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DIFFERENTIAL GEOMETRIC STRUCTURE OF NON-EQUILIBRIUM DYNAMICS IN COMPETITION AND PREDATION: FINSLER GEOMETRY AND KCC THEORY

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Abstract. We considered the differential geometric structure of non-equilibrium dynamics in non-linear interactions, such as competition and predation, based on Kosambi-Cartan-Chern (KCC) theory. The stability of a geodesic flow on a Finslerian manifold is characterized by the deviation curvature (the second invariant in the dynamical system). According to KCC theory, the value of the deviation curvature is constant around the equilibrium point. However, in the non-equilibrium region, not only the value but also the sign of the deviation curvature depend on time. Next, we reapplied KCC theory to the dynamics of the deviation curvature and determined the hierarchical structure of the geometric stability. The dynamics of the deviation curvature in the non-equilibrium region is accompanied by a complex periodic (node) pattern in the predation (competition) system.

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1. Introduction

In biological dynamics, competition and predation are typical interactions between species (e.g., [14, 13, 21]). These interactions can be considered non-linear processes, so there has been some interest in ecological stability against several perturbations, such as environmental change and evolution (e.g., [18, 19, 17]). The Lotka-Volterra equation is one well-known equation that describes non-linear interactions such as intraspecific or interspecific competition, and predation between prey and predators (e.g., [13, 4]). From the Lotka-Volterra equation, we can derive a second-order differential equation that describes ecological interactions (e.g., [5, 8, 30]).

The Lotka-Volterra form, including the second-order differential equation type, enables us to consider the stability of the ecological system from the viewpoint of the Kosambi-Cartan-Chern (KCC) theory (e.g., [6, 9, 10]). This approach allows the evolution of a dynamic system to be described in geometric terms, by considering it as a geodesic in a Finsler space (e.g., [23, 12, 7]). From the geodesic equation, we can obtain the covariant form of the variational equations for the deviation of the whole trajectory to nearby ones, so the KCC theory represents a powerful mathematical method for the analysis of dynamic systems (e.g., [1, 3, 8, 12, 15]).

Generally, the stability of the geodesic flow on a Finslerian manifold is called "Jacobi stability" and is characterized by the deviation curvature (the second invariant in the dynamical system) (e.g., [12, 16, 27]). It has been applied to various non-linear dynamic systems (e.g., [2, 15, 11, 24, 26, 25]) because it clarifies their intrinsic properties using differential geometric concepts such as "connection" and "curvatures." In our previous paper [30], we showed that the stability of competition and predation can be described by geometric objects, such as non-linear connections and deviation curvature. As we will see in this paper, around the equilibrium point, these geometric objects of the ecological system are constant, so the stability of the dynamic system is relatively simple. However, in areas away from an equilibrium point, the geometric objects of the ecological system depend on time, so it is expected that the geometric objects themselves represent another dynamic system. This indicates that applying the KCC theory to the dynamic system of the geometric objects leads us to derive other geometric objects. However, few papers have considered how the dynamic change of geometric objects affects the non-equilibrium region of biological dynamical systems. Thus, the purpose of this study was to consider the dynamics of geometric objects related to the ecological system, based on KCC theory.

The structure of this paper is as follows. In section 2, we review KCC theory from the viewpoint of Jacobi stability and linear stability. In sections 3 and 4, we consider the non-equilibrium properties of ecological interactions, such as competition and predation. In section 5, we consider the dynamics of geometric objects in a simple example.

2. STABILITY ANALYSIS IN KCC THEORY

2.1. **Deviation curvature and stability.** Einstein's summation convention is used throughout this paper. Let M be a real smooth n-dimensional manifold, and (TM, π, M) be its tangent bundle, where $\pi : TM \to M$ is a projection from the total space TM to the base manifold M. A point $x \in M$ has local coordinates (x^i), where $i = 1, \dots, n$. The local chart of a point in TM is denoted by $(x^i, \dot{x}^i)$, where t is time (regarded as an absolute invariant) and $\dot{x}^i = dx/dt$. Let us consider the path equation:

$$\ddot{x}^i + g^i(x, \dot{x}) = 0, \tag{1}$$

where $g^i(x, \dot{x})$ is a smooth function. The trajectory $x^i(t)$ of the system (1) is changed to nearby trajectories according to $\overline{x}^i = x^i + u^i \delta\tau$, where u^i is a vector field and $\delta\tau$ is a small parameter. In this case, Eq. (1) becomes a variational equation for the limit $\delta\tau \to 0$. The covariant form of this variational equation is given by:

$$\frac{D^2 u^i}{Dt^2} = P^i_j u^j, \tag{2}$$

where $D(\cdots)/Dt$ is a covariant differential defined by:

$$\frac{Du^i}{Dt} = \frac{du^i}{dt} + \frac{1}{2}\frac{\partial g^i}{\partial \dot{x}^j} u^j. \tag{3}$$

P^i_j is the deviation curvature tensor given by:

$$\begin{aligned} P^i_j &= -\frac{\partial g^i}{\partial x^j} + \frac{1}{2}\frac{\partial^2 g^i}{\partial \dot{x}^j \partial x^k}\dot{x}^k - \frac{1}{2}\frac{\partial^2 g^i}{\partial \dot{x}^j \partial \dot{x}^k} g^k + \frac{1}{4}\frac{\partial g^i}{\partial \dot{x}^k}\frac{\partial g^k}{\partial \dot{x}^j} \\ &= -\frac{\partial g^i}{\partial x^j} + \frac{\partial N^i_j}{\partial x^k}\dot{x}^k - G^i_{jk} g^k + N^i_k N^k_j, \end{aligned} \tag{4}$$

where $N^i_j = (1/2)\partial g^i / \partial \dot{x}^j$ is a coefficient of the non-linear connection and $G^i_{jk} = \partial N^i_j / \partial \dot{x}^k$ is a Finsler (Berwald) connection.

The deviation curvature tensor P^i_j gives the stability of whole trajectories via the following theorem [5]: *The trajectories of the system (1) are Jacobi-stable if and only if the real parts of the eigenvalues of P^i_j are strictly negative everywhere, and Jacobi-unstable otherwise.* In particular, the trajectories of the one-dimensional system are Jacobi-stable when $P^1_1 < 0$, and Jacobi-unstable when $P^1_1 \geq 0$.

Jacobi stability is a natural generalization of the stability of the geodesic flow on a differentiable manifold endowed with a metric (Riemannian or Finslerian) to the non-metric setting [12]. The Jacobi stability conditions of non-degenerate equilibrium points are the same as the known conditions for Lyapunov stability [1, 2, 3]. When the potential surface of Lyapunov stability can be defined, the deviation curvature related to Jacobi stability corresponds to the Willmore energy density of the potential surface [30, 28, 29].

2.2. **Jacobi stability and linear stability.** Jacobi stability gives a more global stability than linear stability (e.g., [15, 11, 23, 22]). To clarify the relationship between Jacobi stability and linear stability (see also [23, 22]), we can consider a vector field described by $\dot{x}^i = f^i(x)$, where $i = 1, 2, \ldots, n$ and f^i denote a given function. This can be approximated by a linear system around an equilibrium point x^i_0 using the relation $x^i = x^i_0 + \xi^i$, where ξ^i is a small quantity: $\dot{\xi}^i = J^i_j(x_0)\xi^j$, where $J^i_j(x_0)$ is the Jacobian matrix of f^i. When we consider the coordinate system $(\xi^i, \dot{\xi}^i)$ in the two-dimensional case ($i = 1, 2$), we have the following equation for $i = 1$:

$$\ddot{\xi}^1 - tr[J]\dot{\xi}^1 + det[J]\xi^1 = 0. \tag{5}$$

This is a particular case of (1) for $g^1 = -tr[J]\dot{\xi}^1 + det[J]\xi^1$. Thus, (4) give the deviation curvature of the linearized system:

$$P_1^1 = \frac{1}{4}\{(tr[J])^2 - 4det[J]\}. \tag{6}$$

Eq. (6) shows that the deviation curvature corresponds to the discriminant, so the sign of the deviation curvature (i.e., Jacobi stability in the one-dimensional system) distinguishes the node (or saddle) system and the center (or spiral) system, as pointed out by Sabău [23, 22].

