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Non-holonomic geometric structures of rigid body system in Riemann-Cartan space

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A theory of non-Riemannian geometry (Riemann-Cartan geometry) can be applied to a free rotation of a rigid body system. The Euler equations of angular velocities are transformed into equations of the Euler angle. This transformation is geometrically non-holonomic, and the Riemann-Cartan structure is associated with the system of the Euler angles. Then, geometric objects such as torsion and curvature tensors are related to a singularity of the Euler angle. When a pitch angle becomes singular $\pm\pi/2$, components of the torsion tensor diverge for any shape of the rigid body while components of the curvature tensor do not diverge in case of a symmetric rigid body. Therefore, the torsion tensor is related to the singularity of dynamics of the rigid body rather than the curvature tensor. This means that the divergence of the torsion tensor is interpreted as the occurrence of the gimbal lock. Moreover, attitudes of the rigid body for the singular pitch angles $\pm\pi/2$ are distinguished by the condition that a path-dependence vector of the Euler angles diverges or converges.

1. Introduction

The interaction between theoretical physics and geometric concepts has a long history. The geometric approaches have been used to model gravitational phenomena in the general relativity (e.g. [1]). Moreover, the geometric structures can be found within a framework of a classical Newtonian mechanics. For example, a propagation of an elastic wave through anisotropic media is geometrically described in the Finsler space because of a direction-dependence of a crystal [2–6]. Similarly, a fluids flow followed by the Darcy's law through inhomogeneous porous media gives the Finslerian metric function because a hydraulic conductivity depends on a position by an inhomogeneity [7–9]. These studies imply that the geometric approaches will be applied to familiar phenomena followed by the Newtonian mechanics.

A rotational motion of rigid body is one of basic subject in the Newtonian mechanics [10–12]:

$$I_1 \frac{d\omega^1}{dt} = (I_2 - I_3)\omega^2\omega^3, I_2 \frac{d\omega^2}{dt} = (I_3 - I_1)\omega^3\omega^1, I_3 \frac{d\omega^3}{dt} = (I_1 - I_2)\omega^1\omega^2, \quad (1)$$

where I_1, I_2, I_3 are principal moments of inertia that define principal axes. $\omega^1, \omega^2, \omega^3$ are the angular velocities in a body fixed frame. The system of the rigid body is an example in the integrability problems [13, 14] and its applications in physics [15–17].

From a viewpoint of geometry, the system of the free rigid body can be discussed based on both the Lagrangian and the Hamiltonian formalisms [18–20]. For the Lagrangian on the Lie algebra \mathbf{R}^3 , the equations for the rigid body system are derived and have the Euler-Poincaré form:

$$\frac{d}{dt} \frac{\partial L}{\partial \boldsymbol{\omega}} = \frac{\partial L}{\partial \boldsymbol{\omega}} \times \boldsymbol{\omega}, \quad (2)$$

where $\boldsymbol{\omega}$ is the angular velocity vector, and the Lagrangian $L : \mathbf{R}^3 \rightarrow \mathbf{R}$ is

$$L(\boldsymbol{\omega}) = \frac{1}{2}(I_1(\omega^1)^2 + I_2(\omega^2)^2 + I_3(\omega^3)^2). \quad (3)$$

In contrast, there is also the Hamiltonian structure for the rigid body equations when the Poisson brackets are introduced [21–23]. The rigid body equations are written in terms of angular momenta

$$\mathbf{p} = (p_1, p_2, p_3) = (I_1\omega^1, I_2\omega^2, I_3\omega^3):$$

$$\frac{dp_1}{dt} = \frac{I_2 - I_3}{I_2 I_3} p_2 p_3, \quad \frac{dp_2}{dt} = \frac{I_3 - I_1}{I_3 I_1} p_3 p_1, \quad \frac{dp_3}{dt} = \frac{I_1 - I_2}{I_1 I_2} p_1 p_2. \quad (4)$$

Then, the rigid body equations (4) are equivalent to

$$\frac{dF}{dt} = \{F, H\}, \quad (5)$$

where F is any of p_1, p_2, p_3 . The Poisson bracket is defined by $\{F, H\}(\mathbf{p}) = -\mathbf{p} \cdot (\nabla F \times \nabla H)$, and the Hamiltonian H is given by

$$H(\mathbf{p}) = \frac{1}{2} \left(\frac{p_1^2}{I_1} + \frac{p_2^2}{I_2} + \frac{p_3^2}{I_3} \right). \quad (6)$$

The Hamiltonian formulation for the rigid body system is generalized into a construction on the dual space of the Lie algebra. Then, equations of motion are described by the Lie-Poisson equations.

