



Products of composite operators in the exact renormalization group formalism

Pagani, C.

Sonoda, H.

(Citation)

Progress of Theoretical and Experimental Physics, 2018(2):023B02-023B02

(Issue Date)

2018-02

(Resource Type)

journal article

(Version)

Version of Record

(Rights)

© The Author(s) 2018. Published by Oxford University Press on behalf of the Physical Society of Japan.

This is an Open Access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0/>), which permits...

(URL)

<https://hdl.handle.net/20.500.14094/90004764>



Products of composite operators in the exact renormalization group formalism

C. Pagani¹ and H. Sonoda^{2,*}

¹*Institute für Physik (WA THEP) Johannes-Gutenberg-Universität Staudingerweg 7, 55099 Mainz, Germany*

²*Physics Department, Kobe University, Kobe 657-8501, Japan*

*E-mail: hsonoda@kobe-u.ac.jp

Received October 12, 2017; Accepted December 18, 2017; Published February 8, 2018

.....
We discuss a general method of constructing the products of composite operators using the exact renormalization group formalism. Considering mainly the Wilson action at a generic fixed point of the renormalization group, we give an argument for the validity of short-distance expansions of operator products. We show how to compute the expansion coefficients by solving differential equations, and test our method with some simple examples.
.....

Subject Index B30, B32, B39

1. Introduction

In the framework of the Wilsonian renormalization group (RG), the physics of a system is completely characterized by a Wilson action. The momentum cutoff of the action is fixed by rescaling while the corresponding size in physical units diminishes exponentially under the RG transformation. The Wilson action of a critical theory eventually reaches a fixed point which is scale invariant with no characteristic length. It is important to understand how small deformations of the fixed-point Wilson action grow under the RG transformation. For example, the exponential growth of deformations is dictated by the critical exponents. There are also space-dependent deformations with nontrivial rotation properties. These deformations constitute what we call composite operators, and their scaling properties under the RG transformation constitute an essential part of our understanding of critical phenomena and continuum field theory.

The purpose of this paper is to improve our understanding of composite operators using the formalism of the exact renormalization group (ERG) or functional RG. The importance of composite operators in ERG was emphasized early by Becchi [1], and his results have been extended in some later works such as Refs. [2–4].

In this work we use ERG to construct products of composite operators and study their properties. We discuss the insertion of two (or more) composite operators in correlation functions of the elementary fields. Particular attention is paid to the ERG differential equation satisfied by composite operators and their products at the fixed point.

When considering the product of two composite operators, it is natural to ask about its short-distance behavior. At short distances the operator product expansion (OPE) of K. Wilson [5] is expected to be valid. In the past, ERG has been used to provide an alternative proof of the existence of the OPE in perturbation theory [6–13]; the original perturbative proof goes back to Zimmermann [14]. The purpose of examining the OPE within ERG is to fill the gap between ERG and other nonperturbative

approaches to quantum field theory where the OPE forms the backbone structure of the theory. Particularly relevant is the case of conformal field theories, especially in the two-dimensional case. Using ERG we argue the plausibility (if not a proof) of the existence of OPE. In particular, we derive ERG differential equations (a.k.a. flow equations) satisfied by the Wilson coefficients and solve them for some simple examples.

The paper is organized as follows. In Sect. 2 we define composite operators at a fixed point of the RG transformation. We introduce three equivalent approaches using the Wilson action [15,16], the generating functional of connected correlation functions with an infrared cutoff [17], and its Legendre transform (called the effective average action; [18,19]), respectively. The three approaches differ in the natural choice of field variables: ϕ, J, Φ . In Sect. 3 we generalize our construction to the product of two composite operators and consider how the OPE arises. In Sect. 4 some working examples are presented. In Sect. 5 we explain how to generalize the ERG differential equations to consider the insertion of an arbitrary number of composite operators, and in Sect. 6 we discuss the ERG differential equations for composite operators away from the fixed point. We summarize our findings in Sect. 7. We confine some technical parts to three appendices. In Appendix A we review the basics of the ERG formalism that this paper is based on. A best pedagogical effort has been made for those readers familiar with Ref. [16] but not with Ref. [15]. In Appendix B we explain some properties of the simplest composite operator $\Phi(p)$ corresponding to the elementary field $\phi(p)$. In Appendix C we explain how to construct local composite operators in the massive free scalar theories. In Appendix D we derive the asymptotic behavior of a short-range function necessary for the examples of Sect. 4.

We shall work in the dimensionless convention, where all dimensionful quantities have been rescaled via a suitable power of the cutoff. We also adopt the following notation:

$$\int_p = \int \frac{d^D p}{(2\pi)^D}, \quad \delta(p) = (2\pi)^D \delta^{(D)}(p), \quad p \cdot \partial_p = \sum_{\mu=1}^D p_\mu \frac{\partial}{\partial p_\mu}. \quad (1)$$

2. Composite operators at a fixed point

At a fixed point of the exact renormalization group, the Wilson action satisfies the ERG equation [15,20]

$$0 = \int_p \left[\left(-p \cdot \partial_p \ln K(p) + \frac{D+2}{2} - \gamma + p \cdot \partial_p \right) \phi(p) \cdot \frac{\delta}{\delta \phi(p)} + \left(-p \cdot \partial_p \ln R(p) + 2 - 2\gamma \right) \frac{K(p)^2}{R(p)} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right] e^{\mathcal{S}[\phi]}, \quad (2)$$

where γ is the anomalous dimension. (We have prepared Appendix A for readers who are familiar with Ref. [16] but not with Ref. [15].) We have introduced two positive cutoff functions:

- (1) $K(p)$ approaches 1 as $p^2 \rightarrow 0$, and decays rapidly for $p^2 \gg 1$.
- (2) $R(p)$ must be nonvanishing at $p = 0$ and decays rapidly for $p^2 \gg 1$. The inverse transform of $R(p)$ is a function in space that is nonvanishing only over a region of unit size.

For example, the choice $K(p) = e^{-p^2}$, $R(p) = \frac{p^2 K(p)}{1-K(p)} = \frac{p^2}{e^{p^2}-1}$ satisfies the criteria. The original choice made in Ref. [15] was $K(p) = e^{-p^2}$, $R(p) = e^{-2p^2}$.

Equation (2) implies that the modified correlation functions defined by

$$\langle \phi(p_1) \cdots \phi(p_n) \rangle \equiv \prod_{i=1}^n \frac{1}{K(p_i)} \cdot \left\langle \exp \left(- \int_p \frac{K(p)^2}{R(p)} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right) \phi(p_1) \cdots \phi(p_n) \right\rangle_S \quad (3)$$

satisfy the scaling law

$$\langle \phi(p_1 e^t) \cdots \phi(p_n e^t) \rangle = \exp \left(n \left(-\frac{D+2}{2} + \gamma \right) \right) \langle \phi(p_1) \cdots \phi(p_n) \rangle \quad (4)$$

for arbitrary momenta [21].

In our discussion of composite operators we find it more convenient to deal with a functional $W[J]$ defined directly in terms of $S[\phi]$ as [17,22]

$$W[J] \equiv \frac{1}{2} \int_p \frac{J(p)J(-p)}{R(p)} + S[\phi], \quad (5a)$$

$$\text{where } J(p) \equiv \frac{R(p)}{K(p)} \phi(p). \quad (5b)$$

In fact it is even more convenient to deal with the Legendre transform of $W[J]$ [18,23]:

$$-\frac{1}{2} \int_p R(p) \Phi(p) \Phi(-p) + \Gamma[\Phi] \equiv W[J] - \int_p J(-p) \Phi(p), \quad (6a)$$

$$\Phi(p) \equiv \frac{\delta W[J]}{\delta J(-p)}. \quad (6b)$$

$\Gamma[\Phi]$ is often called the effective average action. We can interpret $W[J]$ as the generating functional of connected correlation functions [17,22] and $\Gamma[\Phi]$ as the effective action [18,23], both in the presence of an infrared cutoff. (The same cutoff is called an ultraviolet cutoff for S and an infrared cutoff for W and Γ . This is because we regard S as the weight of functional integration over low momenta to be done, but we regard W and Γ as consequences of functional integration over high momenta already done. It has recently been shown in Ref. [24] that the high-momentum limit of W and Γ gives the corresponding functionals without the infrared cutoff.) The ERG equations satisfied by W and Γ are given (see Ref. [20] and references therein) by

$$0 = \int_p J(-p) \left(-p \cdot \partial_p - \frac{D+2}{2} + \gamma \right) \frac{\delta}{\delta J(-p)} e^{W[J]} + \int_p (-p \cdot \partial_p + 2 - 2\gamma) R(p) \cdot \frac{1}{2} \frac{\delta^2}{\delta J(p) \delta J(-p)} e^{W[J]}, \quad (7)$$

$$0 = \int_p \frac{\delta \Gamma[\Phi]}{\delta \Phi(p)} \left(-p \cdot \partial_p - \frac{D+2}{2} + \gamma \right) \Phi(p) + \int_p (-p \cdot \partial_p + 2 - 2\gamma) R(p) \cdot \frac{1}{2} G_{p,-p}[\Phi], \quad (8)$$

where

$$G_{p,q}[\Phi] \equiv \frac{\delta^2 W[J]}{\delta J(p) \delta J(q)} \quad (9)$$

satisfies

$$\int_q G_{p,q}[\Phi] \left(R(q) \delta(q-r) - \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(-q) \delta \Phi(r)} \right) = \delta(p-r). \quad (10)$$

Now, composite operators can be thought of as infinitesimal changes of S , W , or Γ . Correspondingly, we can regard composite operators as functionals of ϕ , J , or Φ . Let $\mathcal{O}(p)$ be a composite operator of scale dimension $-y$ and momentum p . Regarding it as a functional of J , we obtain the following ERG equation:

$$\begin{aligned} (y + p \cdot \partial_p) \mathcal{O}(p) = \int_q \left\{ J(q) \left(-q \cdot \partial_q - \frac{D+2}{2} + \gamma \right) \frac{\delta}{\delta J(q)} \right. \\ \left. + (-q \cdot \partial_q + 2 - 2\gamma) R(q) \left(\frac{\delta W[J]}{\delta J(-q)} \frac{\delta}{\delta J(q)} + \frac{1}{2} \frac{\delta^2}{\delta J(q) \delta J(-q)} \right) \right\} \mathcal{O}(p). \end{aligned} \quad (11)$$

Similarly, regarding $\mathcal{O}(p)$ as a functional of Φ , we can rewrite the above as

$$\begin{aligned} (y + p \cdot \partial_p) \mathcal{O}(p) = \int_q \left\{ \left(q \cdot \partial_q + \frac{D+2}{2} - \gamma \right) \Phi(q) \cdot \frac{\delta}{\delta \Phi(q)} \right. \\ \left. + (-q \cdot \partial_q + 2 - 2\gamma) R(q) \frac{1}{2} \int_{r,s} G_{q,-r}[\Phi] G_{-q,-s}[\Phi] \frac{\delta^2}{\delta \Phi(r) \delta \Phi(s)} \right\} \mathcal{O}(p), \end{aligned} \quad (12)$$

where G is defined by Eq. (9).

