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Numerical approximation of the basic reproduction number for a class of age-structured epidemic models

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Abstract

We are concerned with the numerical approximation of the basic reproduction number $\mathcal{R}_0$, which is the well-known epidemiological threshold value defined by the spectral radius of the next generation operator. For a class of age-structured epidemic models in infinite-dimensional spaces, $\mathcal{R}_0$ has the abstract form and can not be explicitly calculated in general. We discretize the linearized equation for the infective population into a system of ordinary differential equations in a finite n-dimensional space and obtain a corresponding threshold value $\mathcal{R}_{0,n}$, which can be explicitly calculated as the positive dominant eigenvalue of the next generation matrix. Under the compactness of the next generation operator, we show that $\mathcal{R}_{0,n} \to \mathcal{R}_0$ as $n \to +\infty$ in terms of the spectral approximation theory.

Keywords: Spectral approximation, Epidemic model, Age structure, Basic reproduction number, Next generation operator
2010 MSC: 35Q92, 65N25, 92D30

1. Introduction

In the field of mathematical epidemiology, the age-structure of population has been regarded important since most infectious diseases such as childhood diseases and sexually transmitted diseases have age-dependent characteristics. Age-structured SIR epidemic models, in which the population is divided into three subpopulations called susceptible, infective and recovered, are one of the most basic epidemic models and have attracted much attention of researchers for decades [1–8]. In [3], Greenhalgh conjectured that the spectral radius of a certain linear integral operator would play the role of a threshold value for the existence and stability of each equilibrium in an age-structured SIR epidemic model. In [5], Inaba proved his conjecture: if the threshold value is less than one, then the disease-free equilibrium is globally asymptotically stable and there exists no endemic equilibrium, whereas if it is greater than one, then the disease-free equilibrium is unstable and the locally stable endemic equilibrium uniquely exists under some additional conditions. The threshold value is nowadays called the basic reproduction number $\mathcal{R}_0$, and its epidemiological meaning is the expected value of secondary cases produced by a typical infective individual during its entire infectious period in a fully susceptible population [9].

$\mathcal{R}_0$ is not only mathematically but also epidemiologically important for assessing the disease burden of infectious diseases. However, for age-structured epidemic models in infinite-dimensional spaces, $\mathcal{R}_0$ can not be explicitly calculated in general since it has the abstract form as the spectral radius of the linear integral operator, called the next generation operator. If we discretize the model into a finite dimensional space, then the corresponding threshold value can be explicitly calculated as the positive dominant eigenvalue of the nonnegative irreducible matrix, called the next generation matrix. We can expect that the corresponding threshold value converges to $\mathcal{R}_0$ as the step size of the discretization decreases. However, the convergence of the eigenvalues with preservation of the algebraic multiplicity is not trivial and we need the spectral

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approximation theory [10] to show it mathematically rigorously. In this study, in terms of the spectral approximation theory, we show that the corresponding threshold value $\mathcal{R}_{0,n}$ in the n-dimensional space converges to $\mathcal{R}_0$ as $n \to +\infty$, provided the next generation operator is compact. The compactness holds under relatively weak conditions. By using the demographic data in Japan, we give a numerical example to illustrate the theoretical result. For other studies of age-structured epidemic models from the numerical viewpoints, see [11–14]. Although these studies focused on the convergence of numerical solutions, our focus in this paper is on the convergence of the spectral approximation for the computation of $\mathcal{R}_0$. For another study of the spectral approximation for age-structured population models, see [15]. For another study of the approximation of $\mathcal{R}_0$ for epidemic models in time periodic environment, see [16].

2. Main result

We consider the following equation for the infective population, which is linearized around the disease-free steady state.