In the Lagrangian and Hamilton formulations, the motion of the free rigid body is described by a system of first-order differential equations of angular velocities. Moreover, there is another approach to formulating the rigid body system based on the Euler angles. For example, an attitude of the rigid body can be discussed [24–26]. Since the angular velocity of rigid body is given by a differential of the Euler angles [10], the behavior of the free rigid body is described by a system of second-order differential equations in terms of the Euler angles. In this case, the configuration space for the rotational motion of the rigid body is $SO(3)$, and the velocity phase-space of the free rigid body is $TSO(3)$.

From a viewpoint of differential geometry, the system of second-order differential equations is defined on a tangent bundle, and then the Finslerian geometric objects can be obtained [27–30]. The geometry of second-order system has been applied to dynamical systems in ecology and physical fields, and the stability of system can be discussed by the geometric objects [5, 28–36]. Therefore, the motion of the rigid body system can also be discussed based on the geometric theory of second-order system. However, a coordinate transformation between the angular velocities ω^i and the Euler angles x^i is given by a complicated matrix [10]:

$$\omega^i = X_j^i(x^k) \frac{dx^j}{dt}. \quad (7)$$

This means that the second-order system of the rigid body gives different geometric structures from the results in previous studies [5, 27–36]. In this study, we formulate the geometry of the second-order system of rigid body according to the coordinate transformation (7). Then, the behavior of the rigid body is discussed based on the framework of the geometric objects obtained from the second-order system.

The structure of this paper is as follows. In section 2, a general relationship between the second-order system and the first-order system is shown. In this case, the second-order system is obtained from a non-holonomic transformation. Then, we obtain the geometric objects such as curvature and torsion tensors, and we show that the second-order system is defined on the Riemann-Cartan space as the non-Riemannian space. In section 3, we apply the geometric theory to the dynamical system of the rigid body and discuss an influence of the Riemann-Cartan geometric objects on a behavior of the rigid body. In section 4, we show a relation between a singularity of the Euler angles and the torsion tensor by using a concept of a topological charge. Section 5 is the conclusion. Throughout this paper, the Einstein's summation convention is used.

2. Non-holonomic relation between a system of first-order differential equations and of second-order differential equations

Let us consider a smooth n -dimensional manifold \tilde{M} with a local coordinate system (y^α) . We consider a general form of a system of first-order differential equations on \tilde{M} :

$$\frac{dy^\alpha}{dt} + \tilde{F}_{\beta\gamma}^\alpha y^\beta y^\gamma = 0, \quad (8)$$

where the coefficient $\tilde{F}_{\beta\gamma}^\alpha$ is constant. Here, we assume that $\tilde{F}_{\beta\gamma}^\alpha$ is not symmetric between β and γ ($\tilde{F}_{\beta\gamma}^\alpha \neq \tilde{F}_{\gamma\beta}^\alpha$).

Moreover, we consider another smooth n -dimensional manifold M with a local coordinate system (x^i) . A tangent bundle over M is denoted by TM with a local coordinate system (x^i, \dot{x}^i) , where \dot{x}^i is given by a differential of x^i with respect to a time parameter t : $\dot{x}^i \equiv dx^i/dt$. From a viewpoint of rigid body dynamics, y^α and x^i are regarded as angular velocities and the Euler angles, respectively. In this case, y^α is linked with \dot{x}^i through a matrix of the Euler angles [10]. Based on this physical framework, we consider the case that a relation between (y^α) and (x^i) is given by

$$y^\alpha = P_i^\alpha(x^j) \dot{x}^i, \quad (9)$$

where P_i^α is a transformation matrix. An inverse matrix of P_i^α is denoted by \tilde{P}_α^i .