For example, we can consider the case of the two species coexisting in the Lotka-Volterra system. In this case, the deviation curvature of competition is positive, and that of the predation is negative[30]. However, this is the result around the equilibrium point, so we consider the non-equilibrium region of the Lotka-Volterra system in the following sections.

3. NON-EQUILIBRIUM ANALYSIS IN A PREDATION SYSTEM

3.1. **Geometric structure.** We consider the Lotka-Volterra predation system for the prey $x^1 = x$ and the predator $x^2 = y$. The standard form of the system is given by (e.g., [13, 20]):

$$\dot{x} = rx - axy, \tag{7}$$

$$\dot{y} = bxy - cy, \tag{8}$$

where r is the natural growth rate of the prey, a and b are coefficients of predation, and c is the natural death rate of the predator. These parameters are all positive. This system can be approximated by a linear system around an equilibrium point $(x_0, y_0) = (c/b, r/a)$. In the case when the two species coexist, i.e., $x_0 \neq 0$ and $y_0 \neq 0$, the Jacobian matrix of the system is:

$$J = \begin{pmatrix} 0 & -ac/b \\ br/a & 0 \end{pmatrix}. \tag{9}$$

Then, from (6), the deviation curvature of the prey: P_1^1 and that of the predator: P_2^2 around the equilibrium point $(x_0, y_0) = (c/b, r/a)$ is:

$$P^1_1 = P^2_2 = -cr. \tag{10}$$

Eq. (10) shows that the prey and the predator have the same geometric structure around the equilibrium point.

Because KCC theory can treat small perturbations at any point, we can derive the deviation curvature in the non-equilibrium case. To use Eq. (4) based on Eq. (1), we can rewrite Eqs. (7) and (8) as the equation for the predator $y = x^2$ by removing $x = x^1$ as follows: $\ddot{y} + g^2 = 0$ with $g^2 = -\dot{y}^2/y + (ay - r)\dot{y} + acy^2 - cr$. Thus, from (4), the deviation curvature of the predator in the non-equilibrium case can be obtained:

$$P^2_2 = -\frac{a}{2}\dot{y} - acy + \frac{a^2}{4}\left(y - \frac{r}{a}\right)^2. \tag{11}$$

In a similar fashion, we can derive the deviation curvature of the prey in the non-equilibrium case:

$$P^1_1 = \frac{b}{2}\dot{x} - brx + \frac{b^2}{4}\left(x - \frac{c}{b}\right)^2. \tag{12}$$

Comparison of Eq. (11) and Eq. (12) reveals that the predator and prey have distinct geometric structures in the non-equilibrium case. However, the predator and prey in the equilibrium case ($y = y_0 = r/a$ and $x = x_0 = c/b$) have the same geometric structure, $P^1_1 = P^2_2 = -cr$, in agreement with Eq. (10). This finding suggests that the equilibrium point can be interpreted as the point in which the two deviation curvatures become equal, although the original equations have a non-symmetric variable structure.

3.2. **Numerical simulation.**

Because the geometric structure of the dynamic system is related to the stability of the system, Eqs. (11) and (12) mean that the system has a different and time-dependent stability structure for each species in the non-equilibrium region. Let us rewrite Eq. (12) using (7):

$$P^1_1 = \left(\frac{b}{2}\right)^2 x^2 - \frac{b}{2}(ay + c + r)x + \left(\frac{c}{2}\right)^2. \tag{13}$$

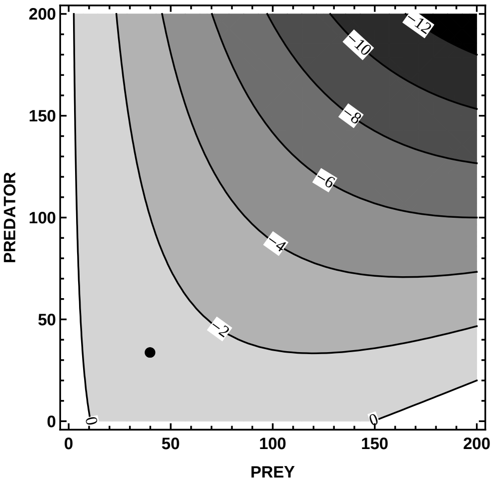

Figure 1. A contour plot of the deviation curvature of the predation system as a function of the number of population of the prey and that of the predator. The filled circle is the equilibrium point.

