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Real Indeterminacy of Stationary Monetary Equilibria in Centralized Economies

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Abstract

We show that real indeterminacy of stationary equilibria, by which the set of stationary equilibria is a continuum and the real allocation varies among equilibria, may arise in some general equilibrium models with fiat money. The conditions under which such equilibria arise are: (i) each household optimally saves a constant amount of money and (ii) at least two households face different budget constraints. We present various models, including a decentralized money search model and a centralized model with a monopoly firm, to explain how these conditions lead to real indeterminacy. Finally, we present a policy that uniquely implements any desirable outcome.

Keywords: real indeterminacy, divisible money, dynamic general equilibrium

Journal of Economic Literature Classification Number: D51, E40

1 Introduction

As described herein, we examine real indeterminacy of stationary equilibrium in dynamic general equilibrium models with a centralized market. The type of equilibrium indeterminacy we study has three special characteristics. First, by indeterminacy of equilibria, we mean situations in which there exists a *continuum* of (as opposed to countably many) stationary equilibria. Second, each equilibrium allocation differs from one another in *real* terms. Third, in terms of what is being indeterminate, the type of indeterminacy we study has real indeterminacy of *stationary* outcomes. This type of indeterminacy differs from the type usually seen among dynamic general equilibrium models. As classified in Table 1, most existing models have indeterminacy of dynamic equilibrium paths (i.e., continuously many paths that lead to a single stationary outcome). Understanding whether real indeterminacy of stationary allocations arises in a wider class of dynamic general equilibrium models is important because this type of indeterminacy is a concern

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among monetary policy makers. That is, if there are continuously many equilibria with different outcomes, it becomes harder to predict the effects of monetary policies accurately.

Table 1: Existing literature with indeterminacy of stationary equilibria.

	Centralized/Walrasian market	Decentralized or random search market
Indeterminate dynamic paths to a stationary allocation	Many models (e.g., Benhabib and Farmer (1999))	Many models (e.g., Trejos and Wright (1995), Lagos and Wright (2003, 2005))
Indeterminate stationary allocations	Overlapping generation models (e.g., Gottaridi (1996), Spear, Srivastava, and Woodford (1990))	Decentralized market only (e.g., Green and Zhou (2002))
	Trade model (Nishimura and Shimomura (2002))	With centralized market (Jean et al. (2010))
	Monetary economy with infinitely lived agents *This paper*	

In contrast to indeterminacy of dynamic paths, indeterminacy of stationary allocations is mainly observed among special types of models in money search and overlapping generation (OLG) literature. For example, Green and Zhou (1998, 2002) show that in a specific money search model, there exists a continuum of stationary equilibria in which the law of one price, a fundamental property of Walrasian equilibria, holds.[1] In addition, Jean et al. (2010) show a continuum of stationary equilibrium using the Lagos–Wright model, which consists of both decentralized and centralized markets. Outside of the money search literature, Gottaridi (1996) and Spear, Srivastava, and Woodford (1990) find a continuum of Markov stationary equilibria in OLG framework. In addition, Nishimura and Shimomura (2002) show a dynamic trade model with a continuum of stationary equilibria.

As described in this paper, we present a new model of a centralized market in which real indeterminacy of a stationary equilibrium arises. The model is novel because our model has no random search and because it differs from the existing centralized market models. We further specify conditions that induce real indeterminacy in a centralized market model. That is, indeterminacy of stationary equilibria arises in any models of centralized market as long as (i) households save a constant amount of money and (ii) at least two households face different budget constraints.

Although our model has no decentralized market, our mechanism of real indeterminacy is closely related to the mechanism seen in one of the money search models, which is Jean et al. (2010). In fact, Jean et al. (2010) can be regarded as a special case in which the two conditions we specify in our paper are met. In Jean et al. (2010), the two conditions are satisfied because of the strategic interaction between buyers and sellers and the random bilateral trade

[1] Other papers show real indeterminacy of stationary equilibria in monetary search models with divisible fiat money. They include Kamiya and Shimizu (2006), Lagos and Wright (2003), and Matsui and Shimizu (2005).

framework. Therefore, the equilibrium prices exist in a continuum. Section 2 presents a comparison of the model reported by Jean et al. and our model, which reveals that their type of indeterminacy can arise without random search.

However, not all money search models with real indeterminacy of stationary equilibria have the same mechanism as that of our model. Particularly, the mechanism that induces real indeterminacy in the Green–Zhou model differs from our model, and is therefore different from that of Jean et al. (2010) as well. The real indeterminacy in the Green–Zhou model can be derived only under a random search framework. Intuitively speaking, our real indeterminacy requires goods of at least two types to be traded by households, whereas the Green–Zhou model includes only a single indivisible good.[2]

The plan of our paper is the following. Section 2 presents three models that induce real indeterminacy, including that presented by Jean et al. (2010), to illustrate the mechanism. Section 3 gives an explanation of the logic of real indeterminacy using a model with a general utility function and a policy that uniquely attains any equilibrium. We conclude in section 4.

2 Illustrative Models

In this section, we present three models in which stationary equilibria are indeterminate. The three models have the following common structures: (i) each period is divided into two sub-periods of day and night; and (ii) at night, households trade goods and money in a centralized Walrasian market. As for the day market, we adopt the following three market structures.

Model 1: Households do not trade and simply pay a fixed amount of money to the government

Model 2: A search market as in Jean et al. (2010)

Model 3: A centralized market with a monopoly firm

The first model is a canonical example illustrating the logic of indeterminacy. The day market is extremely simple: households are required only to pay a fixed amount of money to the government. The fixed payment can be interpreted as a tax, transportation costs of visiting the market, necessary living costs, etc. We show that the fixed payment leads to the real indeterminacy. It induces market clearing in the night market no matter what the price of the good is. We place this ad-hoc assumption to illustrate why indeterminacy arises when households save a fixed amount of money. In the second and the third models, such saving behavior arises endogenously.

The day market in the second model is a random matching market in which agents trade an indivisible good using fiat money, which is identical to the setup described by Jean et al. (2010). In the day market, the households meet randomly and pairwise. In each matching, one is assigned to be a seller; the other is assigned as a buyer. The trading

[2] Kamiya and Shimizu (2006)(2007) show that in the Green–Zhou model, there exists a hidden identity that leads to indeterminacy. This hidden identity never holds under the Walrasian market. Therefore, Green–Zhou indeterminacy can be derived only in bilateral monetary trading models and cannot exist in Walrasian models.

protocol is price posting by the seller.[3] As shown by Jean et al. (2010), there exists real indeterminacy. We show that the source of real indeterminacy is the same as that in the first model; that is, coordination failure endogenously creates the same saving behavior as in the first model. Therefore, we conclude that the real indeterminacy in both models results from the saving behavior of the households in the *centralized market* in the night market. Jean et al. (2010) report that their model inherits important features of the real indeterminacy in Green and Zhou (1998). However, the type of indeterminacy described in Jean et al. and our model requires at least two goods and cannot be applied to the Green–Zhou setup. The Green–Zhou model consists only of one good, which is traded in a decentralized market.[4]

In the third model, we formulate the day market as a centralized market with a monopoly firm. The aim is to show that this type of indeterminacy arises in some economies without a decentralized market. To illustrate this point, we use a model with a monopoly firm which produces and sells a durable indivisible good. The firm only accepts cash as a payment device. Because of this assumption, households have an incentive to save some money so that they can buy the durable good in the next period. A strategic interaction between the monopoly firm and households endogenously induces the constant amount of saving. Therefore, real indeterminacy occurs in this model as well.

2.1 Model 1: Model with a fixed payment

Time is infinite and discrete. There is a continuum of homogeneous households for which the measure is one. Following Lagos and Wright (2005), we assume that each period is divided into two sub-periods: day and night. Goods of two types exist: fiat money with fixed supply $M > 0$ and a general good.

In the day market, the households do not trade. They must pay M units of money to the government at the beginning of the period, which equals the money supply M. The collected money is redistributed to the households at the end of the day market. We assume that the government subsidizes $2M$ units of money each period as a lump-sum transfer to the half of households, and the other half receives nothing. More precisely, a household receives $2M$ units of money with probability $1/2$ and receives nothing with probability $1/2$. Later, we show that this payment plays the same role as the decentralized random matching market in Jean et al. (2010).

