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Sonoda, H.

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Conformal invariance for Wilson actions

H. Sonoda*

Physics Department, Kobe University, Kobe 657-8501, Japan

*E-mail: hsonoda@kobe-u.ac.jp

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We discuss the realization of conformal invariance for Wilson actions using the formalism of the exact renormalization group. This subject has been studied extensively in the recent works of O. J. Rosten. The main purpose of this paper is to reformulate Rosten's formulas for conformal transformations using a method developed earlier for the realization of any continuous symmetry in the exact renormalization group formalism. The merit of the reformulation is simplicity and transparency via the consistent use of equation-of-motion operators. We derive equations that imply the invariance of the Wilson action under infinitesimal conformal transformations which are non-linearly realized but form a closed conformal algebra. The best effort has been made to make the paper self-contained; ample background on the formalism is provided.
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Subject Index B30, B32, B39

1. Introduction

The study of conformally invariant field theories (in dimensions $D > 2$) was initiated long ago by J. Wess [1] with the hope that conformal invariance constrains a theory more than scale invariance, since the latter is implied by the former. The requirement for conformal invariance seemed much stronger than that for scale invariance at first sight, but the difference turned out to be subtle. In the seminal work [2], J. Polchinski showed the equivalence of conformal invariance to the vanishing of the trace of the energy–momentum tensor; scale invariance requires the vanishing of only its integral. The question of whether scale invariance implies conformal invariance has attracted much attention lately, and we would like to refer the reader to a recent review by Y. Nakayama [3] and references therein.

The subject of this paper is realization of conformal symmetry using Wilson actions [4]. This was recently taken up by O. J. Rosten [5], and also by Delamotte, Tissier, and Wschebor [6]. Rosten has extended his work further in Refs. [7,8]. It is the recent works of Rosten (especially [5] and [8]) that we wish to improve upon by using the method of symmetry realization developed and reviewed in Ref. [9]. We aim to add simplicity and transparency to the structure of conformal transformations in the exact renormalization group (ERG) formalism.

Wilson actions come with a finite momentum cutoff, and it is generally accepted that only the physics at scales below the cutoff is effectively described by Wilson actions. This is indeed the case with a generic Wilson action, but there are exceptions. Those Wilson actions flowing out of a fixed point under the renormalization group transformations correspond to a continuum limit, and the physics at all momentum scales are described by the Wilson actions. (In Ref. [4], these Wilson actions form a finite-dimensional space $S(\infty)$.) Hence, if the continuum limit of a theory has symmetry, we can realize the symmetry using its Wilson action. Now, a fixed point of the renormalization group

transformation is a continuum limit. If the limit possesses conformal symmetry, its Wilson action must realize the symmetry, too.

The method of Ref. [9] has recently been applied to the construction of the energy–momentum tensor in Ref. [10]. Our expression of special conformal transformation (15d) was in fact first derived there from the assumption of a vanishing trace. We summarize this derivation in Appendix C.

We organize the paper as follows. In Sect. 2 we introduce infinitesimal conformal transformations of the elementary scalar field in D -dimensional Euclidean space. In Sect. 3, we quickly review how to express continuous symmetry of a Wilson action in terms of equation-of-motion composite operators. Then, in Sect. 4, we construct equation-of-motion composite operators for the conformal symmetry, and subsequently in Sect. 5 we construct the products of the infinitesimal transformations to show the closure of the algebra. Sections 4 and 5 constitute the main part of this paper. In Sect. 6, we rewrite the invariance of the Wilson action as that of the associated generating functional and one-particle irreducible (1PI) action. In Sect. 7 we construct the 1PI action of a Wilson–Fisher fixed point in $D = 4 - \epsilon$ dimensions to first order in ϵ . We extend the conformal transformation to the scalar composite operators in Sect. 8, before we conclude the paper in Sect. 9.

We have kept the main text reasonably short by relegating the technicalities to five appendices. This has been done to make this technical paper an easy read; the first reading of the main text is best done without referring to the appendices. We have adopted the notation

$$\int_p = \int \frac{d^D p}{(2\pi)^D}, \quad \delta(p) = (2\pi)^D \delta^{(D)}(p), \quad p \cdot q = p_\mu q_\mu = \sum_{\mu=1}^D p_\mu q_\mu \quad (1)$$

to simplify the formulas.

2. Conformal algebra

We consider a real scalar field theory in D -dimensional Euclidean space. We first consider the field in coordinate space. Infinitesimal conformal transformations act on the field as follows [1]:

$$D_\mu^T \phi(x) \equiv \frac{1}{i} \partial_\mu \phi(x), \quad (2a)$$

$$D_{\mu\nu}^R \phi(x) \equiv (x_\mu \partial_\nu - x_\nu \partial_\mu) \phi(x), \quad (2b)$$

$$D^S \phi(x) \equiv \left(x_\mu \partial_\mu + \frac{D-2}{2} + \gamma \right) \phi(x), \quad (2c)$$

$$D_\mu^K \phi(x) \equiv \frac{1}{i} \left(x_\mu x_\nu \partial_\nu - \frac{1}{2} x^2 \partial_\mu + \left(\frac{D-2}{2} + \gamma \right) x_\mu \right) \phi(x), \quad (2d)$$

where $\frac{D-2}{2} + \gamma$ is the full scale dimension of the scalar field, including the anomalous dimension γ . We have chosen the superscript T for translation, R for rotation, S for scale transformation, and K for the special conformal transformation that results from the succession of inversion, translation, and inversion. The algebra of the differential operators is closed, and is called the conformal algebra [1]:

$$[D_\mu^T, D_\nu^T] = 0, \quad (3a)$$

$$[D_{\alpha\beta}^R, D_{\gamma\delta}^R] = \delta_{\beta\gamma} D_{\alpha\delta}^R - \delta_{\beta\delta} D_{\alpha\gamma}^R - \delta_{\alpha\gamma} D_{\beta\delta}^R + \delta_{\alpha\delta} D_{\beta\gamma}^R, \quad (3b)$$

$$[D_{\mu\nu}^R, D_\alpha^T] = -\delta_{\mu\alpha} D_\nu^T + \delta_{\nu\alpha} D_\mu^T, \quad (3c)$$

$$[D^S, D_\mu^T] = D_\mu^T, \quad (3d)$$

$$[D^S, D_{\mu\nu}^R] = 0, \quad (3e)$$

$$[D_\mu^K, D_\nu^K] = 0, \quad (3f)$$

$$[D_\mu^K, D_\nu^T] = D^S \delta_{\mu\nu} + D_{\mu\nu}^R, \quad (3g)$$

$$[D_\mu^K, D_{\alpha\beta}^R] = \delta_{\mu\alpha} D_\beta^K - \delta_{\mu\beta} D_\alpha^K, \quad (3h)$$

$$[D_\mu^K, D^S] = -D_\mu^K. \quad (3i)$$

We formulate the Wilson action in momentum space; it is more convenient to rewrite the above transformations in momentum space. Denoting the Fourier transform of the scalar field by

$$\phi(p) \equiv \int d^D x e^{-ipx} \phi(x), \quad (4)$$

we obtain

$$D_\mu^T(p)\phi(p) = p_\mu \phi(p), \quad (5a)$$

$$D_{\mu\nu}^R(p)\phi(p) = \left(p_\mu \frac{\partial}{\partial p_\nu} - p_\nu \frac{\partial}{\partial p_\mu} \right) \phi(p), \quad (5b)$$

$$D^S(p)\phi(p) = \left(-p_\mu \frac{\partial}{\partial p_\mu} - \frac{D+2}{2} + \gamma \right) \phi(p), \quad (5c)$$

$$D_\mu^K(p)\phi(p) = \left(-p_\nu \frac{\partial^2}{\partial p_\mu \partial p_\nu} + \frac{1}{2} p_\mu \frac{\partial^2}{\partial p_\nu \partial p_\nu} + \left(-\frac{D+2}{2} + \gamma \right) \frac{\partial}{\partial p_\mu} \right) \phi(p). \quad (5d)$$

The above $D(p)$'s obey the same conformal algebra as Eq. (3); for example, we obtain

$$[D_\mu^K(p), D_\nu^T(p)] = D^S(p) \delta_{\mu\nu} + D_{\mu\nu}^R(p). \quad (6)$$