In this study, we focus on a dynamical system defined on M induced from (8). By using (9), the equations of motion (8) are rewritten by a system of second-order differential equations for x^i :

$$\frac{d^2 x^i}{dt^2} + F_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} = 0, \quad (10)$$

where F_{jk}^i is given by

$$F_{jk}^i = \tilde{P}_\alpha^i P_j^\beta P_k^\gamma \tilde{F}_{\beta\gamma}^\alpha + \tilde{P}_\alpha^i \frac{\partial P_j^\alpha}{\partial x^k}. \quad (11)$$

As shown in the following, we focus on the geometric objects determined by the connection coefficients F_{jk}^i of the second-order system (10), and we do not consider a concrete representation of metric tensor.

The relation (11) satisfies a non-holonomic coordinate transformation rule between the connection coefficients $\tilde{F}_{\beta\gamma}^\alpha$ and F_{jk}^i . In this case, the following non-holonomic objects are introduced [37, 38]:

$$\Omega_{jk}^\alpha = \frac{\partial P_j^\alpha}{\partial x^k} - \frac{\partial P_k^\alpha}{\partial x^j}, \quad \omega_{h^\alpha jk} = \frac{\partial^2 P_h^\alpha}{\partial x^k \partial x^j} - \frac{\partial^2 P_h^\alpha}{\partial x^j \partial x^k}. \quad (12)$$

Ω_{jk}^α and $\omega_{h^\alpha jk}$ are related to non-integrability conditions [38–42]. If P_j^α is given by a certain multi-valued function ξ^α : $P_j^\alpha = \partial \xi^\alpha / \partial x^j$, the non-holonomic object Ω_{jk}^α is related to non-integrability condition:

$$\Omega_{jk}^\alpha = \left[\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j} \right] \xi^\alpha \neq 0, \quad (13)$$

where the bracket product is defined by

$$\left[\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j} \right] = \frac{\partial}{\partial x^k} \frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^k}. \quad (14)$$

Moreover, when P_h^α is multi-valued, $\omega_{h^\alpha jk}$ is also related to a non-integrability condition:

$$\omega_{h^\alpha jk} = \left[\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j} \right] P_h^\alpha \neq 0. \quad (15)$$

From (11), F_{jk}^i and $\tilde{F}_{\beta\gamma}^\alpha$ satisfy the relation between connection coefficients on M and \tilde{M} , respectively. Therefore, we consider the following covariant derivatives for arbitrary vector fields \tilde{u}^α on \tilde{M} and u^i on M :

$$\tilde{u}_{;\beta}^\alpha \equiv \frac{\partial \tilde{u}^\alpha}{\partial y^\beta} + \tilde{F}_{\eta\beta}^\alpha \tilde{u}^\eta, \quad u_{|j}^i \equiv \frac{\partial u^i}{\partial x^j} + F_{hj}^i u^h. \quad (16)$$

Then, geometric objects on \tilde{M} and M are obtained from the Ricci identity:

$$\tilde{u}_{;\beta;\gamma}^\alpha - \tilde{u}_{;\gamma;\beta}^\alpha = \tilde{R}_{\nu\beta\gamma}^\alpha \tilde{u}^\nu - \tilde{T}_{\beta\gamma}^\nu \tilde{u}_{;\nu}^\alpha, \quad u_{|j|k}^i - u_{|k|j}^i = R_{h^i jk} u^h - T_{jk}^h u_{|h}^i, \quad (17)$$

