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It is well known that a Wilson action reduces to the generating functional of connected correlation functions as we take the momentum cutoff to zero. For a fixed-point Wilson action, this implies that for momenta large compared with the cutoff, the action reduces to the generating functional. We elaborate on this simple observation.
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Subject Index   B32, B39

1.   Introduction

A Wilson action is usually thought of as a functional integral half done: the field with momenta below the ultraviolet (UV) cutoff still needs to be integrated [1]. To obtain full correlation functions from a Wilson action, we have two ways. We can compute the correlation functions by functional integration over the exponentiated Wilson action. Thanks to the UV momentum cutoff incorporated into the action, this functional integration is well defined. Alternatively, we can lower the momentum cutoff all the way to zero, where the Wilson action becomes the generating functional of the connected correlation functions. The two ways are equivalent, and neither is easy.

The exception is given by scale-invariant theories. In the dimensionless convention of the renormalization group (RG), where the momentum cutoff stays fixed, the scale-invariant theories correspond to fixed points under the RG transformation. Given a fixed-point Wilson action (getting it is actually the hard part), we can switch to the dimensionful convention, where the momentum cutoff Λ decreases under the RG flow. The Wilson action now depends on Λ, but the dependence is given by simple scaling. It is trivial to take Λ to zero, obtaining the correlation functions. Transcribing the vanishing cutoff limit into the dimensionless convention, the correlation functions appear as a high-momentum limit of the Wilson action because any finite momentum in units of the vanishing cutoff becomes large.

Considering how simple the idea is, the reader may find the paper too long or even unnecessary. Our excuse is that the purpose of the paper is to provide a technically robust derivation to justify the idea. We use the formalism of the exact renormalization group (ERG) for generic real scalar theories in D-dimensional Euclidean space (Sect. 11 of Ref. [1]).

To reach a wide range of readers, including those who have not been much exposed to the ERG formalism, we have provided plenty of background material. In fact, most of what is written here can be considered a review. To derive the main result of the paper, which is Eq. (65), all we have

to do is to collect the right background material and present it in the right order. It is helpful if the reader is familiar with the idea of ERG through reading the first third [up to Eq. (19)] of Ref. [2].

We organize the paper as follows. In Sect. 2 we review ERG by following the perturbative treatment of Ref. [2]. The goal of this section is to introduce the idea of a generating functional W_Λ with an infrared (IR) cutoff Λ, and to show that it becomes the generating functional $\mathcal{W}$ of the connected correlation functions in the limit that Λ goes to zero. In Sects. 3 and 4 we generalize the ERG formalism just enough for the discussion of fixed points in Sect. 5, where we derive the main result (65) that gives the connected correlation functions of a fixed-point theory as a high-momentum limit of its Wilson action. Section 5 is followed by two short sections: in Sect. 6 we check the consistency of Eq. (65) with potential conformal invariance, and in Sect. 7 we extend Eq. (65) to massive theories. We conclude the paper in Sect. 8. We have prepared three appendices. In Appendix A we show how to derive the diffusion equation satisfied by the generating functional with an IR cutoff, starting from the ERG differential equation of the corresponding Wilson action. In Appendix B we give details of conversion between the dimensionless and dimensionful conventions. In Appendix C we rewrite Eq. (65) for the effective action. Throughout the paper we use shorthand notation such as

$$p \cdot \partial_p = \sum_{\mu=1}^{D} p_\mu \frac{\partial}{\partial p_\mu}, \quad \int_p = \int \frac{d^D p}{(2\pi)^D}, \quad \delta(p) = (2\pi)^D \delta^{(D)}(p). \tag{1}$$

2. Review

We review Wilson's ERG (Sect. 11 of Ref. [1]) following the perturbative treatment by J. Polchinski [2]. We rely on perturbation theory for intuition, but the results we review below should be valid beyond perturbation theory.

We consider the action

$$S_\Lambda[\phi] = -\frac{1}{2} \int_p \frac{p^2 + m^2}{K(p/\Lambda)} \phi(p)\phi(-p) + S_{I\Lambda}[\phi], \tag{2}$$

where $S_{I\Lambda}$ consists of interaction vertices. The free part of the action gives the propagator

$$\frac{K(p/\Lambda)}{p^2 + m^2}, \tag{3}$$

where $K(p/\Lambda)$ is a decreasing positive function of p^2/Λ^2 such as

$$K(p/\Lambda) = e^{-\frac{p^2}{\Lambda^2}}. \tag{4}$$

If $K(p/\Lambda)$ decays fast enough for large $p^2 > \Lambda^2$, and if the interaction part is reasonable, the theory defined by S_Λ is free of UV divergences. We can regard Λ as the UV cutoff of the theory. Thus, we can assume that the correlation functions given by functional integrals

$$\langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_\Lambda} = \int [d\phi] \, e^{S_\Lambda[\phi]} \phi(p_1) \cdots \phi(p_n) \tag{5}$$

are well defined. (Note that we use e^{S_Λ} rather than e^{-S_Λ} as the weight of integration).

Now, how do we determine the Λ-dependence of $S_{I\Lambda}$? The short answer is that we give such Λ-dependence of $S_{I\Lambda}$ that compensates the Λ-dependence of the propagator. Let us elaborate on this. When we lower the cutoff infinitesimally from Λ to $\Lambda e^{-\Delta t} < \Lambda$, the propagator changes by

$$-\Delta t \frac{\Lambda \frac{\partial}{\partial \Lambda} K(p/\Lambda)}{p^2 + m^2}. \tag{6}$$

The functional integrals using the same interaction part $S_{I\Lambda}$ change accordingly. If we wish to keep the same functional integrals, we must change the interaction part to compensate for the effect of Eq. (6). The required compensation comes in two types: two vertices connected by minus (6) and single vertices with a loop given by minus (6). This results in the differential equation

$$
-\Lambda\frac{\partial}{\partial\Lambda}S_{I\Lambda} = \int_p \frac{\Lambda\frac{\partial}{\partial\Lambda}K(p/\Lambda)}{p^2+m^2}\frac{1}{2}\left\{\frac{\delta S_{I\Lambda}}{\delta\phi(p)}\frac{\delta S_{I\Lambda}}{\delta\phi(-p)} + \frac{\delta^2 S_{I\Lambda}}{\delta\phi(p)\delta\phi(-p)}\right\}. \tag{7}
$$

Exponentiating $S_{I\Lambda}$, we can rewrite this as

$$
-\Lambda\frac{\partial}{\partial\Lambda}e^{S_{I\Lambda}} = \int_p \frac{\Lambda\frac{\partial}{\partial\Lambda}K(p/\Lambda)}{p^2+m^2}\frac{1}{2}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}e^{S_{I\Lambda}}, \tag{8}
$$

which is a functional generalization of the diffusion equation.

Hence, as far as the internal propagators go, their cutoff dependence is compensated by the cutoff dependence of $S_{I\Lambda}$. But the external lines still depend on Λ, and the two-point function and the connected part of the higher-point functions acquire the following Λ-dependence:

$$
\langle\phi(p)\phi(q)\rangle_{S_\Lambda} = \frac{K(p/\Lambda)}{p^2+m^2}\delta(p+q) + \frac{K(p/\Lambda)}{p^2+m^2}G_2(p,q)\frac{K(q/\Lambda)}{q^2+m^2}, \tag{9a}
$$

$$
\langle\phi(p_1)\cdots\phi(p_n)\rangle_{S_\Lambda}^{\text{connected}} = \left(\prod_{i=1}^{n}\frac{K(p_i/\Lambda)}{p_i^2+m^2}\right)\cdot G_n^{\text{connected}}(p_1,\ldots,p_n) \quad (n>2), \tag{9b}
$$

where both G_2 and $G_{n>2}^{\text{connected}}$ correspond to the sums of diagrams with the amputated external lines, and they are independent of the cutoff Λ. ($G_2(p,q)$ is proportional to $\delta(p+q)$ in the absence of symmetry breaking.)