3. Invariance of a Wilson action

The infinitesimal conformal transformations are linear transformations of the scalar field. There is no guarantee, however, that they are realized as linear transformations for the Wilson action. Suppose that the Wilson action $S[\phi]$ is “invariant” under an infinitesimal transformation $\Delta\phi(p)$ of the field variable $\phi(p)$. Since the exponentiated Wilson action e^S is the measure of functional integration, the invariance of the theory under the infinitesimal transformation amounts to

$$\int_p \left(\Delta\phi(p) \frac{\delta S}{\delta\phi(p)} + \frac{\delta}{\delta\phi(p)} \Delta\phi(p) \right) = 0, \quad (7)$$

where the second term comes from the Jacobian. This can be written as

$$\int_p \frac{\delta}{\delta\phi(p)} (\Delta\phi(p) e^S) = 0. \quad (8)$$

In the ERG formalism we choose

$$\Delta\phi(p) = K(p) \mathcal{O}(p). \quad (9)$$

$K(p)$ is a positive momentum cutoff function: it depends only on p^2 , is nearly 1 for momenta low compared with the cutoff $p = 1$, and decreases rapidly for $p \gg 1$. $\mathcal{O}(p)$ is a composite operator (i.e., a functional of ϕ) with momentum p . Using the above $\Delta\phi$, we obtain the invariance as

$$\int_p K(p) \frac{\delta}{\delta\phi(p)} (\mathcal{O}(p) e^S) = 0. \quad (10)$$

This is the general form of the equation of motion in the ERG formalism. The equation of motion implies the Ward–Takahashi identity for the correlation functions:

$$\sum_{i=1}^n \langle \phi(p_1) \cdots \mathcal{O}(p_i) \cdots \phi(p_n) \rangle = 0, \quad (11)$$

where the i th ϕ is replaced by \mathcal{O} . Note that we use $\langle \cdots \rangle$ for the continuum limit of correlation functions. We refer the reader to Appendices A and B, where we give technical details on the ERG formalism such as modified correlation functions and equation-of-motion composite operators.

4. Conformal invariance

For infinitesimal conformal transformations, we choose $\mathcal{O}(p)$ of the previous section as $D(p)\Phi(p)$, where $D(p)$ is one of the $D_\mu^T(p)$, $D_{\mu\nu}^R(p)$, $D^S(p)$, or $D_\mu^K(p)$ introduced in Sect. 2. $\Phi(p)$ is a composite operator defined by

$$\Phi(p) \equiv e^{-S} \frac{1}{K(p)} \left(\phi(p) + \frac{k(p)}{p^2} \frac{\delta}{\delta\phi(-p)} \right) e^S, \quad (12)$$

where S is the Wilson action, and K, k are cutoff functions. $\Phi(p)$ has the same correlation functions as the elementary field $\phi(p)$:

$$\langle \Phi(p) \phi(p_1) \cdots \phi(p_n) \rangle = \langle \phi(p) \phi(p_1) \cdots \phi(p_n) \rangle. \quad (13)$$

See Appendix A for the precise definition of both sides. Hence,

$$\begin{aligned} \langle D(p)\Phi(p) \phi(p_1) \cdots \phi(p_n) \rangle &\equiv D(p) \langle \Phi(p) \phi(p_1) \cdots \phi(p_n) \rangle \\ &= D(p) \langle \phi(p) \phi(p_1) \cdots \phi(p_n) \rangle. \end{aligned} \quad (14)$$

We now introduce the following the equation-of-motion composite operators:

$$\Sigma_\mu^T \equiv -e^{-S} \int_p K(p) \frac{\delta}{\delta\phi(p)} (D_\mu^T(p) \Phi(p) e^S), \quad (15a)$$

$$\Sigma_{\mu\nu}^R \equiv -e^{-S} \int_p K(p) \frac{\delta}{\delta\phi(p)} (D_{\mu\nu}^R(p) \Phi(p) e^S), \quad (15b)$$

$$\Sigma^S \equiv -e^{-S} \int_p K(p) \frac{\delta}{\delta\phi(p)} (D^S(p) \Phi(p) e^S), \quad (15c)$$

$$\Sigma_\mu^K \equiv -e^{-S} \int_p K(p) \frac{\delta}{\delta\phi(p)} (D_\mu^K(p) \Phi(p) e^S). \quad (15d)$$

These carry no momentum. The conformal invariance amounts to the vanishing of the above operators:

$$\Sigma_\mu^T = \Sigma_{\mu\nu}^R = \Sigma^S = \Sigma_\mu^K = 0. \quad (16)$$

Substituting these into the correlation functions, we obtain the following Ward–Takahashi identities:

$$\sum_{i=1}^n D_{\mu}^T(p_i) \langle \phi(p_1) \cdots \phi(p_n) \rangle = 0, \quad (17a)$$

$$\sum_{i=1}^n D_{\mu\nu}^R(p_i) \langle \phi(p_1) \cdots \phi(p_n) \rangle = 0, \quad (17b)$$

$$\sum_{i=1}^n D^S(p_i) \langle \phi(p_1) \cdots \phi(p_n) \rangle = 0, \quad (17c)$$

$$\sum_{i=1}^n D_{\mu}^K(p_i) \langle \phi(p_1) \cdots \phi(p_n) \rangle = 0. \quad (17d)$$

Note that the scale invariance, given by $\Sigma^S = 0$, is nothing but the ERG differential equation for a fixed-point Wilson action, which is usually given in the form [4]

$$\begin{aligned} 0 = & \int_p \left(-p \cdot \partial_p \ln K(p) + \frac{D+2}{2} - \gamma + p \cdot \partial_p \right) \phi(p) \cdot \frac{\delta}{\delta \phi(p)} e^S \\ & + \int_p \left(p \cdot \partial_p \ln \frac{k(p)}{K(p)^2} - 2\gamma \right) \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} e^S. \end{aligned} \quad (18)$$

This rewriting is explained in Appendix B of Ref. [10].

As for the special conformal invariance $\Sigma_{\mu}^K = 0$, an equivalent formula was first derived by Rosten as (3.25) in Ref. [5]. The particular form Σ_{μ}^K given by Eq. (15d) was first obtained in Ref. [10]. (This is briefly explained in Appendix C.) Rosten has rewritten his result as (2.79b) in Ref. [7]. We will explain how to derive (a formula similar to) his (2.79b) by rewriting our $\Sigma_{\mu}^K = 0$ in Appendix E.

5. Realization of the conformal algebra

So far we have only discussed the invariance of the Wilson action under infinitesimal conformal transformations. The conformal transformations form a closed algebra, and the algebraic structure must be realized on the Wilson action.

For realization of the algebra, we need the product of two infinitesimal transformations. Let D_i ($i = 1, 2$) be two of the infinitesimal transformations $D_{\mu}^T, D_{\mu\nu}^R, D^S, D_{\mu}^K$, and we denote

$$\Sigma_i \equiv -e^{-S} \int_p K(p) \frac{\delta}{\delta \phi(p)} (D_i(p) \Phi(p) e^S). \quad (19)$$

We construct the product as

$$\Sigma_1 * \Sigma_2 \equiv -e^{-S} \int_p K(p) \frac{\delta}{\delta \phi(p)} \{ D_1(p) [\Phi(p) \Sigma_2] e^S \}, \quad (20)$$

where

$$[\Phi(p) \Sigma_2] \equiv \Phi(p) \Sigma_2 + \frac{k(p)}{p^2 K(p)} \frac{\delta \Sigma_2}{\delta \phi(-p)} \quad (21)$$

is a composite operator corresponding to the product of $\Phi(p)$ and Σ_2 (see Appendix A). Using

$$\left\langle [\Phi(p_i) \Sigma_2] \phi(p_1) \cdots \widehat{\phi(p_i)} \cdots \phi(p_n) \right\rangle = \langle \Sigma_2 \phi(p_1) \cdots \phi(p_n) \rangle \quad (22)$$

