Robust comparative statics for non-monotone shocks in large aggregative games

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Abstract

A policy change that involves a redistribution of income or wealth is typically controversial, affecting some people positively but others negatively. In this paper we extend the "robust comparative statics" result for large aggregative games established by Acemoglu and Jensen (2010) to possibly controversial policy changes. In particular, we show that both the smallest and the largest equilibrium values of an aggregate variable increase in response to a policy change to which individuals' reactions may be mixed but the overall aggregate response is positive. We provide sufficient conditions for such a policy change in terms of distributional changes in parameters.
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1. Introduction

Recently, Acemoglu and Jensen (2010, 2015) developed new comparative statics techniques for large aggregative games, where there are a continuum of individuals interacting with each other only through an aggregate variable. Rather surprisingly, in such games, one can obtain a "robust comparative statics" result without considering the interaction between the aggregate variable and individuals' actions. In particular, Acemoglu and Jensen (2010) defined a positive shock as a positive parameter change that positively affects each individual's action for each value of the aggregate variable. Then they showed that both the smallest and the largest equilibrium values of the aggregate variable increase in response to a positive shock.

Although positive shocks are common in economic models, many important policy changes in reality tend to be controversial, affecting some individuals positively but others negatively. For example, a policy change that involves a redistribution of income necessarily affects some individuals' income positively but others' negatively. Such policy changes of practical importance cannot be positive shocks.

The purpose of this paper is to show that Acemoglu and Jensen's (2010, 2015) analysis can in fact be extended to such policy changes. Using Acemoglu and Jensen's (2010) static framework, we consider possibly controversial policy changes by defining an "overall positive shock" to be a parameter change to which individuals' reactions may be mixed but the overall aggregate response is positive for each value of the aggregate variable. We show that both the smallest and the largest equilibrium values of the aggregate variable increase in response to an overall positive shock. Then we provide sufficient conditions for an overall positive shock in terms of distributional changes in parameters.[1] These conditions enable one to deal with various policy changes, including ones that involve a redistribution of income.

This paper is not the first to study comparative statics for distributional changes. In a general dynamic stochastic model with a continuum of individuals, Acemoglu and Jensen (2015) considered robust comparative statics for changes in the stationary distributions of individuals' idiosyncratic shocks, but their analysis was restricted to positive shocks in the above sense. Jensen (2018) and Nocetti (2016) studied comparative statics for more general distributional changes, but neither of them considered robust comparative statics. This paper bridges the gap between robust comparative statics and distributional comparative statics in large aggregative games.[2]

Before showing our robust comparative statics results, we establish the existence of the smallest and the largest equilibrium values of the aggregate variable. This result is closely related to the literature on the existence of a Nash equilibrium for games with a continuum of players. The seminal result in this literature is Schmeidler's (1973) existence theorem. Mas-Colell (1984) reformulated Schmeidler's model and equilibrium concept in terms of distributions rather than measurable functions, offering an elegant approach to the existence problem. In this paper, while we use measurable functions to obtain our existence result, we consider distributions to develop

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[1] The concept of overall positive shocks is related not only to that of positive shocks but also to Acemoglu and Jensen's (2013) concept of "shocks that hit the aggregator," which were defined as parameter changes that directly affect the "aggregator" positively along with additional restrictions. Such parameter changes are not considered in this paper, but they can easily be incorporated by slightly extending our framework.

[2] See Balbus et al. (2015) for robust comparative statics results on distributional Bayesian Nash equilibria with strategic complementarities.

sufficient conditions for robust comparative statics. Mas-Colell's (1984) distributional approach was extended by Jovanovic and Rosenthal (1988) to sequential games.

Rath (1992) provided a simple proof of Schmeidler's (1973) existence theorem, which was extended by Balder (1995). Although the existence of an equilibrium in this paper follows from one of his results, the existence of the smallest and the largest equilibrium values of the aggregate variable does not directly follow from the existence results available in the literature, including more recent results (e.g., Khan et al., 1997; Khan and Sun, 2002; Carmona and Podczeck, 2009). The existence of extremal equilibria were shown by Vives (1990), Van Zandt and Vives (2007), and Balbus et al. (2015) for different settings.