The above two ERG equations are equivalent, and they imply that the modified correlation functions defined by

$$\begin{aligned} \langle \mathcal{O}(p) \phi(p_1) \cdots \phi(p_n) \rangle \\ \equiv \prod_{i=1}^n \frac{1}{K(p_i)} \left\langle \mathcal{O}(p) \exp \left(- \int_q \frac{K(q)^2}{R(q)} \frac{1}{2} \frac{\delta^2}{\delta \phi(q) \delta \phi(-q)} \right) \phi(p_1) \cdots \phi(p_n) \right\rangle_S \end{aligned} \quad (13)$$

satisfy the scaling law

$$\langle \mathcal{O}(pe^t) \phi(p_1 e^t) \cdots \phi(p_n e^t) \rangle = \exp \left(t \left(-y - n \frac{D+2}{2} \right) \right) \langle \mathcal{O}(p) \phi(p_1) \cdots \phi(p_n) \rangle. \quad (14)$$

The simplest composite operator is $\Phi(p)$, satisfying Eq. (12) with $-y = -\frac{D+2}{2} + \gamma$. From Eqs. (5) and (6b), we obtain $\Phi(p)$ in terms of ϕ as

$$\Phi(p) = \frac{1}{K(p)} \left(\phi(p) + \frac{K(p)^2}{R(p)} \frac{\delta S[\phi]}{\delta \phi(-p)} \right). \quad (15)$$

$\Phi(p)$ is a composite operator corresponding to the elementary field $\phi(p)$ in the sense that they have the same modified correlation functions [3]:

$$\langle \Phi(p) \phi(p_1) \cdots \phi(p_n) \rangle = \langle \phi(p) \phi(p_1) \cdots \phi(p_n) \rangle. \quad (16)$$

We derive this in Appendix B.

3. Products of composite operators

Given two composite operators $\mathcal{O}_1(p)$, $\mathcal{O}_2(p)$ of scale dimensions $-y_1, -y_2$, we wish to define their product as a composite operator of scale dimension $-(y_1 + y_2)$. The naive product $\mathcal{O}_1(p)\mathcal{O}_2(q)$ will not do because it does not satisfy Eqs. (11) or (12). We must define the product by adding a *local* counterterm:

$$[\mathcal{O}_1(p)\mathcal{O}_2(q)] \equiv \mathcal{O}_1(p)\mathcal{O}_2(q) + \mathcal{P}_{12}(p, q); \quad (17)$$

otherwise the product will not satisfy the scaling law:

$$\begin{aligned} & \langle [\mathcal{O}_1(pe^t)\mathcal{O}_2(qe^t)] \phi(p_1 e^t) \cdots \phi(p_n e^t) \rangle \\ &= \exp \left[t \left(-y_1 - y_2 + n \left(-\frac{D+2}{2} + \gamma \right) \right) \right] \langle [\mathcal{O}_1(p)\mathcal{O}_2(q)] \phi(p_1) \cdots \phi(p_n) \rangle. \end{aligned} \quad (18)$$

In the ERG formalism we have a dimensionless cutoff of order 1 (either in momentum space or in coordinate space). In coordinate space the inverse Fourier transform

$$\mathcal{O}(r) = \int_p e^{ipr} \mathcal{O}(p) \quad (19)$$

is expected to have a support of unit size around the coordinate r . We expect the same property for the product of two composite operators. Given $\mathcal{O}_1(p)$ and $\mathcal{O}_2(q)$, we denote their inverse Fourier transforms using the same symbol:

$$\begin{cases} \mathcal{O}_1(r) = \int_p e^{ipr} \mathcal{O}_1(p), \\ \mathcal{O}_2(r') = \int_q e^{iqr'} \mathcal{O}_2(q). \end{cases} \quad (20)$$

Both have a distribution of unit size in coordinate space. The limit

$$\mathcal{O}_1(r)\mathcal{O}_2(r') \xrightarrow{r' \rightarrow r} \mathcal{O}_1(r)\mathcal{O}_2(r) \quad (21)$$

is well defined. If there are short-distance singularities, we cannot find them in $\mathcal{O}_1(p)\mathcal{O}_2(q)$: we must look for them in the counterterm $\mathcal{P}_{12}(p, q)$, which is required by ERG (or equivalently scaling). Even without the help of ERG, we expect the need for the counterterm in defining the Fourier transform; the integration over the case where the two operators are dangerously close together requires special attention, resulting in a local counterterm.

Let us further analyze the nature of $\mathcal{P}_{12}(p, q)$. Regarding composite operators as functionals of Φ , we obtain

$$(y_1 + p \cdot \partial_p - \mathcal{D}) \mathcal{O}_1(p) = 0, \quad (22a)$$

$$(y_2 + q \cdot \partial_q - \mathcal{D}) \mathcal{O}_2(q) = 0, \quad (22b)$$

$$(y_1 + y_2 + p \cdot \partial_p + q \cdot \partial_q - \mathcal{D}) [\mathcal{O}_1(p)\mathcal{O}_2(q)] = 0, \quad (22c)$$

where

$$\begin{aligned} \mathcal{D} \equiv & \int_r \left\{ \left(r \cdot \partial_r + \frac{D+2}{2} - \gamma \right) \Phi(r) \cdot \frac{\delta}{\delta \Phi(r)} \right. \\ & \left. + (-r \cdot \partial_r + 2 - 2\gamma) R(r) \frac{1}{2} \int_{s,t} G_{r,-s}[\Phi] G_{-r,-t}[\Phi] \frac{\delta^2}{\delta \Phi(s) \delta \Phi(t)} \right\}. \end{aligned} \quad (23)$$

Equation (22c) is equivalent to the scaling of Eq. (18).

From Eq. (22), we obtain the following ERG equation for the counterterm:

$$\begin{aligned} & (y_1 + y_2 + p \cdot \partial_p + q \cdot \partial_q - \mathcal{D}) \mathcal{P}_{12}(p, q) \\ &= \int_r (-r \cdot \partial_r + 2 - 2\gamma) R(r) \cdot \int_s G_{r,-s}[\Phi] \frac{\delta \mathcal{O}_1(p)}{\delta \Phi(s)} \int_t G_{-r,-t}[\Phi] \frac{\delta \mathcal{O}_2(q)}{\delta \Phi(t)} \\ &= \int_r (-r \cdot \partial_r + 2 - 2\gamma) R(r) \cdot \frac{\delta \mathcal{O}_1(p)}{\delta J(r)} \frac{\delta \mathcal{O}_2(q)}{\delta J(-r)}, \end{aligned} \quad (24)$$

where we have used

$$G_{r,-s}[\Phi] = \frac{\delta W[J]}{\delta J(r) \delta J(-s)} = \frac{\delta \Phi(s)}{\delta J(r)}. \quad (25)$$

Since R is the Fourier transform of a function nonvanishing only over a region of unit size, Eq. (24) is local in space. This means that the inverse Fourier transform

$$\int_{p,q} e^{ipx+iqx'} \int_r (-r \cdot \partial_r + 2 - 2\gamma) R(r) \cdot \frac{\delta \mathcal{O}_1(p)}{\delta J(r)} \frac{\delta \mathcal{O}_2(q)}{\delta J(-r)}$$

is nonvanishing only when the distance $|x - x'|$ is of order 1 or less.

Therefore, we can expand the counterterm $\mathcal{P}_{12}(p, q)$ using a basis of local composite operators:

$$\mathcal{P}_{12}(p, q) = \sum_i c_{12,i}(p - q) \mathcal{O}_i(p + q), \quad (26)$$

where \mathcal{O}_i is a composite operator of scale dimension $-y_i$, satisfying

$$(y_i + p \cdot \partial_p - \mathcal{D}) \mathcal{O}_i(p) = 0. \quad (27)$$

The coefficient $c_{12,i}$ depends only on $p - q$; we have absorbed all the dependence on $p + q$ into \mathcal{O}_i . Similarly, we can expand the right-hand side of Eq. (24) as

$$\int_r (-r \cdot \partial_r + 2 - 2\gamma) R(r) \cdot \frac{\delta \mathcal{O}_1(p)}{\delta J(r)} \frac{\delta \mathcal{O}_2(q)}{\delta J(-r)} = \sum_i d_{12,i}(p - q) \mathcal{O}_i(p + q). \quad (28)$$

Substituting Eqs. (26) and (28) into Eq. (24), we obtain

$$(p \cdot \partial_p + q \cdot \partial_q + y_1 + y_2 - y_i) c_{12,i}(p - q) = d_{12,i}(p - q),$$

or equivalently

$$(p \cdot \partial_p + y_1 + y_2 - y_i) c_{12,i}(p) = d_{12,i}(p), \quad (29)$$

which determines $c_{12,i}(p)$ in terms of $d_{12,i}(p)$.

Before discussing the short-distance behavior of $c_{12,i}(p - q)$ for large $|p - q|$, we would like to consider the solvability of Eq. (29) and the uniqueness of its solution. We assume analyticity: both $c_{12,i}(p)$ and $d_{12,i}(p)$ are regular functions of p at $p = 0$. Equation (29) can be solved uniquely unless

$$n \equiv -(y_1 + y_2) + y_i = 0, 1, 2, \dots \quad (30)$$

If Eq. (30) holds, and if $d_{12,i}(p)$ contains a constant multiple of p^n , we cannot find an analytic solution. (If $c_{12,i}(p)$ is a scalar, the following discussion applies only for even n .) Even if $d_{12,i}(p)$

has no such a term, $c_{12,i}(p)$ is ambiguous by a constant multiple of p^n . So, what to do if Eq. (30) is the case?

If Eq. (30) holds, we need to modify Eq. (22c) so that

$$(y_1 + y_2 + p \cdot \partial_p + q \cdot \partial_q - \mathcal{D}) [\mathcal{O}_1(p)\mathcal{O}_2(q)] = d_i \cdot (p - q)^n \mathcal{O}_i(p + q), \quad (31)$$

where the constant d is determined so that

$$d_{12,i}(p) + d_i \cdot p^n$$

has no term proportional to p^n . Then, we need to solve

$$(p \cdot \partial_p - n) c_{12,i}(p) = d_{12,i}(p) + d_i \cdot p^n \quad (32)$$

instead of Eq. (29). This can be solved, but the solution is not unique. To fix the coefficient of p^n , we must introduce a convention such as the absence of the p^n term in $c_{12,i}(p)$:

$$\left. \frac{\partial^n}{\partial p^n} c_{12,i}(p) \right|_{p=0} = 0. \quad (33)$$

All this implies that the scale dimension $y_1 + y_2$ is extended to a matrix: the product $[\mathcal{O}_1(p)\mathcal{O}_2(q)]$ mixes with a local operator $\mathcal{O}_i(p + q)$ for which Eq. (30) is satisfied. For example, at the Gaussian fixed point in $D = 4$, the product

$$\left[\frac{1}{2} [\phi^2(p)] \frac{1}{2} [\phi^2(q)] \right]$$

($y_1 = y_2 = 2$) mixes with $\delta(p + q)$. See Example 1 of Sect. 4 for more details. So much for the discussion of Eq. (30).