After the redistribution of money in the day market, the night market opens. In the night market, the households trade a consumption good in a Walrasian market. Their linear technology transforms one unit of labor H into one unit of consumption good X. Consequently, X and H are fundamentally the same good, which we call the general

[3] As explained in Jean et al. (2010), price posting by a seller differs from a take-it-or-leave-it offer by a seller. Specifically, when a seller can observe the matched buyer's money holdings, the two trading protocols differ because, under the take-it-or-leave-it offer framework, the seller can extract all the *ex post* surplus by offering a price that depends on the buyer's money holdings. This extraction does not hold under a price-posting framework because the price is set before the meeting and therefore cannot depend on the matched buyer's money holdings. When a seller cannot observe the matched buyer's money holdings, the two trading protocols are equivalent because a take-it-or-leave-it price offer cannot depend on the buyer's money holdings. We thank an anonymous referee for bringing this to our attention.

[4] We emphasize this point because Jean et al. (2010) write in their introduction that "the same multiplicity of steady state, single-price, monetary equilibria arise" as in Green and Zhou (1998). However, the type of indeterminacy in Jean et al. (2010) differs from the type seen in the Green–Zhou model.

good. Following Lagos and Wright (2005), we assume quasi-linear utility $U(X) - H$, i.e., the labor disutility is linear with the coefficient one and the utility of X is $U(X)$. We assume that $U(\cdot)$ is increasing, strictly concave, differentiable, and there exists X^* satisfying $U'(X^*) = 1$. The discount factor between periods is $\beta \in (0, 1)$, whereas the discount factor is one between day and night markets within the same period.

We specifically examine a stationary monetary equilibrium, which is described by a list of demands for the consumption good, labor supplies, savings, money holding distributions, and prices such that the following are true: i) money holding distributions and prices are constant over time; ii) given the price, all households optimally demand the consumption good, supply labor, and save; and iii) the centralized money market clears and the goods market clears (by Walras' law).

Let $V(m)$ be the discounted utility of a household that has m units of money at the beginning of the day market, and let $W(m)$ be the value function at the beginning of the night market. Then $V(m)$ is written as

$$V(m) = \frac{1}{2}W(m - M + 2M) + \frac{1}{2}W(m - M), \tag{1}$$

where $V(m)$ is the expected value of W because no trade occurs in the day market. Then, $W(m)$, the value at the beginning of the night, is written as

$$\begin{aligned} W(m) = &\max_{X,H,m'} U(X) - H + \beta V(m'), \\ &\text{s.t. } m' = PH - PX + m, \\ &\quad\;\; m' \geq M, \end{aligned} \tag{2}$$

where m' represents the amount of money held by the household at the end of the period. The constraint $m' \geq M$ requires households to pay the fixed amount at the beginning of the day market in the subsequent period.

The optimization problem in the night (2) is rewritten as

$$W(m) = \max_{X,m'} \quad U(X) - \left(\frac{m' - m}{P} + X\right) + \beta V(m') \;\; \text{s.t. } m' \geq M. \tag{3}$$

The first-order condition with respect to X is $U'(X^*) = 1$. Therefore, the amount of the consumption X is independent of P. This is a well-known property of the quasi-linear utility. By substituting the optimal X, it follows that

$$W(m) = U(X^*) - X^* + \frac{m}{P} + \max_{m'} \left\{-\frac{m'}{P} + \beta V(m')\right\} \;\; \text{s.t. } m' \geq M. \tag{4}$$

It is noteworthy that the term in the first bracket is independent of m'. By the envelope theorem, $W'(m) = \frac{1}{P}$ holds. Consequently, from (1), $V'(m) = \frac{W'(m-M+2M)}{2} + \frac{W'(m-M)}{2} = \frac{1}{P}$ holds. Therefore, V is linear in m, i.e., $V(m) = C + \frac{m}{P}$, where C is a constant.

To summarize, each household solves the following problem in the night market:

$$\max_{m'} -\frac{m'}{P} + \beta\left(C + \frac{m'}{P}\right) \;\; \text{s.t. } m' \geq M. \tag{5}$$

The objective function is linear in m' with a negative coefficient on m' of $-(1-\beta)/P$. Then, the household chooses $m' = M$ because the household has no incentive to save extra amounts of money. Intuitively, because households discount the future utility, their utility increases by $1/P$ if she spends one unit of money today, whereas their utility increases by at most β/P if money are spent in the future. Consequently, a household spends as much money as possible today and chooses the minimum required amount $m' = M$.

Because all households save $m' = M$ and because the measure of households is one, the total money demand is M, which equals to the money supply. Consequently, the money market in the night market clears for any P. Then, by Walras' law, the market of general good also clears. Therefore, the price P is indeterminate. The key assumption is that the fixed payment is exactly the same as the total money supply.

It is noteworthy that the indeterminacy is *real* because the real allocation such as labor supply is also indeterminate. For illustration, note that given $m = m' = M$, the budget constraint at the end of the day market is

$$H = \begin{cases} X^* - \frac{M}{P} & \text{if a household receives } 2M, \\ X^* + \frac{M}{P} & \text{if a household receives nothing.} \end{cases} \tag{6}$$

Therefore, because the price P is indeterminate, the allocation of labor supply is also indeterminate. The real effect is created by the heterogeneity of money holdings. At the beginning of the night market, some rich households hold $2M$ units of money; some poor households have no money. Because P is a nominal price, it affects the value of money in terms of the consumption good. For example, as P increases, the advantage of rich households compared with poor households decreases.

To summarize, the indeterminacy arises because all households must save exactly M units of money, which clears the money market in the night market for any value of P. The nominal price indeterminacy of P also affects real allocation because the nominal money holdings of the households are heterogeneous.

2.2 Model 2: Random matching with Price posting

In this subsection, we present the model explained by Jean et al. (2010) in line with our framework that induces a fixed amount of savings. The night market is the same as that in the first model: each household trades labor and consumption good in the centralized market. The day market is a random bilateral matching market. Within a matched pair, one becomes a buyer with probability 1/2, and the other becomes a seller.[5] They trade one unit of specialized good, which is indivisible and non-storable. After the seller posts a price p, the buyer chooses whether to accept it or not. The production cost of the indivisible good in terms of utility is $c > 0$; the utility of the consumption good is $u > 0$. Assume that u is sufficiently larger than c. Each household maximizes the discounted sum of expected utility streams as

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[u I_t^u - c I_t^c + U(X_t) - H_t \right], \tag{7}$$

where $I_t^u \in \{0, 1\}$ is an indicator of whether the household buys the indivisible good, (i.e., $I_t^u = 1$ implies that the household consumes the indivisible good in period t); $I_t^c \in \{0, 1\}$ is an indicator of whether the household produces the indivisible good or not.

[5] Jean et al. (2010) allows for a possibility of not meeting anyone, but we exclude this case for simplicity.

A stationary monetary equilibrium in this model is described by a list of buyers' decision rules for purchasing the indivisible durable good, demands for the consumption good, labor supplies, savings, money holding distributions, the divisible good price and the indivisible durable good price such that the following are true: i) money holding distributions and prices are constant over time; ii) a buyer's decision rule for purchasing the indivisible durable good is optimal; iii) given the buyer's decision rule for purchasing the indivisible durable good, a seller optimally posts the price of the indivisible durable good; iv) given the prices, all households optimally demand the consumption good, supply labor, and save; and v) centralized money market clears and goods market clears (by Walras' law).