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) I(t,a) = S^0(a) \displaystyle\int_0^{a_\dagger} \beta(a,\sigma) I(t,\sigma) d\sigma - (\mu(a) + \gamma(a)) I(t,a), \quad t > 0, \ a \in (0, a_\dagger), \\ \qquad I(t,0) = 0, \quad t > 0, \qquad I(0,a) = I_0(a), \quad a \in (0, a_\dagger), \end{cases} \tag{2.1}$$

where $I(t,a)$ denotes the infective population of age a at time t, $S^0(a)$ denotes the susceptible population of age a in the disease-free steady state, $a_\dagger > 0$ denotes the maximum age, $\beta(a,\sigma)$ denotes the disease-transmission coefficient, $\mu(a)$ denotes the force of mortality, $\gamma(a)$ denotes the recovery rate and $I_0(a)$ denotes the initial age distribution of the infective population. Note that the vertical transmission is excluded since $I(t,0) = 0$ for all $t > 0$. We assume that S^0, β, μ and γ are continuous, strictly positive and uniformly bounded on $[0, a_\dagger]$. We define the following two linear operators on $X := L^1(0, a_\dagger)$.

$$\begin{cases} A\varphi(a) := -\dfrac{d}{da}\varphi(a) - (\mu(a) + \gamma(a))\varphi(a), \quad D(A) := \Big\{\varphi \in X : \ \varphi \text{ is absolutely continuous on } [0, a_\dagger], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \dfrac{d}{da}\varphi \in X \text{ and } \varphi(0) = 0\Big\}, \\ F\varphi(a) := S^0(a) \displaystyle\int_0^{a_\dagger} \beta(a,\sigma)\varphi(\sigma) d\sigma. \end{cases}$$

Using A and F, we can rewrite (2.1) into the following abstract Cauchy problem in X.

$$\frac{d}{dt} I(t) = AI(t) + FI(t), \quad I(0) = I_0 \in D(A). \tag{2.2}$$

It is easy to see that the positive inverse $(-A)^{-1}$ is defined as $(-A)^{-1}\varphi := \int_0^a e^{-\int_\sigma^a (\mu(\eta)+\gamma(\eta))d\eta} \varphi(\sigma) d\sigma$, $\varphi \in X$ and the next generation operator K is defined as follows (see, for instance, [5]).

$$K\varphi(a) := F(-A)^{-1}\varphi(a) = S^0(a) \int_0^{a_\dagger} \beta(a,\sigma) \int_0^\sigma e^{-\int_\rho^\sigma (\mu(\eta)+\gamma(\eta))d\eta} \varphi(\rho) d\rho \, d\sigma, \quad \varphi \in X.$$

Following the definition in [9], we define the basic reproduction number $\mathcal{R}_0$ by $r(K)$, where $r(\cdot)$ denotes the spectral radius of an operator.

Since $\mathcal{R}_0 = r(K)$ has this abstract form, we can not explicitly compute it in general. To overcome this issue, we discretize (2.2) into a system of ordinary differential equations in $X_n := \mathbb{R}^n$, $n \in \mathbb{N}$. Let $\Delta a := a_\dagger/n$, $a_k := k\Delta a$, $S_k^0 := S^0(a_k)$, $\beta_{kj} := \beta(a_k, a_j)$, $\mu_k := \mu(a_k)$ and $\gamma_k := \gamma(a_k)$, $k, j = 1, 2, \cdots, n$. The abstract Cauchy problem (2.2) can be discretized into the following system.

$$\frac{d}{dt} I(t) = A_n I(t) + F_n I(t), \quad I(0) = I_0 \in X_n,$$

where $I(t)$ and I_0 are n-column vectors and A_n and F_n are n-square matrices defined as follows.

$$A_n := \begin{pmatrix} -\mu_1 - \gamma_1 - \frac{1}{\Delta a} & 0 & \cdots & 0 \\ \frac{1}{\Delta a} & -\mu_2 - \gamma_2 - \frac{1}{\Delta a} & & \vdots \\ & \ddots & \ddots & 0 \\ 0 & \cdots & \frac{1}{\Delta a} & -\mu_n - \gamma_n - \frac{1}{\Delta a} \end{pmatrix}, \quad F_n := \begin{pmatrix} S_1^0\beta_{11}\Delta a & \cdots & S_1^0\beta_{1n}\Delta a \\ \vdots & \ddots & \vdots \\ S_n^0\beta_{n1}\Delta a & \cdots & S_n^0\beta_{nn}\Delta a \end{pmatrix}.$$