(where the hat above $\phi(p_i)$ implies omission), we obtain

$$\begin{aligned} & \langle \Sigma_1 * \Sigma_2 \phi(p_1) \cdots \phi(p_n) \rangle \\ &= \sum_{i=1}^n D_1(p_i) \langle \phi(p_1) \cdots [\Phi(p_i) \Sigma_2] \cdots \phi(p_n) \rangle \\ &= \sum_{i=1}^n D_1(p_i) \langle \Sigma_2 \phi(p_1) \cdots \phi(p_n) \rangle \\ &= \sum_{i=1}^n D_1(p_i) \sum_{j=1}^n D_2(p_j) \langle \phi(p_1) \cdots \phi(p_n) \rangle \\ &= \sum_{i=1}^n \left(D_1(p_i) D_2(p_i) + \sum_{j \neq i} D_1(p_i) D_2(p_j) \right) \langle \phi(p_1) \cdots \phi(p_n) \rangle. \end{aligned} \quad (23)$$

Therefore, we obtain

$$\langle (\Sigma_1 * \Sigma_2 - \Sigma_2 * \Sigma_1) \phi(p_1) \cdots \phi(p_n) \rangle = \sum_{i=1}^n [D_1(p_i), D_2(p_i)] \langle \phi(p_1) \cdots \phi(p_n) \rangle. \quad (24)$$

This implies that

$$\Sigma_1 * \Sigma_2 - \Sigma_2 * \Sigma_1 = -e^{-S} \int_p K(p) \frac{\delta}{\delta \phi(p)} ([D_1(p), D_2(p)] \Phi(p) e^S). \quad (25)$$

Hence, the algebra of the D 's translates into the algebra of the Σ 's.

The higher products of the Σ 's can be defined recursively as

$$\Sigma_1 * \Sigma_2 * \cdots * \Sigma_I \equiv - \int_p K(p) \frac{\delta}{\delta \phi(p)} \{ D_1(p) [\Phi(p) \Sigma_2 * \cdots * \Sigma_I] e^S \}, \quad (26)$$

so that

$$\begin{aligned} & \langle \Sigma_1 * \Sigma_2 * \cdots * \Sigma_I \phi(p_1) \cdots \phi(p_n) \rangle \\ &= \sum_{i_1=1}^n D_1(p_{i_1}) \sum_{i_2=1}^n D_2(p_{i_2}) \cdots \sum_{i_I=1}^n D_I(p_{i_I}) \langle \phi(p_1) \cdots \phi(p_n) \rangle. \end{aligned} \quad (27)$$

6. Conformal symmetry for the generating functional and 1PI action

We wish to rewrite the equation-of-motion composite operators (15) in terms of the generating functional $W[J]$ of connected correlations and the 1PI action $\Gamma[\Phi]$ associated with the Wilson action $S[\phi]$. The Wilson action results from the integration of the field with momenta above $p = 1$; the field with momenta below $p = 1$ has not been integrated for the generating functional $W[J]$ and 1PI action $\Gamma[\Phi]$. $W[J]$ and $\Gamma[\Phi]$ depend only on a particular combination of the two cutoff functions,

$$R(p) \equiv \frac{p^2}{k(p)} K(p)^2, \quad (28)$$

which is non-vanishing (if not divergent) at $p = 0$, and decreases rapidly for $p \gg 1$. ($R(p)$ is often called a scale-dependent squared mass in the ERG literature.)

Formulas necessary for rewriting conformal invariance of $S[\phi]$ as that of $W[J]$ have been summarized in Appendix B:

$$J(p) \equiv \frac{R(p)}{K(p)}\phi(p), \quad \frac{\delta W[J]}{\delta J(-p)} = \Phi(p), \quad W[J] = S[\phi] + \frac{1}{2} \int_p \frac{1}{R(p)} J(p) J(-p). \quad (29)$$

Similarly, the formulas

$$\begin{cases} J(p) = -R(p)\Phi(p) + \frac{\delta\Gamma[\Phi]}{\delta\Phi(-p)}, & \frac{\delta^2 W[J]}{\delta J(p)\delta J(q)} = G_{p,q}[\Phi], \\ \Gamma[\Phi] - \frac{1}{2} \int_p R(p)\Phi(p)\Phi(-p) = W[J] - \int_p J(p)\Phi(-p), \end{cases} \quad (30)$$

which are necessary for rewriting the conformal invariance of $W[J]$ as that of $\Gamma[\Phi]$, are also summarized in Appendix B. Assuming the rotational invariance of $R(p)$ and $R(-p) = R(p)$, we obtain the following results:

(1) T (translation invariance)

$$\int_p D_\mu^T(p) J(p) \cdot \frac{\delta W[J]}{\delta J(p)} = 0, \quad (31a)$$

$$\int_p D_\mu^T(p) \Phi(p) \cdot \frac{\delta\Gamma[\Phi]}{\delta\Phi(p)} = 0. \quad (31b)$$

(2) R (rotation invariance)

$$\int_p D_{\mu\nu}^R(p) J(p) \cdot \frac{\delta W[J]}{\delta J(p)} = 0, \quad (32a)$$

$$\int_p D_{\mu\nu}^R(p) \Phi(p) \cdot \frac{\delta\Gamma[\Phi]}{\delta\Phi(p)} = 0. \quad (32b)$$

(3) S (scale invariance)

$$\int_p J(-p) D^S(p) \frac{\delta W[J]}{\delta J(-p)} \quad (33a)$$

$$\begin{aligned} &+ \int_p (-p \cdot \partial_p + 2 - 2\gamma) R(p) \cdot \frac{1}{2} \left\{ \frac{\delta^2 W[J]}{\delta J(p)\delta J(-p)} + \frac{\delta W[J]}{\delta J(p)} \frac{\delta W[J]}{\delta J(-p)} \right\} = 0, \\ &- \int_p \frac{\delta\Gamma[\Phi]}{\delta\Phi(p)} D^S(p) \Phi(p) + \int_p (-p \cdot \partial_p + 2 - 2\gamma) R(p) \cdot \frac{1}{2} G_{p,-p}[\Phi] = 0, \end{aligned} \quad (33b)$$

where the integrals with R have been simplified by partial integration.

(4) K (special conformal invariance)

$$\int_p J(-p) D_\mu^K(p) \frac{\delta W[J]}{\delta J(-p)} \quad (34a)$$

$$\begin{aligned} &+ \frac{1}{2} \int_p (-p \cdot \partial_p + 2 - 2\gamma) R(p) \cdot \frac{\partial}{\partial p_\mu} \left\{ \frac{\delta^2 W[J]}{\delta J(p)\delta J(-q)} + \frac{\delta W[J]}{\delta J(p)} \frac{\delta W[J]}{\delta J(-q)} \right\} \Big|_{q=p} = 0, \\ &- \int_p \frac{\delta\Gamma[\Phi]}{\delta\Phi(p)} D_\mu^K(p) \Phi(p) + \frac{1}{2} \int_p (-p \cdot \partial_p + 2 - 2\gamma) R(p) \cdot \frac{\partial G_{-p,q}[\Phi]}{\partial p_\mu} \Big|_{q=p} = 0, \end{aligned} \quad (34b)$$

where q is set equal to p only after the derivative is taken. The integrals with R have been simplified by partial integration. This step is explained in Appendix D.

The first two types of invariance are free of the cutoff function R . In fact, the invariance of the Wilson action under translation and rotation can also be written without R [10]:

$$\int_p D_\mu^T(p) \phi(p) \cdot \frac{\delta S[\phi]}{\delta \phi(p)} = 0, \quad (35a)$$

$$\int_p D_{\mu\nu}^R(p) \phi(p) \cdot \frac{\delta S[\phi]}{\delta \phi(p)} = 0. \quad (35b)$$

On the other hand, the invariance under the scale and special conformal transformations depends non-trivially on the cutoff function R .

As for the special conformal invariance, Eq. (34b) for Γ was obtained by Rosten as (4.16) in Ref. [8]. A similar expression was also derived as (10) in Ref. [6].