Following Jean et al. (2010), we specifically examine the equilibrium in which the money holding distribution at the end of the night market is degenerate at some point, denoted by m^b. Let $W(m)$ be the value function at the beginning of the night market. Then, the seller's problem is written as

$$\max_{p} \quad -c + W(m^s + p) \quad \text{s.t. } u + W(m^b - p) \geq W(m^b), \tag{8}$$

where m^s is the money holdings of the seller. The constraint assures that the buyer has an incentive to accept the posted price. It might be readily apparent that the seller chooses p so that the constraint is binding. Jean et al. (2010) show that there exists a single price equilibrium: all sellers set the same price and the price will be accepted by all buyers. Below, we specifically examine the single-price equilibrium. Given the equilibrium single price p^*, the discounted utility at the beginning of each period can be written as

$$V(m) = \frac{1}{2}\big[u + W(m - p^*)\big] + \frac{1}{2}\big[-c + W(m + p^*)\big]. \tag{9}$$

Because the night market is the same as that in the first model, each household's problem in the night market is given as (2).

We present the main result of Jean et al. (2010). A coordination failure exists in the day market as described below.

1. *If all households have $m = M$ units of money at the beginning of the day market, then each seller sets $p = M$.*
   Intuitive proof: Given the assumption that u is sufficiently large, all buyers will accept any price if they have a sufficiently large amount of money. To maximize profits, each seller sets the highest price $p = M$ that the buyer can pay.

2. *If all sellers set $p = M$, then each household carries over M units of money to the subsequent period.*
   Intuitive proof: Given that u is sufficiently large, all households choose to consume the indivisible good. If a seller rationally expects that all households saved $m' \geq M$ units of money in the previous night market, then the seller sets $p = M$ in the current day market, and the buyer purchases the indivisible good.[6] By the same

[6]The price posting assumption compels each seller to set a single price $p = M$ irrespective of the matched buyer's money holdings. Under the price posting framework, "each agent sets a p such that in any meeting where a buyer likes his specialized good, [and] he commits to producing a unit of it as long as the buyer hands over p dollars." (p.394 in Jean et al. (2010)) As we explained in footnote 3, as long as a seller cannot observe the matched buyer's money holdings, then this seller's pricing strategy is identical to that under the bargaining with the seller's take-it-or-leave-it-offer framework. In fact, Green and Zhou (1998) rationalize seller setting $p = M$ for any money holding the matched buyer has by assuming that the money holding of a matched buyer is unobservable to a seller.

argument as in the first model, the households have no incentive to save $m' > M$ because they discount future consumption. Therefore, the households save $m' = M$.

Then, saving M units of money and posting $p = M$ are the outcomes of best responses.

The correspondence between the first model and the second model related to the households' behavior that leads to indeterminacy is noteworthy. In the first model, households save a fixed amount of money (i.e., $m' = M$) because of the exogenously given setup. In the second model, the households endogenously choose to save $m' = M$. In both models, because all households save $m' = M$ units of money, the money market clears for any P in the night market. Therefore, P is indeterminate. It also induces real indeterminacy because heterogeneity exists in the money holdings at the beginning of the night market. That is, half of the households have $2M$ units of money at the beginning of the night market because they were sellers in the day market and obtained M unit. The other half of the households are buyers, and they have no money at the beginning of the night market. The indeterminacy in the nominal price P therefore affects the real value of money.

2.3 Model 3: Centralized Market with a Monopoly Firm

In this subsection, we demonstrate that the same type of indeterminacy can occur even in an economy having only centralized markets. In the third model, the night market is the same as in the previous two models. The day market is a centralized market with a monopoly firm. The fixed monetary payment in the day market arises endogenously.[7] This model also clarifies that the indeterminacy in the Green–Zhou model is a different type from that of Jean et al. (2010). That is, the Green–Zhou model has only a decentralized market, whereas the type of indeterminacy explained in Jean et al. (2010) can occur without a decentralized market.

In the day market, the centralized market opens. A household trades an indivisible durable good to labor. Examples of the indivisible durable good include a house, a car, a refrigerator, and health (medical expenses). The monopoly firm has a linear technology to produce one unit of indivisible durable good from one unit of labor provided by the household. The firm has monopoly power only over the indivisible durable good market and takes the wage as given. Figure 1 presents the timing of events.

At the beginning of the day market, the durable good breaks down with a probability $\mu \in (0, 1)$. If that happens, then the household decides whether to buy a new one from the monopoly firm in the day market. We assume that the households are anonymous to serve two purposes: i) it assures households to use money to buy the durable good, and ii) it excludes the possibility of intra-temporal insurance contracts among households and inter-temporal credit contracts for the durable good shock.[8] We also do not allow a household to purchase the good from the firm on

[7] The division of day and night markets is not necessary for the real indeterminacy. In a previous version of the paper, Kamiya, Kubota and Nakajima (2011), we derive the same result with a model in which day and night markets are integrated.

[8] Although many goods are sold using a credit card as a payment device, numerous household expenses require cash as a payment. Telyukova (2013) uses Consumer Expenditure Survey data and reports that a large portion of household expenses requires a cash payment. It includes predictable payments such as mortgage and rent payments, utilities, babysitting, and daycare services, and also unpredictable expenses such as household expenditures and auto repairs, as well as emergencies of other types. Telyukova (2013) in particular emphasizes the role of precautionary demand for money induced by the unexpected expense on cash goods. This evidence supports our model's assumptions that the household must save money in case they need to purchase the durable good.

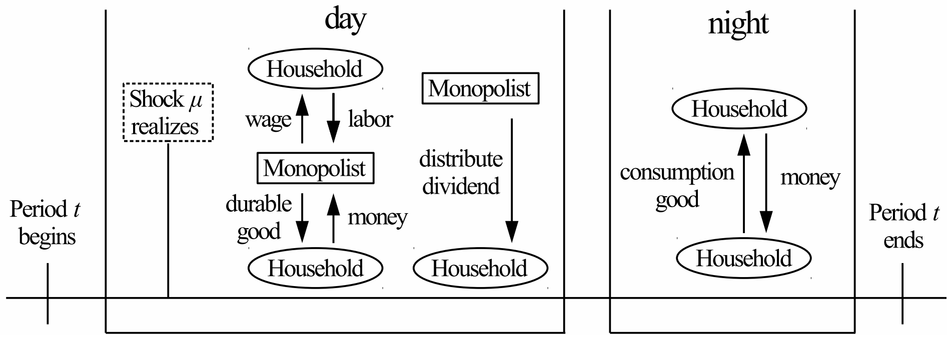

Figure 1: Timing of events in the monopoly firm's model.

credit.[9] After the durable good shock occurs, the day market opens. In the day market, the households supply labor to the monopoly firm. Some households purchase the durable good using money. At the end of the day market, the firm pays the wage and distributes the dividends equally to the households. Finally, households trade a general good (consumption good and labor in the night market) in the night market. We assume that the monopoly firm cannot hold an inventory.

Households

First, consider a household for which the durable good did not break down. Let $V^1(m)$ represent the discounted utility of a household with m units of money immediately after the shock. Let $W^1(m)$ be the value of a household with the durable good and m units of money at the beginning of the night market. Then the Bellman equation of the household with the durable good is written as

$$V^1(m) = \max_{L,s} \quad u - L + W^1(s + \Pi), \tag{10}$$

$$\text{s.t.} \quad s = wL + m, \quad L \geq 0,$$

where $u > 0$ denotes the utility of the durable good, w denotes a wage, L denotes an amount of labor, $s \in \mathbb{R}_+$ denotes savings, and Π denotes the dividend.

Next, consider a household with durable goods that have broken down. Then this household chooses whether to buy the durable good. Let $V^0(m)$ be the value of a household with m units of money immediately after the shock. Let $W^0(m)$ be the value of a household without the durable good and with m units of money at the beginning of

[9] At the end of Section 2, we relax this assumption and show that the indeterminacy results hold, unless the firm allows households to purchase entirely on credit. This requirement—charging a portion of the price to be paid up front using cash—is seen in the real world, such as a down payment for renting/purchasing a house.

the night market. Then the Bellman equation of a household without the durable good is written as

$$V^0(m) = \max_{L,I,s} \quad uI - L + W^I(s+\Pi) \tag{11}$$

$$\text{s.t.} \;\; s = wL - pI + m,$$

$$pI \leq m, \tag{12}$$

$$L \geq 0, \quad I \in \{0,1\},$$

where $I \in \{0,1\}$ is an indicator for the purchase of the durable good (i.e., $I = 1$ implies that the household buys the durable good). The main difference between $V^1(m)$ and $V^0(m)$ is the indivisible good expenditure pI. Equation (12) imposes a restriction that the indivisible good must be purchased using money. It is, at first glance, similar to the cash-in-advance constraint. However, it is derived endogenously by the lack of credit.[10] We show at the end of this section that similar results are obtainable even if the households are allowed to borrow money to purchase the durable good.