When $P_1^1 = -cr$, Eq. (13) gives us the equilibrium point $(x_0, y_0) = (c/b, r/a)$ as described above. Moreover, Eq. (13) also gives us the analytical solution of the "neutral case:" $P_1^1 = 0$, as follows:

$$b(-bx + ay + c + r) \pm \sqrt{b^2(r + ay)(2c + r + ay)} = 0. \tag{14}$$

By crossing this boundary line, the sign of the deviation curvature of the prey system changes. In a case where two species coexist, the negative deviation curvature corresponds to the case where the equilibrium point is a center, and positive deviation curvature corresponds to the case where the equilibrium point is a node[30], so the trajectory of the number of individual changes by crossing the line (14).

To visualize this, we can counter plot P_1^1 of Eq. (13) (Fig. 1). The parameters are as follows: $r = 1$, $a = 0.03$, $b = 0.025$, $c = 1$. The filled circle is the equilibrium point. In most regions, including the equilibrium point, the deviation curvature is negative (i.e., center). In a small region, however, positive deviation curvature exists (i.e., node). Because the number of individuals depends on time, it is expected from (13) that the sign of the deviation curvature will also change with time. Thus, we should analyze the non-equilibrium dynamics of not only the population but also the deviation curvatures. For this, we will consider the numerical simulation based on three initial population cases: (i) equilibrium, (ii) nearly equilibrium, (iii) far equilibrium as follows.

First, we can consider the equilibrium population $x(0) = c/b = 40$, and $y(0) = r/a = 33.3$ as the initial population. Figures 2(a) and 2(b) show the dynamics of the population and that of the deviation curvatures, respectively. As indicated by Eq. (10), both the prey (solid line) and the predator (dashed line) exhibit the same deviation curvature (Fig. 2(b)) despite the population being different (Fig. 2(a)).

Second, we can consider the nearly equilibrium initial population $x(0) = 40$ and $y(0) = 40$. In this case, the value of the deviation curvatures are no longer constant, i.e., dependent on time, but the sign is always negative (Fig. 3(b)). Thus, the dynamics of the population exhibit typical periodic variation (Fig. 3(a)).

Finally, we can consider the far equilibrium initial population $x(0) = 3$ and $y(0) = 3$. In this case, it is expected from Figure 1 that the sign of the deviation curvature is not always negative. In fact, Figure 4(b) shows that the signs of the deviation curvatures depend on time. Moreover, compared with Figures 2(b) and 3(b), the difference in the deviation curvatures between the prey and the predator is relatively large. As a result, the dynamics of the population exhibit a non-typical periodic pattern, as shown in Figure 4(a). In particular, when the sign of the deviation curvature is positive, the dynamics of the population exhibits a distorted periodic pattern.

4. Non-equilibrium analysis in a competition system

4.1. **Geometric structure.** We can consider the Lotka-Volterra competition system for two species: $x^1 = x$ and $x^2 = y$. The standard form of the system is given by (e.g., [13, 20]):

$$\dot{x} = r_1 x\left(1 - \frac{x + a_1 y}{k_1}\right), \tag{15}$$

$$\dot{y} = r_2 y\left(1 - \frac{y + a_2 x}{k_2}\right), \tag{16}$$

where r_i is the natural growth rate, k_i is the carrying capacity, and a_i is the competition coefficient. These parameters are all positive. This system can be approximated

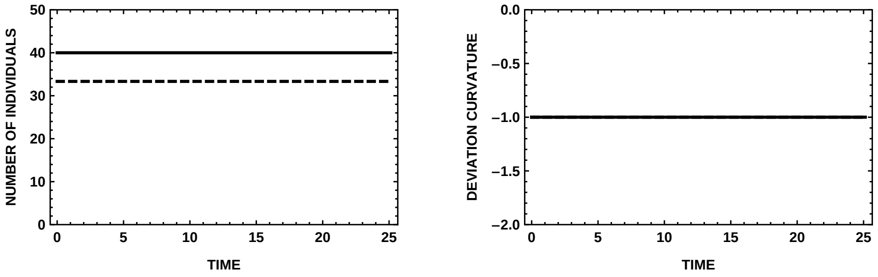

Figure 2. (a) Dynamics of the population number in the case of the initial population number being the equilibrium one. (b) The corresponding deviation curvature in the same case.