To extract the Λ-independent correlation functions, we must remove the cutoff functions from the external lines. For the connected part of the higher-point functions, we can do this simply by factoring out the cutoff functions:

$$
\langle\!\langle\phi(p_1)\cdots\phi(p_n)\rangle\!\rangle^{\text{connected}} \equiv \prod_{i=1}^{n}\frac{1}{K(p_i/\Lambda)}\cdot\langle\phi(p_1)\cdots\phi(p_n)\rangle_{S_\Lambda}^{\text{connected}} \tag{10}
$$

$$
= \prod_{i=1}^{n}\frac{1}{p_i^2+m^2}\cdot G_n^{\text{connected}}(p_1,\ldots,p_n).
$$

For the two-point function, we first subtract a high-momentum propagator to get

$$
\langle\phi(p)\phi(q)\rangle_{S_\Lambda} - \frac{K(p/\Lambda)\,(1-K(p/\Lambda))}{p^2+m^2}\delta(p+q)
$$

$$
= K(p/\Lambda)^2\left(\frac{1}{p^2+m^2}\delta(p+q) + \frac{1}{p^2+m^2}G_2(p,q)\frac{1}{q^2+m^2}\right). \tag{11}
$$

Then, factoring out the cutoff function, we obtain a Λ-independent two-point function:

$$
\langle\!\langle\phi(p)\phi(q)\rangle\!\rangle \equiv \frac{1}{K(p/\Lambda)^2}\left(\langle\phi(p)\phi(q)\rangle_{S_\Lambda} - \frac{K(p/\Lambda)\,(1-K(p/\Lambda))}{p^2+m^2}\delta(p+q)\right)
$$

$$
= \frac{1}{p^2+m^2}\delta(p+q) + \frac{1}{p^2+m^2}G_2(p,q)\frac{1}{q^2+m^2}. \tag{12}
$$

Incorporating the disconnected part, we can express the full correlation functions as

$$
\langle\!\langle \phi(p_1)\cdots\phi(p_n)\rangle\!\rangle \equiv \prod_{i=1}^{n} \frac{1}{K(p_i/\Lambda)}
$$

$$
\times \left\langle \exp\left(-\frac{1}{2}\int_p \frac{K(p/\Lambda)\,(1-K(p/\Lambda))}{p^2+m^2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right) \phi(p_1)\cdots\phi(p_n)\right\rangle_{S_\Lambda} \quad (13)
$$

To show how the exponentiated double differentiation works, we give an example of the four-point function:

$$
\langle\!\langle \phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\rangle\!\rangle
$$

$$
= \prod_{i=1}^{4} \frac{1}{K(p_i/\Lambda)} \left[\langle \phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\rangle_{S_\Lambda} \right.
$$

$$
- \frac{K(p_1/\Lambda)\,(1-K(p_1/\Lambda))}{p_1^2+m^2}\delta(p_1+p_2)\,\langle\phi(p_3)\phi(p_4)\rangle_{S_\Lambda}
$$

$$
- \frac{K(p_3/\Lambda)\,(1-K(p_3/\Lambda))}{p_3^2+m^2}\delta(p_3+p_4)\,\langle\phi(p_1)\phi(p_2)\rangle_{S_\Lambda}
$$

$$
+ \frac{K(p_1/\Lambda)\,(1-K(p_1/\Lambda))}{p_1^2+m^2}\delta(p_1+p_2)\frac{K(p_3/\Lambda)\,(1-K(p_3/\Lambda))}{p_3^2+m^2}\delta(p_3+p_4)
$$

$$
\left. + \text{(t, u-channels)} \right]. \quad (14)
$$

It is commonly taken for granted that only the low-momentum correlation functions are kept invariant under the exact renormalization group transformations, but we have shown more than that: via Eq. (13) we can recover the entire cutoff-independent correlation functions. (This was first pointed out in Ref. [3], and has been used extensively for the realization of symmetry in the ERG formalism [4].) Now, introducing a source $\mathcal{J}(p)$, and summing Eq. (13) over all n, we can express the generating functional $\mathcal{W}[\mathcal{J}]$ of the connected correlation functions as

$$
e^{\mathcal{W}[\mathcal{J}]} \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \int_{p_1,\cdots,p_n} \mathcal{J}(-p_1)\cdots\mathcal{J}(-p_n)\,\langle\!\langle\phi(p_1)\cdots\phi(p_n)\rangle\!\rangle
$$

$$
= \left\langle\!\left\langle \exp\left(\int_p \mathcal{J}(-p)\phi(p)\right)\right\rangle\!\right\rangle
$$

$$
= \left\langle \exp\left(-\frac{1}{2}\int_p \frac{K(p/\Lambda)\,(1-K(p/\Lambda))}{p^2+m^2}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right)\exp\left(\int_p \frac{\mathcal{J}(-p)\phi(p)}{K(p/\Lambda)}\right)\right\rangle_{S_\Lambda}
$$

$$
= \int [d\phi]\exp\left[S_\Lambda[\phi] \right.
$$

$$
\left. + \int_p \left(-\frac{1}{2}\frac{K(p/\Lambda)\,(1-K(p/\Lambda))}{p^2+m^2}\frac{\mathcal{J}(-p)\mathcal{J}(p)}{K(p/\Lambda)^2} + \frac{\mathcal{J}(-p)\phi(p)}{K(p/\Lambda)}\right)\right]. \quad (15)
$$

So far we have regarded Λ as a UV cutoff because the momentum modes with $p > \Lambda$ are suppressed in the functional integration over e^{S_Λ}. Let us note, however, that the interaction part $S_{I\Lambda}$ results from

integrating over the modes with momentum higher than Λ. So, if we regard $S_{I\Lambda}$ as a consequence of functional integration, we may call Λ an IR cutoff. Since the propagator of the high-momentum modes is

$$\frac{1 - K(p/\Lambda)}{p^2 + m^2} \, , \tag{16}$$

the generating functional of the connected correlation functions with an IR cutoff Λ is defined by

$$W_\Lambda[J] \equiv \frac{1}{2} \int_p \frac{1 - K(p/\Lambda)}{p^2 + m^2} J(p)J(-p) + S_{I\Lambda}\left[\frac{1 - K(p/\Lambda)}{p^2 + m^2} J(p)\right], \tag{17}$$

where we have added the free part, and substituted

$$\phi(p) = \frac{1 - K(p/\Lambda)}{p^2 + m^2} J(p) \tag{18}$$

into the interaction part. (The relation between W_Λ and $S_{I\Lambda}$ was first pointed out in Ref. [5] and then derived carefully in Ref. [6].) Using the full action, we can rewrite W_Λ as

$$W_\Lambda[J] = \frac{1}{2} \int_p \frac{J(p)J(-p)}{R_\Lambda(p)} + S_\Lambda\left[\frac{1 - K(p/\Lambda)}{p^2 + m^2} J(p)\right], \tag{19}$$

where

$$R_\Lambda(p) \equiv \frac{K(p/\Lambda)}{1 - K(p/\Lambda)}(p^2 + m^2) \tag{20}$$

is a positive cutoff function that decays rapidly for $p^2 > \Lambda^2$. We note that

$$\lim_{\Lambda \to 0+} R_\Lambda(p) = 0 \, . \tag{21}$$

Now, using $W_\Lambda[J]$ instead of $S_\Lambda[\phi]$ in Eq. (15), and using J instead of ϕ as integration variables, we obtain a simpler expression for $\mathcal{W}$:

$$e^{\mathcal{W}[\mathcal{J}]} = \int [d\phi] \exp\left[S_\Lambda[\phi] + \int_p \left(-\frac{1}{2}\frac{\mathcal{J}(p)\mathcal{J}(-p)}{R_\Lambda(p)} + \frac{\mathcal{J}(-p)\phi(p)}{K(p/\Lambda)}\right)\right]$$

$$= \int [dJ] \exp\left[W_\Lambda[J] - \frac{1}{2}\int_p \frac{1}{R_\Lambda(p)}(J(p) - \mathcal{J}(p))(J(-p) - \mathcal{J}(-p))\right]. \tag{22}$$

It is straightforward to obtain the cutoff dependence of $W_\Lambda[J]$. Since $W_\Lambda[J]$ is defined by Eq. (17), and the Λ-dependence of $S_{I\Lambda}$ is given by Eq. (7), we obtain

$$-\Lambda\frac{\partial}{\partial \Lambda}W_\Lambda[J] = \frac{1}{2}\int_p \Lambda\frac{\partial R_\Lambda(p)}{\partial \Lambda}\left\{\frac{\delta W_\Lambda[J]}{\delta J(-p)}\frac{\delta W_\Lambda[J]}{\delta J(p)} + \frac{\delta^2 W_\Lambda[J]}{\delta J(-p)\delta J(p)}\right\}, \tag{23a}$$

or equivalently,

$$-\Lambda\frac{\partial}{\partial \Lambda}e^{W_\Lambda[J]} = \int_p \Lambda\frac{\partial R_\Lambda(p)}{\partial \Lambda}\frac{1}{2}\frac{\delta^2}{\delta J(p)\delta J(-p)}e^{W_\Lambda[J]} \, . \tag{23b}$$