The Bellman equations at the beginning of the night market are shown below:

$$W^0(m) = \max_{X,H,m'} \quad U(X) - H + \beta V^0(m'), \tag{13}$$

$$\text{s.t.} \;\; m' = P(H - X) + m, \quad X \geq 0, \;\; H \geq 0,$$

and

$$W^1(m) = \max_{X,H,m'} \quad U(X) - H + \beta[\mu V^0(m') + (1-\mu)V^1(m')], \tag{14}$$

$$\text{s.t.} \;\; m' = P(H - X) + m, \quad X \geq 0, \;\; H \geq 0.$$

Under some technical assumptions, $L \geq 0$ in (10) and (11) and $X \geq 0$ and $H \geq 0$ in (13) and (14) never bind.[11] Using quasi-linearity of the utility function, (13) is written as

$$W^0(m) = \max_{X,m'} \quad U(X) - \left[X + \frac{1}{P}(m' - m)\right] + \beta V^0(m'). \tag{15}$$

Then, applying the first-order condition with respect to X yields

$$W^0(m) = \quad U(X^*) - X^* + \frac{m}{P} + \max_{m'} \left\{ -\frac{m'}{P} + \beta V^0(m') \right\}, \tag{16}$$

where X^* is a solution to $U'(X^*) = 1$. Similarly,

$$W^1(m) = \quad U(X^*) - X^* + \frac{m}{P} + \max_{m'} \left\{ -\frac{m'}{P} + \beta \left[\mu V^0(m') + (1-\mu)V^1(m')\right] \right\}. \tag{17}$$

Because of the quasi-linearity, there is no wealth effect, i.e., the decision on m' does not depend on m.

Substituting (17) into (10), the problems in two subperiods are integrated as shown below.

$$V^1(m) = \max_{L} \quad u - L + U(X^*) - X^* + \frac{wL + m + \Pi}{P} + \max_{m'} \left\{ -\frac{m'}{P} + \beta \left[\mu V^0(m') + (1-\mu)V^1(m')\right] \right\}. \tag{18}$$

[10]SeeRocheteau and Wright (2005).

[11]See an earlier version of this paper Kamiya, Kubota and Nakajima (2011) for more detailed exposition. We impose the constraint on maximal amount of money holding so that $H \geq 0$ is always satisfied in equilibrium. Lagos and Wright (2005) impose a similar assumption.

In the equilibrium, $w = P$ must hold, so the coefficient on L is $-1+w/P$. Therefore, w must equal to P in equilibrium because, if $w \neq P$, L takes an extreme value and violates equilibrium condition. If $w < P$, then nobody works in the day market and the indivisible durable good will not be produced. If $w > P$, then households supply labor to the greatest extent possible in the day market so that they can purchase the consumption good in the night market without working. This results in zero production of the consumption good. Under $w = P$, L disappears from the above equation. It becomes

$$V^1(m) = u + U(X^*) - X^* + \frac{m+\Pi}{P} + \max_{m'} \left\{ -\frac{m'}{P} + \beta \left[\mu V^0(m') + (1-\mu)V^1(m') \right] \right\}. \tag{19}$$

In (19), the household only chooses m'. Similarly, the maximization problem of a household with no durable good at the beginning of the day market is

$$\begin{aligned} V^0(m) = \max_I \quad & I \left\{ u + U(X^*) - X^* + \frac{-pI+m+\Pi}{P} + \max_{m'} \left[-\frac{m'}{P} + \beta \left[\mu V^0(m') + (1-\mu)V^1(m') \right] \right] \right\} \\ & + (1-I) \left\{ U(X^*) - X^* + \frac{-pI+m+\Pi}{P} + \max_{m'} \left[-\frac{m'}{P} + \beta V^0(m') \right] \right\}, \end{aligned} \tag{20}$$

$$\text{s.t.} \quad pI \leq m.$$

It is noteworthy that a household decides whether to purchase the product given the price and how much to save. These decisions are identical to a buyer's decisions as explained by Jean et al. (2010).

Monopoly Firm

In the day market, the monopoly firm has a linear technology and produces one unit of durable good from one unit of labor. The firm has a monopoly power in the indivisible good market, while the labor market is competitive. Given wage w, the firm posts the indivisible good price p to maximize the profit.[12] The demand for the indivisible good depends not only on the decision to purchase the good but also on the money holdings distribution.

Therefore, the monopoly firm's problem is

$$\Pi = \max_p \quad py(p) - wy(p), \tag{21}$$

where the demand for the indivisible good $y(\cdot)$ is derived from the household demand function and the money holding distribution.[13] At the end of the period, the monopoly firm distributes the profit Π equally to all households as dividends.

It is noteworthy that similarities exist between our model and Jean et al. (2010). In Jean et al. (2010), the seller in the decentralized market posts a price, which creates the fixed monetary payment by a buyer that eventually

[12] We assume that the monopoly firm posts a single price to all households instead of negotiating prices with each household. Even if negotiating price with each household incurs a small transaction cost, the total cost becomes large when there are many households in the market. On the other hand, posting a single price of a homogeneous product incurs a fixed transaction cost irrespective of how large the number of households is and could largely save the transaction cost. In our model, the monopoly firm sells a homogeneous product to numerous households. Therefore, the firm rationally chooses price posting. It is noteworthy that even if we introduce a transaction cost of choosing price posting in our model, the results remain the same. We are grateful to an anonymous referee for drawing our attention to this point.

[13] We assume that the firm does not save money to pay the wage. The reason is that the firm is not anonymous and can therefore trade on credit.

induces the indeterminacy of the equilibrium price. In our model, the monopoly firm plays a similar role: monopoly pricing is identical to the price posting by a seller because the buyers have no outside option to purchase the durable indivisible good.

A stationary monetary equilibrium in this model is described by a list of demands for the indivisible durable good, labor supplies at the day and the night market, demands for the consumption good, savings, money holding distributions, the divisible good price, the indivisible durable good price, and the wage in the day market such that the following are true: i) money holding distributions and prices are constant over time; ii) given households' demands for the indivisible durable good, the monopoly firm optimally sets the price of indivisible durable good; iii) given the prices, all households optimally demand the consumption good, supply labor, and save; and iv) the centralized money market clears and the goods market clears (by Walras' law).

Equilibria

Below, we show that the equilibria have the following features:

1. all households have $m = M$ units of money at the beginning of the day market;
2. the households purchase the durable good if it breaks down at the beginning of the day market;
3. the monopoly firm sets $p = M$; and
4. the wage in the day market equals to the price of the consumption good in the night market (i.e., $w = P$).

These features induce indeterminacy. Because all households have M units of money and because all want to buy the durable good, the monopoly firm sets the price of the durable good as exactly M. Additionally, following the same argument as in the first model, households do not save more than M units of money. Therefore, $p = M$ and $m' = M$ hold in equilibrium. These features are also found in the work by Jean et al. (2010), where the sellers post the price of M and the buyers save M in equilibrium.

We show below that the above features are indeed satisfied in equilibrium and that there exists a real indeterminacy. As explained above, $w = P$ must hold in the equilibrium from (18). We first assume that all households bring M to the day market to purchase the durable good if the price is affordable. Given the household behavior, we check the optimal response of the monopoly firm. We restrict our attention to the following domain of w:

$$w \leq M. \tag{22}$$

Under this domain, the firm faces a positive profit when it charges $p > w$. The firm has an incentive to raise the price as long as households demand the durable good. Therefore, it chooses the profit-maximizing price of $p = M$.