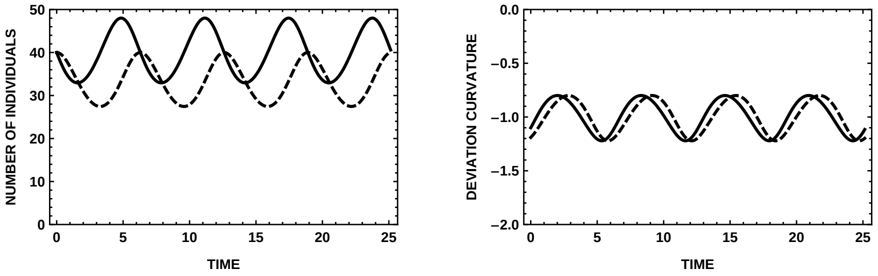

Figure 3. (a) Dynamics of the population number in the case of the initial population number being near the equilibrium one. (b) The corresponding deviation curvature in the same case.

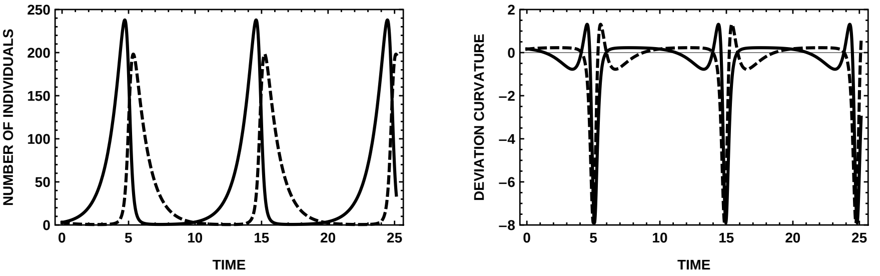

Figure 4. (a) Dynamics of the population number in the case of the initial population number being far from the equilibrium one. (b) The corresponding deviation curvature in the same case

by a linear system around an equilibrium point (x_0, y_0). In the case when the two species coexist, the Jacobian matrix of the system is:

$$J = \begin{pmatrix} A_1 & a_1 A_1 \\ a_2 A_2 & A_2 \end{pmatrix}, \tag{17}$$

where $A_1 = -r_1 x_0/k_1$ and $A_2 = -r_2 y_0/k_2$. Then, from (6), the deviation curvature for the two species coexisting is:

$$P_1^1 = P_2^2 = \frac{1}{4}\left\{(A_1 - A_2)^2 + 4a_1 a_2 A_1 A_2\right\}. \tag{18}$$

Like the predation system in Eq.(10), Eq. (18) shows that the two species in a competition system also have the same geometric structure around the equilibrium point. In a case where two species coexist, the positive deviation curvature corresponds to the case where the equilibrium point is a node [30], so Eq. (18) means that the equilibrium point is a node.

In a similar fashion to Section 3.1, we can obtain the deviation curvature of the competition in the non-equilibrium case for the two-species:

$$\begin{aligned} P_1^1 = {} & \left(-\frac{1}{2k_1} + \frac{2}{a_1 k_2} - \frac{a_2}{2k_2}\right)\dot{x} + \left(\frac{1}{4k_1^2} + \frac{2}{a_1 k_1 k_2} - \frac{3a_2}{k_1 k_2} + \frac{a_2^2}{4k_2^2}\right)x^2 \\ & + \left(\frac{1}{k_1} - \frac{2}{a_1 k_2}\right)x + 1, \end{aligned} \tag{19}$$

$$\begin{aligned} P_2^2 = {} & \left(-\frac{1}{2k_2} + \frac{2}{a_2 k_1} - \frac{a_1}{2k_1}\right)\dot{y} + \left(\frac{1}{4k_2^2} + \frac{2}{a_2 k_1 k_2} - \frac{3a_1}{k_1 k_2} + \frac{a_1^2}{4k_1^2}\right)y^2 \\ & + \left(\frac{1}{k_2} - \frac{2}{a_2 k_1}\right)y + 1. \end{aligned} \tag{20}$$