In this setting, the next generation matrix K_n is given by $F_n(-A_n)^{-1}$ and the threshold value $\mathcal{R}_{0,n} := r(K_n)$, which corresponds to $\mathcal{R}_0$, can be explicitly computed. Note that since $-A_n$ is a nonsingular M-matrix, $(-A_n)^{-1}$ is positive and hence, it follows from the Perron-Frobenius theorem [17] that $\mathcal{R}_{0,n} = r(K_n)$ is the positive dominant eigenvalue with algebraic multiplicity 1. The purpose of this study is to show the convergence $\mathcal{R}_{0,n} \to \mathcal{R}_0$ as $n \to +\infty$ in terms of the spectral approximation theory [10].

To apply the spectral approximation theory, we define the following two bounded linear operators $P_n : X \to X_n$ and $J_n : X_n \to X$ (see, for instance, [18, Section 4.1]).

$$\begin{cases} (P_n\varphi)_k := \dfrac{1}{\Delta a}\displaystyle\int_{a_{k-1}}^{a_k} \varphi(a)da, \quad k = 1, 2, \dots, n, \quad \varphi \in X, \\ (J_n\psi)(a) := \displaystyle\sum_{k=1}^{n} \psi_k \chi_{(a_{k-1},a_k]}(a), \quad \psi = (\psi_1, \psi_2, \dots, \psi_n)^T \in X_n, \end{cases}$$

where the subscript k implies the k-th entry of a vector, $\chi_A(x)$ denotes the indicator function which equals to 1 if $x \in A$ and otherwise 0, the superscript T implies the transpose of a vector and $a_0 = 0$. We define the norm $\|\cdot\|_{X_n}$ in X_n as $\|\psi\|_{X_n} := \Delta a \sum_{k=1}^{n} |\psi_k|$, $\psi = (\psi_1, \psi_2, \cdots, \psi_n)^T \in X_n$. Using [10, Propositions 2.2, 2.3 and 2.11], we prove the following proposition.

Proposition 2.1. *If K is compact and $\lim_{n\to+\infty} \|J_nK_nP_n\varphi - K\varphi\|_X = 0$ for all $\varphi \in X$, then $\mathcal{R}_{0,n} \to \mathcal{R}_0$ as $n \to +\infty$, preserving algebraic multiplicity 1.*

Proof. Under the above assumptions on the parameters, K is strictly positive, irreducible and compact. Hence, it follows from [19, Theorem 3] and the Krein-Rutman theorem [20] that $\mathcal{R}_0 = r(K) > 0$ is an eigenvalue of K. It is easy to see that $\|P_n\| \le 1$ and $\|J_n\| \le 1$ for all $n \in \mathbb{N}$ (see also [18, Section 4.1]). Since

$$(-A_n)^{-1} = \begin{pmatrix} \frac{1}{\mu_1+\gamma_1+1/\Delta a} & 0 & \cdots & 0 \\ \frac{1/\Delta a}{(\mu_1+\gamma_1+1/\Delta a)(\mu_2+\gamma_2+1/\Delta a)} & \frac{1}{\mu_2+\gamma_2+1/\Delta a} & \cdots & \vdots \\ \vdots & & \ddots & \\ \frac{(1/\Delta a)^{n-1}}{\Pi_{k=1}^{n}(\mu_k+\gamma_k+1/\Delta a)} & \frac{(1/\Delta a)^{n-2}}{\Pi_{k=2}^{n}(\mu_k+\gamma_k+1/\Delta a)} & \cdots & \frac{1}{\mu_n+\gamma_n+1/\Delta a} \end{pmatrix},$$