Next, we assume $p = M$ and check the optimal behavior of the households. For households in which a durable good breaks down by the shock, they purchase only if the first term in the two braces in (20) is larger than the second term, i.e.:

$$\begin{aligned} & u + U(X^*) - X^* + \frac{-M + m + \Pi}{P} + \max_{m'} \left[-\frac{m'}{P} + \beta \left[\mu V^0(m') + (1-\mu) V^1(m') \right] \right] \\ & \geq U(X^*) - X^* + \frac{m + \Pi}{P} + \max_{m'} \left[-\frac{m'}{P} + \beta V^0(m') \right]. \end{aligned} \tag{23}$$

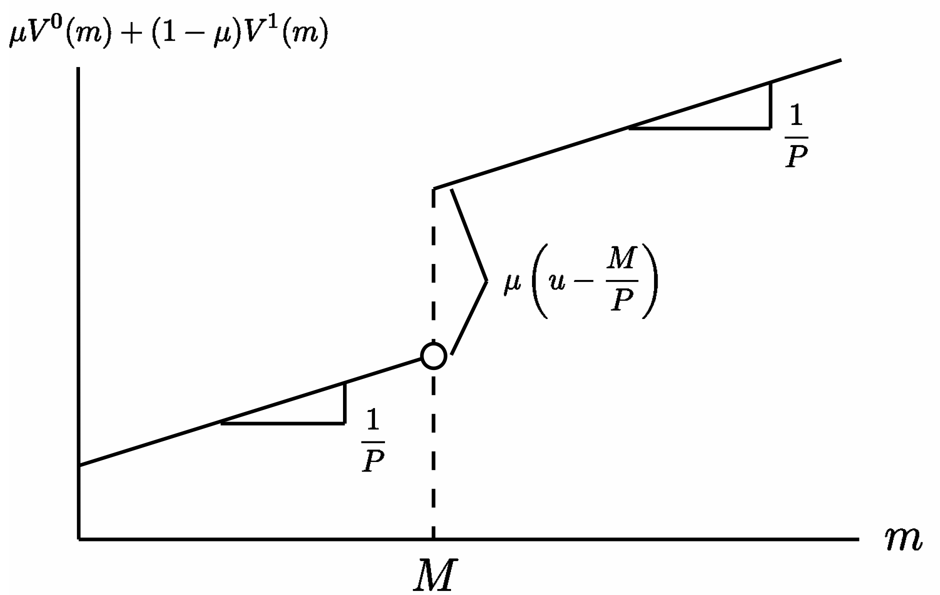

Figure 2: The ex-ante value function.

Because $V^1(m) > V^0(m)$, the sufficient condition for choosing to purchase whenever the durable good breaks down is

$$u \geq \frac{M}{P}. \tag{24}$$

Intuitively, the households choose to purchase if the utility from the durable good u is sufficiently larger than the real price of the indivisible durable good.

Finally, we check the condition for $m' = M$. Except at M, the value functions $V^0(m)$ and $V^1(m)$ are linear and have a common coefficient $\frac{1}{P}$ on m (see Figure 2). Moreover, $V^0(m)$ has a jump at M because holding M or more lets the household purchase the indivisible durable good, which is sold at the price of M. From equations (19) and (20), the objective function of the maximization problem with respect to m' is

$$F(m') \equiv -\frac{m'}{P} + \beta[\mu V^0(m') + (1-\mu)V^1(m')], \tag{25}$$

(see Figure 3). Because $\mu V^0(m') + (1-\mu)V^1(m')$ is linear with the coefficient $1/P$ on m', F is also linear and has the negative coefficient $-(1-\beta)/P$, except at M. Because F has a jump at M, the household chooses $m' = 0$ or $m' = M$. The household chooses $m' = M$ if u is sufficiently large. More precisely, a sufficient condition for saving M when the household has the indivisible durable good at the beginning of the day market is

$$-\frac{M}{P} + \beta[\mu V^0(M) + (1-\mu)V^1(M)] \geq \beta[\mu V^0(0) + (1-\mu)V^1(0)]. \tag{26}$$

A sufficient condition when the household has no good is

$$-\frac{M}{P} + \beta V^0(M) \geq \beta V^0(0). \tag{27}$$

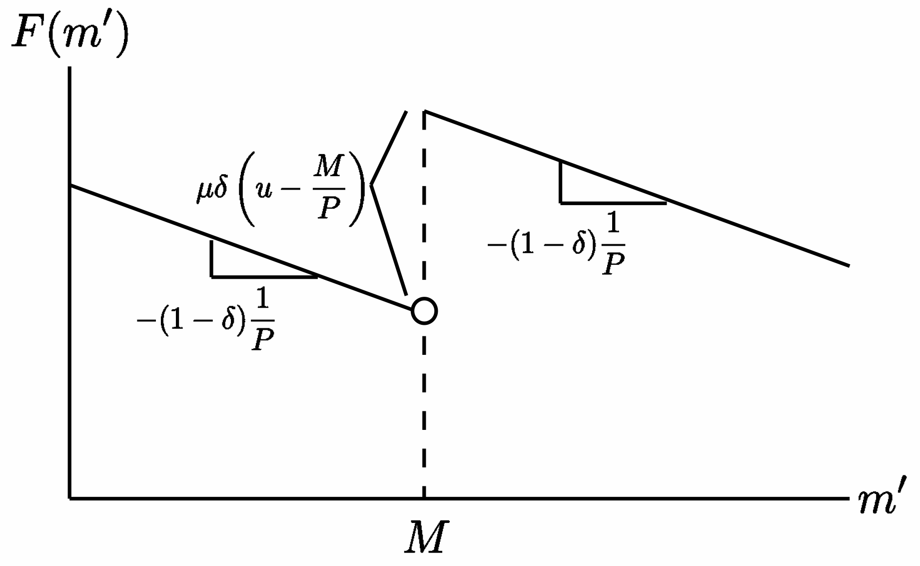

Figure 3: The objective function with respect to saving.

A sufficient condition for (26) and (27) to hold is[14]

$$\frac{M}{P} \leq \frac{\beta\mu u}{1-(1+\beta^2)(1-\mu)}. \tag{28}$$

In summary, if the inequalities (22), (24) and (28) are satisfied, then there exists an equilibrium with the four features listed above. The equilibrium is similar to that of the first model. That is, because (i) all households have an incentive to save M units of money to purchase the durable good and (ii) the money market clears if P satisfies the inequalities above, P is indeterminate. In the equilibrium, the total amount of labor supply of the households with the durable good is

$$H+L = X^* - \frac{\Pi}{P}, \tag{29}$$

That of the households without the durable good is

$$H+L = X^* + \frac{M-\Pi}{P}. \tag{30}$$

[14] We need only to show the inequality (26) because (27) can be derived from (26). Indeed, assuming (24) a household with $V^0(M)$ buys the durable good if

$$\begin{aligned}
&\Leftrightarrow \ \beta\left\{\mu[V^0(M)-V^0(0)]+(1-\mu)[V^1(M)-V^1(0)]\right\} \geq \frac{M}{P} \\
&\Leftrightarrow \ \beta\left\{\mu\left[u+\beta[\mu V^0(M)+(1-\mu)V^1(M)]-\beta V^0(M)\right]+(1-\mu)[V^1(M)-V^1(0)]\right\} \geq \frac{M}{P} \\
&\Leftrightarrow \ \beta\mu u+\beta^2(1-\mu)[V^1(M)-V^0(M)]+(1-\mu)[V^1(M)-V^1(0)] \geq \frac{M}{P} \\
&\Leftrightarrow \ \beta\mu u+\beta^2(1-\mu)\frac{M}{P}+(1-\mu)\frac{M}{P} \geq \frac{M}{P} \\
&\Leftrightarrow \ \frac{M}{P} \leq \frac{\beta\mu u}{1-(1+\beta^2)(1-\mu)}.
\end{aligned}$$

Therefore, the indeterminacy of P also induces the indeterminacy of the labor supply, i.e., and thereby makes the indeterminacy real. The following Proposition summarizes the result.

Proposition 1. *If* $1 < D \equiv u \cdot \min\left\{1, \frac{\beta\mu}{1-(1+\beta^2)(1-\mu)}\right\}$*, then there exists real indeterminacy of stationary equilibrium. The set of nominal equilibrium prices is defined as* $1 \leq \frac{M}{P} \leq D$.