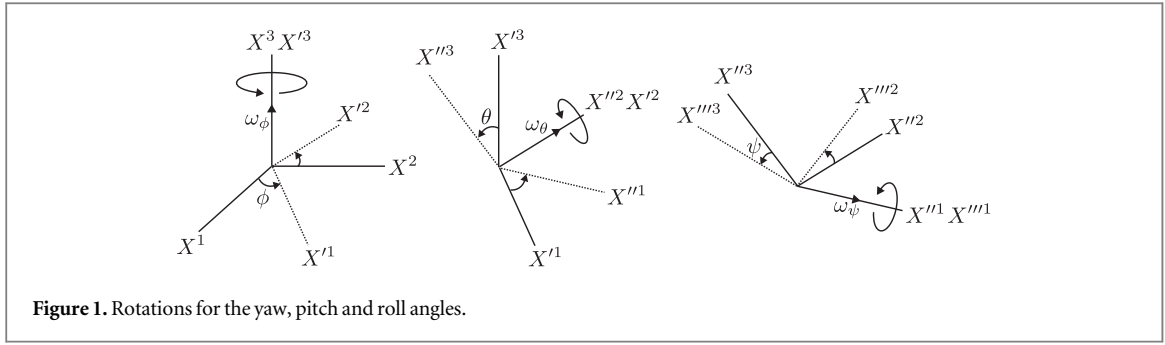
where $\tilde{R}_{\nu\beta\gamma}^\alpha$ and $R_{h^i jk}$ are the curvature tensor, and $\tilde{T}_{\beta\gamma}^\nu$ and T_{jk}^h are the torsion tensor:

$$\tilde{R}_{\nu\beta\gamma}^\alpha = \tilde{F}_{\nu\beta}^\xi \tilde{F}_{\xi\gamma}^\alpha - \tilde{F}_{\nu\gamma}^\xi \tilde{F}_{\xi\beta}^\alpha, \quad \tilde{T}_{\beta\gamma}^\nu = \tilde{F}_{\beta\gamma}^\nu - \tilde{F}_{\gamma\beta}^\nu, \quad (18)$$

$$R_{h^i jk} = \frac{\partial F_{hj}^i}{\partial x^k} - \frac{\partial F_{hk}^i}{\partial x^j} + F_{hj}^m F_{mk}^i - F_{hk}^m F_{mj}^i, \quad T_{jk}^h = F_{jk}^h - F_{kj}^h. \quad (19)$$

Since we assume $\tilde{F}_{\beta\gamma}^\alpha \neq \tilde{F}_{\gamma\beta}^\alpha$ and the existence of Ω_{jk}^α , the torsion tensor T_{jk}^h does not vanish generally. The non-vanishing torsion tensor means that the second-order system (10) is defined on the Riemann-Cartan space.

With respect to (11), the curvature and torsion tensors are divided into the asymmetric connection part $\tilde{F}_{\beta\gamma}^\alpha$ and the non-holonomic object part. The curvature tensor $R_{h^i jk}$ is expressed by



$$R_h^i{}_{jk} = S_h^i{}_{jk} + \omega_h^i{}_{jk}, \quad (20)$$

where we put

$$S_h^i{}_{jk} = \tilde{P}_\alpha^i P_h^\nu (\Omega_{jk}^\gamma \tilde{F}_{\nu\gamma}^\alpha + P_j^\beta P_k^\gamma \tilde{R}_{\nu}{}^\alpha{}_{\beta\gamma}), \quad \omega_h^i{}_{jk} = \tilde{P}_\alpha^i \omega_h^\alpha{}_{jk}. \quad (21)$$

Moreover, the torsion tensor T_{jk}^i is given by

$$T_{jk}^i = L_{jk}^i + \Omega_{jk}^i, \quad (22)$$

where we put

$$L_{jk}^i = \tilde{P}_\alpha^i P_j^\beta P_k^\gamma \tilde{T}_{\beta\gamma}^\alpha, \quad \Omega_{jk}^i = \tilde{P}_\alpha^i \Omega_{jk}^\alpha. \quad (23)$$

In the multivalued field theory [38], the connection coefficient $\tilde{F}_{\beta\gamma}^\alpha$ on \tilde{M} vanishes, and the geometric structures on M are discussed based on only $\omega_h^i{}_{jk}$ and Ω_{jk}^i . In contrast, two different properties given by the asymmetric connection coefficients and the non-holonomic objects give rise to the Riemann-Cartan geometric objects for the second-order system (10).