(See Appendix A for the derivation.) This functional diffusion equation can be solved formally: for $\Lambda' < \Lambda$, we obtain

$$
e^{W_{\Lambda'}[J]} = \exp\left(\int_p (R_\Lambda(p) - R_{\Lambda'}(p)) \frac{1}{2} \frac{\delta^2}{\delta J(p)\delta J(-p)} \right) e^{W_\Lambda[J]}
$$

$$
= \int [dJ'] \exp\Bigg[W_\Lambda[J']
$$

$$
- \frac{1}{2} \int_p \frac{1}{R_\Lambda(p) - R_{\Lambda'}(p)} \left(J'(p) - J(p) \right) \left(J'(-p) - J(-p) \right) \Bigg]. \tag{24}
$$

Comparing this with Eq. (22) and using Eq. (21), we obtain

$$
\mathcal{W}[\mathcal{J}] = \lim_{\Lambda \to 0+} W_\Lambda[\mathcal{J}]. \tag{25}
$$

This is the well-known equality referred to at the beginning of the abstract of the paper. Since W_Λ is directly related to S_Λ by Eq. (19), we can say that Eq. (25) gives the generating functional of the connected correlation functions as the zero-cutoff limit of the Wilson action.

3. Generalization

In the previous section we summarized Wilson's ERG following [2]. We introduced two types of generating functionals: W_Λ with an IR cutoff and $\mathcal{W}$ without. In this section we would like to generalize the formalism in two ways. So far, we have introduced only one cutoff function $K(p/\Lambda)$. Another cutoff function $R_\Lambda(p)$ is given in terms of $K(p/\Lambda)$ by Eq. (20). Our first generalization follows Ref. [7], and we introduce $K_\Lambda(p)$ and $R_\Lambda(p)$ as two independent positive cutoff functions. (This is necessary not only for the second generalization, but also if we wish to include the original formulation of Ref. [1] under the same footing.) The second generalization, to be introduced in the next section, follows Ref. [8], and we introduce an anomalous dimension to the scalar field.

Using two independent cutoff functions, we define the correlation functions by

$$
\langle\langle \phi(p_1) \cdots \phi(p_n) \rangle\rangle_{S_\Lambda}^{K_\Lambda, R_\Lambda} \equiv \prod_{i=1}^n \frac{1}{K_\Lambda(p_i)}
$$

$$
\times \left\langle \exp\left(-\frac{1}{2} \int_p \frac{K_\Lambda(p)^2}{R_\Lambda(p)} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right) \phi(p_1) \cdots \phi(p_n) \right\rangle_{S_\Lambda}. \tag{26}
$$

In the previous section we chose

$$
K_\Lambda(p) = K(p/\Lambda), \quad R_\Lambda(p) = \frac{K(p/\Lambda)}{1 - K(p/\Lambda)}(p^2 + m^2), \tag{27}
$$

for which Eq. (26) reduces to Eq. (13). We assume in general that both K_Λ and R_Λ decay rapidly for large momenta $p^2 > \Lambda^2$. This implies

$$
\lim_{\Lambda \to 0+} R_\Lambda(p) = 0. \tag{28}
$$

For Eq. (26) to be independent of Λ, the Wilson action must satisfy

$$-\Lambda \frac{\partial}{\partial \Lambda} S_\Lambda = \int_p \Lambda \frac{\partial}{\partial \Lambda} \ln K_\Lambda(p) \cdot \phi(p) \frac{\delta S_\Lambda}{\delta \phi(p)}$$

$$+ \int_p \Lambda \frac{\partial}{\partial \Lambda} R_\Lambda(p) \cdot \frac{K_\Lambda(p)^2}{R_\Lambda(p)^2} \frac{1}{2} \left\{ \frac{\delta S_\Lambda}{\delta \phi(p)} \frac{\delta S_\Lambda}{\delta \phi(-p)} + \frac{\delta^2 S_\Lambda}{\delta \phi(p) \delta \phi(-p)} \right\}. \qquad (29)$$

For the choice (27), this reduces to

$$-\Lambda \frac{\partial}{\partial \Lambda} S_\Lambda = \int_p \Lambda \frac{\partial}{\partial \Lambda} \ln K(p/\Lambda) \cdot \phi(p) \frac{\delta S_\Lambda}{\delta \phi(p)}$$

$$+ \int_p \frac{\Lambda \frac{\partial}{\partial \Lambda} K(p/\Lambda)}{p^2 + m^2} \frac{1}{2} \left\{ \frac{\delta S_\Lambda}{\delta \phi(p)} \frac{\delta S_\Lambda}{\delta \phi(-p)} + \frac{\delta^2 S_\Lambda}{\delta \phi(p) \delta \phi(-p)} \right\}, \qquad (30)$$

which is (7) rewritten for the total action. We do not derive Eq. (29) here; we refer the interested reader to Ref. [7] for the derivation.

Now, we define the generating functional with an IR cutoff in the same way as before by

$$W_\Lambda[J] \equiv \frac{1}{2} \int_p \frac{J(p)J(-p)}{R_\Lambda(p)} + S_\Lambda[\phi], \qquad (31a)$$

where

$$J(p) \equiv \frac{R_\Lambda(p)}{K_\Lambda(p)} \phi(p). \qquad (31b)$$

For the choice (27), the above reduces to Eqs. (19) and (20). Using Eq. (29), it is straightforward to show that $W_\Lambda[J]$ satisfies the same equation as Eq. (23):

$$-\Lambda \frac{\partial}{\partial \Lambda} e^{W_\Lambda[J]} = \int_p \Lambda \frac{\partial R_\Lambda(p)}{\partial \Lambda} \frac{1}{2} \frac{\delta^2}{\delta J(p) \delta J(-p)} e^{W_\Lambda[J]}. \qquad (32)$$

(See Appendix A for the derivation.) The rest proceeds the same way as in the previous section. The generating functional of the connected correlation functions, defined by

$$e^{\mathcal{W}[\mathcal{J}]} \equiv \left\langle\!\!\left\langle \exp\left(\int_p \mathcal{J}(-p)\phi(p) \right) \right\rangle\!\!\right\rangle_{S_\Lambda}^{K_\Lambda, R_\Lambda}, \qquad (33)$$

is given by

$$e^{\mathcal{W}[\mathcal{J}]} = \int [dJ] \exp\left(W_\Lambda[J] - \frac{1}{2} \int_p \frac{1}{R_\Lambda(p)} (J(p) - \mathcal{J}(p)) (J(-p) - \mathcal{J}(-p)) \right). \qquad (34)$$

Hence, we obtain the same result as Eq. (25):

$$\mathcal{W}[\mathcal{J}] = \lim_{\Lambda \to 0+} W_\Lambda[\mathcal{J}], \qquad (35)$$

where we have used Eq. (28).

4. Anomalous dimension

In this section, we introduce an anomalous dimension of the scalar field. A nonvanishing anomalous dimension is required by the nontrivial fixed point to be discussed in the next section. Let S_Λ be the Wilson action for which

$$\langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_\Lambda}^{K_\Lambda, R_\Lambda} \tag{36}$$

are independent of Λ. We wish to construct Λ-dependent Wilson actions $\tilde{S}_\Lambda$ so that

$$\langle \phi(p_1) \cdots \phi(p_n) \rangle_{\tilde{S}_\Lambda}^{K_\Lambda, R_\Lambda} = \left(\frac{\mu}{\Lambda} \right)^{n\gamma} \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_\mu}^{K_\mu, R_\mu}. \tag{37}$$

Here, μ is a fixed reference scale chosen arbitrarily. For simplicity, we have chosen the anomalous dimension γ as a constant independent of Λ. At $\Lambda = \mu$, the two actions agree:

$$\tilde{S}_\mu = S_\mu. \tag{38}$$

Unlike S_Λ, the correlation functions of $\tilde{S}_\Lambda$ are Λ dependent, but the Λ dependence is merely a change of normalization of the field. We wish to relate $\tilde{S}_\Lambda$ to S_Λ in the following.