The rest of the paper is organized as follows. In Section 2 we provide a simple motivating example of income redistribution and aggregate labor supply. In Section 3 we present our general framework along with basic assumptions, and show the existence of the smallest and the largest equilibrium values of the aggregate variable. In Section 4 we formally define overall positive shocks. We also introduce a more general definition of "overall monotone shocks." We then present our general robust comparative statics result. In Section 5 we provide sufficient conditions for an overall monotone shock in terms of distributional changes in parameters based on first-order stochastic dominance and mean-preserving spreads. In Section 6 we apply our results to the example of income redistribution.

2. A simple model of income redistribution

Consider an economy with a continuum of agents indexed by $i \in [0, 1]$. Agent i solves the following maximization problem:

$$\max_{c_i, x_i \geq 0} \quad u(c_i) - x_i \tag{2.1}$$

$$\text{s.t.} \quad c_i = w x_i + e_i + s_i, \tag{2.2}$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, strictly concave, and twice continuously differentiable, w is the wage rate, s_i is a lump-sum transfer to agent i, and c_i, x_i, and e_i are agent i's consumption, labor supply, and endowment, respectively. We assume that $e_i + s_i \geq 0$ for all $i \in [0, 1]$. If $s_i < 0$, agent i pays a lump-sum tax of $-s_i$. For simplicity, we assume that the upper bound on x_i is never binding for relevant values of w and is thus not explicitly imposed. This simply means that no agent works 24 hours a day, 7 days a week. The government has no external revenue and satisfies

$$\int_{i \in I} s_i \, di = 0. \tag{2.3}$$

Aggregate demand for labor is given by a demand function $D(w)$ such that $D(0) < \infty$, $D(\overline{w}) = 0$ for some $\overline{w} > 0$, and $D : [0, \overline{w}] \to \mathbb{R}_+$ is continuous and strictly decreasing. The market-clearing condition is

$$D(w) = \int_{i \in I} x_i \, di. \tag{2.4}$$

Given (2.3), any change in the profile of s_i affects some agents' income positively but others' negatively. Hence it cannot be a positive shock in the sense of Acemoglu and Jensen (2010). However, one may still ask, for example, how does a policy change that widens income inequality affect aggregate labor supply and the wage rate?

This question cannot be answered using standard methods such as the implicit function theorem if the policy change in question is a discrete jump from one policy to another. If one insists on applying the implicit function theorem, then one needs to introduce a policy parameter that affects income distribution in a differentiable way, and find a set of equations that characterize aggregate labor supply and the wage rate. Even then, one typically needs to assume the existence of a unique equilibrium and the assumptions of the implicit function theorem.

It turns out that, using our results, one can answer the above and other questions in a "robust" way without introducing these extra assumptions.

3. Large aggregative games

Consider a large aggregative game as defined by Acemoglu and Jensen (2010, Sections II, III). There are a continuum of players indexed by $i \in I \equiv [0, 1]$. Player i's action and action space are denoted by x_i and $X_i \subset \mathbb{R}$, respectively. The assumptions made in this section are maintained throughout the paper.

Assumption 3.1. For each $i \in I$, X_i is nonempty and compact. There exists a compact convex set $K \subset \mathbb{R}$ such that $X_i \subset K$ for all $i \in I$.

Let $X = \prod_{i \in I} X_i$. Let $\mathscr{X}$ be the set of action profiles $x \in X$ such that the mapping $i \in I \mapsto x_i$ is measurable.[3] Let H be a function from K to a subset Ω of $\mathbb{R}$. We define $G : \mathscr{X} \to \Omega$, called the *aggregator*, by

$$G(x) = H \left(\int_{i \in I} x_i \, di \right). \tag{3.1}$$

Assumption 3.2. The set $\Omega \subset \mathbb{R}$ is compact and convex, and $H : K \to \Omega$ is continuous.[4]

Given $x \in \mathscr{X}$ and $i \in I$, player i's payoff takes the form $\pi_i(x_i, G(x), t_i)$, where t_i is player i's parameter. Let T_i be the underlying space for t_i; i.e., $t_i \in T_i$. Let $T \subset \prod_{i \in I} T_i$. We regard T as a set of well-behaved parameter profiles; for example, T can be a set of measurable functions from I to $\mathbb{R}$. We only consider parameter profiles t in T.