Now, we consider the short-distance limit of the product. We consider $[\mathcal{O}_1(p)\mathcal{O}_2(q)]$, taking $|p - q|$ large while fixing $p + q$. As has been explained, a singular behavior is expected not of $\mathcal{O}_1(p)\mathcal{O}_2(q)$ but of $\mathcal{P}_{12}(p, q)$:

$$\langle [\mathcal{O}_1(p)\mathcal{O}_2(q)] \phi(p_1) \cdots \phi(p_n) \rangle \underset{p+q \text{ fixed}}{\overset{|p-q| \text{ large}}{\approx}} \langle \mathcal{P}_{12}(p, q) \phi(p_1) \cdots \phi(p_n) \rangle. \quad (34)$$

We can regard $\mathcal{P}_{12}(p, q)$ as a functional of J with momentum $p + q$. Since we keep the momenta p_1, \dots, p_n finite in the above, we can assume that $\frac{\delta \mathcal{O}_1(p)}{\delta J(r)}$ and $\frac{\delta \mathcal{O}_2(q)}{\delta J(-r)}$ of Eq. (24) depend only on J s with finite momenta. Hence, r in Eq. (24) must be of order p by momentum conservation. Therefore, $R(r)$ becomes extremely small. Hence, from Eq. (28), we expect

$$d_{12,i}(p - q) \xrightarrow{|p-q| \rightarrow \infty} 0. \quad (35)$$

Thus, we obtain

$$(p \cdot \partial_p + y_1 + y_2 - y_i) c_{12,i}(p - q) \xrightarrow{|p-q| \rightarrow \infty} 0. \quad (36)$$

This implies that

$$c_{12,i}(p - q) \xrightarrow[p+q \text{ fixed}]{|p-q| \rightarrow \infty} C_{12,i} \cdot p^{-y_1 - y_2 + y_i}, \quad (37)$$

where $C_{12,i}$ is a constant.

We thus obtain the short-distance expansion (a.k.a. operator product expansion)

$$[\mathcal{O}_1(p)\mathcal{O}_2(q)] \xrightarrow[p+q \text{ fixed}]{|p-q| \rightarrow \infty} \sum_i C_{12,i} \cdot p^{-y_1-y_2+y_i} \mathcal{O}_i(p+q). \quad (38)$$

To be able to neglect the contribution of $\mathcal{O}_1(p)\mathcal{O}_2(q)$, we must restrict the sum over i to

$$-y_1 - y_2 + y_i \geq -D, \quad (39)$$

corresponding to singularities in space. This condition can be rewritten as

$$(D - y_1) + (D - y_2) \geq (D - y_i), \quad (40)$$

where $D - y_i$ is the scale dimension of the inverse Fourier transform of \mathcal{O}_i (operator in coordinate space). The operator \mathcal{O}_i with the lowest scale dimension provides the highest short-distance singularity.

4. Examples

We would like to provide concrete applications of the general theory we have developed. In the first subsection we consider a generic fixed-point action, and in the second we take examples from the Gaussian fixed point in D dimensions ($2 < D \leq 4$).

4.1. $[\mathcal{O}(p)\Phi(q)]$

$\Phi(p)$ is a composite operator corresponding to the elementary field $\phi(p)$. $\Phi(p)$ satisfies Eq. (12) with scale dimension

$$-y_\Phi = -\frac{D+2}{2} + \gamma. \quad (41)$$

Let $\mathcal{O}(p)$ be an arbitrary composite operator of scale dimension $-y$. Its product with $\Phi(q)$ must satisfy

$$\langle [\mathcal{O}(p)\Phi(q)] \phi(p_1) \cdots \phi(p_n) \rangle = \langle \mathcal{O}(p)\phi(q)\phi(p_1) \cdots \phi(p_n) \rangle. \quad (42)$$

This gives [3]

$$\begin{aligned} [\mathcal{O}(p)\Phi(q)] &= \mathcal{O}(p)\Phi(q) + \frac{K(q)}{R(q)} \frac{\delta \mathcal{O}(p)}{\delta \phi(-q)} \\ &= \mathcal{O}(p)\Phi(q) + \frac{\delta \mathcal{O}(p)}{\delta J(-q)}. \end{aligned} \quad (43)$$

(See Appendix B for more explanation.) This implies the counterterm

$$\mathcal{P}_{\mathcal{O}\Phi}(p, q) = \frac{\delta \mathcal{O}(p)}{\delta J(-q)}. \quad (44)$$

For the simplest case of $\mathcal{O} = \Phi$, we use Eqs. (6b) and (44) to obtain

$$\begin{aligned} [\Phi(p)\Phi(q)] &= \frac{\delta W[J]}{\delta J(-p)} \frac{\delta W[J]}{\delta J(-q)} + \frac{\delta^2 W[J]}{\delta J(-p)\delta J(-q)} \\ &= e^{-W[J]} \frac{\delta^2}{\delta J(-p)\delta J(-q)} e^{W[J]}. \end{aligned} \quad (45)$$

This generalizes to [25]

$$[\Phi(p_1) \cdots \Phi(p_n)] = e^{-W[J]} \frac{\delta^n}{\delta J(-p_1) \cdots \delta J(-p_n)} e^{W[J]}, \quad (46)$$

which can be checked to satisfy Eq. (43):

$$\begin{aligned} e^{-W[J]} \frac{\delta^n}{\delta J(-p_1) \cdots \delta J(-p_n)} e^{W[J]} &= \frac{\delta W[J]}{\delta J(-p_n)} \cdot e^{-W[J]} \frac{\delta^{n-1}}{\delta J(-p_1) \cdots \delta J(-p_{n-1})} e^{W[J]} \\ &+ \frac{\delta}{\delta J(-p_n)} \left(e^{-W[J]} \frac{\delta^{n-1}}{\delta J(-p_1) \cdots \delta J(-p_{n-1})} e^{W[J]} \right). \end{aligned} \quad (47)$$

Using Eq. (9) we can rewrite Eq. (45) as

$$[\Phi(p)\Phi(q)] = \Phi(p)\Phi(q) + G_{-q,-p}[\Phi]. \quad (48)$$

Hence, for $p+q$ fixed, we obtain

$$[\Phi(p)\Phi(q)] \xrightarrow[p+q \text{ fixed}]{|p-q| \rightarrow \infty} G_{-p,-q}[\Phi]. \quad (49)$$

From the scaling law

$$\langle \phi(p)\phi(q) \rangle = \text{const} \frac{1}{p^{2(1-\gamma)}} \delta(p+q), \quad (50)$$

we obtain the coefficient of the identity operator as

$$[\Phi(p)\Phi(q)] \xrightarrow[p+q \text{ fixed}]{|p-q| \rightarrow \infty} \text{const} \frac{1}{p^{2(1-\gamma)}} \delta(p+q). \quad (51)$$

Further coefficients can be computed by employing some approximation scheme. For instance, we have checked in the ϕ^4 theory in dimension $D=4$ that the order λ correction is given by

$$[\Phi(p)\Phi(q)] \xrightarrow[p+q \text{ fixed}]{|p-q| \rightarrow \infty} \frac{1}{p^2} \delta(p+q) - \lambda \frac{1}{p^4} \frac{1}{2} [\phi^2(p+q)]. \quad (52)$$

In coordinate space the order λ correction is proportional to the logarithm of the distance.

4.2. Examples from the Gaussian fixed point in D dimensions

We now consider the composite operators at the Gaussian fixed point:

$$\Gamma[\Phi] = -\frac{1}{2} \int_p p^2 \Phi(p) \Phi(-p). \quad (53)$$

There is no anomalous dimension: $\gamma=0$. The high-momentum propagator is given by

$$G_{p,q}[\Phi] = \frac{1}{p^2 + R(p)} \delta(p+q) \equiv h(p) \delta(p+q). \quad (54)$$

For convenience we introduce

$$f(p) \equiv (2 + p \cdot \partial_p) h(p) = \frac{(2 - p \cdot \partial_p) R(p)}{(p^2 + R(p))^2}, \quad (55)$$

which vanishes rapidly for $p \gg 1$. Now, the “differential” operator \mathcal{D} defined by Eq. (23) can be written as

$$\mathcal{D} \equiv \int_r \left\{ \left(r \cdot \partial_r + \frac{D+2}{2} \right) \Phi(r) \cdot \frac{\delta}{\delta \Phi(r)} + f(r) \frac{1}{2} \frac{\delta^2}{\delta \Phi(r) \delta \Phi(-r)} \right\}. \quad (56)$$

In the remaining part of this section we consider the products of the composite operators

$$\frac{1}{2} [\phi^2(p)] \equiv \frac{1}{2} \int_{p_1, p_2} \Phi(p_1) \Phi(p_2) \delta(p_1 + p_2 - p) + \kappa_2 \cdot \delta(p), \quad (57)$$

$$\begin{aligned} \frac{1}{4!} [\phi^4(p)] &\equiv \frac{1}{4!} \int_{p_1, \dots, p_4} \Phi(p_1) \cdots \Phi(p_4) \delta(p_1 + \cdots + p_4 - p) \\ &+ \kappa_2 \frac{1}{2} \int_{p_1, p_2} \Phi(p_1) \Phi(p_2) \delta(p_1 + p_2 - p) + \frac{1}{2} \kappa_2^2 \delta(p), \end{aligned} \quad (58)$$

where the constant κ_2 is defined by

$$\kappa_2 \equiv -\frac{1}{D-2} \frac{1}{2} \int_q f(q). \quad (59)$$

The scale dimensions are -2 and $D-4$, respectively. See Appendix C for the construction of composite operators in the free theory.

Example 1: $[[\frac{1}{2}\phi^2(p)][\frac{1}{2}\phi^2(q)]]$

The scale dimension of the product is $-y = -4$. Hence, in $D = 4$, we expect mixing with the unit operator $\delta(p+q)$. Let

$$\left[\frac{1}{2} [\phi^2(p)] \frac{1}{2} [\phi^2(q)] \right] = \frac{1}{2} [\phi^2(p)] \frac{1}{2} [\phi^2(q)] + \mathcal{P}_{22}(p, q). \quad (60)$$

The counterterm must satisfy

$$\begin{aligned} &(4 + p \cdot \partial_p + q \cdot \partial_q - \mathcal{D}) \mathcal{P}_{22}(p, q) \\ &= \int_r f(r) \frac{\delta}{\delta \Phi(r)} \frac{1}{2} [\phi^2(p)] \frac{\delta}{\delta \Phi(-r)} \frac{1}{2} [\phi^2(q)] \\ &= \int_r f(r) \Phi(p-r) \Phi(q+r). \end{aligned} \quad (61)$$

To solve this, let us expand

$$\begin{aligned} \mathcal{P}_{22}(p, q) &= \frac{1}{2} \int_{p_1, p_2} \Phi(p_1) \Phi(p_2) \delta(p_1 + p_2 - p - q) u_2(p - q; p_1, p_2) \\ &+ u_0(p) \delta(p + q). \end{aligned} \quad (62)$$

Substituting this into Eq. (61), we obtain

$$\left(2 + p \cdot \partial_p + q \cdot \partial_q + \sum_{i=1,2} p_i \cdot \partial_{p_i} \right) u_2(p - q; p_1, p_2) = f(p_1 - p) + f(p_1 - q) \quad (63a)$$

and

$$(4 - D + p \cdot \partial_p) u_0(p) = \frac{1}{2} \int_r f(r) u_2(2p; r, -r). \quad (63b)$$

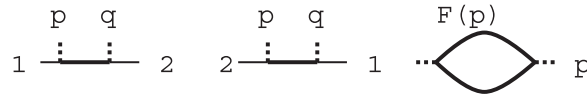


Fig. 1. Graphical representation of $\mathcal{P}_{22}(p, q)$: a dark line for h .

The homogeneous solution of Eq. (63a) is excluded on account of analyticity at zero momentum. Hence, we obtain

$$u_2(p - q; p_1, p_2) = h(p_1 - p) + h(p_1 - q). \quad (64)$$

Substituting this into Eq. (63b), we obtain

$$(4 - D + p \cdot \partial_p) u_0(p) = \int_q f(q) h(q + p). \quad (65)$$

For $2 < D < 4$, this is uniquely solved by

$$u_0(p) = F(p) \equiv \frac{1}{2} \int_q h(q) h(q + p) \quad (66)$$

(see Fig. 1).