3. Applications of non-holonomic transformation to a dynamical system of Euler's rigid body

In this section, the Riemann-Cartan geometry of second-order system given by the non-holonomic transformation (9) is applied to a dynamical system of a free rigid body. The equations of motion expressed by a first-order system of angular velocity are rewritten by a second-order system of the Euler angles. Then, a behavior of the second-order system is discussed based on the torsion and curvature tensors.

3.1. Equations of motion of Euler's rigid body and Euler angles

Let us rewrite the equations of motion of the free rigid body for the angular velocity (1) into the equations of motion of the Euler angles. According to [10], we use the xyz -convention for the Euler angles (figure 1). A yaw, pitch and roll angles are denoted by $0 \leq \phi \leq 2\pi$, $-\pi/2 < \theta < \pi/2$ and $0 \leq \psi \leq 2\pi$, respectively. A fixed frame to a space is denoted by $X^1 X^2 X^3$. The first rotation is by the yaw angle ϕ about the X^3 -axis using a matrix D , and a frame after this rotation is denoted by $X'^1 X'^2 X'^3$. The second rotation is by the pitch angle θ about the X'^2 -axis using a matrix C , and a frame after this rotation is denoted by $X''^1 X''^2 X''^3$. The third rotation is by the roll angle ψ about the X''^1 -axis using a matrix B , and the frame after this rotation is denoted by $X'''^1 X'''^2 X'''^3$. The matrices B , C and D are given by

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & \sin \psi \\ 0 & -\sin \psi & \cos \psi \end{bmatrix}, \quad C = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix}, \quad (24)$$

$$D = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (25)$$

A transformation from the frame fixed to the space $X^1 X^2 X^3$ to the frame fixed to the rigid body $X'''^1 X'''^2 X'''^3$ is given by a matrix $A = BCD$. Here, time derivatives of the Euler angles $\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$ are directed to the X'^3 -axis, X''^2 -axis and X'''^1 -axis, respectively. The angular velocities about these axes are denoted by

$$\omega_\phi = \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}, \quad \omega_\theta = \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix}, \quad \omega_\psi = \begin{bmatrix} \dot{\psi} \\ 0 \\ 0 \end{bmatrix}. \quad (26)$$

By using A , B , C and D , (26) is transformed to the body fixed frame. Since ω_ϕ is directed to the X^3 -axis, the transformation is given by the matrix A :

$$A\omega_\phi = \begin{bmatrix} -\sin \theta \dot{\phi} \\ \cos \theta \sin \psi \dot{\phi} \\ \cos \theta \cos \psi \dot{\phi} \end{bmatrix}. \quad (27)$$

Then, since ω_θ is directed to the X'^2 -axis, the transformation is given by the matrix B :

$$B\omega_\theta = \begin{bmatrix} 0 \\ \cos \psi \dot{\theta} \\ -\sin \psi \dot{\theta} \end{bmatrix}. \quad (28)$$

Finally, ω_ψ is directed to the body fixed frame itself. In this case, the angular velocities $\omega^1, \omega^2, \omega^3$ are expressed by the summation of $A\omega_\phi, B\omega_\theta$ and ω_ψ :

$$\begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{bmatrix} = \begin{bmatrix} \dot{\psi} - \sin \theta \dot{\phi} \\ \cos \psi \dot{\theta} + \cos \theta \sin \psi \dot{\phi} \\ -\sin \psi \dot{\theta} + \cos \theta \cos \psi \dot{\phi} \end{bmatrix}. \quad (29)$$

Since the relation (29) corresponds to the relation (9), the approach of the Riemann-Cartan geometry can be applied to a dynamics of the Euler angles for the rigid body. Therefore, the Riemann-Cartan geometric structures can be obtained for the system of rigid body as a second-order system.

3.2. Equations of motion for Euler angles and its geometric interpretation

We introduce the following geometric definitions in order to discuss geometric structures of the rigid body dynamics. Let \tilde{M} be a manifold for the angular velocity, and the local coordinate system on \tilde{M} is $(y^\alpha) = (y^1, y^2, y^3) = (\omega^1, \omega^2, \omega^3)$. On the other hand, let M be a manifold of the Euler angles, and the local coordinate system on M is $(x^i) = (x^1, x^2, x^3) = (\psi, \theta, \phi)$.