To conclude, real indeterminacy can occur in an economy with centralized markets and no decentralized market. The monopoly pricing and the saving behavior of the households creates a similar coordination failure as described in Jean et al. (2010). Consequently, a decentralized market is not necessary for the existence of this type of real indeterminacy.

2.4 Generalizations of Model 3

The model with the monopoly firm above might not seem realistic because of some restrictive assumptions. For example, the good must be produced by the monopoly firm, it must be indivisible, and it cannot be purchased using credit. However, we show below that the real indeterminacy remains even if these assumptions are relaxed.

Cournot Competition

The real indeterminacy remains even if we change the setup from a monopolistic firm to N firms competing in Cournot fashion.[15] We adopt the same model environment and the firm's production technology as in the monopoly model. We show below that households optimally behave in the same way as they do in the monopoly model. For illustration of this fact, we assume that all households have exactly the same demand function of the indivisible

[15] In models where the firm produces the good using households' labor, the firm must have some market power to attain the indeterminacy result. We find that if the firm is a price-taker, then the equilibrium is unique. To illustrate this point, if the firm takes the price as given and solves the following problem of

$$\max_{y} \quad py - wy,$$

then the equilibrium price is given uniquely as $p = w$, $w = P$ (for households to work at the firm), and $p = M$ (for the money market at the centralized market to clear). However, in some models where the sellers are *endowed* with the good, the equilibrium can be indeterminate even when the sellers take a price as given. To illustrate this point, presume that among the $1 - \mu$ fraction of households who kept their durable indivisible good, fraction μ suddenly endowed with another one unit of durable indivisible good. One can further assume that the second unit is useless to them because of capacity constraints. Therefore, those with two indivisible goods want to sell one at the market. They do so as long as p is non-negative. Under this modified setup, all households still save p under some range of the relative price of money in preparation for purchasing one. The price of the good is pinned down by the money market clearing condition in the centralized market, which is $p = M$. Consequently, the price of money is again indeterminate. Lastly, if buyers have the entire market power and set the price, then the money market will not clear. For example, presuming that sellers and buyers meet randomly in the decentralized market, and assuming that they trade the indivisible durable good according to the chosen price by the buyer, then a buyer determines the price to make the seller indifferent between producing one or not. In this case, the price is not M in general; the money market will not clear.

durable good as in the monopoly model. Because the fraction μ of households lose the good,

$$y(p) = \begin{cases} 0 & \text{if } p > M, \\ \mu & \text{if } p \leq M. \end{cases} \tag{31}$$

Consequently, the inverse demand function is the following:[16]

$$p(y) = \begin{cases} [M, \infty) & \text{if } y = 0, \\ M & \text{if } y \in (0, \mu), \\ [0, M] & \text{if } y = \mu, \\ 0 & \text{if } y > \mu. \end{cases} \tag{32}$$

Then, we can show that the total amount of production is μ in equilibrium. Let y_i denote the output of firm i. Without loss of generality, we specifically examine firm 1. Presuming that $y_{other} \equiv y_2 + \cdots + y_N < \mu$ is given, then if $y_1 > \mu - y_{other}$, the supply exceeds the demand so that the price p jumps down to 0. Consequently, the profit of firm 1 becomes zero. Therefore, the firm has no incentive to produce. If $y_1 < \mu - y_{other}$, then the demand exceeds the supply so that the price rises until it reaches M, which is the total money that the households hold. Under this situation, firm 1 enjoys a positive profit, so firm 1 chooses to produce $y_1 = \mu - y_{other}$, which contradicts the case assumption. Consequently, the only equilibrium is $y_1 = \mu - y_{other}$, and the total output is $\sum_{i=1}^{N} y_i = \mu$. In sum, under Cournot competition, given the same demand function, the firms provide in all of μ units of indivisible durable good with price $p = M$. Given the price of $p = M$, it is optimal for the household to behave in a way that induces $y(p)$. Consequently, the same strategic complementarity between households and firms exists as in the case of households and the monopoly firm, which leads to the real indeterminacy.

Divisible Durable Good

The real indeterminacy remains even if we change the setup from an indivisible durable good to a divisible durable good as long as the utility from consuming that durable good saturates at some sufficiently large amount. The saturation commonly occurs in the consumption of durable goods. For example, the utility gain from purchasing an additional car is most likely to be very small when one already owns more than three cars. Assume that each household can choose any non-negative amount of durable good I_t, and the utility from it is uI_t if $I_t \in [0, 1]$ and u if $I_t > 1$. If u is sufficiently large, then we can obtain the same result as that in the third model, where each household chooses $I_t = 1$.

Allowing the Households to Buy on Credit

The real indeterminacy remains even if we allow households to buy the indivisible durable good on credit.[17] Presume

[16] We assume that the price p increases as long as there is excess demand (i.e., $y < \mu$).

[17] If households are allowed to purchase completely on credit (with no partial payment by cash that they carried over from the last period), they will use their labor income in the current period to purchase the good and no household will bring over money. This violates the money market clearing condition in the centralized market, and there will be no equilibrium.

that all households face an ad-hoc borrowing constraint[18] with a borrowing limit $\underline{m}$. Assume also that the utility from consuming the durable good u is sufficiently high so that households have an incentive to buy the durable good using all the money they can afford. Then, the same equilibrium characteristics as in the third model hold, except for the price of the indivisible durable good, which is now $p = M + \underline{m}$. If the monopolist expects that the households bring M units of money, then the firm optimally charges $M + \underline{m}$ to exploit all the money households can pay. However, if households think that the monopoly firm charges $M + \underline{m}$, then they will carry over M and borrow until $\underline{m}$. Consequently, the money market clears and the equilibrium is indeterminate.

The real indeterminacy remains even if households are permitted to buy the indivisible durable good on credit from the monopoly firm, as long as the firm charges a portion of the price to be paid up front using cash. This type of requirement is apparent in the real world, such as a down payment for renting/purchasing a house. Under this rule, households have an incentive to carry over the required portion of the price to be paid up front using money, and the equilibrium price is indeterminate.

3 Discussion

In this section, we first explain the common features in the previous models which induce the real indeterminacy. Then we discuss how these features lead to the real indeterminacy of equilibrium prices. We also show that the real indeterminacy occurs even under a general utility that makes the real allocation and social welfare depend on equilibrium prices. Finally, we analyze a policy intervention that uniquely attains a desirable equilibrium.

3.1 The Logic of Real Indeterminacy

There are two key features satisfied in all the models in Section 2 that lead to real indeterminacy of stationary equilibria.

1. *Indeterminacy:* Each household optimally saves a constant amount of money at the night market under any price of the general good within some range. This leads to indeterminacy of equilibria because the amount of money savings clears the money market for some range of prices for the general good and the general good market clears by Walras' law.

2. *Real:* Budget constraints differ across households. At the beginning of the night market, there exist at least two types of households facing different budget constraints. Because all households save the same amount of money, the heterogeneity in budget constraints induces some households to supply more labor while other households to supply less labor when the general good price changes.

Below, we extend the fixed-payment model (Model 1) in Section 2.1 to a general setting and show that the conditions above are sufficient to induce real indeterminacy of the stationary equilibria in a general framework. Particularly we

[18] It is a common assumption placed in incomplete market literature in macroeconomics. See, e.g., Chapter 18 of Ljungqvist and Sargent (2012).

use a general utility function instead of quasi-linear utility. Here, we only specifically examine Model 1 for illustrative purposes. We can also extend other models in Section 2 and show sufficiency of the above conditions in a similar fashion.

The basic environment is the same as that presented in Section 2.1: each household must pay M units of money to the government at the beginning of the day market and trade goods in the centralized market at night. There are N types of households with equal fraction $1/N$. A type i households receive π_t^i units of money satisfying $\frac{1}{N}\sum_{i=1}^N \pi_t^i = M$. There are L types of goods. In the night market in period t, a type i household receives an endowment $\omega_t^i \equiv (\omega_{t1}^i, \ldots, \omega_{tL}^i) \in \mathbb{R}_+^L$. A type i household has a general utility function $U^i : \mathbb{R}_+^L \to \mathbb{R}$. We assume that U^i is strictly increasing, differentiable, and strictly concave.