Now, for $D = 4$, Eq. (65) does not admit a solution analytic at $p = 0$. Since the left-hand side vanishes at $p = 0$, we must modify it to

$$p \cdot \partial_p u_0(p) = \int_q f(q) (h(q + p) - h(q)) \quad (D = 4) \quad (67)$$

by subtracting a constant from the right-hand side. The constant can be evaluated as

$$\begin{aligned} \int_q f(q) h(q) &= \int_q h(q) (2 + q \cdot \partial_q) h(q) \\ &= \int_q (4 + q \cdot \partial_q) \frac{1}{2} h(q)^2 \\ &= \frac{1}{2} \int_r \frac{\partial}{\partial q_\mu} (q_\mu h(q)^2) \\ &= \frac{1}{(4\pi)^2}. \end{aligned} \quad (68)$$

Equation (67) determines $u_0(p)$ up to an additive constant. We define $F(p)$ by

$$p \cdot \partial_p F(p) = \int_q f(q) h(q + p) - \frac{1}{(4\pi)^2}, \quad (69a)$$

$$F(0) \equiv 0. \quad (69b)$$

As a convention we adopt the choice $u_0(p) = F(p)$. The subtraction in Eq. (69a) implies the mixing of the product with the identity operator, and the product satisfies the ERG equation

$$(4 + p \cdot \partial_p + q \cdot \partial_q - \mathcal{D}) \left[\frac{1}{2} [\phi^2(p)] \frac{1}{2} [\phi^2(q)] \right] = -\frac{1}{(4\pi)^2} \delta(p + q). \quad (70)$$

Let us find the asymptotic behavior of the product as $p \rightarrow \infty$ for a fixed $p + q$. We find

$$u_2(p - q; p_1, p_2) \xrightarrow[p+q \text{ fixed}]{|p-q| \rightarrow \infty} \frac{2}{p^2}. \quad (71)$$

From Appendix D, we obtain

$$F(p) \xrightarrow{p \rightarrow \infty} \begin{cases} c_F p^{D-4} + \frac{2\kappa_2}{p^2} & (2 < D < 4) \\ -\frac{1}{(4\pi)^2} \ln |p| + \text{const} + \frac{2\kappa_2}{p^2} & (D = 4), \end{cases} \quad (72)$$

where c_F is given by Eq. (D6). Hence, we obtain

$$\left[\frac{1}{2} [\phi^2(p)] \frac{1}{2} [\phi^2(q)] \right] \xrightarrow[p+q \text{ fixed}]{|p-q| \rightarrow \infty} c_F p^{D-4} \delta(p+q) + \frac{2}{p^2} \frac{1}{2} [\phi^2(p+q)] \quad (73)$$

for $2 < D < 4$, and

$$\left[\frac{1}{2} [\phi^2(p)] \frac{1}{2} [\phi^2(q)] \right] \xrightarrow[p+q \text{ fixed}]{|p-q| \rightarrow \infty} \left(-\frac{1}{(4\pi)^2} \ln |p| + \text{const} \right) \delta(p+q) + \frac{2}{p^2} \frac{1}{2} [\phi^2(p+q)] \quad (74)$$

for $D = 4$.

Example 2: $\left[\left[\frac{1}{4!} \phi^4(p) \right] \left[\frac{1}{2} \phi^2(q) \right] \right]$

The scale dimension of the product is $-y = D - 6$. Hence, in $D = 4$, the product mixes with $\frac{1}{2} [\phi^2(p+q)]$. Let

$$\left[\frac{1}{4!} [\phi^4(p)] \frac{1}{2} [\phi^2(q)] \right] = \frac{1}{4!} [\phi^4(p)] \frac{1}{2} [\phi^2(q)] + \mathcal{P}_{42}(p, q). \quad (75)$$

Solving

$$\begin{aligned} & (6 - D + p \cdot \partial_p + q \cdot \partial_q - \mathcal{D}) \mathcal{P}_{42}(p, q) \\ &= \int_r f(r) \frac{\delta}{\delta \Phi(r)} \frac{1}{4!} [\phi^4(p)] \frac{\delta}{\delta \Phi(-r)} \frac{1}{2} [\phi^2(q)] \\ &= \frac{1}{4!} \int_{p_1, \dots, p_4} \Phi(p_1) \cdots \Phi(p_4) \delta(p_1 + \cdots + p_4 - (p+q)) \sum_{i=1}^4 f(p_i - q) \\ & \quad + \kappa_2 \frac{1}{2} \int_{p_1, p_2} \Phi(p_1) \Phi(p_2) \delta(p_1 + p_2 - (p+q)) (f(p_1 - p) + f(p_1 - q)), \end{aligned} \quad (76)$$

we obtain, for $2 < D < 4$,

$$\begin{aligned} \mathcal{P}_{42}(p, q) &= \frac{1}{4!} \int_{p_1, \dots, p_4} \Phi(p_1) \cdots \Phi(p_4) \delta(p_1 + \cdots + p_4 - (p+q)) \sum_{i=1}^4 h(p_i - q) \\ & \quad + \frac{1}{2} \int_{p_1, p_2} \Phi(p_1) \Phi(p_2) \delta(p_1 + p_2 - (p+q)) \left(\kappa_2 \sum_{i=1,2} h(p_i - q) + F(q) \right) \\ & \quad + \kappa_2 F(p) \delta(p+q) \end{aligned} \quad (77)$$

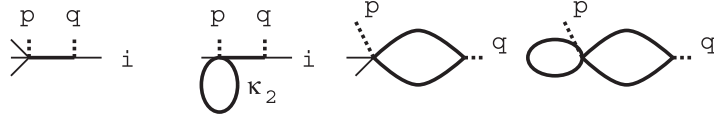


Fig. 2. Graphical representation of $\mathcal{P}_{42}(p, q)$.

(see Fig. 2).

The above expression is also valid for $D = 4$, except that the ERG equation for the product is modified to

$$(6 - D + p \cdot \partial_p + q \cdot \partial_q - D) \left[\frac{1}{4!} [\phi^4(p)] \frac{1}{2} [\phi^2(q)] \right] = -\frac{1}{(4\pi)^2} \frac{1}{2} [\phi^2(p+q)]. \quad (78)$$

Using Eq. (72), we obtain

$$\left[\frac{1}{4!} [\phi^4(p)] \frac{1}{2} [\phi^2(q)] \right] \xrightarrow[p+q \text{ fixed}]{|p-q| \rightarrow \infty} c_F p^{D-4} \frac{1}{2} [\phi^2(p+q)] + \frac{4}{p^2} \frac{1}{4!} [\phi^4(p+q)] \quad (79)$$

for $2 < D < 4$, and

$$\left[\frac{1}{4!} [\phi^4(p)] \frac{1}{2} [\phi^2(q)] \right] \xrightarrow[p+q \text{ fixed}]{|p-q| \rightarrow \infty} \left(-\frac{1}{(4\pi)^2} \ln |p| + \text{const} \right) \frac{1}{2} [\phi^2(p+q)] + \frac{4}{p^2} \frac{1}{4!} [\phi^4(p+q)] \quad (80)$$

for $D = 4$.

5. Multiple products

Multiple products of composite operators are defined just like the product of two composite operators as single nonlocal composite operators. To start with, the product of three composite operators \mathcal{O}_i of scale dimension $-y_i$ ($i = 1, 2, 3$) is defined as a composite operator of scale dimension $-(y_1 + y_2 + y_3)$ as

$$\begin{aligned} [\mathcal{O}_1(p)\mathcal{O}_2(q)\mathcal{O}_3(r)] &\equiv \mathcal{O}_1(p)\mathcal{O}_2(q)\mathcal{O}_3(r) + \mathcal{P}_{12}(p, q)\mathcal{O}_3(r) \\ &\quad + \mathcal{P}_{13}(p, r)\mathcal{O}_2(q) + \mathcal{P}_{23}(q, r)\mathcal{O}_1(p) + \mathcal{P}_{123}(p, q, r), \end{aligned} \quad (81)$$

where the counterterm \mathcal{P}_{12} is the same counterterm that makes

$$[\mathcal{O}_1(p)\mathcal{O}_2(q)] = \mathcal{O}_1(p)\mathcal{O}_2(q) + \mathcal{P}_{12}(p, q)$$

a composite operator. The extra counterterm $\mathcal{P}_{123}(p, q, r)$ has to do with the three operators close to each other simultaneously. We obtain the ERG equation

$$\begin{aligned} &\left(\sum_{i=1}^3 y_i + p \cdot \partial_p + q \cdot \partial_q + r \cdot \partial_r - D \right) \mathcal{P}_{123}(p, q, r) \\ &= \int_s f(s) \cdot \left(\frac{\delta}{\delta \Phi(s)} \mathcal{P}_{12}(p, q) \frac{\delta}{\delta \Phi(-s)} \mathcal{O}_3(r) \right. \\ &\quad \left. + \frac{\delta}{\delta \Phi(s)} \mathcal{P}_{23}(q, r) \frac{\delta}{\delta \Phi(-s)} \mathcal{O}_1(p) + \frac{\delta}{\delta \Phi(s)} \mathcal{P}_{31}(r, p) \frac{\delta}{\delta \Phi(-s)} \mathcal{O}_2(q) \right). \end{aligned} \quad (82)$$

If there is a local composite operator \mathcal{O} whose scale dimension $-y$ satisfies

$$-y = -\sum_{i=1}^3 y_i - n \quad (n = 0, 1, \dots), \quad (83)$$

the product $[\mathcal{O}_1(p)\mathcal{O}_2(q)\mathcal{O}_3(r)]$ may mix with $\mathcal{O}(p+q+r)$, and we obtain

$$\left(\sum_{i=1}^3 y_i + p \cdot \partial_p + q \cdot \partial_q + r \cdot \partial_r - \mathcal{D} \right) [\mathcal{O}_1(p)\mathcal{O}_2(q)\mathcal{O}_3(r)] = d(p-q, q-r) \mathcal{O}(p+q+r), \quad (84)$$

where $d(p-q, q-r)$ is a degree n polynomial of $p-q, q-r$.

Proceeding further, we can define the product of four composite operators as

$$\begin{aligned} [\mathcal{O}_1(p)\mathcal{O}_2(q)\mathcal{O}_3(r)\mathcal{O}_4(s)] &\equiv \mathcal{O}_1(p)\mathcal{O}_2(q)\mathcal{O}_3(r)\mathcal{O}_4(s) \\ &+ \mathcal{P}_{12}(p, q)\mathcal{O}_3(r)\mathcal{O}_4(s) + 5 \text{ more terms} \\ &+ \mathcal{P}_{12}(p, q)\mathcal{P}_{34}(r, s) + \mathcal{P}_{13}(p, r)\mathcal{P}_{24}(q, s) + \mathcal{P}_{14}(p, s)\mathcal{P}_{23}(q, r) \\ &+ \mathcal{P}_{123}(p, q, r)\mathcal{O}_4(s) + 3 \text{ more terms} + \mathcal{P}_{1234}(p, q, r, s). \end{aligned} \quad (85)$$

In the absence of mixing the last counterterm satisfies

$$\begin{aligned} &\left(\sum_{i=1}^4 y_i + p \cdot \partial_p + \dots + s \cdot \partial_s - \mathcal{D} \right) \mathcal{P}_{1234}(p, q, r, s) \\ &= \int_t f(t) \left(\frac{\delta \mathcal{P}_{12}(p, q)}{\delta \Phi(t)} \frac{\delta \mathcal{P}_{34}(r, s)}{\delta \Phi(-t)} + \dots + \frac{\delta \mathcal{P}_{123}(p, q, r)}{\delta \Phi(t)} \frac{\delta \mathcal{O}_4(s)}{\delta \Phi(-t)} + \dots \right). \end{aligned} \quad (86)$$

This can be generalized to higher-order products of composite operators.