Assumption 3.3. For each $i \in I$, player i's payoff function π_i maps each $(k, Q, \tau) \in K \times \Omega \times T_i$ into $\mathbb{R}$.[5] For each $t \in T$, $\pi_i(\cdot, \cdot, t_i)$ is continuous on $K \times \Omega$, and for each $(k, Q) \in K \times \Omega$, $\pi_i(k, Q, t_i)$ is measurable in $i \in I$.

The game here is aggregative since each player's payoff is affected by other players' actions only through the aggregate $G(x)$. Accordingly, each player i's best response correspondence depends only on $Q = G(x)$ and t_i:

$$R_i(Q, t_i) = \arg\max_{x_i \in X_i} \pi_i(x_i, Q, t_i). \tag{3.2}$$

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[3] Unless otherwise specified, measurability means Lebesgue measurability.

[4] Given the assumptions on H and K, the properties of Ω here can be assumed without loss of generality.

[5] If π_i is initially defined only on $X_i \times \Omega \times T_i$, then this means that π_i can be extended to $K \times \Omega \times T_i$ in such a way as to satisfy Assumption 3.3.

The following assumption ensures that given any $Q \in \Omega$, one can find a measurable action profile $x \in \mathscr{X}$ such that $x_i \in R_i(Q, t_i)$ for all $i \in I$.

Assumption 3.4. For each open subset U of K, the set $\{i \in I : X_i \cap U \neq \emptyset\}$ is measurable.

Throughout the paper, we restrict attention to pure-strategy Nash equilibria, which we simply call equilibria. To be more precise, given $t \in T$, an *equilibrium* of the game is an action profile $x \in \mathscr{X}$ such that $x_i \in R_i(G(x), t_i)$ for all $i \in I$. We define an *equilibrium aggregate* as $Q \in \Omega$ such that $Q = G(x)$ for some equilibrium $x \in \mathscr{X}$. The following is a useful observation.

Remark 3.1. Given $t \in T$, $Q \in \Omega$ is an equilibrium aggregate if and only if $Q \in \mathcal{G}(Q, t)$, where

$$\mathcal{G}(Q, t) = \{G(x) : x \in \mathscr{X}, \forall i \in I, x_i \in R_i(Q, t_i)\}. \tag{3.3}$$

For $t \in T$, define $\underline{Q}(t)$ and $\overline{Q}(t)$ as the smallest and largest equilibrium aggregates, respectively, provided that they exist.

Theorem 3.1. *For any $t \in T$, the set of equilibrium aggregates is nonempty and compact. Therefore, both $\underline{Q}(t)$ and $\overline{Q}(t)$ exist.*

Proof. See Appendix A.1. □

Our primary concern here is not the existence of an equilibrium but that of $\underline{Q}(t)$ and $\overline{Q}(t)$. Although the existence of an equilibrium for our model follows from Theorem 3.4.1 in Balder (1995) under more general assumptions,[6] the compactness of the set of equilibrium aggregates does not directly follow from his result or other existence results in the literature, as mentioned in the introduction.

Theorem 3.1 differs from Theorem 1 in Acemoglu and Jensen (2010) in that we assume a continuum of player types rather than a finite number of player types.[7] But our proof follows the basic strategy of their proof.

4. Overall monotone shocks

By a *parameter change*, we mean a change in $t \in T$ from one parameter profile to another. We fix $\underline{t}, \overline{t} \in T$ in Sections 4 and 5.

Definition 4.1 (*Acemoglu and Jensen, 2010*)**.** The parameter change from $\underline{t}$ to $\overline{t}$ is a *positive shock* if (a) T is equipped with a partial order $\prec$, (b) $H(\cdot)$ is an increasing function,[8] (c) $\underline{t} \prec \overline{t}$, and (d) for each $Q \in \Omega$ and $i \in I$, the following properties hold:

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[6] In particular, the continuity requirement in Assumption 3.3 can be relaxed as follows: for each $t \in T$, $\pi_i(\cdot, \cdot, t_i)$ is upper semicontinuous on $K \times \Omega$, and $\pi_i(k, \cdot, t_i)$ is continuous on Ω for each $k \in K$. Furthermore, the aggregator G can be a multidimensional function in a specific way; see Balder (1995, Assumption 3.4.2).

[7] Acemoglu and Jensen (2015) allow for a continuum of player types, which can be a continuum of random variables, by using the Pettis integral in (3.1).

[8] In this paper, "increasing" means "nondecreasing," and "decreasing" means "nonincreasing."