At the beginning of the night market, a type i household has

$$W_t^i = m_t^i - M + \pi_t^i \tag{33}$$

units of money, where m_t^i denotes the amount of money carried over from the last period, M denotes the amount paid to the government in the day market, and π_t^i denotes the amount of transfer received at the end of the day market. Then a type i household maximizes the discounted sum of a utility stream as shown below.

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta^t U^i(X_t^i) \\ &\text{s.t.} \quad P_t \cdot (X_t^i - \omega_t^i) + m_{t+1}^i \leq W_t^i, \quad m_{t+1}^i \geq M, \end{aligned} \tag{34}$$

where $\beta \in (0,1)$ is a discount factor, $X_t^i \equiv (X_{t1}^i, \ldots, X_{tL}^i) \in \mathbb{R}_+^L$ is the consumption vector of the type i household, $P_t \equiv (P_{t1}, \ldots, P_{tL}) \in \mathbb{R}_+^L$ is the price vector in period t, and m_{t+1}^i is the amount of savings.

We first show that $m_{t+1}^i \geq M$ binds in stationary equilibria under standard assumptions: all households save M irrespective of their type. Intuitively, because a household discounts future utility, it is better for the household to carry no more than the required amount.

Let λ_t and ϕ_t respectively denote the Lagrange multipliers of the first and second inequalities. The first-order conditions are those shown below.

$$\frac{\partial U^i}{\partial X_{t\ell}^i} - \lambda_t P_{t\ell} = 0 \quad \forall i, \tag{35}$$

$$-\lambda_t + \phi_t + \beta\lambda_{t+1} = 0, \tag{36}$$

$$\lambda_t[W_t^i - P_t \cdot (X_t^i - \omega_t^i) - m_{t+1}^i] = 0, \tag{37}$$

$$\phi_t(m_{t+1}^i - M) = 0, \tag{38}$$

$$\lambda_t \geq 0, \quad \phi_t \geq 0 \quad \forall t.$$

Below, we specialize the analysis to stationary equilibria and omit subscript t. We first show that $m^i = M$ holds at stationary equilibria. To the contrary, assume $m^i > M$. Then $\phi = 0$ follows from (38). Consequently, by (36), $\lambda = 0$ holds. Then, by (35), $\partial U^i / \partial X^i = 0$. This result contradicts the assumption that U is strictly increasing. Therefore, $m^i = M$ holds at stationary equilibria.

The equations below follow from the above first-order conditions:

$$\frac{\partial U^i}{\partial X_\ell^i} = \frac{P_\ell}{P_1}\frac{\partial U^i}{\partial X_1^i}, \quad \forall \ell = 2, \ldots, L, \tag{39}$$

$$P \cdot X^i = W^i - m^i + P \cdot \omega^i, \tag{40}$$

$$m^i = M. \tag{41}$$

As shown above, the money demand $m^i(P)$ is always equal to M. For a given P, the demand for divisible goods $X^i(P) = (X_1^i(P), \ldots, X_L^i(P))$ is determined uniquely by equations (39) and (40) and the strict concavity of U.

We are ready to show that the equilibrium is indeterminate. The market clearing conditions for the divisible goods and money are presented below.

$$\frac{1}{N}\sum_{i=1}^{N} X_\ell^i(P) = \frac{1}{N}\sum_{i=1}^{N} \omega_\ell^i, \qquad \forall \ell = 1, \ldots, L, \tag{42}$$

$$\frac{1}{N}\sum_{i=1}^{N} m^i(P) = M, \tag{43}$$

The system of equations contains $L+1$ equations and L unknown variables $(P_1, \ldots, P_L)$. By Walras' law, one equation is redundant. Moreover, as we have shown, $m^i(P) = M$ holds for all P. Therefore, equation (43) is independent from P. Then, there exist $L-1$ linearly independent equations and L unknowns. As a result, the system has one degree of freedom. The equilibrium price vector P is indeterminate.

It is worthwhile to contrast our model with the standard dynamic general equilibrium models with cash-in-advance constraints. In such models, the demand for goods $X^i(P)$ is homogeneous of degree zero and the demand for money $m^i(P)$ is homogeneous of degree one. By Walras' law, the number of independent equations is equal to that of unknowns. Unlike such models, (43) is not an equation but an identity in our case. Therefore, the number of equations is one less than that of unknowns. Moreover, a proportional change in P affects the equilibrium consumptions because constant M in the budget constraint makes $X^i(P)$ not homogeneous of degree zero.[19]

Next, we show that the above indeterminacy is real if different types of households face different budget constraints. To illustrate how the heterogeneous budget constraints induce real indeterminacy, we simply assume that $W_i \neq W_j$ for some i, j. In stationary equilibria, $m^i = M$ holds and the budget constraint of household i is

$$P \cdot (X^i - \omega^i) + M = W^i. \tag{44}$$

Because W^i differs among some households, the change in P results in varying effects on the households' consumption X^i. Therefore, the indeterminacy is real.

Finally, under a general utility function, social welfare differs depending on the equilibrium price P. Recall that in section 2, the social welfare is the same across P because the household utility function is assumed to be quasi-linear. For simplicity, we consider a case in which all households have the same utility function, i.e., $U^i = U$, and the same initial endowments, i.e., $\omega^i = \omega$. Furthermore, for simplicity, we presume that there is only one type of good ($L = 1$).

[19] Even in the case of an economy with stochastic shocks such as durable good shocks, as long as the equilibrium conditions are similar to (42) and (43), and $m(P) = M$ holds for all P, equation (43) holds independent of P. Therefore, the equilibrium price becomes indeterminate.

Rearranging the budget constraint, the consumption of household i is

$$X^i = \frac{W^i - M}{P} + \omega. \tag{45}$$

Without loss of generality, we assume that $W^1 \leq W^2 \leq \ldots \leq W^N$ and $W^1 < W^N$. Because $\frac{1}{N}\sum_{i=1}^N X^i = \omega$, there exists some integer $1 \leq n < N$ such that $X^i \leq \omega$ and $W^i - M \leq 0$ for $i \leq n$ and $X^i > \omega$ and $W^i - M > 0$ for $i > n$.

We define social welfare as shown below.

$$\text{Welfare} = \frac{1}{1-\beta}\sum_{i=1}^N U(X^i) = \frac{1}{1-\beta}\sum_{i=1}^N U\left(\frac{W^i - M}{P} + \omega\right). \tag{46}$$

Then, the welfare is strictly increasing in P. Indeed,

$$\begin{aligned}
\frac{\partial \text{ Welfare}}{\partial P} &= \frac{1}{1-\beta}\sum_{i=1}^N \left[-\frac{W^i - M}{P^2}U'\left(\frac{W^i - M}{P} + \omega\right)\right] \\
&= \frac{1}{P^2(1-\beta)}\sum_{i=1}^n \left[(M - W^i)U'\left(X^i\right)\right] - \frac{1}{P^2(1-\beta)}\sum_{i=n+1}^N \left[(W^i - M)U'\left(X^i\right)\right] \\
&> \frac{1}{P^2(1-\beta)}\sum_{i=1}^n \left[(M - W^i)U'\left(X^n\right)\right] - \frac{1}{P^2(1-\beta)}\sum_{i=n+1}^N \left[(W^i - M)U'\left(X^n\right)\right] \\
&= \frac{U'(X^n)}{P(1-\beta)}\left[\sum_{i=1}^n \left(\frac{M - W^i}{P}\right) - \sum_{i=n+1}^N \left(\frac{W^i - M}{P}\right)\right] \\
&= \frac{U'(X^n)}{P(1-\beta)}\sum_{i=1}^N (\omega - X^i) = 0,
\end{aligned} \tag{47}$$

where U' denotes the derivative of U with respect to X. Intuitively, the increase in P diminishes the inequality in consumption among households. That is, an increase in P reduces the consumption of household i if $X^i > \omega$, and increases it for other households. Because each household's utility function is strictly concave, this increases the utilitarian social welfare. In the next section, we present a possible government intervention that leads the economy to the welfare maximizing equilibrium.