Because the natural growth rate is not related to Jacobi stability, we can assume that $r_1 = 1$ and $r_2 = 2$ for simplicity. These equations show that the two species in the competition system have distinct geometric structures in the non-equilibrium case. At the equilibrium point: $x = x_0 = (k_1 - a_1 k_2)/(1 - a_1 a_2)$ and $y = y_0 = (k_2 - a_2 k_1)/(1 - a_1 a_2)$, Eqs. (19) and (20) reduce to Eq. (18).

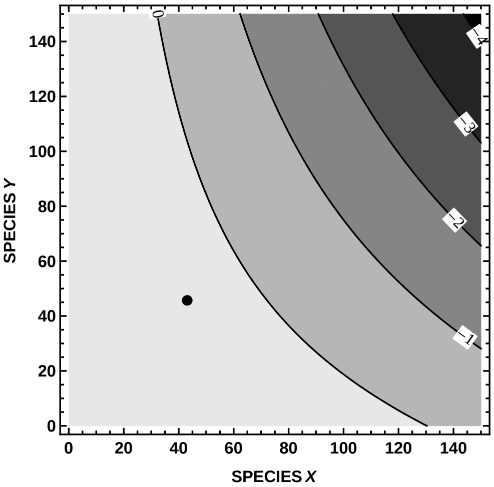

Figure 5. A contour plot of the deviation curvature of the competition system as a function of the number of population of the two species. The filled circle is the equilibrium point.

4.2. **Numerical simulation.** To see the stability structure of the competition, we counter plot P_1^1 of Eq. (19) in (x, y) plane using Eq. (15) (Fig. 5). The parameters are as follows: $r_1 = 1$, $r_2 = 2$, $a = 0.7$, $b = 0.8$, $k_1 = 75$, and $k_2 = 80$. In most regions, including the equilibrium point (filled circle), the sign of the deviation curvature is positive. However, there is a region where the sign of the deviation curvature is negative. This means that the sign of the deviation curvature depends on time. Next, we perform a numerical simulation of the competition system by considering three initial population cases as follows.

First, we can consider the equilibrium population $x(0) = (k_1 - a_1 k_2)/(1 - a_1 a_2) = 43.2$, and $y(0) = (k_2 - a_2 k_1)/(1 - a_1 a_2) = 45.5$ as the initial population. As indicated by Eq. (18), the two species exhibit the same deviation curvature (Fig. 6(b)) despite the populations being different (Fig. 6(a)).

Second, we can consider the nearly equilibrium initial population $x(0) = 50$ and $y(0) = 40$. Because the equilibrium population of y (i.e., 45.5) is larger than that of x (i.e., 43.2), the population of y increases and the population of x decreases toward the equilibrium populations (Fig. 7(a))). That is, this partially crossing pattern is due to the magnitude correlation of the equilibrium population, so the pattern is not part of a true periodic pattern. In fact, Figure 7(b) shows that the

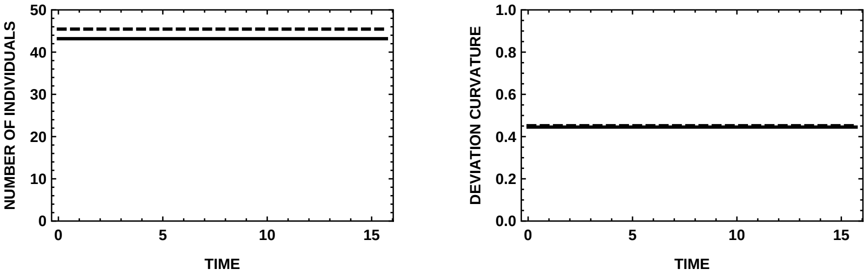

FIGURE 6. (a) Dynamics of the population number in the case of the initial population number being the equilibrium one. (b) The corresponding deviation curvature in the same case.