(i) For each $\underline{x}_i \in R_i(Q, \underline{t}_i)$ there exists $\overline{x}_i \in R_i(Q, \overline{t}_i)$ such that $\underline{x}_i \leq \overline{x}_i$.

(ii) For each $\overline{y}_i \in R_i(Q, \overline{t}_i)$ there exists $\underline{y}_i \in R_i(Q, \underline{t}_i)$ such that $\underline{y}_i \leq \overline{y}_i$.

For comparison purposes, Acemoglu and Jensen's (2010) key assumptions are included in the above definition. We introduce additional definitions.

Definition 4.2. The parameter change from $\underline{t}$ to $\overline{t}$ is a *negative shock* if the parameter change from $\overline{t}$ to $\underline{t}$ is a positive shock. A parameter change is a *monotone shock* if it is a positive shock or a negative shock.

Acemoglu and Jensen (2010, Theorem 2) show that if the parameter change from $\underline{t}$ to $\overline{t}$ is a positive shock, then the following inequalities hold:

$$\underline{Q}(\underline{t}) \leq \underline{Q}(\overline{t}), \quad \overline{Q}(\underline{t}) \leq \overline{Q}(\overline{t}). \tag{4.1}$$

The following definitions allow us to show that the above inequalities hold for a substantially larger class of parameter changes.

Definition 4.3. The parameter change from $\underline{t}$ to $\overline{t}$ is an *overall positive shock* if for each $Q \in \Omega$ the following properties hold:

(i) For each $\underline{q} \in \mathcal{G}(Q, \underline{t})$ there exists $\overline{q} \in \mathcal{G}(Q, \overline{t})$ such that $\underline{q} \leq \overline{q}$.

(ii) For each $\overline{r} \in \mathcal{G}(Q, \overline{t})$ there exists $\underline{r} \in \mathcal{G}(Q, \underline{t})$ such that $\underline{r} \leq \overline{r}$.

Definition 4.4. The parameter change from $\underline{t}$ to $\overline{t}$ is an *overall negative shock* if the parameter change from $\overline{t}$ to $\underline{t}$ is an overall positive shock. A parameter change is an *overall monotone shock* if it is an overall positive shock or an overall negative shock.

It is easy to see that a positive shock is an overall positive shock under Acemoglu and Jensen's (2010) assumption that there are only a finite number of player types. We are ready to state our general result on robust comparative statics:

Theorem 4.1. *Suppose that the parameter change from $\underline{t}$ to $\overline{t}$ is an overall positive shock. Then both inequalities in (4.1) hold. The reserve inequalities hold if the parameter change is an overall negative shock.*

Proof. See Appendix A.2. □

The proof of this result is similar to that of Theorem 2 in Acemoglu and Jensen (2010). The latter result is immediate from Theorem 4.1 under their assumptions, which imply that a positive shock is an overall positive shock. The dynamic version of their result established by Acemoglu and Jensen (2015, Theorem 5) can also be extended to overall monotone shocks in a similar way.

5. Sufficient conditions

In this section we provide sufficient conditions for overall monotone shocks by assuming that players differ only in their parameters t_i. To be more specific, we assume the following for the rest of the paper.

Assumption 5.1. There exists a Borel-measurable convex set $\mathcal{T} \subset \mathbb{R}^n$ (equipped with the usual partial order) with $n \in \mathbb{N}$ such that $T_i \subset \mathcal{T}$ for all $i \in I$. There exists a convex-valued correspondence $\mathcal{X} : \mathcal{T} \to 2^{\mathcal{T}}$ such that $X_i = \mathcal{X}(t_i)$ for all $i \in I$ and $t_i \in T_i$. Moreover, there exists a function $\pi : K \times \Omega \times \mathcal{T} \to \mathbb{R}$ such that

$$\forall i \in I, \forall(k, Q, \tau) \in K \times \Omega \times \mathcal{T}, \quad \pi_i(k, Q, \tau) = \pi(k, Q, \tau). \tag{5.1}$$

This assumption implies that player i's best response correspondence $R_i(Q, \tau)$ does not directly depend on i; we denote this correspondence by $R(Q, \tau)$. For $(Q, \tau) \in \Omega \times \mathcal{T}$, we define

$$\underline{R}(Q, \tau) = \min R(Q, \tau), \quad \overline{R}(Q, \tau) = \max R(Q, \tau). \tag{5.2}$$

Both $\underline{R}(Q, \tau)$ and $\overline{R}(Q, \tau)$ are well-defined since $R(Q, \tau)$ is a compact set for each $(Q, \tau) \in (\Omega, \mathcal{T})$ (see Camacho et al., 2016, Lemma A.1). We assume the following for the rest of the paper.