7. Wilson–Fisher fixed point to order ϵ

As a concrete example, we consider the Wilson-Fisher fixed point in $D = 4 - \epsilon$ dimensions, and construct a conformally invariant 1PI action Γ to first order in ϵ . Assuming $\gamma = 0$ at this order, we obtain the following equations from Eqs. (33) and (34):

(1) Scale invariance

$$\int_p \left(\frac{D+2}{2} + p \cdot \partial_p \right) \Phi(p) \cdot \frac{\delta \Gamma[\Phi]}{\delta \Phi(p)} + \int_p (2 - p \cdot \partial_p) R(p) \cdot \frac{1}{2} G_{p,-p}[\Phi] = 0. \quad (36)$$

(2) Special conformal invariance

$$\begin{aligned} & \int_p \left(p_\nu \frac{\partial^2}{\partial p_\mu \partial p_\nu} - \frac{1}{2} p^\mu \frac{\partial^2}{\partial p_\nu \partial p_\nu} + \frac{D+2}{2} \frac{\partial}{\partial p_\mu} \right) \Phi(p) \cdot \frac{\delta \Gamma[\Phi]}{\delta \Phi(p)} \\ & + \int_p (2 - p \cdot \partial_p) R(p) \cdot \frac{1}{2} \frac{\partial}{\partial p_\mu} G_{-p,q}[\Phi] \Big|_{q=p} = 0. \end{aligned} \quad (37)$$

We will solve these equations with the ansatz

$$\Gamma[\Phi] = -\frac{1}{2} \int_p (p^2 + m^2) \Phi(p) \Phi(-p) - \lambda \frac{1}{4!} \int_{p_1, \dots, p_4} \Phi(p_1) \cdots \Phi(p_4) \delta(p_1 + \cdots + p_4), \quad (38)$$

where m^2, λ are both of order ϵ . Note that this is automatically invariant under translation and rotation.

The high-momentum propagator $G_{-p,q}[\Phi]$ is now defined by

$$\begin{aligned} & \int_q G_{-p,q}[\Phi] \left((q^2 + m^2 + R(q)) \delta(q - r) + \frac{\lambda}{2} \int_{p_1, p_2} \Phi(p_1) \Phi(p_2) \delta(p_1 + p_2 - q + r) \right) \\ & = \delta(p - r), \end{aligned} \quad (39)$$

and it is obtained as

$$\begin{aligned} G_{-p,q}[\Phi] &= \frac{1}{p^2 + R(p)} \delta(p - q) - \frac{m^2}{(p^2 + R(p))^2} \delta(p - q) \\ & - \lambda \frac{1}{p^2 + R(p)} \frac{1}{q^2 + R(q)} \frac{1}{2} \int_{p_1, p_2} \Phi(p_1) \Phi(p_2) \delta(p_1 + p_2 - p + q) \end{aligned} \quad (40)$$

up to first order in ϵ .

7.1. Scale invariance

Substituting Eq. (38) into Eq. (36), we obtain two equations, one quadratic in Φ and the other quartic in Φ . The latter is given by

$$\frac{\lambda}{4!} \int_{p_1, \dots, p_4} \Phi(p_1) \cdots \Phi(p_4) \sum_{i=1}^4 \left(\frac{D-2}{2} + p_i \cdot \partial_{p_i} \right) \delta(p_1 + \cdots + p_4) = 0. \quad (41)$$

This gives

$$\lambda(4-D) = 0, \quad (42)$$

which is trivially satisfied to order ϵ . We are now left with

$$\frac{1}{2} \int_p \Phi(p) \Phi(-p) \left\{ (2 - p \cdot \partial_p) (p^2 + m^2) + \frac{\lambda}{2} \int_q \frac{(2 - q \cdot \partial_q) R(q)}{(q^2 + R(q))^2} \right\} = 0. \quad (43)$$

This is solved by

$$m^2 = -\frac{\lambda}{4} \int_q \frac{(2 - q \cdot \partial_q) R(q)}{(q^2 + R(q))^2}. \quad (44)$$

7.2. Special conformal invariance

Substituting Eq. (38) into Eq. (37), we obtain two equations, one quadratic in Φ and the other quartic in Φ . The latter is given by

$$\begin{aligned} & \frac{\lambda}{4!} \int_{p_1, \dots, p_4} \delta(p_1 + \cdots + p_4) \\ & \times \sum_{i=1}^4 \left(p_{iv} \frac{\partial^2}{\partial p_{i\mu} \partial p_{iv}} - \frac{1}{2} p_{i\mu} \frac{\partial^2}{\partial p_{iv} \partial p_{i\mu}} + \frac{D+2}{2} \frac{\partial}{\partial p_{i\mu}} \right) \cdot \Phi(p_1) \cdots \Phi(p_4) = 0. \end{aligned} \quad (45)$$

Using

$$\frac{\partial}{\partial p_{i\mu}} \delta(p_1 + \cdots + p_4) = \frac{\partial}{\partial p_{1\mu}} \delta(p_1 + \cdots + p_4) \quad (46)$$

(independent of i), we obtain

$$\frac{\lambda}{4!} \int_{p_1, \dots, p_4} \Phi(p_1) \cdots \Phi(p_4) (D-4) \frac{\partial}{\partial p_{1\mu}} \delta(p_1 + \cdots + p_4) = 0, \quad (47)$$

which gives Eq. (42) again. The equation quadratic in Φ is given by

$$\begin{aligned} & \int_p \Phi(-p) (p^2 + m^2) \left(p_v \frac{\partial^2}{\partial p_\mu \partial p_v} - \frac{1}{2} p_\mu \frac{\partial^2}{\partial p_v \partial p_\mu} + \frac{D+2}{2} \frac{\partial}{\partial p_\mu} \right) \Phi(p) \\ & - \lambda \frac{1}{2} \int_p (-p \cdot \partial_p + 2) R(p) \cdot \frac{1}{(p^2 + R(p))^2} \frac{1}{2} \int_{p_1, p_2} \Phi(p_1) \Phi(p_2) \frac{\partial}{\partial p_{1\mu}} \delta(p_1 + p_2) = 0. \end{aligned} \quad (48)$$

Integration by parts reduces this to

$$\int_p \frac{\partial}{\partial p_\mu} \Phi(p) \cdot \Phi(-p) \left(2m^2 + \lambda \frac{1}{2} \int_q \frac{(2 - q \cdot \partial_q) R(q)}{(q^2 + R(q))^2} \right) = 0. \quad (49)$$

Hence, we obtain Eq. (44) again. We have thus seen that scale invariance automatically leads to conformal invariance.

We need a second-order calculation to fix λ to order ϵ . (It turns out that $\frac{\lambda}{(4\pi)^2} = \frac{\epsilon}{3}$ [4].)

8. Conformal transformation of composite operators

Let $\mathcal{O}(p)$ be a scalar composite operator of scale dimension $-y$ with momentum p . (In coordinate space the scale dimension is $-y + D$.) Translations and rotations act on $\mathcal{O}(p)$ the same way as on $\phi(p)$; we only need to generalize $D^S(p)$ and $D_\mu^K(p)$ as

$$D^S(p)\mathcal{O}(p) \equiv \left(-p_\mu \frac{\partial}{\partial p_\mu} - y\right) \mathcal{O}(p), \quad (50a)$$

$$D_\mu^K(p)\mathcal{O}(p) \equiv \left(-p_\nu \frac{\partial^2}{\partial p_\mu \partial p_\nu} + \frac{1}{2} p_\mu \frac{\partial^2}{\partial p_\nu \partial p_\nu} - y \frac{\partial}{\partial p_\mu}\right) \mathcal{O}(p). \quad (50b)$$

The invariance under scale and special conformal transformations is now given by

$$D^S(p)\mathcal{O}(p) - e^{-S} \int_q K(q) \frac{\delta}{\delta \phi(q)} (D^S(q) [\mathcal{O}(p) \Phi(q)] e^S) = 0, \quad (51a)$$

$$D_\mu^K(p)\mathcal{O}(p) - e^{-S} \int_q K(q) \frac{\delta}{\delta \phi(q)} (D_\mu^K(q) [\mathcal{O}(p) \Phi(q)] e^S) = 0, \quad (51b)$$

where the product of composite operators is defined by

$$[\mathcal{O}(p) \Phi(q)] \equiv \mathcal{O}(p) \Phi(q) + \frac{k(q)}{q^2 K(q)} \frac{\delta \mathcal{O}(p)}{\delta \phi(-q)}. \quad (52)$$