We rewrite Eq. (37) as

$$\langle \phi(p_1) \cdots \phi(p_n) \rangle_{\tilde{S}_\Lambda}^{(\frac{\mu}{\Lambda})^\gamma K_\Lambda, (\frac{\mu}{\Lambda})^{2\gamma} R_\Lambda} = \left(\frac{\Lambda}{\mu} \right)^{n\gamma} \langle \phi(p_1) \cdots \phi(p_n) \rangle_{\tilde{S}_\Lambda}^{K_\Lambda, R_\Lambda}$$

$$= \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_\mu}^{K_\mu, R_\mu}. \tag{39}$$

Since this is independent of Λ, $\tilde{S}_\Lambda$ must satisfy

$$-\Lambda \frac{\partial}{\partial \Lambda} \tilde{S}_\Lambda[\phi] = \int_p \left(\Lambda \frac{\partial}{\partial \Lambda} \ln K_\Lambda(p) - \gamma \right) \phi(p) \frac{\delta \tilde{S}_\Lambda}{\delta \phi(p)}$$

$$+ \int_p \left(\Lambda \frac{\partial}{\partial \Lambda} R_\Lambda(p) - 2\gamma R_\Lambda(p) \right) \frac{K_\Lambda(p)^2}{R_\Lambda(p)^2} \frac{1}{2} \left\{ \frac{\delta \tilde{S}_\Lambda}{\delta \phi(-p)} \frac{\delta \tilde{S}_\Lambda}{\delta \phi(p)} + \frac{\delta^2 \tilde{S}_\Lambda}{\delta \phi(p) \delta \phi(-p)} \right\}. \tag{40}$$

(We obtain this from Eq. (29) by substituting $\left(\frac{\mu}{\Lambda} \right)^\gamma K_\Lambda$ and $\left(\frac{\mu}{\Lambda} \right)^{2\gamma} R_\Lambda$ into K_Λ and R_Λ, respectively.) We define the generating functional $\tilde{W}_\Lambda$ with an IR cutoff for $\tilde{S}_\Lambda$, using the same cutoff functions as for S_Λ:

$$\tilde{W}_\Lambda[J] \equiv \frac{1}{2} \int_p \frac{J(p)J(-p)}{R_\Lambda(p)} + \tilde{S}_\Lambda[\phi], \tag{41}$$

where

$$J(p) \equiv \frac{R_\Lambda(p)}{K_\Lambda(p)} \phi(p). \tag{42}$$

Using Eq. (40), we can derive the cutoff dependence of $\tilde{W}_\Lambda$ as

$$-\Lambda \frac{\partial}{\partial \Lambda} e^{\tilde{W}_\Lambda[J]} = \int_p \left[\gamma J(p) \frac{\delta}{\delta J(p)} + \left(\Lambda \frac{\partial R_\Lambda(p)}{\partial \Lambda} - 2\gamma R_\Lambda(p) \right) \frac{1}{2} \frac{\delta^2}{\delta J(p) \delta J(-p)} \right] e^{\tilde{W}_\Lambda[J]}. \tag{43}$$

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(See Appendix A for the derivation.) To solve this under the initial condition

$$\tilde{W}_\mu[J] = W_\mu[J], \tag{44}$$

we first rewrite the equation as

$$-\Lambda \frac{\partial}{\partial \Lambda} \exp\left(\tilde{W}_\Lambda\left[\left(\frac{\Lambda}{\mu}\right)^\gamma J\right]\right)$$

$$= \int_p \Lambda \frac{\partial}{\partial \Lambda}\left(\left(\frac{\Lambda}{\mu}\right)^{-2\gamma} R_\Lambda(p)\right) \frac{1}{2} \frac{\delta^2}{\delta J(p)\delta J(-p)} \exp\left(\tilde{W}_\Lambda\left[\left(\frac{\Lambda}{\mu}\right)^\gamma J\right]\right). \tag{45}$$

This is solved by

$$\exp\left(\tilde{W}_\Lambda\left[\left(\frac{\Lambda}{\mu}\right)^\gamma J\right]\right)$$

$$= \exp\left[\int_p \left(R_\mu(p) - \left(\frac{\Lambda}{\mu}\right)^{-2\gamma} R_\Lambda(p)\right) \frac{1}{2} \frac{\delta^2}{\delta J(p)\delta J(-p)}\right] e^{W_\mu[J]}. \tag{46}$$

To relate W_Λ to $\tilde{W}_\Lambda$, we compare the above solution with

$$e^{W_\Lambda[J]} = \exp\left[\int_p \left(R_\mu(p) - R_\Lambda(p)\right) \frac{1}{2} \frac{\delta^2}{\delta J(p)\delta J(-p)}\right] e^{W_\mu[J]}, \tag{47}$$

which is obtained from the first line of Eq. (24). We easily obtain

$$e^{W_\Lambda[J]} = \exp\left[\left(\left(\frac{\Lambda}{\mu}\right)^{-2\gamma} - 1\right) \int_p R_\Lambda(p) \frac{1}{2} \frac{\delta^2}{\delta J(p)\delta J(-p)}\right] \exp\left(\tilde{W}_\Lambda\left[\left(\frac{\Lambda}{\mu}\right)^\gamma J\right]\right). \tag{48}$$

We could rewrite this as a relation between S_Λ and $\tilde{S}_\Lambda$, but we do not need it.

We end this section by giving $\mathcal{W}[\mathcal{J}]$ as a limit of $\tilde{W}_\Lambda$. We assume

$$\gamma > 0 \tag{49}$$

so that $(\Lambda/\mu)^{-2\gamma}$ dominates over 1 as $\Lambda \to 0+$. If we assume further that

$$\lim_{\Lambda \to 0+} \Lambda^{-2\gamma} R_\Lambda(p) = 0, \tag{50}$$

which is a little stronger than Eq. (28), we obtain, from Eqs. (35) and (48),

$$\mathcal{W}[\mathcal{J}] = \lim_{\Lambda \to 0+} W_\Lambda[\mathcal{J}] = \lim_{\Lambda \to 0+} \tilde{W}_\Lambda\left[\left(\frac{\Lambda}{\mu}\right)^\gamma \mathcal{J}\right]. \tag{51}$$

5.   Fixed points

The differential equation (40), or equivalently (43), does not have a fixed-point solution for an obvious reason: the cutoff Λ keeps changing. We need to adopt the dimensionless convention in which we measure all the physical quantities in units of appropriate powers of the cutoff Λ. We give a table

of conversion with the dimensionful quantities on the left, and the corresponding dimensionless quantities (with bars except for K, R) on the right:

$$\Lambda = \mu e^{-t}, \tag{52a}$$

$$\phi(p) = \Lambda^{-\frac{D+2}{2}} \bar{\phi}(p/\Lambda), \tag{52b}$$

$$J(p) = \Lambda^{-\frac{D-2}{2}} \bar{J}(p/\Lambda), \tag{52c}$$

$$K_\Lambda(p) = K(p/\Lambda), \tag{52d}$$

$$R_\Lambda(p) = \Lambda^2 R(p/\Lambda), \tag{52e}$$

$$\tilde{S}_\Lambda[\phi] = \bar{S}_t[\bar{\phi}], \tag{52f}$$

$$\tilde{W}_\Lambda[J] = \bar{W}_t[\bar{J}]. \tag{52g}$$

We assume that the dimensionless cutoff functions satisfy

$$\lim_{\Lambda \to 0+} K(p/\Lambda) = \lim_{\Lambda \to 0+} R(p/\Lambda) = 0. \tag{53}$$

Hence, if the anomalous dimension satisfies

$$0 \leq \gamma \leq 1, \tag{54}$$

we obtain Eq. (50).