Assumption 5.2. T is a set of measurable functions from I to $\mathcal{T}$, and $H : K \to \Omega$ is an increasing function.

For any $t \in T$, let $F_t : \mathbb{R}^n \to I$ denote the distribution function of t:

$$F_t(z) = \int\limits_{i \in I} 1\{t_i \le z\} di, \tag{5.3}$$

where $1\{\cdot\}$ is the indicator function; i.e., $1\{t_i \le z\} = 1$ if $t_i \le z$, and $= 0$ otherwise. Note that $F_t(z)$ is the proportion of players $i \in I$ with $t_i \le z$.

For the rest of this section, we take $\underline{t}, \overline{t} \in T$ as given.

5.1. First-order stochastic dominance

Given two distributions $\underline{F}, \overline{F} : \mathbb{R}^n \to I$, $\overline{F}$ is said to *(first-order) stochastically dominate $\underline{F}$* if

$$\int \phi(z) \, d\underline{F}(z) \le \int \phi(z) \, d\overline{F}(z) \tag{5.4}$$

for any increasing bounded Borel function $\phi : \mathbb{R}^n \to \mathbb{R}$, where $\mathbb{R}^n$ is equipped with the usual partial order $\le$. As is well known (e.g., Müller and Stoyan, 2002, Section 1), in case $n = 1$, $\overline{F}$ stochastically dominates $\underline{F}$ if and only if

$$\forall z \in \mathbb{R}, \quad \underline{F}(z) \ge \overline{F}(z). \tag{5.5}$$

The following result provides a sufficient condition for an overall monotone shock based on stochastic dominance.

Theorem 5.1. *Suppose that $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$, and that both $\underline{R}(Q, \tau)$ and $\overline{R}(Q, \tau)$ are increasing (resp. decreasing) Borel functions of $\tau \in \mathcal{T}$ for each $Q \in \Omega$. Then the parameter change from $\underline{t}$ to $\overline{t}$ is an overall positive (resp. negative) shock.*

Proof. We only consider the increasing case; the decreasing case is symmetric. Let $\underline{q} \in \mathcal{G}(Q, \underline{t})$. Then there exists $x \in \mathcal{X}$ such that $\underline{q} = H(\int_{i \in I} x_i di)$ and $x_i \in R(Q, \underline{t}_i)$ for all $i \in I$. Since
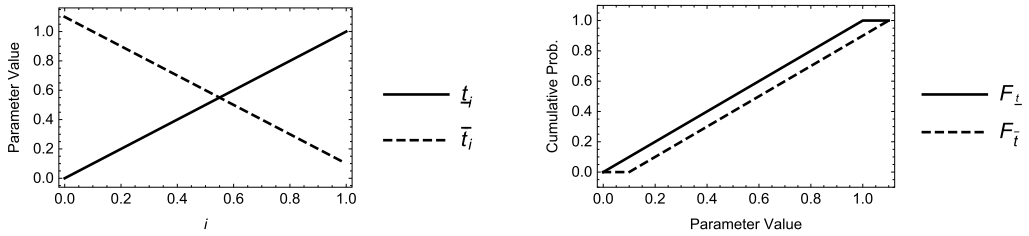
Fig. 1. The parameter change from $\underline{t}$ to $\overline{t}$ is not a monotone shock (left panel), but $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$ (right panel).

$x_i \leq \overline{R}(Q, \underline{t}_i)$ for all $i \in I$ by (5.2), and since H is an increasing function by Assumption 5.2, we have

$$\underline{q} \leq H\left(\int_{i \in I} \overline{R}(Q, \underline{t}_i)di\right) = H\left(\int \overline{R}(Q, z)dF_{\underline{t}}(z)\right) \tag{5.6}$$

$$\leq H\left(\int \overline{R}(Q, z)dF_{\overline{t}}(z)\right) = H\left(\int_{i \in I} \overline{R}(Q, \overline{t}_i)di\right) \in \mathcal{G}(Q, \overline{t}), \tag{5.7}$$

where the inequality in (5.7) holds since $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$ and $\overline{R}(Q, \cdot)$ is an increasing function. It follows that condition (i) of Definition 4.3 holds. By a similar argument, condition (ii) also holds. Hence the parameter change from $\underline{t}$ to $\overline{t}$ is an overall positive shock. □