Equations (51) imply

$$D^S(p) \langle \mathcal{O}(p) \phi(p_1) \cdots \phi(p_n) \rangle + \sum_{i=1}^n D^S(p_i) \langle \mathcal{O}(p) \phi(p_1) \cdots \phi(p_n) \rangle = 0, \quad (53a)$$

$$D_\mu^K(p) \langle \mathcal{O}(p) \phi(p_1) \cdots \phi(p_n) \rangle + \sum_{i=1}^n D_\mu^K(p_i) \langle \mathcal{O}(p) \phi(p_1) \cdots \phi(p_n) \rangle = 0. \quad (53b)$$

For completeness, let us rewrite Eq. (51) in terms of $W[J]$ and $\Gamma[\Phi]$. Regarding $\mathcal{O}(p)$ as a functional of J , we obtain

$$\begin{aligned} & (-p \cdot \partial_p - y) \mathcal{O}(p) + \int_q J(q) \left(-q \cdot \partial_q - \frac{D+2}{2} + \gamma \right) \frac{\delta \mathcal{O}(p)}{\delta J(q)} \\ & + \int_q (-q \cdot \partial_q + 2 - 2\gamma) R(q) \cdot \left\{ \frac{\delta W[J]}{\delta J(-q)} \frac{\delta \mathcal{O}(p)}{\delta J(q)} + \frac{1}{2} \frac{\delta^2 \mathcal{O}(p)}{\delta J(-q) \delta J(q)} \right\} = 0, \end{aligned} \quad (54a)$$

$$\begin{aligned} & D_\mu^K(p) \mathcal{O}(p) + \int_q J(-q) D_\mu^K(q) \frac{\delta \mathcal{O}(p)}{\delta J(-q)} + \frac{1}{2} \int_q (-q \cdot \partial_q + 2 - 2\gamma) R(q) \\ & \cdot \frac{\partial}{\partial q_\mu} \left(\frac{\delta^2 \mathcal{O}(p)}{\delta J(-q) \delta J(q')} + \frac{\delta W}{\delta J(-q)} \frac{\delta \mathcal{O}(p)}{\delta J(q')} + \frac{\delta W}{\delta J(q')} \frac{\delta \mathcal{O}(q)}{\delta J(-q)} \right) \Big|_{q'=q} = 0. \end{aligned} \quad (54b)$$

Alternatively, regarding $\mathcal{O}(p)$ as a functional of Φ , we obtain

$$\begin{aligned} & (-p \cdot \partial_p - y) \mathcal{O}(p) - \int_q \left(-q \cdot \partial_q - \frac{D+2}{2} + \gamma \right) \Phi(q) \cdot \frac{\delta \mathcal{O}(p)}{\delta \Phi(q)} \\ & + \int_q (-q \cdot \partial_q + 2 - 2\gamma) R(q) \cdot \frac{1}{2} \int_{r,s} G_{q,-r} \frac{\delta^2 \mathcal{O}(p)}{\delta \Phi(r) \delta \Phi(-s)} G_{s,-q} = 0, \end{aligned} \quad (55a)$$

$$\begin{aligned} & D_\mu^K(p) \mathcal{O}(p) - \int_q D_\mu^K(q) \Phi(q) \cdot \frac{\delta \mathcal{O}(p)}{\delta \Phi(q)} \\ & + \frac{1}{2} \int_q (-q \cdot \partial_q + 2 - 2\gamma) R(q) \cdot \int_{r,s} \frac{\partial G_{-q,r}}{\partial q_\mu} \frac{\delta^2 \mathcal{O}(p)}{\delta \Phi(-r) \delta \Phi(s)} G_{-s,q} = 0. \end{aligned} \quad (55b)$$

It is most straightforward to obtain the above results by varying either W or Γ infinitesimally by $\mathcal{O}(p)$ in Eqs. (33) and (34).

A concrete example is

$$\left[\frac{1}{2} \phi^2(p) \right] \equiv \frac{1}{2} \int_{p_1, p_2} \Phi(p_1) \Phi(p_2) \delta(p_1 + p_2 - p) + \kappa_2 \delta(p) \quad (56)$$

at the Gaussian fixed point in $D > 2$. With $y = 2$, both of Eq. (51) are satisfied if the constant κ_2 is chosen as

$$\kappa_2 = -\frac{1}{2(D-2)} \int_p \frac{(2 - p \cdot \partial_p) R(p)}{(p^2 + R(p))^2}. \quad (57)$$

9. Conclusion

The main purpose of this paper is to reformulate the recent results of Rosten [5,7,8] using the method of equation-of-motion composite operators advocated in Ref. [9]. The Wilson action of the continuum limit of a theory has all the symmetry intact despite the presence of a finite momentum cutoff. We hope that we have convinced the reader that a finite UV cutoff does not stand in the way of making a Wilson action invariant under conformal transformations.

Note added: Rosten extends his work further in a recent article [11].

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Appendix A. Quick summary of the ERG formalism

The purpose of this and the following appendices is to give a reader without a working knowledge of ERG just enough to follow the flow of the present paper. For further details we recommend [12] and the references cited therein.

As in the main text, we use the dimensionless notation in which dimensionful quantities are measured in units of an appropriate power of the momentum cutoff. Hence, the momentum cutoff becomes 1 in this convention.

The renormalization group flow of the Wilson action $S_t[\phi]$ is given by the exact renormalization group equation [4]

$$\begin{aligned} \partial_t e^{S_t} = & \int_p \left(-p_\mu \frac{\partial}{\partial p_\mu} \ln K(p) + \frac{D+2}{2} - \gamma_t + p_\mu \frac{\partial}{\partial p_\mu} \right) \phi(p) \cdot \frac{\delta}{\delta \phi(p)} e^{S_t} \\ & + \int_p \left(-p_\mu \frac{\partial}{\partial p_\mu} \ln \frac{K(p)^2}{k(p)} - 2\gamma_t \right) \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(-p) \delta \phi(p)} e^{S_t}, \end{aligned} \quad (\text{A.1})$$

where t is the logarithmic scale factor. This is a generalized version with two cutoff functions $K(p)$, $k(p)$ [13]: $K(p)$ approaches 1 as $p \rightarrow 0$, and decreases rapidly for $p \gg 1$, and $k(p)$ vanishes at $p = 0$. In the popular adaptation by Polchinski [14], $k(p)$ is taken as

$$k(p) = K(p) (1 - K(p)). \quad (\text{A.2})$$

To obtain $S_{t+\Delta t}$ from S_t , we first integrate over the field with momenta between 1 and $e^{-\Delta t}$. We then rescale the momentum by the factor $e^{\Delta t}$ to restore the cutoff at 1, and renormalize the field so that, for example, the kinetic term is canonically normalized. It is remarkable that this whole procedure can be expressed as a functional differential equation.

In this paper we are not interested in t -dependent actions, but only in a fixed-point solution $S[\phi]$ satisfying

$$\begin{aligned} 0 = & \int_p \left(-p_\mu \frac{\partial}{\partial p_\mu} \ln K(p) + \frac{D+2}{2} - \gamma + p_\mu \frac{\partial}{\partial p_\mu} \right) \phi(p) \cdot \frac{\delta}{\delta \phi(p)} e^S \\ & + \int_p \left(-p_\mu \frac{\partial}{\partial p_\mu} \ln \frac{K(p)^2}{k(p)} - 2\gamma \right) \frac{k(p)}{p^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(-p) \delta \phi(p)} e^S, \end{aligned} \quad (\text{A.3})$$

where γ is a constant anomalous dimension. This S has a UV cutoff $p = 1$, just like a generic bare action with the same cutoff $p = 1$, but it corresponds to a massless continuum theory. The fields with momenta $p > 1$ have already been integrated, and the Wilson action can provide the continuum limit of correlation functions with only a little modification [13]:

$$\begin{aligned} \langle \phi(p_1) \cdots \phi(p_n) \rangle & \equiv \prod_{i=1}^n \frac{1}{K(p_i)} \cdot \left\langle \exp \left(-\frac{1}{2} \int_p \frac{k(p)}{p^2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right) \phi(p_1) \cdots \phi(p_n) \right\rangle \\ & = \prod_{i=1}^n \frac{1}{K(p_i)} \cdot \int [d\phi] e^S \exp \left(-\frac{1}{2} \int_p \frac{k(p)}{p^2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right) \phi(p_1) \cdots \phi(p_n). \end{aligned} \quad (\text{A.4})$$