Let $\mathcal{O}(p)$ be a composite operator with scale dimension $-y < 0$. (If it is a scalar, it is a relevant operator.) The n th-order product has scale dimension $-ny$. We can introduce a source $\mathcal{J}(p)$ so that

$$[\mathcal{O}(p_1) \dots \mathcal{O}(p_n)] = \frac{\delta^n}{\delta \mathcal{J}(-p_1) \dots \delta \mathcal{J}(-p_n)} e^{\mathcal{W}[\mathcal{J}]} \Big|_{\mathcal{J}=0}, \quad (87)$$

where

$$\begin{aligned} \mathcal{W}[\mathcal{J}] &= \int_p \mathcal{J}(-p) \mathcal{O}(p) + \frac{1}{2} \int_{p_1, p_2} \mathcal{J}(-p_1) \mathcal{J}(-p_2) \mathcal{P}_2(p_1, p_2) \\ &+ \frac{1}{3!} \int_{p_1, p_2, p_3} \mathcal{J}(-p_1) \mathcal{J}(-p_2) \mathcal{J}(-p_3) \mathcal{P}_3(p_1, p_2, p_3) + \dots \end{aligned} \quad (88)$$

If no local composite operator has scale dimension as low as $-2y$, there is no mixing between multiple products and local composite operators. (Since $\delta(p)$ has the lowest scale dimension $-D$, there is no mixing if $-2y < -D$.) In the absence of mixing, $\mathcal{W}[\mathcal{J}]$ satisfies the ERG equation:

$$\begin{aligned} &\int_p \mathcal{J}(-p) (y + p \cdot \partial_p) \frac{\delta}{\delta \mathcal{J}(-p)} e^{\mathcal{W}[\mathcal{J}]} \\ &= \int_p J(-p) \left(-p \cdot \partial_p - \frac{D+2}{2} + \gamma \right) \frac{\delta}{\delta J(-p)} e^{\mathcal{W}[\mathcal{J}]} \\ &+ \int_p (-p \cdot \partial_p + 2 - 2\gamma) R(p) \cdot \left(\frac{\delta \mathcal{W}[\mathcal{J}]}{\delta J(-p)} \frac{\delta}{\delta J(p)} + \frac{1}{2} \frac{\delta^2}{\delta J(p) \delta J(-p)} \right) e^{\mathcal{W}[\mathcal{J}]}. \end{aligned} \quad (89)$$

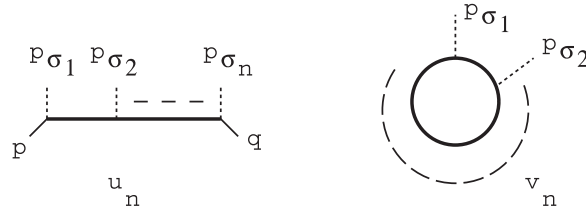


Fig. 3. Graphical representation of u_n and v_n ; σ is a permutation of $1, \dots, n$.

As the simplest example, consider $\mathcal{O}(p) = \Phi(p)$ with $-y = -\frac{D+2}{2} + \gamma$. Equation (46) implies [25]

$$\mathcal{W}[\mathcal{J}] = W[J + \mathcal{J}] - W[J]. \quad (90)$$

Another simple example is $\mathcal{O}(p) = \frac{1}{2} [\phi^2(p)]$ with $-y = -2$ at the Gaussian fixed point ($2 < D \leq 4$). \mathcal{P}_2 is \mathcal{P}_{22} , obtained in Example 1 of Sect. 4. \mathcal{P}_n ($n \geq 3$) are given in the form

$$\begin{aligned} \mathcal{P}_n(p_1, \dots, p_n) = & \frac{1}{2} \int_{p,q} \Phi(p) \Phi(q) \delta \left(p + q - \sum_{i=1}^n p_i \right) u_n(p_1, \dots, p_n; p, q) \\ & + v_n(p_1, \dots, p_n) \delta \left(\sum_{i=1}^n p_i \right), \end{aligned} \quad (91)$$

where u_n, v_n are expressed graphically in Fig. 3.

For example, we obtain

$$u_3(p_1, \dots, p_3; p, q) = h(p - p_1) (h(p - p_1 - p_2) + h(p - p_1 - p_3)) + \dots, \quad (92)$$

$$v_3(p_1, \dots, p_3) = \int_p h(p) h(p - p_1) h(p - p_1 - p_2). \quad (93)$$

At $D = 4$, $\mathcal{P}_2(p_1, p_2)$ mixes with $\delta(p_1 + p_2)$, and we must change the left-hand side of Eq. (89) to

$$\left(\int_p \mathcal{J}(-p) (2 + p \cdot \partial_p) \frac{\delta}{\delta \mathcal{J}(-p)} + \frac{1}{(4\pi)^2} \frac{1}{2} \int_p \mathcal{J}(p) \mathcal{J}(-p) \right) e^{\mathcal{W}[\mathcal{J}]}. \quad (94)$$

6. Away from a fixed point

Let g be a parameter with scale dimension $y_E > 0$. The Wilson action is parametrized by g . Assuming the anomalous dimension is independent of γ , we obtain the ERG equation of the action as

$$\begin{aligned} y_E g \frac{\partial}{\partial g} e^{\mathcal{S}(g)[\phi]} = & \int_p \left(-p \cdot \partial_p \ln K(p) + \frac{D+2}{2} - \gamma + p \cdot \partial_p \right) \phi(p) \cdot \frac{\delta}{\delta \phi(p)} e^{\mathcal{S}(g)[\phi]} \\ & + \int_p (-p \cdot \partial_p \ln R(p) + 2 - 2\gamma) \frac{K(p)^2}{R(p)} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} e^{\mathcal{S}(g)[\phi]}. \end{aligned} \quad (95)$$

The modified correlation functions defined by

$$\langle \phi(p_1) \cdots \phi(p_n) \rangle_g \equiv \prod_{i=1}^n \frac{1}{K(p_i)} \cdot \left\langle \exp \left(- \int_p \frac{K(p)^2}{R(p)} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right) \phi(p_1) \cdots \phi(p_n) \right\rangle_{\mathcal{S}(g)} \quad (96)$$

satisfy the scaling law:

$$\langle \phi(p_1 e^t) \cdots \phi(p_n e^t) \rangle_{g e^{y_E t}} = \exp \left(nt \left(-\frac{D+2}{2} + \gamma \right) \right) \langle \phi(p_1) \cdots \phi(p_n) \rangle_g. \quad (97)$$

Rewriting the ERG equation for

$$W(g)[J] \equiv \frac{1}{2} \int_p \frac{J(p)J(-p)}{R(p)} + S(g)[\phi], \quad (98a)$$

$$J(p) \equiv \frac{R(p)}{K(p)} \phi(p), \quad (98b)$$

we obtain

$$\begin{aligned} y_E g \frac{\partial}{\partial g} e^{W(g)[J]} &= \int_p J(-p) \left(-p \cdot \partial_p - \frac{D+2}{2} + \gamma \right) \frac{\delta}{\delta J(-p)} e^{W(g)[J]} \\ &\quad + \int_p (-p \cdot \partial_p + 2 - 2\gamma) R(p) \cdot \frac{1}{2} \frac{\delta^2}{\delta J(p) \delta J(-p)} e^{W(g)[J]}. \end{aligned} \quad (99)$$

Comparing this with Eq. (89), we find that for the constant source

$$\mathcal{J}(p) = g \delta(p), \quad (100)$$

the sum

$$W(g)[J] \equiv W[J] + \mathcal{W}[\mathcal{J}] \quad (101)$$

satisfies Eq. (99). Therefore,

$$e^{-W[J]} \frac{\partial^n}{\partial g^n} e^{W(g)[J]} \Big|_{g=0} = [\mathcal{Q}(0) \cdots \mathcal{Q}(0)] \quad (102)$$

is the n th order product of the zero-momentum composite operator

$$\mathcal{Q}(0) \equiv \frac{\partial W(g)[J]}{\partial g} \Big|_{g=0}$$

of scale dimension $-y_E$, defined at the fixed point.

Considering operator products, it is even more convenient to introduce the effective action with an infrared cutoff:

$$\Gamma(g)[\Phi] \equiv \frac{1}{2} \int_p R(p) \Phi(p) \Phi(-p) + W(g)[J] - \int_p J(-p) \Phi(p), \quad (103a)$$

$$\Phi(p) \equiv \frac{\delta W(g)[J]}{\delta J(-p)}. \quad (103b)$$

Then, a composite operator of scale dimension $-y$ satisfies

$$(y + y_E g \partial_g + p \cdot \partial_p - \mathcal{D}) \mathcal{O}(p) = 0, \quad (104)$$

where \mathcal{D} is given by Eq. (23) except that G is now defined with the g -dependent W as

$$G(g)_{p,q}[\Phi] \equiv \frac{\delta^2 W(g)[J]}{\delta J(p) \delta J(q)}. \quad (105)$$

The discussion of Sect. 3 goes through as long as we use the g -dependent \mathcal{D} . We introduce g -dependent coefficients $c_{12,i}(g;p)$ and $d_{12,i}(g;p)$ via Eqs. (26) and (28), respectively. Equation (29) is replaced by

$$\left(p \cdot \partial_p + y_E g \frac{\partial}{\partial g} + y_1 + y_2 - y_i\right) c_{12,i}(g;p) = d_{12,i}(g;p). \quad (106)$$

Since the locality of the cutoff function R gives

$$d_{12,i}(g;p) \xrightarrow{p \rightarrow \infty} 0, \quad (107)$$

we obtain the asymptotic behavior

$$c_{12,i}(g;p-q) \xrightarrow[p+q \text{ fixed}]{|p-q| \rightarrow \infty} p^{-y_1-y_2+y_i} \left(C_{12,i} + \frac{g}{p^{y_E}} C'_{12,i} + \mathcal{O}\left(\frac{g^2}{p^{2y_E}}\right) \right), \quad (108)$$

assuming the analyticity of $c_{12,i}(g;p)$ in g .