If the parameter change from $\underline{t}$ to $\overline{t}$ is a positive shock, then it is easy to see from (5.3) and (5.5) that $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$. However, there are many other ways in which $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$. Fig. 1 shows a simple example. In this example, the parameter change from $\underline{t}$ to $\overline{t}$ is not a monotone shock, but $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$ by (5.5). Thus the parameter change here is an overall positive shock by Theorem 5.1 if both $\underline{R}(Q, \tau)$ and $\overline{R}(Q, \tau)$ are increasing in τ.

There are well known sufficient conditions for both $\underline{R}(Q, \tau)$ and $\overline{R}(Q, \tau)$ to be increasing or decreasing; see Milgrom and Shannon (1994, Theorem 4), Topkis (1998, Theorem 2.8.3), Vives (1999, p. 35), Amir (2005, Theorems 1, 2), and Roy and Sabarwal (2010, Theorem 2).[9] Any of those conditions can be combined with Theorem 5.1. Here we state a simple result based on Amir (2005, Lemma 1, Theorems 1, 2).

Corollary 5.1. *Assume the following:* (i) $F_{\overline{t}}$ *stochastically dominates* $F_{\underline{t}}$*;* (ii) $\mathcal{T} \subset \mathbb{R}$*;* (iii) *the upper and lower boundaries of* $\mathcal{X}(\tau)$ *are increasing (resp. decreasing) functions of* $\tau \in \mathcal{T}$*; and* (iv) *for each* $Q \in \Omega$*,* $\pi(k, Q, \tau)$ *is twice continuously differentiable in* $(k, \tau) \in K \times \mathcal{T}$ *and* $\partial^2\pi(k, Q, \tau)/\partial k \partial \tau \geq 0$ *(resp.* ≤ 0*) for all* $(k, \tau) \in K \times \mathcal{T}$*. Then the parameter change from* $\underline{t}$ *to* $\overline{t}$ *is an overall positive (resp. negative) shock.*

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[9] These results originate from games with strategic complementarities, which were popularized by Vives (1990) and Milgrom and Roberts (1990). Other related studies include Roy and Sabarwal (2008), Van Zandt and Vives (2007), and Balbus et al. (2015).
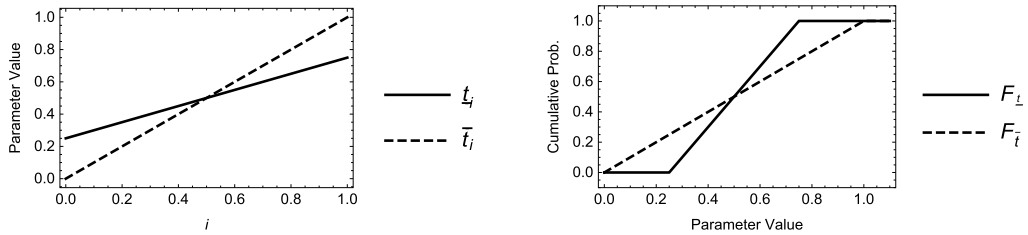
Fig. 2. The parameter change from $\underline{t}$ to $\bar{t}$ is not a monotone shock (left panel), but $F_{\bar{t}}$ is a mean-preserving spread of $F_{\underline{t}}$ (right panel).

5.2. Mean-preserving spreads

Following Acemoglu and Jensen (2015), we say that $F_{\bar{t}}$ is a *mean-preserving spread* of $F_{\underline{t}}$ if (5.4) holds for any Borel convex function $\phi : \mathcal{T} \to \mathbb{R}$.[10] Rothschild and Stiglitz (1970, p. 231) and Machina and Pratt (1997, Theorem 3) show that in case $n = 1$, $F_{\bar{t}}$ is a mean-preserving spread of $F_{\underline{t}}$ if

$$\int F_{\underline{t}}(z)dz = \int F_{\bar{t}}(z)dz, \tag{5.8}$$

and if there exists $\tilde{z} \in \mathbb{R}$ such that

$$F_{\underline{t}}(z) - F_{\bar{t}}(z) \begin{cases} \leq 0 & \text{if } z \leq \tilde{z}, \\ \geq 0 & \text{if } z > \tilde{z}. \end{cases} \tag{5.9}$$

The following result provides a sufficient condition for an overall monotone shock based on mean-preserving spreads.