we have

$$\|K_n\psi\|_{X_n} = \left\|F_n(-A_n)^{-1}\psi\right\|_{X_n} \le \Delta a \sum_{k=1}^{n} \frac{\bar{S}^0\bar{\beta}\Delta a}{\underline{\mu}+\underline{\gamma}} \sum_{j=1}^{k} |\psi_j| \le a_\dagger \frac{\bar{S}^0\bar{\beta}}{\underline{\mu}+\underline{\gamma}} \|\psi\|_{X_n},$$

where $\bar{S}^0$ and $\bar{\beta}$ denote the finite upper bounds for S^0 and β, respectively, and $\underline{\mu}$ and $\underline{\gamma}$ denote the positive lower bounds for μ and γ, respectively. Hence, we see that $\|K_n\| \le a_\dagger\bar{S}^0\bar{\beta}/(\underline{\mu}+\underline{\gamma})$ and thus, it follows from the compactness of K that $\cup_{n\in\mathbb{N}} (J_nK_nP_n - K) B$ is a relatively compact set, where $B := \{\varphi \in X : \|\varphi\|_X = 1\}$ denotes the unit ball in X. Thus, the pointwise convergence $\lim_{n\to+\infty} \|J_nK_nP_n\varphi - K\varphi\|_X = 0$ implies the collectively compact convergence and therefore, it follows from [10, Proposition 2.11] that the approximation $\{J_nK_nP_n\}$ is strongly stable in $\Delta\backslash\{\mathcal{R}_0\}$, where Δ denotes the interior of a closed Jordan curve which isolates $\mathcal{R}_0$ in $\mathbb{C}$. From [10, Propositions 2.2 and 2.3], this implies that $\mathcal{R}_{0,n} \to \mathcal{R}_0$ as $n \to +\infty$, preserving the multiplicity 1. $\square$

For the compactness of K, we make the following additional assumption.

$$\lim_{h\to 0} \int_0^{\omega} |S^0(a+h)\beta(a+h,\sigma) - S^0(a)\beta(a,\sigma)|da = 0 \ \text{ uniformly for } \sigma \in \mathbb{R}, \tag{2.3}$$

where the domain of $S^0\beta$ is extended by $S^0(a)\beta(a,\cdot) = 0$ for all $a \in (-\infty, 0) \cap (a_\dagger, \infty)$. Under (2.3), K is compact (see [5, Assumption 4.4]). On the pointwise convergence of $J_nK_nP_n$ to K, we prove the following proposition.

Proposition 2.2. $\lim_{n\to+\infty} \|J_nK_nP_n\varphi - K\varphi\|_X = 0$ *for all* $\varphi \in X$.

Proof. For any $\varphi \in X$, we have

$$\begin{aligned}
\|J_nK_nP_n\varphi - K\varphi\|_X =& \left\|J_nF_n(-A_n)^{-1}P_n\varphi - F(-A)^{-1}\varphi\right\|_X \\
\le& \left\|J_nF_n(-A_n)^{-1}P_n\varphi - J_nF_nP_n(-A)^{-1}\varphi\right\|_X + \left\|J_nF_nP_n(-A)^{-1}\varphi - F(-A)^{-1}\varphi\right\|_X \\
\le& \|J_n\| \|F_n\| \left\|(-A_n)^{-1}P_n\varphi - P_n(-A)^{-1}\varphi\right\|_{X_n} + \left\|J_nF_nP_n(-A)^{-1}\varphi - F(-A)^{-1}\varphi\right\|_X \\
\le& a_\dagger \bar{S^0}\bar{\beta} \left\|(-A_n)^{-1}P_n\varphi - P_n(-A)^{-1}\varphi\right\|_{X_n} + \left\|J_nF_nP_n(-A)^{-1}\varphi - F(-A)^{-1}\varphi\right\|_X, \qquad (2.4)
\end{aligned}$$