The simplest example is given by the massive Gaussian theory

$$\Gamma(m^2)[\Phi] = -\frac{1}{2} \int_p (p^2 + m^2) \Phi(p) \Phi(-p), \quad (109)$$

where the squared mass m^2 plays the role of g . (See Appendix C for the construction of composite operators.) We can show, for $3 < D < 4$ (the term proportional to m^2 is not singular for $2 < D \leq 3$),

$$\left[\frac{1}{2} [\phi^2(p)] \frac{1}{2} [\phi^2(q)] \right] \xrightarrow[p+q \text{ fixed}]{|p-q| \rightarrow \infty} c_F \left(1 + 2(D-3) \frac{m^2}{p^2} \right) p^{D-4} \delta(p+q) + \frac{2}{p^2} \frac{1}{2} [\phi^2(p+q)], \quad (110)$$

and for $D = 4$,

$$\begin{aligned} \left[\frac{1}{2} [\phi^2(p)] \frac{1}{2} [\phi^2(q)] \right] \xrightarrow[p+q \text{ fixed}]{|p-q| \rightarrow \infty} & \left(-\frac{1}{(4\pi)^2} \left(1 + \frac{2m^2}{p^2} \right) \ln |p| + \text{const} \right) \delta(p+q) \\ & + \frac{2}{p^2} \frac{1}{2} [\phi^2(p+q)]. \end{aligned} \quad (111)$$

7. Conclusions

In this work we have studied the OPE (operator product expansions) in the framework of ERG (the exact renormalization group). The key concepts underlying our analysis are the composite operators and their products, which we define respectively in Sects. 2 and 3. We have argued that the ERG differential equation associated with the product of two operators can be expanded in a local basis of composite operators, leading to the ERG differential equations for the Wilson OPE coefficients. Particular attention has been paid to the form of the equation at a fixed point. It is important to stress that the Wilson coefficients are defined in the large-momentum limit, i.e., $p \rightarrow \infty$. Taking this limit allows us to eliminate spurious contributions dependent on the cutoff.

We have tested our method by considering some explicit examples in Sect. 4. At the technical level, we have found it convenient to first solve the ERG differential equation by taking into account the analyticity of the equation at zero momentum before taking the large-momentum limit. In Sect. 5 we have generalized our discussion to include the definition of multiple products of a composite

operator, and in Sect. 6 we have considered the ERG differential equations away from the fixed point.

Although the examples we have given are for the Gaussian fixed point, we would like to stress that the ERG differential equations discussed in Sect. 3 are nonperturbative and can be employed for non-perturbative, albeit approximate, computations. In this sense suitable approximation schemes should be devised. Nonperturbative approximation schemes, such as the BMW [26], may be employed to solve the ERG differential equations for the Wilson coefficients.

Note added: After completion of the present work, we learned that Prof. H. Osborn had similar ideas to ours about the products of composite operators (Sect. 3.3 of H. Osborn, unpublished¹).

Funding

Open Access funding: SCOAP³.

Appendix A. More background on ERG

To make the content of Sect. 2 easier to understand for readers already familiar with Ref. [16] but not with Ref. [15] (and the subsequent extensions done more recently), we would like to summarize the basics of the ERG formalism adopted in this paper. We rely on perturbation theory for intuition.

In Ref. [16] the Wilson action is given in the form

$$S_\Lambda[\phi] = -\frac{1}{2} \int_p \frac{p^2}{K(p/\Lambda)} \phi(p) \phi(-p) + S_{\Lambda,I}[\phi], \quad (\text{A1})$$

where the first term gives the propagator

$$\frac{K(p/\Lambda)}{p^2}$$

that damps rapidly for $p > \Lambda$, and the second term gives interactions. To preserve physics below the momentum scale Λ , the interaction part must obey the following differential equation [16]:

$$-\Lambda \frac{\partial}{\partial \Lambda} S_{\Lambda,I}[\phi] = \int_p \frac{\Lambda \frac{\partial K(p/\Lambda)}{\partial \Lambda}}{p^2} \frac{1}{2} \left\{ \frac{\delta S_{\Lambda,I}[\phi]}{\delta \phi(-p)} \frac{\delta S_{\Lambda,I}[\phi]}{\delta \phi(p)} + \frac{\delta^2 S_{\Lambda,I}[\phi]}{\delta \phi(-p) \delta \phi(p)} \right\}. \quad (\text{A2})$$

We can rewrite this equation for the whole action as

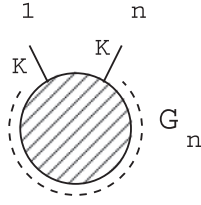
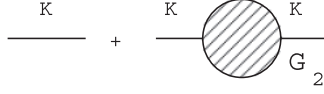
$$\begin{aligned} -\Lambda \frac{\partial}{\partial \Lambda} S_\Lambda[\phi] &= \int_p \Lambda \frac{\partial}{\partial \Lambda} \ln K(p/\Lambda) \cdot \phi(p) \frac{\delta}{\delta \phi(p)} S_\Lambda[\phi] \\ &\quad + \int_p \frac{\Lambda \frac{\partial K(p/\Lambda)}{\partial \Lambda}}{p^2} \frac{1}{2} \left\{ \frac{\delta S_\Lambda}{\delta \phi(p)} \frac{\delta S_\Lambda}{\delta \phi(-p)} + \frac{\delta^2 S_\Lambda}{\delta \phi(p) \delta \phi(-p)} \right\}. \end{aligned} \quad (\text{A3})$$

The correlation functions calculated with S_Λ ,

$$\langle \phi(p_1) \cdots \phi(p_n) \rangle_\Lambda \equiv \int [d\phi] e^{S_\Lambda[\phi]} \phi(p_1) \cdots \phi(p_n), \quad (\text{A4})$$

are not entirely independent of Λ . Take the sum of diagrams contributing to the **connected** part of the n -point function for $n > 2$ (Fig. A1).

¹ Available from <http://www.damtp.cam.ac.uk/user/ho/Norm.pdf>.

**Fig. A1.** Graph for the connected n -point function ($n > 2$).**Fig. A2.** Two-point function.

Polchinski's equation, Eq. (A2), guarantees that the shaded blob (denoted G_n) of Fig. A1 is independent of Λ . But the external propagators are multiplied by K , and we obtain

$$\langle \phi(p_1) \cdots \phi(p_n) \rangle_{\Lambda}^{\text{connected}} = G_n(p_1, \dots, p_n) \prod_{i=1}^n \frac{K(p_i/\Lambda)}{p_i^2}. \quad (\text{A5})$$

For small momenta p_i , the cutoff function $K(p_i/\Lambda)$ is almost 1. We only need to divide the correlation function by a product of K 's to make this strictly Λ independent:

$$\prod_{i=1}^n \frac{1}{K(p_i/\Lambda)} \langle \phi(p_1) \cdots \phi(p_n) \rangle_{\Lambda}^{\text{connected}}. \quad (\text{A6})$$

We are left with the two-point function which is given by

$$\langle \phi(p)\phi(q) \rangle_{\Lambda} = \left[\frac{K(p/\Lambda)}{p^2} + \frac{K(p/\Lambda)}{p^2} G_2(p) \frac{K(p/\Lambda)}{p^2} \right] \delta(p+q), \quad (\text{A7})$$

where $G_2(p)$ is independent of Λ (Fig. A2).

Again, this is almost independent of Λ for $p < \Lambda$. To make the two-point function strictly Λ independent, we first modify it by subtracting a high-momentum propagator,

$$\frac{K(p/\Lambda) (1 - K(p/\Lambda))}{p^2}, \quad (\text{A8})$$

and then divide the result by $K(p/\Lambda)^2$:

$$\begin{aligned} & \frac{1}{K(p/\Lambda)^2} \left(\langle \phi(p)\phi(q) \rangle_{\Lambda} - \frac{K(p/\Lambda) (1 - K(p/\Lambda))}{p^2} \delta(p+q) \right) \\ &= \left(\frac{1}{p^2} + G_2(p) \right) \delta(p+q). \end{aligned} \quad (\text{A9})$$

This is independent of Λ .

We have thus explained that Λ -independent connected correlation functions are given by

$$\langle \phi(p)\phi(q) \rangle \equiv \frac{1}{K(p/\Lambda)^2} \left(\langle \phi(p)\phi(q) \rangle_{\Lambda} - \frac{K(p/\Lambda) (1 - K(p/\Lambda))}{p^2} \delta(p+q) \right), \quad (\text{A10a})$$

$$\langle \phi(p_1) \cdots \phi(p_n) \rangle^{\text{connected}} \equiv \prod_{i=1}^n \frac{1}{K(p_i/\Lambda)} \times \langle \phi(p_1) \cdots \phi(p_n) \rangle_{\Lambda}^{\text{connected}}, \quad (\text{A10b})$$

where $n > 2$. Including the disconnected parts, the Λ -independent four-point correlation function is given by

$$\begin{aligned}
 \langle \phi(p_1) \cdots \phi(p_4) \rangle &\equiv \langle \phi(p_1) \cdots \phi(p_4) \rangle^{\text{connected}} + \langle \phi(p_1)\phi(p_2) \rangle \langle \phi(p_3)\phi(p_4) \rangle + (\text{t, u-channels}) \\
 &= \prod_{i=1}^4 \frac{1}{K(p_i/\Lambda)} \left[\langle \phi(p_1) \cdots \phi(p_4) \rangle_{\Lambda}^{\text{connected}} \right. \\
 &\quad - \frac{K(p_1/\Lambda)(1-K(p_1/\Lambda))}{p_1^2} \delta(p_1+p_2) \langle \phi(p_3)\phi(p_4) \rangle_{\Lambda} \\
 &\quad - \frac{K(p_3/\Lambda)(1-K(p_3/\Lambda))}{p_3^2} \delta(p_3+p_4) \langle \phi(p_1)\phi(p_2) \rangle_{\Lambda} \\
 &\quad + \frac{K(p_1/\Lambda)(1-K(p_1/\Lambda))}{p_1^2} \delta(p_1+p_2) \frac{K(p_3/\Lambda)(1-K(p_3/\Lambda))}{p_3^2} \delta(p_3+p_4) \\
 &\quad \left. + (\text{t, u-channels}) \right]. \tag{A11}
 \end{aligned}$$

This structure generalizes to higher-point functions. Using a formal but more convenient notation, we can express Λ -independent correlation functions by

$$\begin{aligned}
 \langle \phi(p_1) \cdots \phi(p_n) \rangle_{\Lambda} &\equiv \prod_{i=1}^n \frac{1}{K(p_i/\Lambda)} \\
 &\times \left\langle \exp \left(- \int_p \frac{K(p/\Lambda)(1-K(p/\Lambda))}{p^2} \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right) \phi(p_1) \cdots \phi(p_n) \right\rangle_{\Lambda}. \tag{A12}
 \end{aligned}$$

These Λ -independent correlation functions were first introduced in Ref. [21] and termed *modified* correlation functions. Two Wilson actions are called equivalent there if their modified correlation functions are the same. We find the word “modified” somewhat misleading since these are the proper correlation functions given by the Wilson action.

The ERG differential equation in Eq. (A3) still differs from the ERG equations of the main text. In order to discuss a fixed point of the renormalization group, we need to introduce an anomalous dimension and adopt the dimensionless convention to fix the momentum cutoff. Let us explain this one by one.