Equations of motion for the angular velocity (1) are written in terms of (8) when $\tilde{F}_{\beta\gamma}^\alpha$ is expressed by the moments of inertia:

$$\tilde{F}_{\beta\gamma}^\alpha = \varepsilon_{\beta\gamma}^\alpha \frac{I_\gamma}{I_\alpha} \quad (\text{not sum between } \alpha \text{ and } \gamma), \quad (30)$$

where $\varepsilon_{\beta\gamma}^\alpha$ is a permutation symbol. $\tilde{F}_{\beta\gamma}^\alpha$ corresponds to a structure constant of the Lie algebra for the Euler-Poincaré form or the Lie-Poisson bracket for the Hamiltonian form [19, 20].

From (9) and (29), P_i^α and \tilde{P}_α^i are given by

$$P_i^\alpha = \begin{bmatrix} 1 & 0 & -\sin x^2 \\ 0 & \cos x^1 & \sin x^1 \cos x^2 \\ 0 & -\sin x^1 & \cos x^1 \cos x^2 \end{bmatrix}, \quad (31)$$

$$\tilde{P}_\alpha^i = \begin{bmatrix} 1 & \sin x^1 \tan x^2 & \cos x^1 \tan x^2 \\ 0 & \cos x^1 & -\sin x^1 \\ 0 & \sin x^1 \sec x^2 & \cos x^1 \sec x^2 \end{bmatrix}. \quad (32)$$

It is remarked that the independency of x^3 for P_i^α shows that F_{j3}^i is a tensor.

3.2.1. Torsion tensor and singularity of the Euler angles

The torsion tensor T_{jk}^i and the curvature tensor $R_h^i{}_{jk}$ for the Euler angles are given by the connection coefficients F_{jk}^i and $\tilde{F}_{\beta\gamma}^\alpha$. Since the moments of inertia are positive, L_{jk}^i does not vanish for the rigid body. Therefore, the torsion tensor is expressed by both the structure constant and the non-holonomic object. From (30), (31) and (32), non-zero components of the torsion tensor (22) are given by

$$T_{12}^1 = T_{23}^3 = (1 + \tilde{T}_{31}^2 \sin^2 x^1 + \tilde{T}_{12}^3 \cos^2 x^1) \tan x^2, \quad (33)$$

$$T_{13}^1 = T_{32}^2 = \frac{1}{2}(\tilde{T}_{12}^3 - \tilde{T}_{31}^2) \sin 2x^1 \sin x^2, \quad (34)$$

$$T_{23}^1 = \{1 + \tilde{T}_{23}^1 \cos^2 x^2 + (\tilde{T}_{31}^2 \sin^2 x^1 + \tilde{T}_{12}^3 \cos^2 x^1) \sin^2 x^2\} \sec x^2, \quad (35)$$

$$T_{12}^2 = T_{31}^3 = \frac{1}{2}(\tilde{T}_{31}^2 - \tilde{T}_{12}^3)\sin 2x^1, \quad (36)$$

$$T_{13}^2 = -(1 + \tilde{T}_{31}^2 \cos^2 x^1 + \tilde{T}_{12}^3 \sin^2 x^1)\cos x^2, \quad (37)$$

$$T_{12}^3 = (1 + \tilde{T}_{31}^2 \sin^2 x^1 + \tilde{T}_{12}^3 \cos^2 x^1)\sec x^2. \quad (38)$$

When $x^2 \rightarrow \pm\pi/2$, the Euler angle becomes singular because the x^1 -axis is coincide with the x^3 -axis [43]. This singularity is geometrically described by the divergence of the torsion tensor since T_{12}^1 , T_{23}^1 , T_{12}^3 and T_{23}^3 diverge for $x^2 \rightarrow \pm\pi/2$