where note that $\|J_n\| \le 1$ and $\|F_n\| \le a_\dagger \bar{S^0}\bar{\beta}$ for all $n \in \mathbb{N}$. For the first term in the right-hand side of (2.4), we have

$$\begin{aligned}
\left\|(-A_n)^{-1}P_n\varphi - P_n(-A)^{-1}\varphi\right\|_{X_n} =& \left\|(-A_n)^{-1}P_n(-A)(-A)^{-1}\varphi - (-A_n)^{-1}(-A_n)P_n(-A)^{-1}\varphi\right\|_{X_n} \\
\le& \left\|(-A_n)^{-1}\right\| \left\|P_n(-A)(-A)^{-1}\varphi - (-A_n)P_n(-A)^{-1}\varphi\right\|_{X_n} \\
\le& a_\dagger \|P_n(-A)\phi - (-A_n)P_n\phi\|_{X_n},
\end{aligned}$$

where $\phi := (-A)^{-1}\varphi \in D(A)$ and note that $\left\|(-A_n)^{-1}\right\| \le a_\dagger$ since

$$\left\|(-A_n)^{-1}\psi\right\|_{X_n} \le \Delta a \sum_{k=1}^{n} \frac{1}{\underline{\mu} + \underline{\gamma} + 1/\Delta a} \sum_{j=1}^{k} |\psi_j| \le \Delta a \sum_{k=1}^{n} \Delta a \sum_{j=1}^{n} |\psi_j| = a_\dagger \|\psi\|_{X_n}$$

for all $\psi = (\psi_1, \psi_2, \cdots, \psi_n)^T \in X_n$. Hence, we have

$$\begin{aligned}
\left\|(-A_n)^{-1}P_n\varphi - P_n(-A)^{-1}\varphi\right\|_{X_n} \le& a_\dagger \Delta a \sum_{k=1}^{n} \Bigg| \frac{1}{\Delta a} \int_{a_{k-1}}^{a_k} \left(\frac{d}{da}\phi(a) + (\mu(a) + \gamma(a))\phi(a) \right) da \\
& - \frac{\frac{1}{\Delta a}\int_{a_{k-1}}^{a_k} \phi(a)da - \frac{1}{\Delta a}\int_{a_{k-2}}^{a_{k-1}} \phi(a)da}{\Delta a} - \frac{\mu_k + \gamma_k}{\Delta a} \int_{a_{k-1}}^{a_k} \phi(a)da \Bigg|,
\end{aligned}$$

where $a_0 = a_{-1} = 0$. By using the mean value theorem and the Taylor's formula, we have

$$\begin{aligned}
\left\|(-A_n)^{-1}P_n\varphi - P_n(-A)^{-1}\varphi\right\|_{X_n} \le& a_\dagger \Delta a \sum_{k=1}^{n} \Big| \frac{d}{da}\phi(\eta_k) + (\mu(\eta_k) + \gamma(\eta_k))\phi(\eta_k) \\
& - \frac{1}{\Delta a}(\phi(\xi_k) - \phi(\xi_{k-1})) - (\mu_k + \gamma_k)\phi(\zeta_k) \Big| \\
\le a_\dagger \Delta a \sum_{k=1}^{n} & \left(\left| \frac{d}{da}\phi(\eta_k) - \frac{d}{da}\phi(\sigma_k) \right| + |(\mu(\eta_k) + \gamma(\eta_k))\phi(\eta_k) - (\mu_k + \gamma_k)\phi(\zeta_k)| \right) \\
\le a_\dagger^2 & \left[\omega(\phi', 2\Delta a) + \omega(\mu + \gamma, \Delta a)\omega(\phi, \Delta a) \right] \to 0 \quad \text{as} \quad n \to +\infty,
\end{aligned}$$