$k(p)$ modifies the two-point functions trivially at high momenta, and $K(p)$ corrects the normalization of the field. As befits the continuum limit, the modified correlation functions are defined for arbitrary momenta, and satisfy the scaling law

$$\langle \phi(p_1 e^t) \cdots \phi(p_n e^t) \rangle = \exp \left(n \left(-\frac{D+2}{2} + \gamma \right) t \right) \langle \phi(p_1) \cdots \phi(p_n) \rangle. \quad (\text{A.5})$$

Hence, the two-point function is given by

$$\langle \phi(p) \phi(q) \rangle = \frac{\text{const}}{p^{2(1-\gamma)}} \delta(p+q). \quad (\text{A.6})$$

We next introduce the concept of composite operators. (For more details than given here, see Sect. 4 of Ref. [9].) A composite operator $\mathcal{O}(p)$ is a functional of ϕ , and it can be regarded as an infinitesimal variation of the action. We define its modified correlation functions by

$$\langle \mathcal{O}(p) \phi(p_1) \cdots \phi(p_n) \rangle \equiv \prod_{i=1}^n \frac{1}{K(p_i)} \cdot \left\langle \mathcal{O}(p) \exp \left(-\frac{1}{2} \int_q \frac{k(q)}{q^2} \frac{\delta^2}{\delta \phi(-q) \delta \phi(q)} \right) \phi(p_1) \cdots \phi(p_n) \right\rangle. \quad (\text{A.7})$$

Note the absence of $K(p)$ for the composite operator. There are two special composite operators playing important roles in this paper. One is

$$\Phi(p) \equiv \frac{1}{K(p)} \left(\phi(p) + \frac{k(p)}{p^2} \frac{\delta S}{\delta \phi(-p)} \right), \quad (\text{A.8})$$

which has the correlation functions

$$\langle \Phi(p) \phi(p_1) \cdots \phi(p_n) \rangle = \langle \phi(p) \phi(p_1) \cdots \phi(p_n) \rangle. \quad (\text{A.9})$$

$\Phi(p)$ is a composite operator, but it shares the same modified correlation functions as the elementary field $\phi(p)$. The other is a special class of composite operators, called equation-of-motion composite operators (a.k.a. redundant operators). They are given in the form

$$\mathcal{E}_{\mathcal{O}} \equiv -e^{-S} \int_p K(p) \frac{\delta}{\delta \phi(p)} (\mathcal{O}(p) e^S), \quad (\text{A.10})$$

where $\mathcal{O}(p)$ is a composite operator. $\mathcal{E}_{\mathcal{O}}$ has the correlation functions

$$\langle \mathcal{E}_{\mathcal{O}} \phi(p_1) \cdots \phi(p_n) \rangle = \sum_{i=1}^n \langle \phi(p_1) \cdots \mathcal{O}(p_i) \cdots \phi(p_n) \rangle. \quad (\text{A.11})$$

(Derivation) Using Eq. (A.7), we obtain

$$\begin{aligned} \langle \mathcal{E}_{\mathcal{O}} \phi(p_1) \cdots \phi(p_n) \rangle &\equiv \prod_{i=1}^n \frac{1}{K(p_i)} \int_p K(p) \\ &\times \left\langle e^{-S}(-) \frac{\delta}{\delta \phi(p)} (\mathcal{O}(p) e^S) \exp \left(-\frac{1}{2} \int_q \frac{k(q)}{q^2} \frac{\delta^2}{\delta \phi(q) \delta \phi(-q)} \right) \phi(p_1) \cdots \phi(p_n) \right\rangle. \end{aligned} \quad (\text{A.12})$$

Functionally integrating this by parts, we obtain

$$\begin{aligned} &\langle \mathcal{E}_{\mathcal{O}} \phi(p_1) \cdots \phi(p_n) \rangle \\ &= \prod_{i=1}^n \frac{1}{K(p_i)} \int_p K(p) \left\langle \mathcal{O}(p) \exp \left(-\frac{1}{2} \int_q \frac{k(q)}{q^2} \frac{\delta^2}{\delta \phi(q) \delta \phi(-q)} \right) \frac{\delta}{\delta \phi(p)} \{ \phi(p_1) \cdots \phi(p_n) \} \right\rangle \\ &= \prod_{i=1}^n \frac{1}{K(p_i)} \sum_{j=1}^n K(p_j) \left\langle \mathcal{O}(p_j) \exp \left(-\frac{1}{2} \int_q \frac{k(q)}{q^2} \frac{\delta^2}{\delta \phi(q) \delta \phi(-q)} \right) \phi(p_1) \cdots \widehat{\phi(p_j)} \cdots \phi(p_n) \right\rangle \\ &= \sum_{i=1}^n \left\langle \mathcal{O}(p_i) \phi(p_1) \cdots \widehat{\phi(p_i)} \cdots \phi(p_n) \right\rangle, \end{aligned} \quad (\text{A.13})$$

where the hat above ϕ implies the omission. (End of derivation)

Given two composite operators $\mathcal{O}_1(p)$, $\mathcal{O}_2(q)$, their product $\mathcal{O}_1(p)\mathcal{O}_2(q)$ is not necessarily a composite operator. When one of them is $\Phi(p)$, however, its product with an arbitrary $\mathcal{O}(q)$ is easy to construct:

$$\begin{aligned} [\Phi(p)\mathcal{O}(q)] &\equiv \Phi(p)\mathcal{O}(q) + \frac{k(p)}{p^2 K(p)} \frac{\delta \mathcal{O}(q)}{\delta \phi(-p)} \\ &= e^{-S} \frac{1}{K(p)} \left(\phi(p) + \frac{k(p)}{p^2} \frac{\delta}{\delta \phi(-p)} \right) (\mathcal{O}(q) e^S). \end{aligned} \quad (\text{A.14})$$

The product has the correlation functions

$$\langle [\Phi(p)\mathcal{O}(q)] \phi(p_1) \cdots \phi(p_n) \rangle = \langle \mathcal{O}(q) \phi(p) \phi(p_1) \cdots \phi(p_n) \rangle. \quad (\text{A.15})$$

Appendix B. Generating functional $W[J]$ and 1PI action $\Gamma[\Phi]$

We can interpret a Wilson action $S[\phi]$ as a generating functional of the connected correlation functions of the scalar field for which only the field with momentum higher than the cutoff $p = 1$ has been integrated. Regarding

$$J(p) \equiv \frac{R(p)}{K(p)} \phi(p) \quad (\text{B.1})$$

as the source, we obtain the generating functional as

$$W[J] \equiv S[\phi] + \frac{1}{2} \int_p \frac{1}{R(p)} J(p) J(-p), \quad (\text{B.2})$$

where

$$R(p) \equiv \frac{p^2}{k(p)} K(p)^2. \quad (\text{B.3})$$

Recall that $S[\phi]$ depends on two cutoff functions K and k , but $W[J]$ and $\Gamma[\Phi]$, to be defined shortly, depend only on this R .