A.1. Anomalous dimension

To keep the kinetic term of S_{Λ} independent of Λ , we must introduce an appropriate Λ dependence to the normalization of ϕ . This introduces an anomalous dimension γ_{Λ} so that

$$\langle \phi(p_1) \cdots \phi(p_n) \rangle_{\Lambda} = \left(\frac{Z_{\Lambda}}{Z_{\Lambda'}} \right)^{\frac{n}{2}} \langle \phi(p_1) \cdots \phi(p_n) \rangle_{\Lambda'}, \tag{A13}$$

where

$$-\Lambda \frac{\partial}{\partial \Lambda} \ln Z_{\Lambda} = 2\gamma_{\Lambda}. \tag{A14}$$

To obtain this we must change Eq. (A3) to

$$\begin{aligned}
 -\Lambda \frac{\partial S_\Lambda}{\partial \Lambda} &= \int_p \left(\Lambda \frac{\partial}{\partial \Lambda} \ln K(p/\Lambda) - \gamma_\Lambda \right) \phi(p) \frac{\delta S_\Lambda}{\delta \phi(p)} \\
 &\quad + \int_p \frac{1}{p^2} \left(\Lambda \frac{\partial K(p/\Lambda)}{\partial \Lambda} - 2\gamma_\Lambda K(p/\Lambda) (1 - K(p/\Lambda)) \right) \\
 &\quad \times \frac{1}{2} \left\{ \frac{\delta S_\Lambda}{\delta \phi(-p)} \frac{\delta S_\Lambda}{\delta \phi(p)} + \frac{\delta^2 S_\Lambda}{\delta \phi(p) \delta \phi(-p)} \right\}.
 \end{aligned} \tag{A15}$$

(There are other ways of introducing γ_Λ , such as the one given in Ref. [27]. Here we have followed Refs. [20,21].) Rewriting the second integral of the right-hand side, we obtain

$$\begin{aligned}
 -\Lambda \frac{\partial S_\Lambda}{\partial \Lambda} &= \int_p \left(\Lambda \frac{\partial}{\partial \Lambda} \ln K(p/\Lambda) - \gamma_\Lambda \right) \phi(p) \frac{\delta S_\Lambda}{\delta \phi(p)} \\
 &\quad + \int_p \left(\Lambda \frac{\partial}{\partial \Lambda} \ln \frac{K(p/\Lambda)}{1 - K(p/\Lambda)} - 2\gamma_\Lambda \right) \frac{K(p/\Lambda) (1 - K(p/\Lambda))}{p^2} \\
 &\quad \times \frac{1}{2} \left\{ \frac{\delta S_\Lambda}{\delta \phi(-p)} \frac{\delta S_\Lambda}{\delta \phi(p)} + \frac{\delta^2 S_\Lambda}{\delta \phi(p) \delta \phi(-p)} \right\}.
 \end{aligned} \tag{A16}$$

A.2. Dimensionless convention

Finally, to obtain a fixed point we must adopt the dimensionless convention by measuring physical quantities in powers of appropriate powers of the cutoff Λ . This serves the purpose of rescaling, fixing the momentum cutoff at an arbitrary but fixed scale. We make the following replacements:

$$\begin{aligned}
 \frac{\Lambda}{\mu} &\longrightarrow e^{-t}, \\
 \gamma_\Lambda &\longrightarrow \gamma_t, \\
 \frac{p}{\Lambda} &\longrightarrow p, \\
 \Lambda^{\frac{D+2}{2}} \phi(p) &\longrightarrow \phi(p), \\
 S_\Lambda &\longrightarrow S_t,
 \end{aligned} \tag{A17}$$

where μ is an arbitrary momentum scale corresponding to the origin $t = 0$ of the logarithmic momentum scale. The ERG equation becomes

$$\begin{aligned}
 \partial_t S_t[\phi] &= \int_p \left(-p \cdot \partial_p \ln K(p) + \frac{D+2}{2} - \gamma_t + p \cdot \partial_p \right) \phi(p) \cdot \frac{\delta S_t[\phi]}{\delta \phi(p)} \\
 &\quad + \int_p \frac{1}{p^2} \left(-p \cdot \partial_p \ln \frac{K(p)}{1 - K(p)} - 2\gamma_t \right) \frac{K(p) (1 - K(p))}{p^2} \\
 &\quad \times \frac{1}{2} \left\{ \frac{\delta S_t[\phi]}{\delta \phi(-p)} \frac{\delta S_t[\phi]}{\delta \phi(p)} + \frac{\delta^2 S_t[\phi]}{\delta \phi(p) \delta \phi(-p)} \right\}.
 \end{aligned} \tag{A18}$$

Introducing

$$R(p) \equiv p^2 \frac{K(p)}{1 - K(p)}, \tag{A19}$$

we can rewrite the above as

$$\begin{aligned} \partial_t S_t[\phi] &= \int_p \left(-p \cdot \partial_p \ln K(p) + \frac{D+2}{2} - \gamma_t + p \cdot \partial_p \right) \phi(p) \cdot \frac{\delta S_t[\phi]}{\delta \phi(p)} \\ &\quad + \int_p \frac{1}{p^2} \left(-p \cdot \partial_p \ln R(p) + 2 - 2\gamma_t \right) \frac{K(p)^2}{R(p)} \\ &\quad \times \frac{1}{2} \left\{ \frac{\delta S_t[\phi]}{\delta \phi(-p)} \frac{\delta S_t[\phi]}{\delta \phi(p)} + \frac{\delta^2 S_t[\phi]}{\delta \phi(p) \delta \phi(-p)} \right\}. \end{aligned} \quad (\text{A20})$$

At the fixed point, the left-hand side vanishes, and we obtain Eq. (2) at the beginning of Sect. 2. Though the cutoff function R is given in terms of K in the above summary, we can take K and R independently as long as they satisfy the conditions listed in Sect. 2 (Ref. [21]).

Appendix B. Composite operator $\Phi(p)$

Given Eq. (15), we wish to derive Eq. (16). Following Eq. (13), we obtain

$$\begin{aligned} \langle \Phi(p) \phi(p_1) \cdots \phi(p_n) \rangle &= \frac{1}{K(p)} \prod_{i=1}^n \frac{1}{K(p_i)} \left\langle \left(\phi(p) + e^{-S[\phi]} \frac{K(p)^2}{R(p)} \frac{\delta}{\delta \phi(-p)} e^{S[\phi]} \right) \right. \\ &\quad \times \exp \left(- \int_q \frac{K(q)^2}{R(q)} \frac{1}{2} \frac{\delta^2}{\delta \phi(q) \delta \phi(-q)} \right) \phi(p_1) \cdots \phi(p_n) \Bigg\rangle_S. \end{aligned} \quad (\text{B1})$$

Integrating by parts, we obtain

$$\begin{aligned} \langle \Phi(p) \phi(p_1) \cdots \phi(p_n) \rangle &= \frac{1}{K(p)} \prod_{i=1}^n \frac{1}{K(p_i)} \left\langle \left(\phi(p) - \frac{K(p)^2}{R(p)} \frac{\delta}{\delta \phi(-p)} \right) \right. \\ &\quad \times \exp \left(- \int_q \frac{K(q)^2}{R(q)} \frac{1}{2} \frac{\delta^2}{\delta \phi(q) \delta \phi(-q)} \right) \phi(p_1) \cdots \phi(p_n) \Bigg\rangle_S \\ &= \frac{1}{K(p)} \prod_{i=1}^n \frac{1}{K(p_i)} \left\langle \exp \left(- \int_q \frac{K(q)^2}{R(q)} \frac{1}{2} \frac{\delta^2}{\delta \phi(q) \delta \phi(-q)} \right) \phi(p) \phi(p_1) \cdots \phi(p_n) \right\rangle_S \\ &= \langle \phi(p) \phi(p_1) \cdots \phi(p_n) \rangle. \end{aligned} \quad (\text{B2})$$

Analogously, we can obtain Eq. (43) from Eq. (42) by partial integration. Alternatively, we can obtain Eq. (43) from Eq. (6b) by regarding $\mathcal{O}(p)$ as an infinitesimal deformation of $W[J]$:

$$\frac{\delta}{\delta J(-q)} e^{W[J] + \epsilon \mathcal{O}(p)} = \left(\frac{\delta W[J]}{\delta J(-q)} (1 + \epsilon \mathcal{O}(p)) + \epsilon \frac{\delta \mathcal{O}(p)}{\delta J(-q)} \right) e^{W[J]}. \quad (\text{B3})$$

Regarding $[\mathcal{O}(p) \Phi(q)] e^{W[J]}$ as an infinitesimal deformation of $\Phi(q) e^{W[J]}$, we should obtain

$$\frac{\delta}{\delta J(-q)} e^{W[J] + \epsilon \mathcal{O}(p)} = (\Phi(q) + \epsilon [\mathcal{O}(p) \Phi(q)]) e^{W[J]}. \quad (\text{B4})$$

This gives Eq. (43).

Appendix C. Composite operators for the massive free theory

We consider the massive free theory in $D > 2$:

$$\Gamma[\Phi] = -\frac{1}{2} \int_p (p^2 + m^2) \Phi(p) \Phi(-p). \quad (\text{C1})$$

The high-momentum propagator is given by

$$G_{p,q}[\Phi] \equiv \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi(p) \delta \Phi(q)} = h(m^2, p) \delta(p + q), \quad (\text{C2})$$

where

$$h(m^2, p) \equiv \frac{1}{p^2 + m^2 + R(p)}. \quad (\text{C3})$$

We define

$$f(m^2, p) \equiv (2 + 2m^2 \partial_{m^2} + p \cdot \partial_p) h(m^2, p) = \frac{(2 - p \cdot \partial_p) R(p)}{(p^2 + m^2 + R(p))^2}. \quad (\text{C4})$$

A composite operator of scale dimension $(-y)$ satisfies

$$\left(y + 2m^2 \frac{\partial}{\partial m^2} + p \cdot \partial_p - \mathcal{D} \right) \mathcal{O}(p) = 0, \quad (\text{C5})$$

where \mathcal{D} is given by

$$\mathcal{D} = \int_q \left\{ \left(q \cdot \partial_q + \frac{D+2}{2} \right) \Phi(q) \cdot \frac{\delta}{\delta \Phi(q)} + f(m^2, q) \frac{1}{2} \frac{\delta^2}{\delta \Phi(q) \delta \Phi(-q)} \right\}. \quad (\text{C6})$$

A generic (even) scalar composite operator is written as

$$\mathcal{O}(p) = \sum_{n=0}^N \frac{1}{(2n)!} \int_{p_1, \dots, p_{2n}} \Phi(p_1) \cdots \Phi(p_{2n}) \delta \left(\sum_{i=1}^{2n} p_i - p \right) O_{2n}(p_1, \dots, p_{2n}). \quad (\text{C7})$$

Substituting this into Eq. (C5), we obtain

$$\left(2m^2 \partial_{m^2} + \sum_{i=1}^{2N} p_i \cdot \partial_{p_i} + y + N(D-2) - D \right) O_{2N}(p_1, \dots, p_{2N}) = 0 \quad (\text{C8a})$$

$$\begin{aligned} & \left(2m^2 \partial_{m^2} + \sum_{i=1}^{2n} p_i \cdot \partial_{p_i} + y + n(D-2) - D \right) O_{2n}(p_1, \dots, p_{2n}) \\ &= \frac{1}{2} \int_q f(m^2, q) O_{2(n+1)}(p_1, \dots, p_{2n}, q, -q) \quad (n \leq N-1). \end{aligned} \quad (\text{C8b})$$

The operator with the lowest scale dimension is the identity operator $\delta(p)$ with $y = D$. The second lowest dimensional operator, with $y = 2$, is given by

$$\frac{1}{2} [\phi^2(p)] = \frac{1}{2} \int_{p_1, p_2} \Phi(p_1) \Phi(p_2) \delta(p_1 + p_2 - p) + \kappa_2(m^2) \delta(p), \quad (\text{C9})$$

where $\kappa_2(m^2)$ satisfies

$$\left(2 - D + 2m^2 \frac{d}{dm^2}\right) \kappa_2(m^2) = \frac{1}{2} \int_p f(m^2, p). \quad (\text{C10})$$