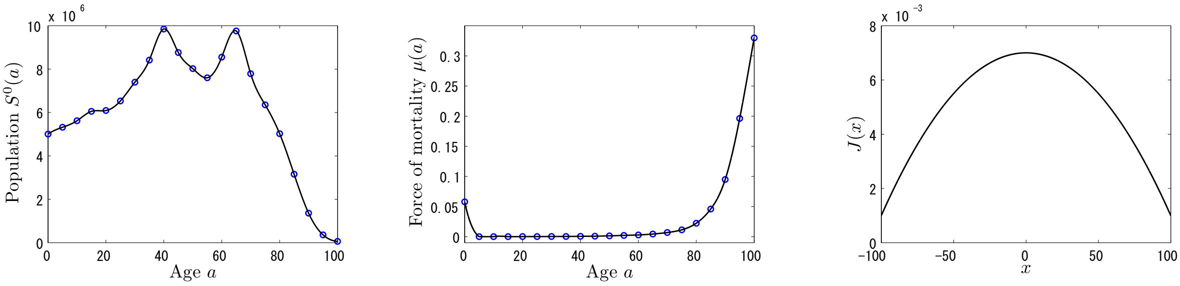
where $\omega(f, r)$ denotes the modulus of continuity defined by $\sup_{|x-y|\le r} |f(x) - f(y)|$. For the second term in the right-hand side of (2.4), we have

$$\begin{aligned}
\left\|J_nF_nP_n(-A)^{-1}\varphi - F(-A)^{-1}\varphi\right\|_X =& \|J_nF_nP_n\phi - F\phi\|_X \\
=& \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} \left| \sum_{j=1}^{n} S_k^0 \beta_{kj} \int_{a_{j-1}}^{a_j} \phi(\sigma)d\sigma - \int_0^{a_\dagger} S^0(a)\beta(a,\sigma)\phi(\sigma)d\sigma \right| da \\
\le& \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} \left|S_k^0\beta_{kj} - S^0(a)\beta(a,\sigma)\right| |\phi(\sigma)| d\sigma da \\
\le& a_\dagger \omega(S^0, \Delta a)\omega(\beta, \Delta a) \|\phi\|_X \to 0 \quad \text{as} \quad n \to +\infty.
\end{aligned}$$

Hence, from (2.4), we see that $\lim_{n\to+\infty} \|J_nK_nP_n\varphi - K\varphi\|_X = 0$. □

In conclusion, from Propositions 2.1 and 2.2, we establish the following main theorem.

Theorem 2.1. *Suppose that the additional assumption* (2.3) *holds. Then,* $\mathcal{R}_{0,n} \to \mathcal{R}_0$ *as* $n \to +\infty$, *preserving algebraic multiplicity* 1.



(a) Total (susceptible) population $S^0(a)$ determined as the spline curve interpolating the data of the population estimates by age in Japan, December 1, 2015 [21].

(b) Force of mortality $\mu(a)$ determined as the spline curve interpolating the data of the 22nd complete life tables in Japan, 2015 [22].

(c) Normalized distance function $J(x) = 0.6\left(-x^2 + 100^2\right) \times 10^{-6} + 0.001$ by which $\beta(a, \sigma) = kJ(a - \sigma)$ implies that the disease transmission is more likely to occur between individuals with similar age.

Figure 1: Parameters used in the numerical example.

3. Numerical example

In this section, we give a numerical example to illustrate Theorem 2.1. We consider a situation where an emerging infectious disease invades a fully susceptible population. For $S^0(a)$, we use the population estimates by age (5-year age group) in Japan, December 1, 2015 [21]. We determine $S^0(a)$ as the spline curve interpolating the data (see Figure 1 (a)). For $\mu(a)$, we use the 22nd complete life tables in Japan, 2015 [22]. Similar to $S^0(a)$, we determine $\mu(a)$ as the spline curve interpolating the data (see Figure 1 (b)). Let the unit of time be a year. In this example, we do not specify the disease and assume the invasion of an influenza-like disease, which $\mathcal{R}_0$ is in the range of $2 - 3$ [23] and average infectious period is 1 week $= 1/52$ year. Thus, $\gamma(a) = \gamma = 52$ since $1/\gamma$ is the average infectious period. Under the assumption that the disease transmission is more likely to occur between individuals with similar age, we set $\beta(a, \sigma) = kJ(a - \sigma)$, where $k > 0$ is a fitting parameter and $J(x)$ is a normalized distance function given by $J(x) = 0.6\left(-x^2 + 100^2\right) \times 10^{-6} + 0.001$, which integration $\int_{-100}^{100} J(x)dx$ is approximately equal to 1 (see Figure 1 (c)). In this setting, we can easily check that the assumption (2.3) holds. In fact, from the continuity of $S^0(a)$, we have