The components of the torsion tensor T_{jk}^i represent the discontinuity of the Euler angles x^i with respect to $x^j x^k$ -plane. This means that T_{12}^1 and T_{23}^1 represent the discontinuity along x^1 -axis with respect to $x^1 x^2$ - and $x^2 x^3$ -planes while T_{12}^3 and T_{23}^3 represent the discontinuity along x^3 -axis with respect to $x^1 x^2$ - and $x^2 x^3$ -planes. Therefore, the singularity of the Euler angle due to the consistency of x^1 and x^3 -axes is geometrically interpreted as the divergence of discrepancy along x^1 and x^3 -axes. Moreover, from a viewpoint of rigid body motion, the singularity of the Euler angle is linked with a gimbal lock because of decreasing the number of degrees of freedom. The above relations between the torsion tensor and the Euler angles imply that the divergence of the torsion tensor is physically interpreted as the occurrence of the gimbal lock.

3.2.2. Curvature tensor and singularity of the Euler angles

Let us consider the curvature tensor (20) on the Euler angle space M . From the transformation matrices (31) and (32), $\omega_h^i{}_{jk} = 0$. Therefore, the curvature tensor is expressed by only the structure constants. As well as the torsion tensor, some components of the curvature tensor diverge when the Euler angle becomes singular $x^2 \rightarrow \pm\pi/2$, for example,

$$R_{112}^1 = \frac{1}{2}\{\tilde{F}_{13}^2(1 + \tilde{F}_{21}^3) + \tilde{F}_{12}^3(1 - \tilde{F}_{31}^2)\}\sin 2x^1 \tan x^2. \quad (39)$$

However, when a shape of the rigid body is symmetric ($I_1 = I_2 = I_3$), non-zero components of the curvature tensor are given by

$$R_{113}^1 = R_{123}^2 = R_{321}^2 = R_{331}^3 = -2\sin x^2, \quad (40)$$

$$R_{212}^1 = R_{313}^1 = R_{121}^2 = R_{323}^2 = R_{131}^3 = R_{232}^3 = 2. \quad (41)$$

In this case, the components of the curvature tensor do not diverge for the symmetric rigid body. In contrast, the components of the torsion tensor diverge for a general shape of rigid body with different moments of inertia. This means that the torsion tensor is more closely connected with the singularity of the Euler angle and the gimbal lock rather than the curvature tensor.

4. Discussion (topological singularity and torsion of the Euler angles)

The singular Euler angles and the gimbal lock are related to the torsion tensor as discussed in the previous section. In this case, the two kinds of singular angles $x^2 = \pm\pi/2$ imply that there are two cases of the gimbal lock. However, the difference between the gimbal locks is not clear by focusing on only the torsion tensor. In contrast, an existence of singularity is related to topological quantities, for example a topological charge in the quantum field theory [44, 45] and the Burgers vector in the continuum theory of defect [38, 40, 42]. Moreover, a singularity is geometrically described by non-Riemannian quantities. For example, a dislocation in crystals is related to the torsion tensor [46–50]. Then, a path-dependency of the topological singularity is linked with the torsion and curvature tensors through the integrability conditions (13) and (15) [51]. This relation implies that the two cases of the gimbal locks may be characterized by using the path-dependency of the Euler angles. In this section, we consider a topological aspect of the Euler angles and clear a role of the torsion tensor for the behavior of rigid body.

Let us consider a total discrepancy of the Euler angles b^i along a closed curve C on M . Similarly to the torsion tensor, b^i is comprised of not only the non-holonomic object but also the structure constant of the Euler-Poincaré form. The non-holonomic part b_{nh}^i is given by the Stokes's theorem:

$$b_{nh}^i = \oint_C dx^i = \oint_C \tilde{P}_\alpha^i y^\alpha dt = \frac{1}{2} \int_S \Omega_{jk}^i dx^j \wedge dx^k, \quad (42)$$

where S is an oriented smooth surface on M . Moreover, we consider a discrepancy b_{EP}^i given by the torsion of the Euler-Poincaré form:

$$b_{EP}^i = \frac{1}{2} \int_S L_{jk}^i dx^j \wedge dx^k. \quad (43)$$

Then, the total discrepancy of the Euler angles is

$$b^i = b_{\text{nh}}^i + b_{\text{EP}}^i = \frac{1}{2} \int_S \Omega_{jk}^i dx^j \wedge dx^k + \frac{1}{2} \int_S L_{jk}^i dx^j \wedge dx^k. \quad (44)$$