It is straightforward to check that the composite operator $\Phi(p)$, defined by Eq. (A.8), is obtained as

$$\Phi(p) = \frac{\delta W[J]}{\delta J(-p)}. \quad (\text{B.4})$$

The 1PI action $\Gamma[\Phi]$ is now defined as the Legendre transform of the generating functional $W[J]$ as

$$\Gamma[\Phi] - \frac{1}{2} \int_p R(p) \Phi(p) \Phi(-p) = W[J] - \int_p J(p) \Phi(-p). \quad (\text{B.5})$$

Differentiating this with respect to $\Phi(-p)$, we obtain

$$J(p) = R(p) \Phi(p) - \frac{\delta \Gamma[\Phi]}{\delta \Phi(-p)}. \quad (\text{B.6})$$

The high-momentum propagator, defined by

$$G_{p,q}[\Phi] \equiv \frac{\delta^2 W[J]}{\delta J(p) \delta J(q)}, \quad (\text{B.7})$$

is symmetric with respect to p and q , and satisfies

$$\int_q G_{p,q}[\Phi] \left(R(q) \delta(q-r) - \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(-q) \delta \Phi(-r)} \right) = \delta(p-r). \quad (\text{B.8})$$

Consider the simplest example of the Gaussian fixed point:

$$S_G[\phi] \equiv -\frac{1}{2} \int_p \frac{p^2}{K(p)^2 + k(p)} \phi(p) \phi(-p). \quad (\text{B.9})$$

We obtain

$$W_G[J] = \frac{1}{2} \int_p \frac{1}{p^2 + R(p)} J(p) J(-p), \quad (\text{B.10})$$

$$\Gamma_G[\Phi] = -\frac{1}{2} \int_p p^2 \Phi(p) \Phi(-p). \quad (\text{B.11})$$

Hence, the high-momentum propagator is given by

$$G_{p,q}[\Phi] = \frac{1}{p^2 + R(p)} \delta(p+q). \quad (\text{B.12})$$

It is trivial to check that

$$\langle \phi(p) \phi(q) \rangle_G = \frac{K(p)^2 + k(p)}{p^2} \delta(p+q), \quad (\text{B.13a})$$

$$\begin{aligned} \langle \phi(p) \phi(q) \rangle_G &= \frac{1}{K(p)K(q)} \left(\langle \phi(p) \phi(q) \rangle - \frac{k(p)}{p^2} \delta(p+q) \right) \\ &= \frac{1}{p^2} \delta(p+q). \end{aligned} \quad (\text{B.13b})$$

Appendix C. Derivation of (16) from the energy–momentum tensor

As has been shown in Ref. [2], conformal invariance is equivalent to the vanishing of the trace of the energy–momentum tensor; scale invariance equivalent to the vanishing of its integral. It is therefore natural that the author of Ref. [5] was led to consider the energy–momentum tensor in the realization of conformal algebra for Wilson actions. In this appendix we wish to summarize how to derive Eq. (16) from the relevant properties of the energy–momentum tensor. We will follow Ref. [10], since we can obtain the particular form of the Σ 's given by Eq. (15) without any effort.

Now, in Ref. [10] we have assumed the invariance of the Wilson action under translations and rotations,

$$\Sigma_\mu^T \equiv -e^{-S} \int_p K(p) \frac{\delta}{\delta \phi(p)} (D_\mu^T(p) \phi(p) e^S) = 0, \quad (\text{C.1a})$$

$$\Sigma_{\mu\nu}^R \equiv -e^{-S} \int_p K(p) \frac{\delta}{\delta \phi(p)} (D_{\mu\nu}^R(p) \phi(p) e^S) = 0, \quad (\text{C.1b})$$

where $D_\mu^T, D_{\mu\nu}^R$ are defined in Eq. (5). We have then shown the existence of the energy–momentum tensor $\Theta_{\mu\nu}(p)$ satisfying

$$p_\mu \Theta_{\mu\nu}(p) = \int_q K(q) e^{-S} \frac{\delta}{\delta\phi(q)} ((p+q)_\nu \Phi(p+q) e^S), \quad (\text{C.2a})$$

$$\Theta_{\mu\nu}(p) = \Theta_{\nu\mu}(p). \quad (\text{C.2b})$$

It is straightforward to go backward, and derive Eq. (C.1) from Eq. (C.2). To obtain Eq. (C.1a), we simply set $p = 0$ in Eq. (C.2a). To obtain Eq. (C.1b), differentiate Eq. (C.2a) with respect to p_α , antisymmetrize the result with respect to ν and α , and set $p = 0$.

The invariance under scale and special conformal transformations is given respectively by

$$\Sigma^S \equiv -e^{-S} \int_p K(p) \frac{\delta}{\delta\phi(p)} (D^S(p) \Phi(p) e^S) = 0, \quad (\text{C.3a})$$

$$\Sigma_\mu^K \equiv -e^{-S} \int_p K(p) \frac{\delta}{\delta\phi(p)} (D_\mu^K(p) \Phi(p) e^S) = 0, \quad (\text{C.3b})$$

where D^S, D_μ^K are defined in Eq. (5). We wish to show how to obtain these from the trace condition:

$$\Theta(p) \equiv \Theta_{\mu\mu}(p) = \left(\frac{D-2}{2} + \gamma \right) \int_q K(q) e^{-S} \frac{\delta}{\delta\phi(q)} (\Phi(p+q) e^S). \quad (\text{C.4})$$

In Ref. [10] it is shown that a fixed-point Wilson action, satisfying Eq. (C.3a), also satisfies Eq. (C.4) at $p = 0$. Conversely, to obtain Eq. (C.3a) from Eq. (C.4), we differentiate Eq. (C.4) with respect to p_ν , sum over ν , and set $p = 0$.

Getting Eq. (C.3b) from Eqs. (C.2) and (C.4) is a little more involved. (This was done in Sect. VI of Ref. [10], where Eq. (C.4) is assumed up to a two-derivative term $p_\mu p_\nu L_{\mu\nu}(p)$. For simplicity, we have removed the two-derivative term by redefining $\Theta_{\mu\nu}(p)$.) We apply

$$\frac{\partial^2}{\partial p_\alpha \partial p_\nu} - \frac{1}{2} \delta_{\alpha\nu} \frac{\partial^2}{\partial p_\beta \partial p_\beta}$$

on Eq. (C.2a) and set $p = 0$. Using Eq. (C.4), we can write the left-hand side as

$$\frac{\partial}{\partial p_\alpha} \Theta(p) \Big|_{p=0} = \left(\frac{D-2}{2} + \gamma \right) \int_q K(q) e^{-S} \frac{\delta}{\delta\phi(q)} \left(\frac{\partial}{\partial q_\alpha} \Phi(q) e^S \right). \quad (\text{C.5})$$

The right-hand side gives

$$\int_q K(q) e^{-S} \frac{\delta}{\delta\phi(q)} \left\{ \left(q_\nu \frac{\partial^2}{\partial q_\alpha \partial q_\nu} - \frac{1}{2} q_\alpha \frac{\partial^2}{\partial q_\beta \partial q_\beta} + D \frac{\partial}{\partial q_\alpha} \right) \Phi(q) e^S \right\}. \quad (\text{C.6})$$

Equating the two sides, we obtain $\Sigma_\alpha^K = 0$.

In a recent work [7], Rosten regards Eqs. (C.2) and (C.4) as fundamental equations from which he attempts to construct a conformally invariant Wilson action.

Appendix D. Derivation of Eq. (34)

We wish to rewrite the special conformal invariance $\Sigma_\mu^K = 0$, where Σ_μ^K is defined by Eq. (15d), in terms of the generating functional $W[J]$ and 1PI action $\Gamma[\Phi]$. (The content of this appendix overlaps

with the main subject of Ref. [8]. Our Wilson action is more straightforwardly related to Γ , resulting in a simpler derivation.) We first expand Σ_μ^K as

$$\Sigma_\mu^K = - \int_p K(p) D_\mu^K(p) \left(\frac{\delta \Phi(p)}{\delta \phi(q)} + \Phi(p) \frac{\delta S[\phi]}{\delta \phi(q)} \right) \Big|_{q=p}, \quad (\text{D.1})$$

where we set $q = p$ only after the action of $D_\mu^K(p)$. Then, using

$$\Phi(p) = \frac{\delta W[J]}{\delta J(-p)}, \quad J(p) = \frac{R(p)}{K(p)} \phi(p), \quad S[\phi] = W[J] - \frac{1}{2} \int_p \frac{J(p)J(-p)}{R(p)}, \quad (\text{D.2})$$

we obtain

$$\begin{aligned} \Sigma_\mu^K &= - \int_p R(p) \left\{ D_\mu^K(p) \frac{\delta^2 W[J]}{\delta J(q) \delta J(-p)} \Big|_{q=p} + D_\mu^K(p) \frac{\delta W[J]}{\delta J(-p)} \cdot \left(\frac{\delta W[J]}{\delta J(p)} - \frac{J(-p)}{R(p)} \right) \right\} \\ &= \int_p J(-p) D_\mu^K(p) \frac{\delta W[J]}{\delta J(-p)} \\ &\quad - \int_p R(p) D_\mu^K(p) \left\{ \frac{\delta^2 W[J]}{\delta J(q) \delta J(-p)} + \frac{\delta W[J]}{\delta J(q)} \frac{\delta W[J]}{\delta J(-p)} \right\}_{q=p}. \end{aligned} \quad (\text{D.3})$$