$$
\begin{aligned}
&\left|S^0(a+h)\beta(a+h,\sigma) - S^0(a)\beta(a,\sigma)\right| \\
&\le 0.6k \times 10^{-6} \left|-S^0(a+h)(a+h-\sigma)^2 + S^0(a)(a-\sigma)^2 + 100^2(S^0(a+h) - S^0(a))\right| + 0.001k\left|S^0(a+h) - S^0(a)\right| \\
&\le 0.6k \times 10^{-6} \left|(-S^0(a+h) + S^0(a))(a-\sigma)^2 - S^0(a+h)2h(a-\sigma) - h^2 S^0(a+h)\right| + 0.007k\left|S^0(a+h) - S^0(a)\right| \\
&\le \left(0.6k \times 10^{-6}\omega^2 + 0.007k\right)\left|S^0(a+h) - S^0(a)\right| + \left(2\omega h + h^2\right)\bar{S^0} \to 0 \quad \text{as} \quad h \to 0 \quad \text{uniformly for } \sigma \in \mathbb{R},
\end{aligned}
$$

and (2.3) holds. Hence, from Theorem 2.1, we can expect that $\mathcal{R}_{0,n} \to \mathcal{R}_0$ as $n \to +\infty$. In fact, when $k = 3 \times 10^{-5}$, we calculate $\mathcal{R}_{0,n}$ for each n, and obtain Figure 2 and Table 1. In this example, we chose $\mathcal{R}_{0,10000} \approx 2.288887722281441 =: \mathcal{R}^*$ as a reference value for $\mathcal{R}_0$, and we see that the error $\mathcal{R}^* - \mathcal{R}_{0,n}$ converges to zero as n increases. This implies that $\mathcal{R}_0 \approx 2.288887722281441$.

Remark. In the discretization in this study, we have employed the basic Euler method for the sake of simplicity. To employ higher order schemes would make the convergence faster. For such schemes, the proof of the pointwise convergence as in Proposition 2.2 would become more challenging. We leave it as an important future work.

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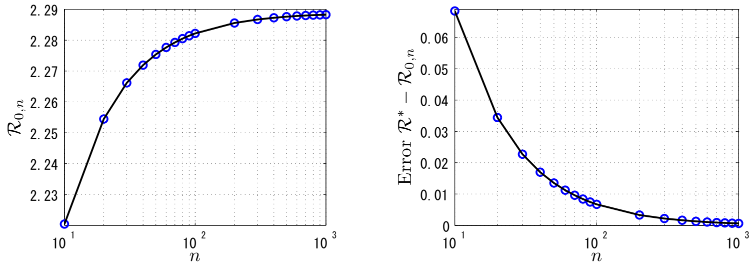

Figure 2: Logarithmic plots of the threshold value $\mathcal{R}_{0,n}$ for the discretized system (left) and the error $\mathcal{R}^* - \mathcal{R}_{0,n}$ with respect to the reference value $\mathcal{R}^* = 2.288887722281441$ (right).

n	$\mathcal{R}_{0,n}$
1	0.024876292690764
5	2.226400640236108
10	2.220410578405053
50	2.275364513811083
100	2.282192345750525
500	2.287607948110380
1000	2.288281821694171
$\mathcal{R}^*$	2.288887722281441

Table 1: Numerical value (with 15 digits) of $\mathcal{R}_{0,n}$.

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