The correlation functions in the dimensionless convention are related to those in the dimensionful convention by

$$\langle \bar{\phi}(p_1) \cdots \bar{\phi}(p_n) \rangle_{\bar{S}_t}^{K,R} = \Lambda^{n\frac{D+2}{2}} \langle \phi(p_1\Lambda) \cdots \phi(p_n\Lambda) \rangle_{\tilde{S}_\Lambda}^{K_\Lambda, R_\Lambda}. \tag{55a}$$

Using Eq. (37), we can rewrite the right-hand side using Λ-independent correlation functions as

$$\langle \bar{\phi}(p_1) \cdots \bar{\phi}(p_n) \rangle_{\bar{S}_t}^{K,R} = \Lambda^{n\frac{D+2}{2}} \left(\frac{\mu}{\Lambda}\right)^{n\gamma} \langle \phi(p_1\Lambda) \cdots \phi(p_n\Lambda) \rangle_{S_\Lambda}^{K_\Lambda, R_\Lambda}. \tag{55b}$$

Hence, in the dimensionless convention the correlation functions satisfy the following scaling relation:

$$\langle \bar{\phi}(p_1 e^{\Delta t}) \cdots \bar{\phi}(p_n e^{\Delta t}) \rangle_{\bar{S}_{t+\Delta t}}^{K,R} = \exp\left(n\left(-\frac{D+2}{2} + \gamma\right)\Delta t\right) \langle \bar{\phi}(p_1) \cdots \bar{\phi}(p_n) \rangle_{\bar{S}_t}^{K,R}. \tag{56}$$

Note that we are comparing the correlation functions for different sets of momenta at two different Wilson actions which are related by ERG.

It is straightforward to obtain the ERG differential equations for $\bar{S}_t$ and $\bar{W}_t$ by rewriting the equations for $\tilde{S}_\Lambda$ and $\tilde{W}_\Lambda$. For the rewriting we use

$$\frac{\delta}{\delta\phi(p)} = \int_q \frac{\delta\bar{\phi}(q)}{\delta\phi(p)} \frac{\delta}{\delta\bar{\phi}(q)} = \int_q \frac{\delta\left(\Lambda^{\frac{D+2}{2}}\phi(q\Lambda)\right)}{\delta\phi(p)} \frac{\delta}{\delta\bar{\phi}(q)}$$

$$= \int_q \Lambda^{\frac{D+2}{2}} \delta(q\Lambda - p) \frac{\delta}{\delta\bar{\phi}(q)} = \Lambda^{-\frac{D-2}{2}} \frac{\delta}{\delta\bar{\phi}(p/\Lambda)}, \tag{57}$$

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and the analogous

$$\frac{\delta}{\delta J(p)} = \Lambda^{-\frac{D+2}{2}} \frac{\delta}{\delta \bar{J}(p/\Lambda)} . \tag{58}$$

We need only the equation for $\bar{W}_t$ here (see Appendix B for the derivation):

$$\partial_t e^{\bar{W}_t[\bar{J}]} = \int_p \left[\left(p \cdot \partial_p + \frac{D-2}{2} + \gamma \right) \bar{J}(p) \cdot \frac{\delta}{\delta \bar{J}(p)} \right.$$
$$\left. + \left(-p \cdot \partial_p + 2 - 2\gamma \right) R(p) \cdot \frac{1}{2} \frac{\delta^2}{\delta \bar{J}(p) \delta \bar{J}(-p)} \right] e^{\bar{W}_t[\bar{J}]} . \tag{59}$$

For this to have a fixed-point solution, we must choose γ appropriately. With $\gamma = 0$, we only get the Gaussian fixed point:

$$\bar{W}_G[\bar{J}] = \frac{1}{2} \int_p \frac{\bar{J}(p)\bar{J}(-p)}{p^2 + R(p)} . \tag{60}$$

By choosing $0 < \gamma < 1$ appropriately, we can obtain a nontrivial fixed point $\bar{W}^*[\bar{J}]$ that satisfies

$$0 = \int_p \left[\left(p \cdot \partial_p + \frac{D-2}{2} + \gamma \right) \bar{J}(p) \cdot \frac{\delta}{\delta \bar{J}(p)} \right.$$
$$\left. + \left(-p \cdot \partial_p + 2 - 2\gamma \right) R(p) \cdot \frac{1}{2} \frac{\delta^2}{\delta \bar{J}(p) \delta \bar{J}(-p)} \right] e^{\bar{W}^*[\bar{J}]} . \tag{61}$$

For the fixed point, the scaling relation (56) relates the correlation functions for the same fixed-point Wilson action $\bar{S}^*$:

$$\langle \bar{\phi}(p_1 e^{\Delta t}) \cdots \bar{\phi}(p_n e^{\Delta t}) \rangle_{\bar{S}^*}^{K,R} = \exp\left(n\left(-\frac{D+2}{2} + \gamma \right) \Delta t \right) \langle \bar{\phi}(p_1) \cdots \bar{\phi}(p_n) \rangle_{\bar{S}^*}^{K,R} . \tag{62}$$

Now we are ready to derive the main result of this paper. For a general theory, we get

$$\tilde{W}_\Lambda \left[\left(\frac{\Lambda}{\mu} \right)^\gamma \mathcal{J} \right] = \bar{W}_{t=-\ln \Lambda/\mu} \left[\bar{J}(p) = \left(\frac{\Lambda}{\mu} \right)^\gamma \Lambda^{\frac{D-2}{2}} \mathcal{J}(p\Lambda) \right] . \tag{63}$$

Unless we know $\bar{W}_t$ for very large t, we cannot use Eq. (51) to obtain $\mathcal{W}[\mathcal{J}]$. At a fixed point, however, $\bar{W}_t$ does not depend on t, and the Λ-dependence of $\tilde{W}_\Lambda[J]$ solely comes from the scaling of variables:

$$\tilde{W}_\Lambda[J] = \bar{W}^* \left[\bar{J}(p) = \Lambda^{\frac{D-2}{2}} J(p\Lambda) \right] , \tag{64}$$

where $\bar{W}^*$ is a fixed-point functional satisfying Eq. (61). Substituting Eq. (64) into Eq. (51), we obtain the main result of this paper,

$$\mathcal{W}[\mathcal{J}] = \lim_{\Lambda \to 0+} \bar{W}^* \left[\bar{J}(p) = \left(\frac{\Lambda}{\mu} \right)^\gamma \Lambda^{\frac{D-2}{2}} \mathcal{J}(p\Lambda) \right] , \tag{65}$$

which gives the connected correlation functions as a high-momentum limit of $\bar{W}^*$.

In Sect. 1 we briefly explained why we call Eq. (65) a high-momentum limit: any momentum in units of Λ gets large as we take $\Lambda \to 0+$. To make this explanation more concrete, expand $\bar{W}^*$ in powers of $\bar{J}$:

$$\bar{W}^*[\bar{J}] = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{p_1, \cdots, p_n} \bar{J}(-p_1) \cdots \bar{J}(-p_n) \, \delta(p_1 + \cdots + p_n) \, w_n(p_1, \ldots, p_n) \,. \tag{66}$$

We then obtain

$$\bar{W}^* \left[\left(\frac{\Lambda}{\mu} \right)^{\gamma} \Lambda^{\frac{D-2}{2}} \mathcal{J}(p\Lambda) \right]$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{p_1, \ldots, p_n} \left(\frac{\Lambda}{\mu} \right)^{n\gamma} \Lambda^{n \frac{D-2}{2}} \mathcal{J}(-p_1 \Lambda) \cdots \mathcal{J}(-p_n \Lambda) \, \delta(p_1 + \cdots + p_n)$$

$$\times w_n(p_1, \ldots, p_n)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{p_1, \ldots, p_n} \mathcal{J}(-p_1) \cdots \mathcal{J}(-p_n) \, \delta(p_1 + \cdots + p_n)$$

$$\times \left(\frac{\Lambda}{\mu} \right)^{n\gamma} \Lambda^{-n \frac{D+2}{2} + D} w_n(p_1/\Lambda, \ldots, p_n/\Lambda) \,. \tag{67}$$

This implies that

$$\langle\!\langle \phi(p_1) \cdots \phi(p_n) \rangle\!\rangle^{\text{connected}}$$

$$= \lim_{\Lambda \to 0+} \left(\frac{\Lambda}{\mu} \right)^{n\gamma} \Lambda^{-n \frac{D+2}{2} + D} \cdot w_n(p_1/\Lambda, \ldots, p_n/\Lambda) \, \delta(p_1 + \cdots + p_n) \,. \tag{68}$$

Thus, the connected correlation functions are obtained as the high-momentum limit of $w_n(p_1, \ldots, p_n)$. Especially for $n = 2$, we obtain

$$\langle\!\langle \phi(p)\phi(q) \rangle\!\rangle = \lim_{\Lambda \to 0+} \left(\frac{\Lambda}{\mu} \right)^{2\gamma} \frac{1}{\Lambda^2} w_2(p/\Lambda, -p/\Lambda) \, \delta(p + q) \,. \tag{69}$$

This implies that

$$w_2(p, -p) \xrightarrow{p \to \infty} \text{const} \, \frac{1}{(p^2)^{1-\gamma}} \,. \tag{70}$$

6. Conformal invariance

We would like to discuss the invariance properties of $\bar{W}^*$ and $\mathcal{W}$. The fixed-point theory has scale invariance, and we expect $\mathcal{W}[\mathcal{J}]$ to have naive scale invariance

$$\int_p \mathcal{J}(-p) D^S(p) \frac{\delta \mathcal{W}[\mathcal{J}]}{\delta \mathcal{J}(-p)} = 0 \,, \tag{71a}$$

where

$$D^S(p) \equiv -p \cdot \partial_p - \frac{D+2}{2} + \gamma \tag{71b}$$

is the generator of scale transformation. This is a direct consequence of Eq. (65); the very existence of the limit implies (71).