To transform the last integral, we use a formula of partial integration,

$$\begin{aligned} &\int_p R(p) \left\{ p_\nu \frac{\partial^2}{\partial p_\mu \partial p_\nu} - \frac{1}{2} p_\mu \frac{\partial^2}{\partial p_\nu \partial p_\nu} \right\} F(-p, q) \Big|_{q=p} \\ &= -\frac{1}{2} \int_p (D + p \cdot \partial_p) R(p) \cdot \frac{\partial}{\partial p_\mu} F(-p, q) \Big|_{q=p}, \end{aligned} \quad (\text{D.4})$$

which is valid for any symmetric $F(-p, q)$ satisfying

$$F(-p, q) = F(q, -p). \quad (\text{D.5})$$

We then obtain

$$\begin{aligned} &- \int_p R(p) D_\mu^K(p) \left\{ \frac{\delta^2 W[J]}{\delta J(q) \delta J(-p)} + \frac{\delta W[J]}{\delta J(q)} \frac{\delta W[J]}{\delta J(-p)} \right\}_{q=p} \\ &= \frac{1}{2} \int_p (-p \cdot \partial_p + 2 - 2\gamma) R(p) \cdot \frac{\partial}{\partial p_\mu} \left\{ \frac{\delta^2 W[J]}{\delta J(q) \delta J(-p)} + \frac{\delta W[J]}{\delta J(q)} \frac{\delta W[J]}{\delta J(-p)} \right\}_{q=p}. \end{aligned} \quad (\text{D.6})$$

Hence, Eq. (34a) is obtained:

$$\begin{aligned} &\int_p J(-p) D_\mu^K(p) \frac{\delta W[J]}{\delta J(-p)} \\ &+ \frac{1}{2} \int_p (-p \cdot \partial_p + 2 - 2\gamma) R(p) \cdot \frac{\partial}{\partial p_\mu} \left\{ \frac{\delta^2 W[J]}{\delta J(q) \delta J(-p)} + \frac{\delta W[J]}{\delta J(q)} \frac{\delta W[J]}{\delta J(-p)} \right\}_{q=p} = 0. \end{aligned} \quad (\text{D.7})$$

It is now easy to rewrite this in terms of Γ ; we substitute

$$J(p) = R(p) \Phi(p) - \frac{\delta \Gamma[\Phi]}{\delta \Phi(-p)}, \quad \frac{\delta W[J]}{\delta J(-p)} = \Phi(p), \quad \frac{\delta^2 W[J]}{\delta J(p) \delta J(q)} = G_{p,q}[\Phi] \quad (\text{D.8})$$

to obtain Eq. (34b):

$$-\int_p \frac{\delta\Gamma[\Phi]}{\delta\Phi(p)} D_\mu^K(p) \Phi(p) + \frac{1}{2} \int_p (-p \cdot \partial_p + 2 - 2\gamma) R(p) \cdot \frac{\partial}{\partial p_\mu} G_{-p,q}[\Phi] \Big|_{q=p} = 0. \quad (\text{D.9})$$

Appendix E. Rewriting $\Sigma_\mu^K = 0$ for S

In Sect. 4 we have written the invariance of the Wilson action S under the special conformal transformation as

$$\Sigma_\mu^K \equiv -e^{-S} \int_p K(p) \frac{\delta}{\delta\phi(p)} (D_\mu^K(p) \Phi(p) e^S) = 0, \quad (\text{E.1})$$

where $\Phi(p)$ is given by Eq. (12):

$$\Phi(p) \equiv \frac{1}{K(p)} \left(\phi(p) + \frac{k(p)}{p^2} \frac{\delta S}{\delta\phi(-p)} \right). \quad (\text{E.2})$$

We wish to rewrite the invariance more explicitly in terms of S and ϕ . Expanding Σ_μ^K , we obtain

$$\begin{aligned} -\Sigma_\mu^K &= \int_p K(p) \left(D_\mu^K(p) \frac{\delta\Phi(p)}{\delta\phi(q)} \Big|_{q=p} + D_\mu^K(p) \Phi(p) \cdot \frac{\delta S}{\delta\phi(p)} \right) \\ &= \int_p K(p) \left[D_\mu^K(p) \left(\frac{k(p)}{p^2 K(p)} \frac{\delta^2 S}{\delta\phi(-p) \delta\phi(q)} \right) \right]_{q=p} \\ &\quad + \left\{ D_\mu^K(p) \left(\frac{1}{K(p)} \phi(p) \right) + D_\mu^K(p) \left(\frac{k(p)}{p^2 K(p)} \frac{\delta S}{\delta\phi(-p)} \right) \right\} \frac{\delta S}{\delta\phi(p)}, \end{aligned} \quad (\text{E.3})$$

where we have dropped the field-independent part. Using $R(p) = \frac{p^2 K(p)^2}{k(p)}$, we rewrite this as

$$\begin{aligned} -\Sigma_\mu^K &= \int K(p) D_\mu^K(p) \left(\frac{1}{K(p)} \phi(p) \right) \frac{\delta S}{\delta\phi(p)} \\ &\quad + \int_p R(p) D_\mu^K(p) \left\{ \frac{K(p)}{R(p)} \left(\frac{\delta^2 S}{\delta\phi(-p) \delta\phi(q)} + \frac{\delta S}{\delta\phi(-p)} \frac{\delta S}{\delta\phi(q)} \right) \frac{K(q)}{R(q)} \right\} \Big|_{q=p}. \end{aligned} \quad (\text{E.4})$$

We can expand

$$K(p) D_\mu^K(p) \frac{\phi(p)}{K(p)} = D_\mu^K(p) \phi(p) + p \cdot \partial_p \ln K(p) \cdot \frac{\partial\phi(p)}{\partial p_\mu} + K(p) D_\mu^K(p) \frac{1}{K(p)} \cdot \phi(p). \quad (\text{E.5})$$

Using Eq. (D.4), we can rewrite the second integral of $-\Sigma_\mu^K$ as

$$\begin{aligned} &\frac{1}{2} \int_p (p \cdot \partial_p - 2 + 2\gamma) R(p) \cdot \frac{\partial}{\partial p_\mu} \left\{ \frac{K(p)}{R(p)} \left(\frac{\delta^2 S}{\delta\phi(-p) \delta\phi(q)} + \frac{\delta S}{\delta\phi(-p)} \frac{\delta S}{\delta\phi(q)} \right) \frac{K(q)}{R(q)} \right\} \Big|_{q=p} \\ &= \frac{1}{2} \int_p (p \cdot \partial_p - 2 + 2\gamma) R(p) \cdot \left(\frac{K(p)}{R(p)} \right)^2 \frac{\partial}{\partial p_\mu} \left\{ \frac{\delta^2 S}{\delta\phi(-p) \delta\phi(q)} + \frac{\delta S}{\delta\phi(-p)} \frac{\delta S}{\delta\phi(q)} \right\} \Big|_{q=p} \end{aligned} \quad (\text{E.6})$$

(Note that $K(p)$ and $R(p)$ depend only on p^2 .) Hence, we can rewrite $\Sigma_\mu^K = 0$ as

$$\begin{aligned} & \int_p \left(D_\mu^K(p) \phi(p) + p \cdot \partial_p \ln K(p) \cdot \frac{\partial \phi(p)}{\partial p_\mu} + K(p) D_\mu^K(p) \frac{1}{K(p)} \cdot \phi(p) \right) \frac{\delta S[\phi]}{\delta \phi(p)} \\ & + \frac{1}{2} \int_p \frac{k(p)}{p^2} (p \cdot \partial_p \ln R(p) - 2 + 2\gamma) \frac{\partial}{\partial p_\mu} \left\{ \frac{\delta^2 S}{\delta \phi(-p) \delta \phi(q)} + \frac{\delta S}{\delta \phi(-p)} \frac{\delta S}{\delta \phi(q)} \right\}_{q=p} = 0. \end{aligned} \quad (\text{E.7})$$

This corresponds to (2.79b) of [7], which differs slightly from Eq. (E.7) due to a difference in the choice of cutoff functions. The similar difference between (2.79a) of Ref. [7] and our ERG differential equation (18) is explained in Appendix C of Ref. [12].

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