From (33)–(38), each components of b^i are given by

$$b^1 = \pi \left\{ \alpha \log \cos x^2 + 2(1 + \beta) \log \tan \left(\frac{x^2}{2} + \frac{\pi}{4} \right) + 2(\tilde{F}_{23}^1 - \beta) \sin x^2 + \frac{\gamma}{2} \cos 2x^1 \sin x^2 \right\}, \quad (45)$$

$$b^2 = -\frac{1}{2} \pi \left\{ 2\alpha x^1 \cos x^2 + \gamma \left(\frac{1}{2} \cos 2x^1 + 3 \sin 2x^1 \cos x^2 \right) \right\}, \quad (46)$$

$$b^3 = -\pi \left\{ \alpha \log \tan \left(\frac{x^2}{2} + \frac{\pi}{4} \right) + 2(1 + \beta) \log \cos x^2 - \frac{\gamma}{2} \cos 2x^2 \right\}, \quad (47)$$

where we put

$$\alpha \equiv 2 + \tilde{T}_{31}^2 + \tilde{T}_{12}^3, \beta \equiv \tilde{T}_{31}^2 \sin^2 x^1 + \tilde{T}_{12}^3 \cos^2 x^1, \gamma \equiv \tilde{T}_{31}^2 - \tilde{T}_{12}^3. \quad (48)$$

Especially, when a shape of the rigid body is symmetric ($I_1 = I_2 = I_3$), the total path-dependency vectors are written by

$$b^1 = -b^3 = 6\pi \log \left\{ \cos x^2 \tan \left(\frac{x^2}{2} + \frac{\pi}{4} \right) \right\}, \quad b^2 = -6\pi x^1 \cos x^2. \quad (49)$$

In the symmetric case, the path-dependency vectors are independent of the moment of inertia. b^1 and b^3 take a finite value when $x^2 \rightarrow \pi/2$:

$$\lim_{x^2 \rightarrow \pi/2} b^1 = \lim_{x^2 \rightarrow \pi/2} (-b^3) = 6\pi \log 2. \quad (50)$$

In contrast, b^1 and b^3 take an infinite value when $x^2 \rightarrow -\pi/2$:

$$\lim_{x^2 \rightarrow -\pi/2} b^1 = \lim_{x^2 \rightarrow -\pi/2} (-b^3) = -\infty. \quad (51)$$

From (50) and (51), two kinds of singular angles $\pm\pi/2$ give different results for the path-dependency vectors. As mentioned before, the rigid body is in the gimbal lock for these singular angles. Therefore, (50) and (51) show that the gimbal lock comprises different types in the sense of differential geometry. This result means that the attitudes of the rigid body with the singular angles can be quantitatively distinguished between the divergence and the convergence of the path-dependency vector b^i .

5. Conclusion

In this paper, the behavior of the free rigid body is discussed based on the non-Riemannian geometry. The system of the rigid body is usually expressed by the first-order system of angular velocity. We transform the first-order system of angular velocity into the second-order system of the Euler angles. Then, the coordinate transformation between the angular velocity and the Euler angles is non-holonomic, and the second-order system contains an asymmetric connection. This means that the system of Euler angles has the Riemann-Cartan structure. The torsion and curvature tensors as the Riemann-Cartan geometric objects for the system of Euler angles are expressed by the non-holonomic and asymmetric connection parts.

The geometric objects are related to the singularity of the Euler angles. The non-zero components of the torsion tensor always diverge for any moment of inertia while those of the curvature tensor do not diverge in case of the symmetric rigid body when the pitch angle becomes singular ($\pm\pi/2$). This means that the torsion tensor is more closely related to the singularity of the Euler angle. The divergence of the torsion tensor corresponds to the gimbal lock of the rigid body. Moreover, the different attitude of the rigid body for $\pm\pi/2$ is geometrically distinguished by either divergence or convergence of the total discrepancy of the torsion tensor.

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