If the fixed-point theory also has conformal invariance, we expect

$$\int_p \mathcal{J}(-p) D_\mu^K(p) \frac{\delta \mathcal{W}[\mathcal{J}]}{\delta \mathcal{J}(-p)} = 0 \,, \tag{72a}$$

where

$$D_\mu^K(p) \equiv -p_\nu \frac{\partial^2}{\partial p_\mu \partial p_\nu} + \frac{1}{2} p_\mu \frac{\partial^2}{\partial p_\nu \partial p_\nu} + \left(-\frac{D+2}{2} + \gamma \right) \frac{\partial}{\partial p_\mu} \tag{72b}$$

is the generator of special conformal transformation. On the other hand, it is known [9–14] that the conformal invariance of the fixed-point theory is realized as

$$\int_p \bar{J}(-p) D_\mu^K(p) \frac{\delta \bar{W}^*[\bar{J}]}{\delta \bar{J}(-p)} + \int_p \left(-p \cdot \partial_p + 2 - 2\gamma \right) R(p)$$

$$\times \frac{1}{2} \int_q \delta(q-p) \frac{\partial}{\partial p_\mu} \left\{ \frac{\delta^2 \bar{W}^*[\bar{J}]}{\delta \bar{J}(p) \delta \bar{J}(-q)} + \frac{\delta \bar{W}^*[\bar{J}]}{\delta \bar{J}(p)} \frac{\delta \bar{W}^*[\bar{J}]}{\delta \bar{J}(-q)} \right\} = 0 \tag{73}$$

in terms of the fixed-point functional. As a consistency check of Eq. (65), we wish to use Eq. (65) to derive Eq. (72a) from Eq. (73).

Substituting

$$\bar{J}(p) = \left(\frac{\Lambda}{\mu} \right)^\gamma \Lambda^{\frac{D-2}{2}} \mathcal{J}(p\Lambda) \tag{74}$$

into Eq. (73), and using

$$\frac{\delta}{\delta \bar{J}(p)} = \left(\frac{\Lambda}{\mu} \right)^{-\gamma} \Lambda^{\frac{D+2}{2}} \frac{\delta}{\delta \mathcal{J}(p\Lambda)} \,, \tag{75}$$

we obtain

$$\int_p \mathcal{J}(-p\Lambda) D_\mu^K(p) \Lambda^D \frac{\delta \mathcal{W}[\mathcal{J}]}{\delta \mathcal{J}(-p\Lambda)} + \int_p \left(-p \cdot \partial_p + 2 - 2\gamma \right) R(p) \tag{76}$$

$$\times \frac{1}{2} \int_q \delta(q-p) \Lambda^{D+2} \left(\frac{\Lambda}{\mu} \right)^{-2\gamma} \frac{\partial}{\partial p_\mu} \left\{ \frac{\delta^2 \mathcal{W}[\mathcal{J}]}{\delta \mathcal{J}(p\Lambda) \delta \mathcal{J}(-q\Lambda)} + \frac{\delta \mathcal{W}[\mathcal{J}]}{\delta \mathcal{J}(p\Lambda)} \frac{\delta \mathcal{W}[\mathcal{J}]}{\delta \mathcal{J}(-q\Lambda)} \right\} = 0 \,.$$

Replacing $p\Lambda$ by p, and dividing the whole thing by Λ, we obtain

$$\int_p \mathcal{J}(-p) D_\mu^K(p) \frac{\delta \mathcal{W}[\mathcal{J}]}{\delta \mathcal{J}(-p)} + \int_p \Lambda \frac{\partial}{\partial \Lambda} \left(R_\Lambda(p) \left(\frac{\Lambda}{\mu} \right)^{-2\gamma} \right)$$

$$\times \frac{1}{2} \int_q \delta(q-p) \frac{\partial}{\partial p_\mu} \left\{ \frac{\delta^2 \mathcal{W}[\mathcal{J}]}{\delta \mathcal{J}(p) \delta \mathcal{J}(-q)} + \frac{\delta \mathcal{W}[\mathcal{J}]}{\delta \mathcal{J}(p)} \frac{\delta \mathcal{W}[\mathcal{J}]}{\delta \mathcal{J}(-q)} \right\} = 0. \tag{77}$$

Since

$$\lim_{\Lambda \to 0+} \Lambda \frac{\partial}{\partial \Lambda} \left(R_\Lambda(p) \left(\frac{\Lambda}{\mu} \right)^{-2\gamma} \right) = 0, \tag{78}$$

we obtain Eq. (72a) in the limit $\Lambda \to 0+$.

7. Extension to massive theories

The main result (65) can be extended to massive theories, but the extension is less interesting for the reason we give at the end of the section.

Let us consider a theory with a mass parameter g with mass dimension, say, 2. In the dimensionful convention, the generating functional of the connected correlation functions is given by

$$\mathcal{W}(g)[\mathcal{J}] = \lim_{\Lambda \to 0+} W_\Lambda(g)[\mathcal{J}] = \lim_{\Lambda \to 0+} \tilde{W}_\Lambda(g)\left[\left(\frac{\Lambda}{\mu}\right)^\gamma \mathcal{J}\right] \tag{79}$$

from Eq. (51). Note that g is a constant independent of Λ, and we have assumed that the anomalous dimension is independent of g. At $g = 0$ we recover the fixed-point theory considered in the previous section. Let $y > 0$ be the scale dimension of the mass parameter in the dimensionless convention. Then the dimensionless mass parameter is related to g by

$$\bar{g} \equiv \frac{g}{\mu^2}\left(\frac{\mu}{\Lambda}\right)^y. \tag{80}$$

Since

$$\partial_t \bar{g} = -\Lambda \frac{\partial}{\partial \Lambda}\bar{g}\Big|_g = y\,\bar{g}, \tag{81}$$

we can trade ∂_t for $y\bar{g}\frac{\partial}{\partial\bar{g}}$. Then, $\bar{W}(\bar{g})$ satisfies

$$y\bar{g}\frac{\partial}{\partial\bar{g}}e^{\bar{W}(\bar{g})[\bar{J}]} = \int_p \left[\left(p \cdot \partial_p + \frac{D-2}{2} + \gamma\right)\bar{J}(p) \cdot \frac{\delta}{\delta\bar{J}(p)}\right.$$
$$\left. + \left(-p \cdot \partial_p + 2 - 2\gamma\right)R(p)\frac{1}{2}\frac{\delta^2}{\delta\bar{J}(p)\delta\bar{J}(-p)}\right]e^{\bar{W}(g)[J]}. \tag{82}$$

Since

$$\bar{W}(\bar{g})[\bar{J}] = \tilde{W}_\Lambda(g)[J], \tag{83}$$

we obtain, from Eq. (51),

$$\mathcal{W}(g)[\mathcal{J}] = \lim_{\Lambda \to 0} \bar{W}(\bar{g})\left[\bar{J}(p) = \left(\frac{\Lambda}{\mu}\right)^\gamma \Lambda^{\frac{D-2}{2}}\mathcal{J}(p\Lambda)\right], \tag{84}$$

where the Λ dependence of $\bar{g}$ is given by Eq. (80). Note that $\bar{g}$ diverges as we take $\Lambda \to 0+$.