Theorem 5.2. *Suppose that $F_{\bar{t}}$ is a mean-preserving spread of $F_{\underline{t}}$, and that both $\underline{R}(Q, \tau)$ and $\overline{R}(Q, \tau)$ are Borel convex (resp. concave) functions of $\tau \in \mathcal{T}$ for each $Q \in \Omega$. Then the parameter change from $\underline{t}$ to $\bar{t}$ is an overall positive (resp. negative) shock.*

Proof. The proof is essentially the same as that of Theorem 5.1 except that the inequality in (5.7) holds since $F_{\bar{t}}$ is a mean-preserving spread of $F_{\underline{t}}$ and $\overline{R}(Q, \tau)$ is convex in τ. $\quad\square$

Fig. 2 shows a simple example of a mean-preserving spread. As can be seen in the left panel, the parameter change from $\underline{t}$ to $\bar{t}$ is not a monotone shock. However, it is a mean-preserving spread by (5.8) and (5.9), as can be seen in the right panel. Thus the parameter change here is an overall positive shock by Theorem 5.2 if both $\underline{R}(Q, \tau)$ and $\overline{R}(Q, \tau)$ are convex in $\tau \in \mathcal{T}$.

Sufficient conditions for $\underline{R}(Q, \tau)$ or $\overline{R}(Q, \tau)$ to be convex or concave are established by Jensen (2018). The following result is based on Jensen (2018, Lemmas 1, 2, Theorem 2, Corollary 2).

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[10] Our approach differs from that of Acemoglu and Jensen (2015) in that while they consider positive shocks induced by applying a mean-preserving spread to the stationary distribution of each player's idiosyncratic shock, we consider non-monotone shocks induced by applying a mean-preserving spread to the entire distribution of parameters.

Corollary 5.2. *Assume the following:* (i) $F_{\bar{t}}$ *is a mean-preserving spread of* $F_{\underline{t}}$; (ii) *the upper and lower boundaries of* $\mathcal{X}(\tau)$ *are convex (resp. concave) continuous functions of* $\tau \in \mathcal{T}$; (iii) *for each* $(Q, \tau) \in \Omega \times \mathcal{T}$, $\pi(k, Q, \tau)$ *is strictly quasi-concave and continuously differentiable in* $k \in K$; (iv) $\overline{R}(Q, \tau) < \max \mathcal{X}(\tau)$ *(resp.* $\underline{R}(Q, \tau) > \min \mathcal{X}(\tau)$*); and* (v) *for each* $Q \in \Omega$, $\partial \pi(k, Q, \tau)/\partial k$ *is quasi-convex (resp. quasi-concave) in* $(k, \tau) \in K \times \mathcal{T}$. *Then the parameter change from* $\underline{t}$ *to* $\bar{t}$ *is an overall positive (resp. negative) shock.*

6. Applications

Recall the model of Section 2. Let $t_i = e_i + s_i$ for $i \in I$. The first-order condition for the maximization problem (2.1)–(2.2) is written as

$$u'(wx_i + t_i)w \begin{cases} \leq 1 & \text{if } x_i = 0, \\ = 1 & \text{if } x_i > 0. \end{cases} \tag{6.1}$$

Let $x(w, t_i)$ denote the solution for x_i as a function of w and t_i. Let $Q = \int_{i \in I} x(w, t_i) di$. Then (2.4) implies that $w = D^{-1}(Q)$. Let $\bar{\tau} > 0$ and $\mathcal{T} = [0, \bar{\tau}]$. The model here is a special case of the game in Section 5 with

$$\pi(k, Q, \tau) = u(D^{-1}(Q)k + \tau) - k, \quad \mathcal{X}(\tau) = K = \Omega = [0, \bar{k}], \tag{6.2}$$

where $\bar{k}$ is a constant satisfying $\bar{k} > \max_{(w, \tau) \in [0, \bar{w}] \times \mathcal{T}} x(w, \tau)$.

