b\in S}\lim_{d\to\infty}\varlimsup_{N\to\infty}\frac{1}{\sqrt{2N\log\log N}}\left|\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{a,b}^{(d)}(n_{\sigma(k)}x)\right| \quad a.e.,$$

where $\widetilde{\mathbf{1}}_{a,b}^{(d)}(x)$ denotes the d-th sum of the Fourier series of $\mathbf{1}_{[a,b)}(\langle x\rangle)-(b-a)$.

By changing values of the first finitely many terms of $\{n_k\}$, we may assume $|n_{k+1}/n_k|\ge 2$. By applying (9) for $f=\widetilde{\mathbf{1}}_{a,b}^{(d)}$, by $\lim_{d\to\infty}\|\widetilde{\mathbf{1}}_{a,b}^{(d)}\|_2=\|\widetilde{\mathbf{1}}_{a,b}\|_2\le\|\widetilde{\mathbf{1}}_{0,1/2}\|_2=\frac{1}{2}$ and Theorem 5, we have (2).

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Department of Mathematics,
Graduate School of Science,
Kobe University,
Kobe 657-8501,
JAPAN

E-mail: fukuyama@math.kobe-u.ac.jp
ytnoda@math.kobe-u.ac.jp