For example, consider the simplest example of the massive Gaussian theory, corresponding to $y = 2$. We obtain

$$\bar{W}\left(\frac{m^2}{\Lambda^2}\right)[\bar{J}] = \frac{1}{2}\int_p \frac{\bar{J}(p)\bar{J}(-p)}{p^2 + m^2/\Lambda^2 + R(p)}. \tag{85}$$

We then find

$$\bar{W}\left(\frac{m^2}{\Lambda^2}\right)\left[\Lambda^{\frac{D-2}{2}}\mathcal{J}(p\Lambda)\right] = W_\Lambda(m^2)[\mathcal{J}] = \frac{1}{2}\int_p \frac{\mathcal{J}(p)\mathcal{J}(-p)}{p^2 + m^2 + R_\Lambda(p)}$$
$$\xrightarrow{\Lambda \to 0} \frac{1}{2}\int_p \frac{\mathcal{J}(p)\mathcal{J}(-p)}{p^2 + m^2}. \tag{86}$$

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The crucial difference of Eq. (84) from Eq. (65) is that the right-hand side is not the high-momentum limit of a fixed $\bar{W}^*$: $\bar{W}(\bar{g})$ depends on the exponentially large parameter $\bar{g}$. This is expected. Take a fixed momentum p for the left-hand side of Eq. (84). The mass scale is of order $\mu(g/\mu^2)^{\frac{1}{y}}$. Now, for the right-hand side, the corresponding dimensionless momentum is p/Λ. Since the ratio to the mass scale must be the same,

$$\frac{p}{\mu(g/\mu^2)^{\frac{1}{y}}} = \frac{p/\Lambda}{\bar{g}^{\frac{1}{y}}}\,, \tag{87}$$

we reproduce Eq. (80):

$$\bar{g} = \frac{g}{\mu^2}\left(\frac{\mu}{\Lambda}\right)^{\frac{1}{y}} \stackrel{\Lambda\to 0+}{\longrightarrow} +\infty\,. \tag{88}$$

To obtain $\bar{W}(\bar{g})$ for large $\bar{g}$, we must solve the ERG equation for a wide range of $\bar{g}$. We have nothing to gain by switching to the dimensionless convention.

8.   Conclusion

In this paper we have shown that the high-momentum limit of a fixed-point Wilson action contains the connected correlation functions of the corresponding massless theory. This is given explicitly by Eq. (65), where $\mathcal{W}$ is the generating functional of the connected correlation functions, and $\bar{W}^*$ is the generating functional with an IR cutoff associated with the fixed-point Wilson action $\bar{S}^*$. $\bar{W}^*$ is directly related to $\bar{S}^*$ by

$$\bar{W}^*[\bar{J}] = \frac{1}{2}\int_p \frac{\bar{J}(-p)\bar{J}(p)}{R(p)} + \bar{S}^*[\bar{\phi}]\,, \tag{89a}$$

$$\bar{J}(p) \equiv \frac{R(p)}{K(p)}\bar{\phi}(p)\,, \tag{89b}$$

where K, R are cutoff functions. In deriving Eq. (65), we have used two equivalent conventions for ERG: one with a dimensionful cutoff Λ, and the other with a fixed dimensionless cutoff 1. In the dimensionful convention, the correlation functions are obtained from the Wilson action in the limit of the vanishing cutoff, as given by Eqs. (35) and (51). On the other hand, in the dimensionless convention, the correlation functions are obtained as the high-momentum limit of the Wilson action. We have used both conventions to derive Eq. (65).

Recently, in Ref. [14], a classical limit has been introduced as the limit of an infinite momentum cutoff where the naive scale and conformal invariance may be restored in the Wilson action. We have discussed the opposite limit of the vanishing cutoff in this paper.

Appendix A.   Derivation of the diffusion equation

We have derived a variant of the diffusion equation three times from the corresponding ERG differential equation: Eq. (23) from Eq. (7), Eq. (32) from Eq. (29), and Eq. (43) from Eq. (40). The derivation is essentially the same; let us show how to derive Eq. (43) from Eq. (40) here.

Differentiating $\tilde{W}_\Lambda[J]$ with respect to Λ while fixing J, we obtain

$$-\Lambda \frac{\partial}{\partial \Lambda} \tilde{W}_\Lambda[J] = \frac{1}{2} \int_p \frac{\Lambda \frac{\partial}{\partial \Lambda} R_\Lambda(p)}{R_\Lambda(p)^2} J(p) J(-p) - \Lambda \frac{\partial}{\partial \Lambda} \tilde{S}_\Lambda[\phi]\Big|_J . \tag{A.1}$$

Since

$$\phi(p) = \frac{K_\Lambda(p)}{R_\Lambda(p)} J(p) , \tag{A.2}$$

we obtain

$$-\Lambda \frac{\partial}{\partial \Lambda} \tilde{S}_\Lambda[\phi]\Big|_J = -\Lambda \frac{\partial}{\partial \Lambda} \tilde{S}_\Lambda[\phi] - \int_p \Lambda \frac{\partial}{\partial \Lambda} \ln \frac{K_\Lambda(p)}{R_\Lambda(p)} \cdot \phi(p) \frac{\delta \tilde{S}_\Lambda[\phi]}{\delta \phi(p)} . \tag{A.3}$$

Using Eq. (40), we obtain

$$-\Lambda \frac{\partial}{\partial \Lambda} \tilde{W}_\Lambda[J] = \frac{1}{2} \int_p \frac{\Lambda \frac{\partial}{\partial \Lambda} R_\Lambda(p)}{R_\Lambda(p)^2} J(p) J(-p) + \int_p \left(\Lambda \frac{\partial}{\partial \Lambda} \ln R_\Lambda(p) - \gamma \right) \phi(p) \frac{\delta \tilde{S}_\Lambda[\phi]}{\delta \phi(p)}$$

$$+ \int_p \left(\Lambda \frac{\partial}{\partial \Lambda} R_\Lambda(p) - 2\gamma R_\Lambda(p) \right) \frac{K_\Lambda(p)^2}{R_\Lambda(p)^2} \frac{1}{2} \left\{ \frac{\delta \tilde{S}_\Lambda}{\delta \phi(p)} \frac{\delta \tilde{S}_\Lambda}{\delta \phi(-p)} + \frac{\delta^2 \tilde{S}_\Lambda}{\delta \phi(p) \delta \phi(-p)} \right\} . \tag{A.4}$$

Using

$$\frac{\delta \tilde{S}_\Lambda[\phi]}{\delta \phi(p)} = \frac{R_\Lambda(p)}{K_\Lambda(p)} \frac{\delta}{\delta J(p)} \left(\tilde{W}_\Lambda[J] - \frac{1}{2} \int_p \frac{J(p) J(-p)}{R_\Lambda(p)} \right) , \tag{A.5}$$

and ignoring the J-independent terms, we obtain

$$-\Lambda \frac{\partial}{\partial \Lambda} \tilde{W}_\Lambda[J] = \int_p \left[\gamma J(p) \frac{\delta \tilde{W}_\Lambda[J]}{\delta J(p)} \right.$$

$$\left. + \left(\Lambda \frac{\partial R_\Lambda(p)}{\partial \Lambda} - 2\gamma R_\Lambda(p) \right) \frac{1}{2} \left\{ \frac{\delta^2 \tilde{W}_\Lambda[J]}{\delta J(p) \delta J(-p)} + \frac{\delta \tilde{W}_\Lambda[J]}{\delta J(p)} \frac{\delta \tilde{W}_\Lambda[J]}{\delta J(-p)} \right\} \right], \tag{A.6}$$

which is Eq. (43).