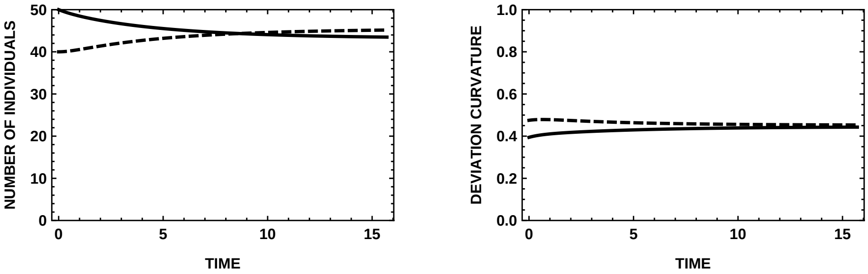

FIGURE 7. (a) Dynamics of the population number in the case of the initial population number being near the equilibrium one. (b) The corresponding deviation curvature in the same case

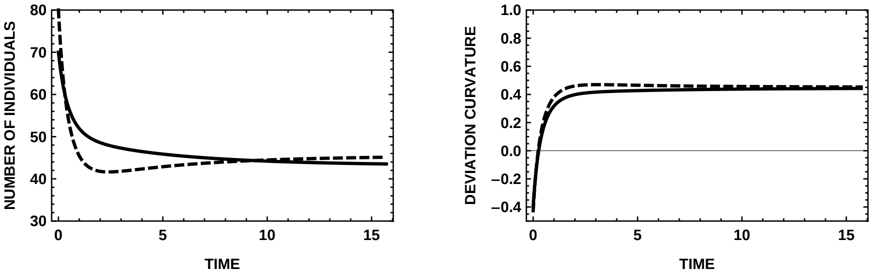

FIGURE 8. (a) Dynamics of the population number in the case of the initial population number being far from the equilibrium one. (b) The corresponding deviation curvature in the same case.

deviation curvatures are always positive. This means that the deviation curvature can distinguish the apparent and the true periodic patterns.

Finally, we can consider the far equilibrium initial population $x(0) = 70$ and $y(0) = 80$. In this case, it is expected from Figure 5 that the sign of the deviation curvature is not always positive. In fact, the beginning of the time transition in Figure 8(b) shows that the sign of the deviation curvature is negative. This negative span is accompanied by interesting dynamics in the population number (Fig. 8(a)): the population of y is smaller than that of x, although the equilibrium population of y is larger than that of x. This means that when the population is large (i.e., the sign of the deviation curvature is negative), the dynamic of the population is not a purely node pattern, but a node with a spiral-like pattern. After the middle of the time transition, an apparently periodic pattern appears in which the sign of the deviation curvature is positive, as in Figures 7(a) and (b).

5. Dynamics of the geometric objects

In this paper, we have shown that the dynamics of an ecological system can be characterized by the deviation curvature. For example, as Eq. (10) indicated, the stability structure of the ecological system at the equilibrium point is simple because the deviation curvature is constant. However, the stability structure of the ecological system becomes relatively complex in the non-equilibrium region, because the deviation curvature depends on the variables, i.e., changes over time (e.g., Eqs. (11) and (12)). That is, the deviation curvature itself forms another dynamic system in the non-equilibrium region. This finding suggests that we can reapply the KCC theory to this dynamic system, and derive another deviation curvature that characterizes the dynamics of the geometric objects.

Next, we considered a simple example: a predation system near the equilibrium point. In this case, Eqs. (11) and (12) show that $x \approx -P^1_1/br$ and $y \approx -P^2_2/ac$. Substitution of these approximated relations in Eqs. (7) and (8) leads to the dynamic equation of the deviation curvature:

$$\dot{P}^1 = rP^1 + \frac{1}{c}P^1P^2, \tag{21}$$

$$\dot{P}^2 = -\frac{1}{r}P^1P^2 - cP^2, \tag{22}$$

where $P_1^1 = P^1$ and $P_2^2 = P^2$ for simplicity. From Eqs. (21) and (22), we can derive the second-order differential equation for the deviation curvature:

$$\ddot{P}^i + \hat{g}^i(P, \dot{P}) = 0, \tag{23}$$

with

$$\hat{g}^1 = -crP^1 - (P^1)^2 + c\dot{P}^1 + \frac{P^1\dot{P}^1}{r} - \frac{(\dot{P}^1)^2}{P^1}, \tag{24}$$

$$\hat{g}^2 = -crP^2 - (P^2)^2 - r\dot{P}^2 - \frac{P^2\dot{P}^2}{c} - \frac{(\dot{P}^2)^2}{P^2}. \tag{25}$$