We expect $\kappa_2(m^2)$ to be analytic at $m^2 = 0$ from the locality of the composite operator $\frac{1}{2} [\phi^2(p)]$; any non-analytic behavior should result from the integration over the momentum modes below the cutoff. Now, Eq. (C10) has a homogeneous solution proportional to $(m^2)^{\frac{D-2}{2}}$. For $2 < D < 4$, this is not analytic, and the equation has a unique analytic solution satisfying

$$\kappa_2(0) = \frac{1}{2-D} \frac{1}{2} \int_p f(0, p). \quad (\text{C11})$$

For $D = 4$ (and higher even dimensions), we have a problem: Eq. (C10) has no analytic solution because the right-hand side has a term proportional to m^2 . We must subtract the linear term and instead solve

$$\left(-2 + 2m^2 \frac{d}{dm^2}\right) \kappa_2(m^2) = \frac{1}{2} \int_p f(m^2, p) - m^2 \frac{1}{2} \int_p \frac{\partial}{\partial m^2} f(m^2, p) \Big|_{m^2=0}, \quad (\text{C12a})$$

where

$$-\frac{1}{2} \int_p \frac{\partial}{\partial m^2} f(m^2, p) \Big|_{m^2=0} = \int_p f(0, p) h(0, p) = \frac{1}{(4\pi)^2} \quad (\text{C12b})$$

as calculated in Eq. (68). Equation (C12a) has an analytic solution, but the solution is not unique due to the analytic homogeneous solution m^2 . This simply means that $\frac{1}{2} [\phi^2](p)$ mixes with $m^2 \delta(p)$ under scaling. We remove the ambiguity by adopting an arbitrary convention such as

$$\frac{d}{dm^2} \kappa_2(m^2) \Big|_{m^2=0} = 0. \quad (\text{C12c})$$

The operator thus defined satisfies

$$\left(2 + 2m^2 \frac{\partial}{\partial m^2} + p \cdot \partial_p - \mathcal{D}\right) \frac{1}{2} [\phi^2(p)] = \frac{m^2}{(4\pi)^2} \delta(p), \quad (\text{C13})$$

where the right-hand side implies mixing. Any alternative choice

$$\frac{1}{2} [\phi^2(p)] + \text{const} \times m^2 \delta(p) \quad (\text{C14})$$

is equally good as an element of a basis of composite operators.

The operator with $y = 4 - D$ can be constructed similarly:

$$\begin{aligned} \frac{1}{4!} [\phi^4(p)] &= \frac{1}{4!} \int_{p_1, \dots, p_4} \Phi(p_1) \cdots \Phi(p_4) \delta\left(\sum_{i=1}^4 p_i - p\right) \\ &+ \kappa_2(m^2) \frac{1}{2} \int_{p_1, p_2} \Phi(p_1) \Phi(p_2) \delta(p_1 + p_2 - p) + \kappa_4(m^2) \delta(p), \end{aligned} \quad (\text{C15})$$

where $\kappa_4(m^2)$ is defined by

$$\left(2(2-D) + 2m^2 \frac{d}{dm^2}\right) \kappa_4(m^2) = \kappa_2(m^2) \frac{1}{2} \int_p f(m^2, p). \quad (\text{C16})$$

For $2 < D < 4$, we obtain

$$\kappa_4(m^2) = \frac{1}{2}\kappa_2(m^2)^2. \quad (\text{C17})$$

For $D = 3$, the solution is ambiguous by a constant multiple of m^2 . (No subtraction is necessary; the right-hand side of Eq. (C16) has no term linear in m^2 .) For $D = 4$, the analyticity at $m^2 = 0$ demands that we modify the equation to

$$\left(-4 + 2m^2 \frac{d}{dm^2}\right) \kappa_4(m^2) = \kappa_2(m^2) \left(\frac{1}{2} \int_p f(m^2, p) + \frac{m^2}{(4\pi)^2}\right), \quad (\text{C18})$$

where $\kappa_2(m^2)$ is determined by Eq. (C12). The solution is given by

$$\kappa_4(m^2) = \frac{1}{2}\kappa_2(m^2)^2 + \text{const} \cdot m^4. \quad (\text{C19})$$

Again as a convention, we may impose

$$\left(\frac{d}{dm^2}\right)^2 \kappa_4(m^2) \Big|_{m^2=0} = 0 \quad (\text{C20})$$

to fix the constant. The operator thus defined satisfies

$$\left(2m^2 \frac{\partial}{\partial m^2} + p \cdot \partial_p - \mathcal{D}\right) \frac{1}{4!} [\phi^4(p)] = \frac{m^2}{(4\pi)^2} \frac{1}{2} [\phi^2(p)], \quad (\text{C21})$$

which implies that $\frac{1}{4!} [\phi^4(p)]$ mixes with $m^2 \frac{1}{2} [\phi^2(p)]$ under scaling.

The operator $p^2 \frac{1}{2} [\phi^2(p)]$ has $y = 0$. The other operator with $y = 0$ is the equation-of-motion composite operator [1,3,25] given by

$$\begin{aligned} \mathcal{E}(p) &\equiv -e^{-S} \int_q K(q) \frac{\delta}{\delta \phi(q)} (\Phi(q+p) e^S) \\ &= \frac{1}{2} \int_{p_1, p_2} \Phi(p_1) \Phi(p_2) \delta(p_1 + p_2 - p) \sum_{i=1}^2 (p_i^2 + m^2) \\ &\quad - \int_q \frac{R(q)}{q^2 + m^2 + R(q)} \delta(p). \end{aligned} \quad (\text{C22})$$

For $2 < D \leq 4$, all the other scalar operators have $y < 0$.

Appendix D. Asymptotic behavior of $F(p)$

For $2 < D < 4$,

$$F(p) \equiv \frac{1}{2} \int_q h(q) h(q+p) \quad (\text{D1})$$

satisfies the differential equation

$$(4 - D + p \cdot \partial_p) F(p) = \int_q f(q) h(q+p). \quad (\text{D2})$$

Since

$$\int_q f(q) h(q+p) \xrightarrow{p \rightarrow \infty} \frac{1}{p^2} \int_q f(q), \quad (\text{D3})$$

we obtain

$$F(p) \xrightarrow{p \rightarrow \infty} c_F p^{D-4} + \frac{1}{p^2} \frac{1}{2-D} \int_q f(q) = c_F p^{D-4} + \frac{2\kappa_2}{p^2}, \quad (\text{D4})$$

where c_F is a constant. From

$$c_F p^{D-4} = \frac{1}{2} \int_q \frac{1}{q^2(p+q)^2} \quad (\text{D5})$$

we obtain

$$c_F = \frac{1}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(2 - \frac{D}{2}) \Gamma(\frac{D}{2} - 1)^2}{2\Gamma(D-2)}. \quad (\text{D6})$$

For $D = 4$, $F(p)$ is determined by

$$p \cdot \partial_p F(p) = \int_q f(q) h(q+p) - \frac{1}{(4\pi)^2} \quad (\text{D7})$$

and

$$F(0) = 0. \quad (\text{D8})$$

For large p , the differential equation becomes

$$p \cdot \partial_p F(p) \xrightarrow{p \rightarrow \infty} -\frac{1}{(4\pi)^2} + \frac{1}{p^2} \int_q f(q), \quad (\text{D9})$$

which gives

$$F(p) \xrightarrow{p \rightarrow \infty} -\frac{1}{(4\pi)^2} \ln |p| + \text{const} + \frac{2\kappa_2}{p^2}. \quad (\text{D10})$$

The constant is determined by the initial condition $F(0) = 0$, but it depends on the choice of a cutoff function R .

References

- [1] C. Becchi, [arXiv:hep-th/9607188](#) [[Search INSPIRE](#)].
- [2] J. M. Pawłowski, *Ann. Phys.* **322**, 2831 (2007) [[arXiv:hep-th/0512261](#)] [[Search INSPIRE](#)].
- [3] Y. Igarashi, K. Itoh, and H. Sonoda, *Prog. Theor. Phys. Suppl.* **181**, 1 (2009) [[arXiv:0909.0327](#) [hep-th]] [[Search INSPIRE](#)].
- [4] C. Pagani, *Phys. Rev. D* **94**, 045001 (2016) [[arXiv:1603.07250](#) [hep-th]] [[Search INSPIRE](#)].
- [5] K. G. Wilson, *Phys. Rev.* **179**, 1499 (1969).
- [6] J. Hughes, *Nucl. Phys. B* **312**, 125 (1989).
- [7] G. Keller and C. Kopper, *Commun. Math. Phys.* **148**, 445 (1992).

- [8] G. Keller and C. Kopper, Commun. Math. Phys. **153**, 245 (1993).
- [9] S. Hollands and C. Kopper, Commun. Math. Phys. **313**, 257 (2012) [[arXiv:1105.3375](#) [hep-th]] [[Search INSPIRE](#)].
- [10] J. Holland and S. Hollands, Commun. Math. Phys. **336**, 1555 (2015) [[arXiv:1401.3144](#) [math-ph]] [[Search INSPIRE](#)].
- [11] J. Holland, S. Hollands, and C. Kopper, Commun. Math. Phys. **342**, 385 (2016) [[arXiv:1411.1785](#) [hep-th]] [[Search INSPIRE](#)].
- [12] M. B. Fröb, J. Holland, and S. Hollands, J. Math. Phys. **57**, 122301 (2016) [[arXiv:1511.09425](#) [math-ph]] [[Search INSPIRE](#)].
- [13] M. B. Fröb and J. Holland, [arXiv:1603.08012](#) [math-ph] [[Search INSPIRE](#)].
- [14] W. Zimmermann, Ann. Phys. **77**, 570 (1973).
- [15] K. G. Wilson and J. Kogut, Phys. Rept. **12**, 75 (1974).
- [16] J. Polchinski, Nucl. Phys. B **231**, 269 (1984).
- [17] T. R. Morris, Int. J. Mod. Phys. A **9**, 2411 (1994) [[arXiv:hep-ph/9308265](#)] [[Search INSPIRE](#)].
- [18] C. Wetterich, Phys. Lett. B **301**, 90 (1993).
- [19] U. Ellwanger, Z. Phys. C **62**, 503 (1994) [[arXiv:hep-ph/9308260](#)] [[Search INSPIRE](#)].
- [20] Y. Igarashi, K. Itoh, and H. Sonoda, Prog. Theor. Exp. Phys. **2016**, 093B04 (2016) [[arXiv:1607.01521](#) [hep-th]] [[Search INSPIRE](#)].
- [21] H. Sonoda, Prog. Theor. Exp. Phys. **2015**, 103B01 (2015) [[arXiv:1503.08578](#) [hep-th]] [[Search INSPIRE](#)].
- [22] T. R. Morris, Phys. Lett. B **329**, 241 (1994) [[arXiv:hep-ph/9403340](#)] [[Search INSPIRE](#)].
- [23] J. Berges, N. Tetradis, and C. Wetterich, Phys. Rept. **363**, 223 (2002) [[arXiv:hep-ph/0005122](#)] [[Search INSPIRE](#)].
- [24] H. Sonoda, [arXiv:1706.00198](#) [hep-th] [[Search INSPIRE](#)].
- [25] O. J. Rosten, Phys. Rept. **511**, 177 (2012) [[arXiv:1003.1366](#) [hep-th]] [[Search INSPIRE](#)].
- [26] J.-P. Blaizot, R. Mendez-Galain, and N. Wschebor, Phys. Rev. E **74**, 051116 (2006) [[arXiv:hep-th/0512317](#)] [[Search INSPIRE](#)].
- [27] R. D. Ball, P. E. Haagensen, J. I. Latorre, and E. Moreno, Phys. Lett. B **347**, 80 (1995) [[arXiv:hep-th/9411122](#)] [[Search INSPIRE](#)].