Appendix B.   Conversion between the dimensionful and the dimensionless conventions

Let us derive the dimensionless diffusion equation (59) from the dimensionful diffusion equation (43), where $\bar{W}_t[\bar{J}]$ and $\tilde{W}_\Lambda[J]$ are related by Eq. (52). Differentiating $\bar{W}_t[\bar{J}]$ with respect to t, we are fixing $\bar{J}$:

$$\partial_t \bar{W}_t[\bar{J}] = -\Lambda \frac{\partial}{\partial \Lambda} \tilde{W}_\Lambda[J]\Big|_{\bar{J}} . \tag{B.1}$$

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Since J and $\bar{J}$ are related by Eq. (52c), we obtain

$$\partial_t \bar{W}_t[\bar{J}] = -\Lambda \frac{\partial}{\partial \Lambda} \tilde{W}_\Lambda[J] + \int_p \left(\frac{D-2}{2} + p \cdot \partial_p \right) J(p) \cdot \frac{\delta}{\delta J(p)} \tilde{W}_\Lambda[J] \,. \tag{B.2}$$

Using Eqs. (43) and (58), we obtain

$$\begin{aligned}
\partial_t e^{\bar{W}_t[\bar{J}]} &= \int_p \left(\frac{D-2}{2} + p \cdot \partial_p + \gamma \right) \bar{J}(p) \cdot \frac{\delta}{\delta \bar{J}(p)} e^{\bar{W}_t[\bar{J}]} \\
&\quad + \int_p \left(\Lambda \frac{\partial}{\partial \Lambda} - 2\gamma \right) \left(\Lambda^2 R(p/\Lambda) \right) \cdot \frac{1}{2} \frac{\delta^2}{\delta J(p) \delta J(-p)} e^{\bar{W}_t[\bar{J}]} \\
&= \int_p \left(\frac{D-2}{2} + p \cdot \partial_p + \gamma \right) \bar{J}(p) \cdot \frac{\delta}{\delta \bar{J}(p)} e^{\bar{W}_t[\bar{J}]} \\
&\quad + \int_p \Lambda^2 \left(-p \cdot \partial_p + 2 - 2\gamma \right) R(p/\Lambda) \cdot \Lambda^{-D-2} \frac{1}{2} \frac{\delta^2}{\delta \bar{J}(p/\Lambda) \delta \bar{J}(-p/\Lambda)} e^{\bar{W}_t[\bar{J}]} \\
&= \int_p \left[\left(p \cdot \partial_p + \frac{D-2}{2} + \gamma \right) \bar{J}(p) \cdot \frac{\delta}{\delta \bar{J}(p)} \right. \\
&\quad \left. + \left(-p \cdot \partial_p + 2 - 2\gamma \right) R(p) \cdot \frac{1}{2} \frac{\delta^2}{\delta \bar{J}(p) \delta \bar{J}(-p)} \right] e^{\bar{W}_t[\bar{J}]} \,,
\end{aligned} \tag{B.3}$$

which is Eq. (59).

Appendix C.   Effective action

The effective action is defined as the Legendre transform of the generating functional of connected correlation functions:

$$\Gamma_{\text{eff}}[\Phi] \equiv \mathcal{W}[\mathcal{J}] - \int_p \mathcal{J}(-p)\Phi(p) \,, \tag{C.1a}$$

where

$$\Phi(p) \equiv \frac{\delta \mathcal{W}[\mathcal{J}]}{\delta \mathcal{J}(-p)} \,. \tag{C.1b}$$

On the other hand, the so-called effective average action $\bar{\Gamma}$ is defined as the analogous Legendre transform:

$$\bar{\Gamma}[\bar{\Phi}] - \frac{1}{2} \int_p R(p) \bar{\Phi}(-p) \bar{\Phi}(p) \equiv \bar{W}[\bar{J}] - \int_p \bar{J}(-p)\bar{\Phi}(p) \,, \tag{C.2a}$$

where

$$\bar{\Phi}(p) \equiv \frac{\delta \bar{W}[\bar{J}]}{\delta \bar{J}(-p)} \,. \tag{C.2b}$$

We have omitted the $*$ from $\bar{\Gamma}$ and $\bar{W}$ to simplify the expression. We wish to express Γ_{eff} as the IR limit of $\bar{\Gamma}$ by rewriting the main result (65).

Recall that Eq. (65) is the IR limit of

$$\mathcal{W}[\mathcal{J}] = \bar{W}[\bar{J}], \tag{C.3}$$

where

$$\bar{J}(p) = \left(\frac{\Lambda}{\mu}\right)^{\gamma} \Lambda^{\frac{D-2}{2}} \mathcal{J}(p\Lambda). \tag{C.4}$$

Correcting Eq. (58) by the anomalous dimension, we obtain

$$\frac{\delta}{\delta \mathcal{J}(-p)} = \left(\frac{\Lambda}{\mu}\right)^{\gamma} \Lambda^{-\frac{D+2}{2}} \frac{\delta}{\delta \bar{J}(-p/\Lambda)}. \tag{C.5}$$

Hence, we obtain

$$\Phi(p) = \frac{\delta \mathcal{W}[\mathcal{J}]}{\delta \mathcal{J}(-p)} = \left(\frac{\Lambda}{\mu}\right)^{\gamma} \Lambda^{-\frac{D+2}{2}} \frac{\delta \bar{W}[\bar{J}]}{\delta \bar{J}(-p/\Lambda)} = \left(\frac{\Lambda}{\mu}\right)^{\gamma} \Lambda^{-\frac{D+2}{2}} \bar{\Phi}(p/\Lambda). \tag{C.6}$$

Thus, from Eq. (C.1), we obtain

$$\begin{aligned}
\Gamma_{\mathrm{eff}}[\Phi] &= \bar{W}[\bar{J}] - \int_p \mathcal{J}(-p)\Phi(p) \\
&= \bar{W}[\bar{J}] - \int_p \left(\frac{\Lambda}{\mu}\right)^{-\gamma} \Lambda^{-\frac{D-2}{2}} \bar{J}(-p/\Lambda) \left(\frac{\Lambda}{\mu}\right)^{\gamma} \Lambda^{-\frac{D+2}{2}} \bar{\Phi}(p/\Lambda) \\
&= \bar{W}[\bar{J}] - \int_p \bar{J}(-p)\bar{\Phi}(p) \\
&= \bar{\Gamma}[\bar{\Phi}] - \frac{1}{2} \int_p R(p)\bar{\Phi}(-p)\bar{\Phi}(p).
\end{aligned} \tag{C.7}$$

Since

$$\int_p R(p)\bar{\Phi}(-p)\bar{\Phi}(p) = \int_p \left(\frac{\Lambda}{\mu}\right)^{-2\gamma} \Lambda^2 R(p/\Lambda)\Phi(-p)\Phi(p) \tag{C.8}$$

vanishes in the limit $\Lambda \to 0+$ as a functional of Φ, we obtain

$$\Gamma_{\mathrm{eff}}[\Phi] = \lim_{\Lambda \to 0+} \bar{\Gamma}[\bar{\Phi}], \tag{C.9a}$$

where

$$\bar{\Phi}(p) = \left(\frac{\Lambda}{\mu}\right)^{-\gamma} \Lambda^{\frac{D+2}{2}} \Phi(p\Lambda). \tag{C.9b}$$

This is the desired result.

References

[1] K. G. Wilson and J. B. Kogut, Phys. Rept. **12**, 75 (1974).
[2] J. Polchinski, Nucl. Phys. B **231**, 269 (1984).
[3] H. Sonoda, J. Phys. A, **40**, 9675 (2007) [arXiv:hep-th/0703167] [Search INSPIRE].

[4] Y. Igarashi, K. Itoh, and H. Sonoda, Prog. Theor. Phys. Suppl. **181**, 1 (2010) [arXiv:0909.0327 [hep-th]] [Search INSPIRE].

[5] G. Keller, C. Kopper, and M. Salmhofer, Helv. Phys. Acta **65**, 32 (1992).

[6] T. R. Morris, Int. J. Mod. Phys. A **9**, 2411 (1994) [arXiv:hep-ph/9308265] [Search INSPIRE].

[7] H. Sonoda, Prog. Theor. Exp. Phys. **2015**, 103B01 (2015) [arXiv:1503.08578 [hep-th]] [Search INSPIRE].

[8] Y. Igarashi, K. Itoh, and H. Sonoda, Prog. Theor. Exp. Phys. **2016**, 093B04 (2016) [arXiv:1607.01521 [hep-th]] [Search INSPIRE].

[9] O. J. Rosten, Eur. Phys. J. C **77**, 477 (2017) [arXiv:1411.2603 [hep-th]] [Search INSPIRE].

[10] B. Delamotte, M. Tissier, and N. Wschebor, Phys. Rev. E **93**, 012144 (2016) [arXiv:1501.01776 [cond-mat.stat-mech]] [Search INSPIRE].

[11] O. J. Rosten (2016), arXiv:1605.01055 [hep-th] [Search INSPIRE].

[12] O. J. Rosten (2016), arXiv:1605.01729 [hep-th] [Search INSPIRE].

[13] H. Sonoda, Prog. Theor. Exp. Phys. **2017**, 083B05 (2017) [arXiv:1705.01239 [hep-th]] [Search INSPIRE].

[14] O. J. Rosten (2017), arXiv:1705.05837 [hep-th] [Search INSPIRE].