When we recognize (23) as the path equation, the new deviation curvature, characterizes the dynamics of the deviation curvature, can be defined by

$$\hat{P}_j^i = -\frac{\partial \hat{g}^i}{\partial P^j} + \frac{\partial \hat{N}_j^i}{\partial P^k}\dot{P}^k - \hat{G}_{jk}^i\hat{g}^k + \hat{N}_k^i\hat{N}_j^k, \tag{26}$$

where $\hat{N}_j^i$ is a coefficient of the non-linear connection and $\hat{G}_{jk}^i$ is that of a Finsler connection related to (23). That is

$$\hat{N}_j^i = \frac{1}{2}\frac{\partial \hat{g}^i}{\partial \dot{P}^j}, \tag{27}$$

and

$$\hat{G}_{jk}^i = \frac{\partial \hat{N}_j^i}{\partial \dot{P}^k}. \tag{28}$$

From (24) and (25), the concrete forms of (26) and (27) are given by

$$\hat{N}_1^1 = \frac{P^1}{2r} - \frac{\dot{P}^1}{P^1} + \frac{c}{2} = \frac{P^1}{2r} - \frac{P^2}{c} + \frac{c}{2} - r, \tag{29}$$

$$\hat{N}_2^2 = -\frac{P^2}{2c} - \frac{\dot{P}^2}{P^2} - \frac{r}{2} = \frac{P^1}{r} - \frac{P^2}{2c} - \frac{r}{2} + c, \tag{30}$$

$$\hat{P}^1_1 = \left(\frac{P^1}{2r}\right)^2 + P^1\left(\frac{1}{2} + \frac{c}{2r} - \frac{P^2}{2cr}\right) + \left(\frac{c}{2}\right)^2, \tag{31}$$

$$\hat{P}^2_2 = \left(\frac{P^2}{2c}\right)^2 + P^2\left(\frac{1}{2} + \frac{r}{2c} - \frac{P^1}{2cr}\right) + \left(\frac{r}{2}\right)^2. \tag{32}$$

We use Eqs. (21) and (22) in the last step. According to the previous papers, the linear stability and the Jacobi stability of the system is given by the non-linear connection and the deviation curvature, respectively. Therefore, Eqs. (29) to (32) show the hierarchical structure of the stability. That is, the linear stability of the Jacobi stability, described by $\hat{N}^i_j$, and the Jacobi stability of the Jacobi stability, described by $\hat{P}^i_j$, are related to the Jacobi stability of the prey x^1 and the predator x^2, described by P^i. Moreover, it is found that the not only the sign, but also the magnitude of the deviation curvature is related to the stability. For instance, since $r > 0$ and $c > 0$, we have $\partial\hat{N}^i_j/\partial P^1 > 0$ and $\partial\hat{N}^i_j/\partial P^2 < 0$, which mean that the increase of the deviation curvature for the prey (the predator) contributes the linear stable (unstable) state of the geometrical objects. From $\partial^2\hat{P}^i_j/\partial P^1\partial P^2 < 0$, the increase of the deviation curvatures contribute the Jacobi stable of the geometrical objects.

6. CONCLUSIONS

Our main conclusions are as follows.

1. Around the equilibrium point, the values of the deviation curvatures of the two species are equal, and their signs are always negative (positive) in the predation (competition) system. However, in the non-equilibrium region, the values and the sign of the deviation curvatures depend on time. In this case, the dynamics of the predation system exhibit a distorted periodic pattern, and the dynamics of the competition system also exhibit a non-typical node: a node with a spiral-like pattern.
2. The dynamics of predation and competition are characterized by geometric objects. In the non-equilibrium region, the geometric objects themselves represent another dynamic systems, so we can reapply KCC theory to the dynamic system to derive other geometric objects that characterize the previous geometric objects. We considered a simple example and showed

that the dynamics of the deviation curvature is expressed in terms of the advanced deviation curvature.

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