3.2 Policy Intervention and Equilibrium Selection

In general, the real indeterminacy of stationary equilibria results in welfare to differ across equilibria. Consequently, in this section, we introduce a policy intervention to achieve a desirable equilibrium. We continue to consider the simple model with N types of households and one divisible consumption good as in the previous section. We propose the following policy:

- The government subsidizes $\varepsilon > 0$ units of money to type 1 households.[20]

[20] It is crucially important to assume that the government subsidizes some but not all households. For example, in the monopoly firm's model in Section 2.3, if the government distributes ε units of money to all households, then the firm increases the price to $M + \varepsilon$, which result in indeterminacy. In contrast, under our proposed policy, only one household has $M + \varepsilon$ units of money. Therefore, the monopoly firm has no incentive to raise the price.

- The government announces that it will sell one unit of consumption good in exchange for Q units of money, but not vice versa (i.e., the government will not give Q units of money in exchange for a unit of a good).[21],[22]
- The government levies a tax of τ units of consumption good to type 1 household to sell the consumption good in exchange for money.

Under this policy, each household chooses to purchase the consumption good either from the government at price Q or from the centralized market at price P. Let G_t^i and X_t^i respectively denote the demand for the consumption good of household i in period t from the government and the centralized market. Then the utility maximization problems are

$$\max_{X_t^1, G_t^1, m_{t+1}^1} \sum_{t=0}^{\infty} \beta^t U(X_t^1 + G_t^1) \tag{48}$$
$$\text{s.t. } P_t(X_t^1 - \omega) + Q_t G_t^1 + m_{t+1}^1 + P_t\tau \leq W_t^1 + \varepsilon, \quad m_{t+1}^1 \geq M$$

for type 1 household, and

$$\max_{X_t^i, G_t^i, m_{t+1}^i} \sum_{t=0}^{\infty} \beta^t U(X_t^i + G_t^i) \tag{49}$$
$$\text{s.t. } P_t(X_t^i - \omega) + Q_t G_t^i + m_{t+1}^i \leq W_t^i, \quad m_{t+1}^i \geq M$$

for type $i \neq 1$ household. We assume that all households have the same initial endowment ω. Presume government budget is balanced each period, i.e., $\sum_{i=1}^N G_t^i = \tau/N \ \forall t$.

Let $X^{i*}(P)$ denote the demand of household i of the consumption good without policy intervention under the market price P. The Proposition below shows that $P = Q$ is the unique equilibrium price. The implication of the Proposition is the following. Presuming that the government seeks to implement an allocation $\{X^{i*}(P^*)\}_{i=1}^N$, which is the equilibrium allocation under the equilibrium price P^* without policy intervention, then, by setting $Q = P^*$ and $\tau = \frac{\varepsilon}{P^*}$, the amount of consumption for each household becomes equal to the amount of consumption under the market price P^* without policy intervention. It is noteworthy that the result still holds if $\varepsilon \to 0$. Therefore, the government can implement the policy at almost no cost.

Proposition 2: (i) The stationary equilibrium price is uniquely pinned down as $P = Q$. (ii) Under $Q = P^*$ and $\tau = \frac{\varepsilon}{P^*}$, the equilibrium consumption (X^i, G^i) satisfies $X^i + G^i = X^{i*}(P^*)$ for all i.

Proof. We prove (i) by consider three cases (i.e., $P > Q$, $P < Q$ and $P = Q$) and show that only $P = Q$ is the stationary equilibrium. First, if $P > Q$, then the price offered by the government is lower than that of the market. Therefore, $X^i = 0$ for all i. Consequently, the demand for the consumption good is zero at the market. However, households with $W^i + \varepsilon \mathbf{1}_{\{i=1\}} < M$ (i.e., $PX^i + QG^i < P\omega^i$) sell the consumption good at the market to prepare

[21] Here, we assume for simplicity that the purchase is not restricted by cash-in-advance constraints. The real indeterminacy vanishes as long as there exists one household who can arbitrage its transactions between the government good and the market good. In the monopoly firm model in Section 2.3, the household who kept the durable good can play this role.

[22] This rule is similar to a currency conversion in gold standard era in the sense that the government guarantees that the money can be exchanged for a certain amount of a good. Sims (1994, Section IV) considers a similar policy in the context of fiscal theory of price level literature.

$m^i \geq M$ for the next period's fixed payment. For that reason, the goods market does not clear under $P > Q$. Second, if $P < Q$, then $G^i = 0$ for all i because the price offered by the government is higher than the market price. As shown in Section 3.1, the equilibrium money demand is $\frac{1}{N}\sum_{i=1}^N m^i = M$. However, the total money supply is $M + \varepsilon$. Therefore, the money market does not clear.

Finally, when $P = Q$, each household is indifferent between purchasing from the government and purchasing from the central market. Therefore, the demand correspondence (X^i, G^i) given $P = Q$ is

$$(X^1, G^1) \in \left\{ (X, G) \in \mathbb{R}_+^2 \;\middle|\; X + G = \frac{W^1 + \varepsilon - M}{P} - \tau + \omega \right\}. \tag{50}$$

for household 1 and

$$(X^i, G^i) \in \left\{ (X, G) \in \mathbb{R}_+^2 \;\middle|\; X + G = \frac{W^i - M}{P} + \omega \right\} \tag{51}$$

for household $i \neq 1$. The aggregate demand for money is

$$\sum_{i=1}^N (m^i + PG^i) = M + \sum_{i=1}^N PG^i. \tag{52}$$

Therefore, we can choose $G^1 \in \left[0, \frac{W^1+\varepsilon-M}{P} - \tau + \omega\right]$ and $G^i \in \left[0, \frac{W^i-M}{P} + \omega\right]$ for $i \neq 1$ such that $\sum_{i=1}^N G^i = \varepsilon$, and make money market clear. By Walras' law, the good market clears as well. Therefore, the unique stationary equilibrium price is $P = Q$.

The proof of (ii) is given as follows. Recall that without policy intervention, the utility maximization problem of any household i is

$$\max_{X_t^i, m_{t+1}^i} \sum_{t=0}^\infty \beta^t U(X_t^i) \tag{53}$$
$$\text{s.t. } P_t(X_t^i - \omega) + m_{t+1}^i \leq W_t^i, \quad m_{t+1}^i \geq M.$$

Consequently,

$$P^*(X^{i*}(P^*) - \omega) + M = W^i \quad \forall i. \tag{54}$$

Under the policy $Q = P^*$, the equilibrium market price is $P = P^*$, as shown in (i), so that the equilibrium allocation (X^i, G^i) for $i \neq 1$ must satisfy the budget constraint

$$P(X^i - \omega) + QG^i + M = W^i \iff P^*(X^i - \omega) + P^*G^i + M = W^i. \tag{55}$$

By comparing with the budget constraint without policy intervention, it follows that $X^i + G^i = X^{i*}(P^*)$. For a type 1 household, the equilibrium allocation (X^1, G^1) must satisfy the budget constraint

$$\begin{aligned} & P(X^1 - \omega) + QG^1 + M + P\tau = W^1 + \varepsilon \\ \iff & P(X^1 - \omega) + QG^1 + M = W^1 \\ \iff & P^*(X^1 - \omega) + P^*G^1 + M = W^1. \end{aligned} \tag{56}$$

By comparing with the budget constraint without policy intervention, it follows that $X^1 + G^1 = X^{1*}(P^*)$. □

4 Conclusion

We presented a new model of a centralized market in which real indeterminacy of stationary equilibria arises. When it is optimal for the households to bring over a fixed amount of money to the next period and the budget constraints are heterogeneous among households, then the demand function of the divisible good violates homogeneity of degree zero in prices. However, at least one degree of freedom always exists for characterization of an equilibrium price for the divisible good. This creates the real indeterminacy of stationary equilibria. We showed the above type of real indeterminacy occurs in three models including Jean et al. (2010). By doing so, we presented that the real indeterminacy seen under the random search framework in Jean et al. (2010) might arise in a wide class of dynamic centralized market models. Finally, we considered a model with a general utility function and proposed a possible policy intervention to implement a desirable outcome.

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