First suppose that $s_i = 0$ and $t_i = e_i$ for all $i \in I$. Let $\underline{t}_i = \underline{e}_i$ and $\bar{t}_i = \bar{e}_i$ be as in Fig. 1. Then the parameter change from $\underline{t}$ to $\bar{t}$ is not a monotone shock. However, it is straightforward to verify the conditions of Corollary 5.1 to conclude that the parameter change is an overall negative shock. Hence the smallest and largest equilibrium values of aggregate labor supply decrease in response to this parameter change, which implies that the smallest and largest equilibrium values of the wage rate increase.

Now suppose that $e_i = e$ and $t_i = s_i$ for all $i \in I$ for some $e > 0$. Let $\underline{t}_i = e + \underline{s}_i$ and $\bar{t}_i = e + \bar{s}_i$ be as in Fig. 2. Then $F_{\bar{t}}$ is a mean-preserving spread of $F_{\underline{t}}$. Thus the parameter change from $\underline{t}$ to $\bar{t}$ widens income inequality, and is not a monotone shock. However, it is straightforward to verify the conditions of Corollary 5.2 to conclude that the parameter change is an overall positive shock. Hence the smallest and largest equilibrium values of aggregate labor supply increase in response to this parameter change, which implies that the smallest and largest equilibrium values of the wage rate decrease.

The above comparative statics results can also be confirmed by solving (6.1) for $x_i = x(w, t_i)$:

$$x(w, t_i) = \begin{cases} \max\{[u'^{-1}(1/w) - t_i]/w, 0\} & \text{if } w > 0, \\ 0 & \text{if } w = 0. \end{cases} \tag{6.3}$$

This function is decreasing, piecewise linear, and convex in t_i; see Fig. 3. Thus the above results directly follow from Theorems 5.1 and 5.2 under (6.3).

Appendix A. Proofs

A.1. Proof of Theorem 3.1

Fix $t \in T$. The existence of an equilibrium follows from Balder (1995, Theorem 3.4.1); thus the set of equilibrium aggregates is nonempty. Recalling Remark 3.1, it remains to verify that the
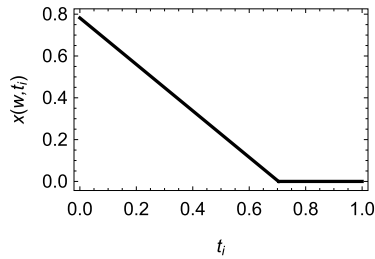
Fig. 3. Individual labor supply as a function of t_i with $u(c) = c^{0.7}/0.7$ and $w = 0.9$.

set of fixed points of $\mathcal{G}(\cdot, t)$ is compact. The following result is shown in Camacho et al. (2016, Lemma A.3).

Lemma A.1. *The correspondence $Q \mapsto \mathcal{G}(Q, t)$ has a compact graph.*

By this result and Lemma 17.51 in Aliprantis and Border (2006), the set of fixed points of $\mathcal{G}(\cdot, t)$ is compact, as desired.

A.2. Proof of Theorem 4.1

The following result is shown in Camacho et al. (2016, Lemma A.2).

Lemma A.2. *The correspondence $Q \mapsto \mathcal{G}(Q, t)$ has nonempty convex values.*

Let $t \in T$ and $Q \in \Omega$. Let $\underline{\mathcal{G}}(Q, t) = \min \mathcal{G}(Q, t)$ and $\overline{\mathcal{G}}(Q, t) = \max \mathcal{G}(Q, t)$. Both exist by Lemma A.1, and $\mathcal{G}(Q, t) = [\underline{\mathcal{G}}(Q, t), \overline{\mathcal{G}}(Q, t)]$ by Lemma A.2. This together with Lemma A.1 implies that $\mathcal{G}(\cdot, t)$ is "continuous but for upward jumps" in the sense of Milgrom and Roberts (1994, p. 447). Suppose that the parameter change from $\underline{t}$ to $\overline{t}$ is an overall positive shock. Then Definition 4.3 implies that $\underline{\mathcal{G}}(Q, \underline{t}) \le \underline{\mathcal{G}}(Q, \overline{t})$ and $\overline{\mathcal{G}}(Q, \underline{t}) \le \overline{\mathcal{G}}(Q, \overline{t})$. Thus both inequalities in (4.1) follow from Milgrom and Roberts (1994, Corollary 2). If the parameter change is an overall negative shock, then the reverse inequalities hold similarly.

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